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Calibrations in Contact and Symplectic Geometry

In a celebrated paper published in 1982, F. Reese Harvey and H. Blaine Lawson introduced four types of calibrated geometries. Special Lagrangian submanifolds of Calabi-Yau manifolds, associative and coassociative submanifolds of G_2 manifolds and Cayley submanifolds of Spin(7) manifolds. Calibrated geometries have been of growing interest over the past few years and represent one of the most mysterious classes of minimal submanifolds.

In this talk, I will first give brief introductions to G_2 manifolds, and then discuss relations between G_2 and contact structures.

If time permits, I will also show that techniques from symplectic geometry can be adapted to the G_2 setting. These are joint projects with Hyunjoo Cho, Firat Arikan and Albert Todd.

Let X^n be a Riemannian manifold. Given $\alpha \in H_k(X, \mathbb{Z})$ we define the set

$\mathcal{H} = \{M$: compact,oriented submanifolds of $X | [M] = \alpha\}$ and the volume functional $V : \mathcal{H} \to \mathbb{R}_{(\geq \mathbf{0})}$ such that

$$V(M) = \int_M dvol_M$$

Goal: Study the geometry of $V(M)$

a) Local minima: Minimal submanifolds

b) Global minima: Calibrated Geometries

In 1982, Harvey and Lawson introduced the *calibrated geometries* and gave four examples.

- Special Lagrangian (in $\mathbb{R}^{2n}$ and Calabi-Yau manifolds) (String Theory)

- Associative (in $\mathbb{R}^7$ and G_2 manifolds) (M-Theory)

- Coassociative (in $\mathbb{R}^7$ and G_2 manifolds) (M-Theory)

- Cayley (in $\mathbb{R}^8$ and spin(7) manifolds)

Calibrated Geometries

Definition: A *calibration* is a closed p-form ϕ on a Riemannian manifold X^n such that ϕ restricts to each oriented tangent p-plane of X^n to be less than or equal to the volume form of that p-plane.

Definition: The submanifolds of X^n for which the p-form ϕ restricts to be equal to the Riemannian volume form are called to be *calibrated* by the form ϕ.

The term *calibrated geometry* represents the ambient manifold X, the calibration ϕ, and the collection of submanifolds calibrated by ϕ.

Calibrated submanifolds are volume minimizing submanifolds in their homology classes.

Examples of Calibrated Geometries:

1. Complex submanifolds of a Kähler manifold are volume minimizing in their homology classes, so if ω is the Kähler form and if $\phi = \dfrac{\omega^m}{m!}$ then ϕ is the calibration and the collection of complex submanifolds are the submanifolds calibrated by ϕ.

2. Let $(X^{2n}, \omega, J, g, \Omega)$ be a *Calabi-Yau* manifold where

$\omega =$ Kähler 2-form,

$J =$ complex structure,

$g =$ compatible Riemannian metric,

$\Omega =$ nowhere vanishing holomorphic $(n, 0)$-form.

Then special Lagrangian submanifolds of X are calibrated by $Re(\Omega)$.

Definition: An n-dimensional submanifold L of a Calabi-Yau manifold X is *special Lagrangian* if $\omega|_L \equiv 0$ and $Im(\Omega)|_L \equiv 0$.

Equivalently, $Re(\Omega)$ restricts to be the volume form on L with respect to the induced metric. Hence $Re(\Omega)$ is the calibration for special Lagrangian geometries.

Examples:

- Complex Lagrangian submanifolds of hyperkähler manifolds.

- Fixed point set of anti-holomorphic involutions of Calabi-Yau manifolds (R. Bryant).

• Calibrated Geometries
(Harvey-Lawson '82)

• String Theory
Mirror Symmetry (80's)

$$\Downarrow$$

• Deformations of SLags (R.C.McLean '91)

$$\Downarrow$$

• Hom. Mirror Sym. Conj. (Kontsevich '94)

AND

• Strominger-Yau-Zaslow Conj. ('96)

$$\Downarrow$$

???

