Recent progress in G_2 geometry

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1. Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds, arXiv:1206.2277.

2. G_2-manifolds and associative submanifolds via semi-Fano 3-folds, arXiv:1207.4470.

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- Exhibit "geometric transitions" between G_2-metrics on different 7-manifolds.

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$\exists$ *close relations between G_2 holonomy and Calabi-Yau geometries in* 2 *and* 3 *dimensions.*

- Write $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$ with $(\mathbb{C}^3, \omega, \Omega)$ the std SU(3) structure then

  $$\phi_0 = dt \wedge \omega + \mathrm{Re}\,\Omega$$

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G_2 structures and G_2 holonomy metrics

What is a G_2 structure?

- A G_2 structure is a 3-form ϕ on an oriented 7-mfd M such that $\forall\ p \in M$ $\exists$ an oriented isomorphism

$$i : T_pM \to \mathbb{R}^7, \text{ such that } i^*\phi_0 = \phi.$$

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- $\dim(\mathrm{GL}_+(7,\mathbb{R})/G_2) = 35 = \dim \Lambda^3\mathbb{R}^7$.
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NB (3) is nonlinear in ϕ because metric g depends nonlinearly on ϕ.

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II. Perturb to a torsion-free G_2 structure ϕ' close to ϕ.

II was understood in some generality by Dominic Joyce (if $d\phi = 0$).

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Oriented 4-planes calibrated by $*\phi_0$ are called *coassociative*. 4-plane is coassociative iff its orthogonal complement is associative.

Holonomy/parallel tensor correspondence $\Rightarrow$

- on any mfd (M, g) with $\mathrm{Hol}(g) \subset G_2$ we have parallel 3 and 4-forms ϕ and $*_g\phi$ modelled on ϕ_0 and $*\phi_0$.
- associative (coassociative) calibration exists on any G_2-manifold.

$1 + 2 = 3$ and $\mathbb{S}^1 \times$ *holomorphic* $=$ *associative*

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We also have $\mathbb{S}^1 \times L \subset \mathbb{S}^1 \times V$ is coassociative iff L is a special Lagrangian 3-fold in X.

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i. Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Calabi-Yau 3-folds V with one end $\sim \mathbb{C}^* \times D$, with D a smooth $K3$.

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Kovalev (2003) carried out Donaldson's proposal for AC CY 3-folds arising from Fano 3-folds.

Twisted connect sum and hyperkahler rotation

Product G_2 structure on $M_\pm = \mathbb{S}^1 \times V_\pm$ asymptotic to

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to identify end of M_- with M_+ we need $f : D_- \to D_+$ to satisfy

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- Constructing such hyperkähler rotations is nontrivial and a major part of the construction.

Twisted connect sum and hyperkahler rotation

Product G_2 structure on $M_\pm = \mathbb{S}^1 \times V_\pm$ asymptotic to

$$d\theta_1 \wedge d\theta_2 \wedge dt + d\theta_1 \wedge \omega_I^\pm + d\theta_2 \wedge \omega_J^\pm + dt \wedge \omega_K^\pm$$

$\omega_I^\pm$, $\omega_J^\pm + i\,\omega_K^\pm$ denote Ricci-flat Kähler metric, parallel $(2,0)$-form on $D_\pm$.

To get a well-defined G_2 structure using

$$F : [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_- \to [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_+$$

given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T - 1 - t, \theta_2, \theta_1, f(y))$$

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- Some problems in Kovalev's original paper here.

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$\Rightarrow$ have reduced solving nonlinear PDEs for G_2-metric to two problems about complex projective 3-folds.

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Recently Hein-Haskins-Nordström gave simpler direct proof using ideas in Hein's thesis (and showed all "asymptotically split" ACyl CY 3-folds arise from such a construction).

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  - A holomorphic line bundle L on X is *nef* if
    $$c_1(L).C = \int_C c_1(L) \geq 0$$
    for every irreducible holomorphic curve $C \subset X$.
  - A holomorphic line bundle L on X is *big* if
    $$h^0(L^{\otimes m}) \geq Cm^n, \quad \text{for } m \gg 1, \quad n = \dim_{\mathbb{C}} X.$$

i.e. we replace condition K_X^{-1} is positive with sufficiently "semi-positive".

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Kovalev used ACyl Calabi-Yau 3-folds of *Fano type* for his twisted connect sum G_2-manifolds; we generalise to (certain classes of) **weak Fano** 3-folds.

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   For each smooth rigid $\mathbb{P}^1$ in a weak Fano 3-fold X any G_2 manifold built from X contains a *rigid associative submanifold* w/ topology $S^1 \times S^2$.

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$\Rightarrow$ can use them to construct compact twisted connect sum G_2 manifolds.

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$$Y = (x_0 a_3 + x_1 b_3 = 0) \subset \mathbb{P}^4$$

where a_3 and b_3 are homogeneous cubic forms in $(x_0, \ldots, x_4)$.

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- Classification results $\Rightarrow$ any Fano 3-fold has Picard rank ≤ 10. In fact, Picard rank ≥ 6 forces X to be $\mathbb{P}^1 \times dP$ for some del Pezzo surface.

Toric semi-Fano 3-folds

Theorem (Coates-Haskins-Kasprzyk)

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    - Most admit *many* nonisomorphic projective small resolutions. Can enumerate those completely in terms of geometry of the polytopes.
- Not every toric semi-Fano is rigid; rigidity is determined by polytope.

G_2-manifolds and toric semi-Fano 3-folds

Theorem (CHNP+CHK)

There exist over 50 million matching pairs of ACyl CY 3-folds of semi-Fano type for which the resulting G_2-manifold is 2-connected.

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Observation: $4|p_1$ and $p_1|48$ for any twisted connect sum.
Strategy to pin-down divisibility: Relate divisibility of p_1 of twisted connect sum to divisibility of c_2 on the pair of building blocks. In the best case:

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In many cases can understand $div(c_2)$ e.g. any Fano w/ $b^2 = 1$.

Look for building block with $\quad div(c_2^-) = 2 \qquad (**)$
e.g. happens for 6 out of 17 Fanos with $b^2 = 1$
(*) + observation $\Rightarrow$ $div(p_1) = 4$ whatever happens for c_2^+.

Diffeomorphism type of twisted connect sums II

Simplest setting: M is 2-connected and H^4M is torsion-free.

Wilkens $\Rightarrow$ M classified by $b = b^4(M)$ and $p_1(M) \in H^4M$.

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$\Rightarrow$ only one diffeo type in almost-diffeo class for any 2-connected twisted connect sum with one side satisfying (**)

Diffeomorphic G_2 manifolds

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Easy to find many examples satisfying i–iii from *toric* semi-Fanos
(but lots of other ways of doing this too..)