On a Class of non-local Conformal Invariants in Asympotic hyperbolic setting

by

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Non-local operators

$$P_{2\gamma} f = (-\Delta)^{\gamma} + \cdots$$

γ, fractional no.

Outline

(1) On $\mathbb{R}^n$, $P_{2\gamma} f = (-\Delta)^{\gamma}$, $0 < \gamma < 1$

Caffarelli - Silvestre Extension Thm

(2). Conformal compact Einstein setting

Chang - Gonzalez $0 < \gamma \leq \frac{n}{2}$

(3). Geometric applications $0 < \gamma < 1$

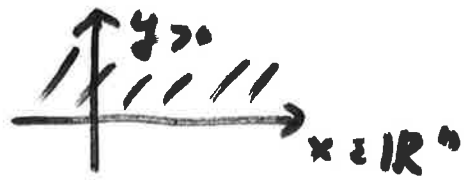
(4). C - S extension $\gamma > 1$

(5) Connection to Q - curvature, renormalized Volume.

§ Classical Setting (Caffarelli-Silvestre)

Well known result: f smooth on $\mathbb{R}^n$

$$(*)\begin{cases} \Delta u(x,y) = 0 \\ u|_{\mathbb{R}^n} = f \end{cases}$$



then $\quad -u_y(x,0) = (-\Delta)^{1/2} f(x)$

<u>Pf</u> Call $Tf = -u_y$, then

$$T(Tf) = u_{yy} = -\Delta_x u\big|_{\mathbb{R}^n} = (-\Delta_x) f$$

$*$

<u>Def</u> $$(-\Delta)^{\gamma} f = c_{n,\gamma} \int_{\mathbb{R}^n} \frac{f(x)-f(t)}{|x-t|^{n+2\gamma}}\, dt$$

then $0<\gamma<1$ $\quad \widehat{(-\Delta)^{\gamma} f} = |\xi|^{2\gamma} \hat{f}(\xi)$

<u>Thm</u> (Caffarelli-Silvestre) '06, $0<\gamma<1$

$$(*)\begin{cases} \operatorname{div}(y^a \nabla u) = 0 \quad \text{on } \mathbb{R}^{n+1} \\ \qquad a = 1-2\gamma \\ u|_{\mathbb{R}^n} = f \end{cases}$$

then.

$$(-\Delta)^{\gamma} f = c_{n,\gamma} \lim_{y\to 0} (y^a u_y) \quad \text{on } \mathbb{R}^n.$$

Actually $\quad 0<\gamma<1$

$$f \in \dot{H}^{\gamma}(\mathbb{R}^n) = \left(\dot{W}^{\gamma,2}(\mathbb{R}^n) \right)$$

then

$$\int_{y>0} |\nabla u|^2 \, y^a \, dx\, dy = \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\hat{f}(\xi)|^2 \, d\xi$$

$$= \int_{\mathbb{R}^n} (-\Delta)^{\gamma} f \cdot f \, dx$$

which implies $\quad (a = 1-2\gamma)$

$$(-\Delta)^{\gamma} f = C_{n,\gamma} \lim_{y \to 0} y^a \frac{\partial u}{\partial n}$$

$$= C_{n,\gamma} \lim_{y \to 0} y^a u_y$$

- Applications to free-boundary problems

  Study of fractional ~~mean-curvature~~ minimal surface ~~curvature~~ etc

§ 2. Conformal Compact Einstein setting

- A class of conformal covariant operators

$P_{2\gamma}$ exists for $0 < 2\gamma \leq n$ (n odd)

all $\gamma > 0$ (n even)

- $P_{2\gamma} = (-\Delta)^{\gamma}$ in special setting of $(H^{n+1}_{+}, \mathbb{R}^n)$

<u>Def</u> (X^{n+1}, M^n, g_+) $M = \partial X$

g_+ Poincare - Einstein $Ric\, g_+ = -n\, g_+$

if $r^2 g_+$ is A H for some distance function r

$r > 0$ on X

$r = 0$ on M

then $\exists\ \rho$ special defining function

$$\begin{cases} \rho > 0 \text{ on } X, \ \rho = 0 \text{ on } M \\ |\nabla_{\rho^2 g_+} \rho| = 1 \quad \text{on } M \times (0, \varepsilon) \end{cases}$$

$$g_+ = \frac{d\rho^2 + g_\rho}{\rho^2} \quad \text{in } M \times (0, \varepsilon)$$

$$\bar{g}| = \rho^2 g_+| = g_\rho|_{\rho=0} = g_0 \quad \text{on } M$$

M conformal infinity of X

Given $f \in C^\infty(M)$ ...