3. Let (M, ϕ, g) be a 7-manifold with the holonomy group of its Levi-Civita connection is inside G_2, where

$\phi =$ closed and co-closed 3-form,

$g =$ compatible Riemannian metric.

Then M is called a G_2 manifold and associative 3-folds and coassociative 4-folds are calibrated by ϕ and $*\phi$, respectively.

An equivalent way of describing the G_2 manifold is as follows:

• Let $\mathbb{O}$ denote the octonions, (i.e Cayley numbers).

• Let $\mathbb{O} \cong \mathbb{R}\langle 1 \rangle \oplus \mathrm{Im}(\mathbb{O})$, where $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ are the imaginary octonions.

• For $u, v \in \mathrm{Im}(\mathbb{O})$, we can define the cross product structure $u \times v = \mathrm{Im}(u\overline{v})$. This cross product structure is defined on $\mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$, similar to the cross product structure defined on $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$, the imaginary quaternions.

- The cross product $\times$ satisfies

$$u \times v = -v \times u \quad \text{and} \quad < u \times v, u >= 0$$

- Now, we can define a 3-form ϕ by

$$\phi(u, v, w) =< u \times v, w >$$

Definition: A 7-dimensional manifold $(M, g, \times, \phi)$ is called a manifold with G_2 structure if each tangent space of M can be identified with Im($\mathbb{O}$).

Definition: Let $(M, g, \times, \phi)$ be a manifold with G_2 structure. Then it is called a G_2 manifold if $\nabla\phi = 0$. (i.e ϕ satisfies an integrability condition)

• There are some rigid relations on a manifold with G_2 structure:

$$(i_u\phi) \wedge (i_v\phi) \wedge \phi = C g_\phi(u, v) vol_\phi$$

$$\phi(u, v, w) = g(u \times v, w) =< u \times v, w >_\phi$$

• This is very different than K$\ddot{a}$hler geometry, where the K$\ddot{a}$hler form ω and the complex structure J are independent and determine the metric $\omega(u, v) = g(Ju, v)$.

• Open problem: Understand the sufficient topological conditions for existence of an integrable G_2 structure.

Existence of contact structures on G_2 manifolds

Problems:

Let $(M, g, \times, \phi)$ be a manifold with G_2 structure.

• Does it admit a contact structure?

• If so, is every contact structure on M either $\mathcal{A}$- or $\mathcal{B}$- compatible with G_2 ?

Contact and Almost Contact Manifolds

Let M be a $(2n+1)$-dimensional smooth manifold. A plane field (or hyperplane distribution) ξ on M can (locally) be given as the kernel of 1-form α : $\xi_x = \ker(\alpha_x)$, $x \in M$.

Definition: A contact structure on M is a hyperplane field ξ that is (locally) given by the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. The pair (M, ξ) is called a contact manifold.

Definition: (Sasaki) An almost contact structure on a differentiable manifold M is a triple (J, R, α), which consists of a field J of endomorphism of the tangent spaces, a vector field R, and a 1-form α satisfying

(i) $\alpha(R) = 1$, and

(ii) J^2 =-id$+\alpha \otimes R$,

where id denotes the identity transformation.

Lemma: Suppose M^{2n+1} has a (J, R, α) structure. Then $J(R) = 0$ and $\alpha \circ J = 0$.

Idea of the proof: Note that

$J^2(R) = -R + \alpha(R)R = -R + 1.R = 0$ and $0 = J^2(J(R)) = -J(R) + \alpha(J(R)).R$,

so we have $J(R) = 0$ or $J(R)$ is a nonzero vector field whose image is 0. Suppose $J(R)$ is a nonzero vector field that maps to 0: Then

$$0 = J^2(R) = J(J(R)) = J(\alpha(J(R))R)$$

$$= \alpha(J(R)).J(R) = \alpha(J(R)).\alpha(J(R)).R$$

$$= (\alpha(J(R)))^2.R \neq 0$$

for nonzero $\alpha(J(R))$ and R.