Consider

$$(*)' \qquad -\Delta_+ u - \overbrace{s(n-s)}^{\lambda} u = 0 \quad \text{on } X$$

- Mazzeo - Melrose

  Except for finite no. of pts λ, no pt spectrum for $\lambda \in [\frac{n^2}{4}, \infty)$.

- So for $\operatorname{Re} s > \frac{n}{2}$, except finite no. of s

  ρ^{n-s}, ρ^s are asympototic solution

- $$u = F\rho^{n-s} + H\rho^s, \quad F, G \in C^\infty(X)$$

  $$F = f + f_2(x)\rho^2 + f_4\rho^4 + \cdots$$

  $$F|_M = F|_{\rho=0} = f \qquad f_{2i} \in C^\infty(M)$$

  $$(\, u = \mathcal{P}_s f \quad \text{Poisson}\,)$$

- Define <u>Scattering matrix</u>

  $$\mathcal{S}(s) : C^\infty(M) \to C^\infty(M)$$

  $$f \to H|_M$$

- $s = \frac{n}{2} + \gamma \quad \gamma \notin \mathbb{Z}^+$

Define $P_{2\gamma} = S(\frac{n}{2} + \gamma)$

is a ΨDO of symbol $|\xi|^{2\gamma}$

- $s = \frac{n}{2} + k \quad S$ has a simple pole

Define $P_{2k} \doteq C_{n,k} \operatorname{Res}_{s=\frac{n}{2}+k} S(\frac{n}{2}+k)$

$k = 1, 2, \ldots$ is Differential operator of order $2k$.

- Conformal Covariant Property:

$$g_0 \to \hat{g}_0 = e^{2w} g_0$$

$$e^{\frac{n+2k}{2}w} P_{2k}(\hat{g}_0)(\phi) = P_{2k}(g_0)(e^{\frac{n-2k}{2}w}\phi)$$

$k = 1$: $\quad P_2(g) = (-\Delta)_g + \frac{d-2}{4(d-1)} R_g$

$\uparrow$ Scalar on (N^d, g)

conformal Laplace, (Yamabe operator)

$k = 2$ (Paneitz operator) '1983

$$P_4(g) = (-\Delta)_g^2 + \delta(a_d R g + b_d \operatorname{Ric}_g) d + \frac{d-4}{2} Q_g$$

In general: P_{2k} are GJMS '85 operators.

$$\begin{cases} 2k \le n & n \text{ odd} \\ \forall k & n \text{ even} \end{cases}$$

- $P_{2\gamma}$ also are conformal covariant.

$\forall\, \phi \in C^\infty(M)$, $w \in C^\infty(\bar{X})$

$$e^{\frac{n+2\gamma}{2}w} P_{2\gamma}(g_+, e^{2w} g_0)(\phi) = P_{2\gamma}(g_+, g_0)\left(e^{\frac{n-2\gamma}{2}w} \phi\right)$$

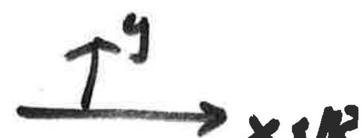
Normally write $e^{2w} g_0 = v^{\frac{4}{n-2\gamma}} g_0$

$$v^{\frac{n+2\gamma}{n-2\gamma}} P_{2\gamma}(g_+, v^{\frac{4}{n-2\gamma}} g_0) = P_{2\gamma}(g_+, g_0)(v\phi)$$

In the special case

$(\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_H)$ $\qquad g_H = \dfrac{dy^2 + dx^2}{y^2}$

$\bar{g} = y^2 g_H = dy^2 + dx^2$
flat



<u>Thm</u> (C + Gonzalez '11) . Given $f \in C^\infty(M)$

On (X^{n+1}, M^n, g_+) C.C.E. setting

$(*)'$ $\quad -\Delta_+ u - s(n-s) u = 0$ on X

$$s = \frac{n}{2} + \gamma$$

$\Updownarrow$

$(*)$ $\quad L_{\bar g} U + \rho^{-a} \nabla_{\bar g} U \cdot \nabla_{\bar g}(\rho^a) = 0$ on X

$$U = \rho^{s-n} u \qquad U| = f$$

$0 < \gamma \le \frac{n}{2}$ $\qquad$ ρ: totally geodesic defining function

And $\quad P_{2\gamma} f \doteq \Delta(\frac{n}{2}+\gamma) f$

$$= C_\gamma \lim_{\rho \to 0} \rho^{a_0} \partial_\rho (\underbrace{\rho^{-1}\partial_\rho \circ \rho^{-1}\partial_\rho \circ \cdots}_{m \text{ times}} \circ U)$$

$$m = [\gamma] \qquad a_0 = 1 - 2(\gamma - m)$$

In flat setting $\quad P_{2\gamma} f = (-\Delta)^\gamma$

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e.g. $\quad 0 < \gamma < 1 \quad P_\gamma f \doteq \rho^{1-2\gamma} \partial_\rho U \qquad \rho \to 0$

$\quad 1 < \gamma < 2 \quad P_\gamma f \doteq \rho^{3-2\gamma} \partial_\rho (\rho^{-1} \partial_\rho U), \ \rho \to 0$

§ Some Geometric Applications

(A). Study of (real) Scattering pole

- $S(\frac{n}{2}+\gamma)$ has simple pole at $\gamma \in \mathbb{Z}^+$

Consider $\hat{S}(\frac{n}{2}+\gamma) = 2^{2\gamma} \dfrac{\Gamma(\gamma)}{\Gamma(-\gamma)} S(\frac{n}{2}+\gamma)$

(Real) Scattering Pole, $s = \frac{n}{2}+\gamma$

Mazzeo-Melrose : $s \geq \frac{n}{2}$ no pole when $\mathrm{Re}\, s \geq \frac{n}{2}$

Ex • $X = H^{n+1}/\Gamma$ Γ convex, co-compact, torsion free

$\Omega(\Gamma) \subset S^n$ domain of discontinuity of Γ

$M = \Omega(\Gamma)/\Gamma$ locally conformally compact

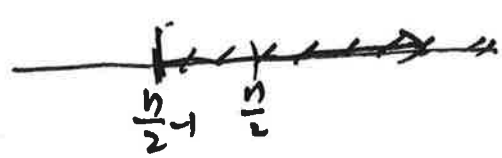
Schoen-Yau. If M is of positive scalar curvature

then $\delta(\Gamma) \doteq$ Hausdorff dim of $S^n \setminus \Omega(\Gamma)$

$$\leq \frac{n}{2} - 1.$$

- Suillvan, Patterson, $\delta(\Gamma) =$ Poincare exponent of Γ
- P. Perry $\delta(\Gamma) =$ the largest (real) scattering pole

Hence $(H^{n+1}/\Gamma,\ \Omega(\Gamma)/\Gamma,\ g_H)$



no pole $s \geq \frac{n}{2}-1$

<u>Thm</u> (J. Qing + Guillarmou) '10

(X^{n+1}, M^n, g^+) C. C. E. $n+1 > 3$

$Y(M^n, g_0) > 0$

$\Longleftrightarrow$ the first real scattering pole $\leq \frac{n}{2} - 1$.

- Equivalent statement

$$\underbrace{S\left(\tfrac{n}{2} - \gamma\right)}_{\text{not pole}} S\left(\tfrac{n}{2} + \gamma\right) = Id$$

not pole $\Longleftrightarrow$ not zero.

(Different) $(P_{2\gamma} f, f) \geq 0$, $0 < \gamma < 1$

<u>Proof</u> (Based on energy identity of C–S)

$Y(M^n_n, g_0) > 0$ on (X^{n+1}, M^n, g_+)

<u>J. Lee</u> '95 $\Downarrow$ no L^2 eigenvalues of Δ_+ on $\left(0, \frac{n^2}{4}\right)$

But his proof : Solves $\begin{cases} -\Delta_+ v = n+1 & \text{on } X \\ v = \frac{1}{\rho}(1 + c\rho^2 + \cdots) & \text{near } M \end{cases}$

$\Rightarrow$ $\tilde{g} = v^{-2} g_+$ is of positive scalar curvature on X.