(If $\alpha(J(R)) = 0$ then $J(R) = 0$ which contradicts to assumption). Hence, $J(R) = 0$.

Now for any vector X,

$0 = J^3(X) = J^2(J(X)) = -J(X) + \alpha(J(X))R$

and

$J^3(X) = J(J^2(X)) = J(-X) + J(\alpha(X)R) = -J(X) + J(\alpha(X)R)$

obtained by applying J to $J^2(X) = -X+\alpha(X)R$.

So we have

$\alpha(J(X))R = J^3(X)+J(X) = -J(X)+J(\alpha(X)R)+J(X) = 0$ because the fact $J(R) = 0$ gives $J(\alpha(X)R) = \alpha(X)J(R) = 0$.

Therefore $\alpha \circ J = 0$ for any vector X.

Definition: (Sasaki) An almost contact metric structure on a differentiable manifold M^{2n+1} is a quadruple (J, R, α, g) where (J, R, α) is an almost contact structure on M and g is a Riemannian metric on M satisfying

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$$

for all vector fields u, v in TM. Such a g is called a compatible metric.

Theorem: (A-C-S) Let (M^7, ϕ) be a manifold with G_2 structure. Then M admits an almost contact structure.

Moreover, for any non-vanishing vector field X_0 on M, $(J, X_0, \alpha_X, <,>_\phi = g_\phi)$ is an almost contact metric structure on M.

Idea of the proof:

- For a non-vanishing vector field X_0, define an associated 1-form α such that $\alpha(\cdot) = g_\phi(X_0, \cdot)$.

- Define $J \in End(TM)$ by $J(u) = X_0 \times u$. Then $(J, X_0, \alpha_X, g_\phi)$ is an almost contact metric structure on (M, ϕ).

Suppose that (M, ϕ) is a manifold with G_2-structure. As M is 7-dimensional, we know that there exists a nowhere vanishing vector field R on M. Denote the Riemannian metric and the cross product (determined by ϕ) by $< \cdot, \cdot >_\phi$ and $\times_\phi$, respectively. Using the metric, we define the 1-form α as the metric dual of R, that is,

$$\alpha(u) =< R, u >_\phi .$$

Moreover, using the cross product and R, we can define an endomorphism $J_R : TM \to TM$ of the tangent spaces by

$$J_R(u) = R \times_\phi u.$$

Note that $J_R(R) = 0$, and so J_R, indeed, defines a complex structure on the orthogonal complement $R^\perp$ of R with respect to $< \cdot, \cdot >_\phi$. By straightforward computations, one easily check that the conditions (i) and (ii) of definition (of being almost contact structure) are satisfied by the triple (J_R, R, α), and so (J_R, R, α) is an almost contact structure on M.

In order to see the existence of compatible metric for our almost contact structure (J, R, α) with metric $< \cdot, \cdot >_\phi$, we compute

$< J_R R, J_R v >_\phi = < 0, J_R v >_\phi = 0$ and also

$< R, v >_\phi - \alpha(R)\alpha(v) = \alpha(v) - \alpha(v) = 0$

Therefore

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$$

holds if $u = R$ or $v = R$.

If u, v are both taken from the orthogonal complement $R^{\perp}$ (wrt $< \cdot, \cdot >_\phi$), then we compute

$$< J_R u, J_R v >_\phi = < R \times_\phi u, R \times_\phi v >_\phi$$

$$= \phi(R, u, R \times_\phi v) = -\phi(R, R \times_\phi v, u)$$

$$= - < R \times_\phi (R \times_\phi v), u >_\phi$$

$$= - < -|R|^2 v + < R, v >_\phi R, u >_\phi$$

$$= - < -v, u >_\phi = < u, v >_\phi$$

Again, $g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v)$ is satisfied.