$(*)'$ $\quad -\Delta_+ u - s(n-s)u = 0 \qquad s = \frac{n}{2}+\gamma$

$\updownarrow$

<u>Say $\gamma = \frac{1}{2}$</u> $\quad \tilde{g} = v^{-2} g_+$ satisfies $L_{\tilde{g}} U = 0$

for $U = v^{\frac{1}{2}} u$

$$0 = \int_X (L_{\tilde{g}} U) U \, dV_{\tilde{g}} = \int_X |\nabla_{\tilde{g}} \tilde{U}|^2 + c R_{\tilde{g}} U^2$$

$$- \underbrace{\int_M \frac{\partial \tilde{U}}{\partial n_{\tilde{g}}} U \, d\sigma_{g_0}}_{\displaystyle \int_M (P_1 f) f \, d\sigma}$$

So $R_{\tilde{g}} \geq 0 \Rightarrow \int (P_1 f) f \geq 0$

<u>In general</u> $\quad 0 < \gamma < 1$, Play with weights

$$\int (-\Delta_\varphi U) U e^{-\varphi} = \int |\nabla U|^2 e^{-\varphi} + \text{boundary term}$$

$$\Delta_\varphi = \Delta - \nabla\varphi \cdot \nabla$$

Take $\quad e^{-\varphi} = \rho^a \qquad\qquad a = 1-2\gamma$

etc.

(B) M. Gonzalez - J. Qing '12

"Fractional Yamabe Problem" on AH manifold

Solve $P_{2\gamma} f = Q_\gamma f^{\frac{n+2\gamma}{n-2\gamma}}$ on M

with $f > 0$, $Q_\gamma = \text{constant}$

e.g. When $\gamma = \frac{1}{2}$, $\left\{ \begin{array}{l} P_1 f = -\partial_\rho U + \frac{n-1}{2} H f \\ \text{where } L_{\bar{g}} U = 0, \quad U\big|_M = f \end{array} \right.$

$Q_{\frac{1}{2}} = \frac{n-1}{2} H$

One Key Step in proof: Sobolev-Trace inequality: $a = 1-2\gamma$

$$(**) \quad \left(\int_{\mathbb{R}^n} f^{\frac{2n}{n-2\gamma}} dx \right)^{\frac{n-2\gamma}{n}} \leq \bar{C}_\gamma \int_{\mathbb{R}^n} (-\Delta)^\gamma f \, f$$

$$\overset{C-S}{\leq} \bar{C}_\gamma \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 y^a \, dx \, dy$$

$$\uparrow$$

$\forall U$ with $U\big|_{\mathbb{R}^n} = f$

The trace-Sobolev inequality set the problem as a "boundary" variational problem

- Hopf boundary maximal principle for $P_{2\gamma}$
- Positive Mass Thm for $P_{2\gamma}$

§ 4. What happens when $\gamma > 1$

(why we are interested)

<u>Recall</u> $0 < \gamma < 1$ $\quad f \in \mathring{H}^{\gamma}$ $\quad a = 1 - 2\gamma$

$$\int_{\mathbb{R}^n} (P_\gamma f) f = \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\hat{f}(\xi)|^2 \underset{\substack{\uparrow \\ \text{fails}}}{=} \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 y^a$$

Recent Work of R. Yang

<u>Thm</u> (Special case $\gamma = \frac{3}{2}$)

$U \in W^{2,2}(\mathbb{R}^{n+1}_+)$, $f \in \mathring{H}^{\gamma}(\mathbb{R}^n)$

$$(**)' \begin{cases} \Delta^2 U(x,y) = 0 & \text{on } \mathbb{R}^{n+1}_+ \\ U(x,0) = f(x) & \text{on } \mathbb{R}^n \\ U_y(x,0) = 0 & \text{on } \mathbb{R}^n \end{cases}$$

then

① $$\int_{\mathbb{R}^n} |\xi|^{2\gamma} |\hat{f}(\xi)|^2 = C_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} (\Delta U)^2 \, dx\, dy$$

Hence $$(-\Delta)^{3/2} f = C_n \lim_{y \to 0} \frac{\partial}{\partial y} \Delta U(x,y)$$

② "Regularized energy"

$$\lim_{\varepsilon \to 0} \left(\int_{y \geq \varepsilon} |\nabla U|^2 y^{-2} - \frac{1}{\varepsilon} \int_{y=0} |\nabla_x U|^2 dx \right) = \frac{1}{2} \int_{y \geq 0} (\Delta U)^2$$