In general,

$$< J_R u, J_R v >_\phi = < R \times_\phi u, R \times_\phi v >_\phi$$

$$= \phi(R, u, R \times_\phi v) = -\phi(R, R \times_\phi v, u)$$

$$= - < R \times_\phi (R \times_\phi v), u >_\phi$$

$$= - < -|R|^2 v + < R, v >_\phi R, u >_\phi$$

$$= < |R|^2 v, u >_\phi - << R, v >_\phi R, u >_\phi$$

$$= < |R|^2 v, u >_\phi - < \alpha(v) R, u >_\phi$$

$$= < u, |R|^2 v >_\phi - \alpha(v) < R, u >_\phi$$

$$= < u, |R|^2 v >_\phi - \alpha(v)\alpha(u)$$

Definition: A contact structure ξ on (M^7, ϕ) is said to be $\mathcal{A}$-compatible with G_2 structure ϕ if $d\alpha = i_R\phi$ where α is a contact form for ξ and R is the Reeb vector field of $f\alpha$ for some nonzero function $f : M \to \mathbb{R}$.

Theorem: (A-C-S) Let (M^7, ϕ) be any manifold with integrable G_2-structure where M is closed (i.e., compact and $\partial M = \emptyset$). Then there is no contact structure on M which is $\mathcal{A}$-compatible with ϕ.

Idea of the proof:

Suppose ξ is an $\mathcal{A}$-compatible contact structure on (M, ϕ). Therefore, $d\alpha = \iota_R \phi$ for some contact form α for ξ and the associated Reeb vector field R. We also have

$$d\alpha \wedge d\alpha \wedge \phi = (\iota_R \phi) \wedge (\iota_R \phi) \wedge \phi = 6|R|^2 dVol.$$

Since $d\phi = 0$, we have $d\alpha \wedge d\alpha \wedge \phi = d(\alpha \wedge d\alpha \wedge \phi)$.

Now by Stoke's Theorem,

$$0 \lneqq \int_M 6|R|^2 dVol = \int_M d(\alpha \wedge d\alpha \wedge \phi)$$

$$= \int_{\partial M} \alpha \wedge d\alpha \wedge \phi = 0$$

(as $\partial M = \emptyset$). This gives a contradiction.

<u>**Exciting Questions:**</u> What if we work with manifolds with boundary ??? Are these new invariants of Calabi-Yau manifolds ???

We need to show that

1) These integrals provide nontrivial values.

2) Every Calabi-Yau manifold bounds a G_2 manifold.

3) If both (1) and (2) are correct then understand what these invariants measure.

<u>**Conjecture:**</u> If X and $\overline{X}$ are mirror pairs then these invariants will be the same.

Definition: A contact structure ξ on (M^7, ϕ) is said to be $\mathcal{B}$-compatible with G_2 structure ϕ if there are (global) vector fields X, Y on M such that $\alpha = i_Y i_X \phi$ where α is a contact form for ξ.

Theorem: (A-C-S) The standard contact structure on $\mathbb{R}^7$ is both $\mathcal{A}$- and $\mathcal{B}$-compatible with the standard G_2 structure ϕ_0.

Idea of the proof:

Fix the coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ on $\mathbb{R}^7$. In these coordinates, one can take

$$\phi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where e^{ijk} denotes the 3-form $dx_i \wedge dx_j \wedge dx_k$. Consider the standard contact structure ξ_0 on $\mathbb{R}^7$ as the kernel of the 1-form

$$\alpha_0 = dx_1 - x_3 dx_2 - x_5 dx_4 - x_7 dx_6.$$

For simplicity we will denote $\partial/\partial x_i$ by ∂x_i (so we have $dx_i(\partial x_j) = \delta_{ij}$). Consider the vector fields

$$R = \partial x_1, \ X = \partial x_7 \text{ and}$$
$$Y = -x_7 \partial x_1 + x_5 \partial x_3 - x_3 \partial x_5 - \partial x_6 + f \partial x_7$$

where $f : \mathbb{R}^7 \to \mathbb{R}$ is any smooth function (in fact, it is enough to take $f \equiv 0$ for our purpose). By a straightforward computation, we see that

$$d\alpha_0 = \iota_R(\phi_0), \quad \alpha_0 = \iota_Y \iota_X(\phi_0).$$

Also observe that R is the Reeb vector field of α_0.