Thm (R. Yang) For all $0 < \gamma \le \frac{n}{2}$, similar result

For example $1 < \gamma < 2$, $f \in \mathring{H}^{\gamma}$

$$(**)\quad \begin{cases} \Delta_b^2 U = 0, & \text{where } \Delta_b U = \Delta U + \frac{b}{y} U_y \\ U(x,0) = f & \text{on } \mathbb{R}^n \end{cases}$$

$$\lim_{y\to 0} y^b U_y(x,y) = 0 \quad \text{on } \mathbb{R}^n$$

then $$\int |\xi|^{2\gamma} |\hat{f}(\xi)|^2 = \int_{y\ge 0} (\Delta_b U)^2 y^b \, dx\, dy$$

And $$(-\Delta)^{\gamma} f = C_{\gamma,n} \lim_{y\to 0} y^b (\Delta_b U)$$

Cor $1 < \gamma < 2$ $\quad \forall\, U$ with boundary constraint.

$$\left(\int_{\mathbb{R}^n} f^{\frac{2n}{n-2\gamma}} \right)^{\frac{n-2\gamma}{n}} \le C_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} y^b (\Delta_b U)^2$$

Remark:

A Hidden fact in the proof is that

($\gamma = \frac{3}{2}$ case)

$$(*) \quad \operatorname{div}(y^a \nabla U) = 0 \quad \text{on } \mathbb{R}^{n+1}_+ \quad \text{when } a = 1-2\gamma = -2$$

$$\Downarrow$$

$$\Delta^2 U = 0 \quad \text{on } \mathbb{R}^{n+1}_+$$

This fact turns out to be true also for C.C.E setting

i.e. when $s = \frac{n}{2} + \frac{3}{2}$

$$(*)' \quad -\Delta_+ u - s(n-s)u = -\Delta_+ u - \frac{n^2-9}{4} u = 0 \quad \text{on } X$$

$$\Downarrow$$

$$(P_4)_{g_+} u = \left(-\Delta_+ - \frac{n^2-9}{4}\right) \circ (P_2)_{g_+} u = 0$$

$$\text{Ric}\, g_+ = -n g_+$$

$$\Downarrow$$

$$(P_4)_{\bar{g}} U = 0 \qquad U = \rho^{3/2} u$$

§5. Other Reasons to study P_4, P_3 and Q_4, Q_3

Recall 2nd order:

Flat setting $(\mathbb{R}^{n+1}_+, \mathbb{R}^n)$	$(X^{n+1}, M^n = \partial X, g_+)$	curvature	
$-\Delta$	$L = -\Delta_g + c_n R$	R	scalar
$\frac{\partial}{\partial n}$	$B = -\frac{\partial}{\partial n} + c_n H$	H	mean curvature

In Conformal geometry $\quad g \mapsto \int_{X^d} R_g + \int_{\partial X} H$

$d=2$ $\quad g \mapsto \underbrace{\int_{X^2} K_g}_{\text{Gauss' curvature}} + \underbrace{\int_{\partial X} k_g}_{\text{geodesic curvature}}$

$d=4$ $\quad g \mapsto \int_{X^4} Q_4 + \int_{\partial X} Q_3$

Indeed

$d = 4 \qquad Q_4^{(d)} \doteq Q_4^{(4)} \overset{\text{denote}}{\doteq} Q_4$ (Branson)

$$= + \frac{1}{12} \left(\Delta R + R^2 - |Ric|^2 \right)$$

$$P_4^{(4)} = (-\Delta)^2 + \delta (a R^2 + b \, Ric \, g) + \frac{d-4}{2} \underset{\substack{\| \\ 0}}{Q_4^{(d)}} \quad d = 4$$

On (N^4, g) closed, compact

$$8\pi^2 \chi(N^4) = \int \underbrace{|W|_g^2}_{} dV_g + c \int Q_4 \, dV_g$$

pt wise conformal invariant
under $g \to \hat{g} = e^{2w} g$

On $(N^4, \partial N, g)$ compact.

$$4\pi^2 \chi(N^4, \partial N) = \int_N |W|_g^2 \, dV_g + c_1 \int_N Q_4 \, dV_g + \int_{\partial N} \mathcal{L}_g \, d\sigma_g + c_2 \int_{\partial N} Q_3 \, d\sigma_g$$

$\mathcal{L}_g \, d\sigma_g$ pointwise conformal invariant

(C+J.Qing)

$$(Q_3)_g = + \frac{1}{12} \frac{\partial R}{\partial n} + \frac{1}{6} R H - R_{\alpha n \beta n} II_{\alpha \beta} + \frac{1}{3} H^3 - \frac{1}{3} \mathrm{Tr}\, II^3 - \frac{1}{3} \tilde{O} H$$

- In case of (X^4, M^3, g_+) C.C.E setting

$\bar{g}$ a compactification with totally geodesic boundary

$$c_1 \int_{X^4} Q_4(\bar{g})\, dV_{\bar{g}} + c_2 \int_{\partial X} Q_3(\bar{g})\, d\sigma_{\bar{g}} = c \int_X \left(R_{\bar{g}}^2 - 3\,|Ric|_{\bar{g}}^2\right) dV_{\bar{g}}$$

Def Renormalized Volume (X^{n+1}, M^n, g_+)

n odd

$$Vol_{g_+}\{\rho > \varepsilon\} = \varepsilon^{-n} c_0 + \varepsilon^{-n+2} c_2 + \cdots + \varepsilon^{-1} c_n + V_+ + o(\varepsilon)$$

V_+ is independent of $[\rho^2 g_+]$

Thm (M. Anderson 2001)

On (X^4, M^3, g_+)

$$8\pi^2 \chi(X^4) = \int_X |W|_g^2\, dV_g + 6\, V_+$$

- C + Qing + P. Yang ('05) $\forall\, \bar{g}$ totally geodesic compactification

$$V_+ = c \int_X \left(R_{\bar{g}}^2 - 3\,|Ric|_{\bar{g}}^2\right) dV_{\bar{g}}$$

work on other dim n (odd)

<u>Thm</u> (C + Qing + P. Yang)

On (X^4, M^3, g_+) c. c. E.

$Y(M^3, g_0) > 0$

(a) $V_+ > \frac{1}{3} \frac{4\pi^2}{3} \chi(X^4)$

then X is homeomorphic to 4-ball B^4
up to a finite cover

(b) $V_+ > \frac{1}{2} \frac{4\pi^2}{3} \chi(X^4)$

then X is diffeomorphic to B^4
M is diffeomorphic to S^3

- Based on works of Graham - Zworski;
  Fefferman - Graham, Graham - Jul, C + Qing + P. Yang
  & C + Fang ....

We derive a "local formula" for V_+ $\forall n$ odd

Notation:
$$g_+ = \frac{d\rho^2 + g_\rho}{\rho^2} \quad \text{near } M$$
$$g_\rho = g_0 + g_{(2)}\rho^2 + \cdots$$
$$\left(\frac{\det g_\rho}{\det g_0}\right)^{1/2} \sim \sum_{k=0}^{N} v_{(2k)}\rho^{2k} + o(\rho^{n+1})$$
$$2N \leq n$$

Thm (C + FANG + Graham)

On (X^{n+1}, M^n, g_+) C. C. E.

n odd
$$V_+ = c_n \int_{X^{n+1}} v_{(n+1)}(\bar{g})\, dV_{\bar{g}}$$
$\forall$ totally geodesic compactificatn $\bar{g}$

- $n = 3$ on (X^4, M^3, g_+)

$$V_{(4)}(\bar{g}) = c\,(R_{\bar{g}}^2 - 3\,|Ric|_{\bar{g}}^2)$$

$$= c \quad \sigma_2(A_{\bar{g}})$$

$\sigma_2(A_{ij}) = \sum_{i<j} \lambda_i \lambda_j$ 2nd symmetric function of eigen values of A

where $A_{\bar{g}} = \frac{1}{d-2}\left(Ric_{\bar{g}} - \frac{1}{2(d-1)} R_{\bar{g}}\, \bar{g}\right)$

- $n = 5$ on (X^6, M^5, g_+)

$$V_{(6)}(\bar{g}) = \sigma_3(A_{\bar{g}}) + \frac{1}{6} B_{\bar{g}}^{ij} (A_{\bar{g}})_{ij}$$

where $B_{\bar{g}}$ 4-th order generalized Bach tensor in dim 6

$$\{B_{\bar{g}}^{(d)}\}_{ij} = \frac{1}{d-3} \nabla^k \nabla^l W_{kilj} + \frac{1}{d-2} R^{kl} W_{kilj}.$$

- The formula was derived through the connection of

$V_{(n+1)}$ to $Q_{n+1}^{(n+1)}(X)$ & $Q_n^n(M)$,

It remains to see if the sign V_+ etc influences the topology & geometry of (X^{n+1}, M^n).

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