# Recent progress in $G_2$ geometry

Alessio Corti, Mark Haskins,
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1. Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds, arXiv:1206.2277.

2. $G_2$-manifolds and associative submanifolds via semi-Fano 3-folds, arXiv:1207.4470.

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- Exhibit "geometric transitions" between $G_2$-metrics on different 7-manifolds.

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  where $\omega_I$ and $\Omega=\omega_J+i\omega_K$ are the standard Kahler and holo $(2,0)$ forms on $\mathbb{C}^2$. Hence subgroup of $G_2$ fixing $\mathbb{R}^3\subset\mathbb{R}^3\times\mathbb{C}^2$ is SU(2) $\subset G_2$.

# $G_2$ structures and $G_2$ holonomy metrics

What is a $G_2$ structure?

- A $G_2$ structure is a 3-form $\phi$ on an oriented 7-mfd $M$ such that $\forall\ p \in M$ $\exists$ an oriented isomorphism

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- $\dim(\mathrm{GL}_+(7,\mathbb{R})/G_2) = 35 = \dim \Lambda^3\mathbb{R}^7$.
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NB (3) is nonlinear in $\phi$ because metric $g$ depends nonlinearly on $\phi$.

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II. Perturb to a torsion-free $G_2$ structure $\phi'$ close to $\phi$.

II was understood in some generality by Dominic Joyce (if $d\phi = 0$).

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Oriented 4-planes calibrated by $*\phi_0$ are called *coassociative*. 4-plane is coassociative iff its orthogonal complement is associative.

Holonomy/parallel tensor correspondence $\Rightarrow$

- on any mfd $(M, g)$ with $\mathrm{Hol}(g) \subset G_2$ we have parallel 3 and 4-forms $\phi$ and $*_g\phi$ modelled on $\phi_0$ and $*\phi_0$.
- associative (coassociative) calibration exists on any $G_2$-manifold.

# $1 + 2 = 3$ and $\mathbb{S}^1 \times$ *holomorphic* $=$ *associative*

Recall when we decomposed $\mathbb{R}^7$ as $\mathbb{R} \times \mathbb{C}^3$ we had

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We also have $\mathbb{S}^1 \times L \subset \mathbb{S}^1 \times V$ is coassociative iff $L$ is a special Lagrangian 3-fold in $X$.

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**i.** Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Calabi-Yau 3-folds $V$ with one end $\sim \mathbb{C}^* \times D$, with $D$ a smooth $K3$.

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**iii.** Take a *twisted connect sum* of a pair of $M_\pm = \mathbb{S}^1 \times V_\pm$

**iv.** For $T >> 1$ construct a $G_2$-structure w/ small torsion (exponentially small in $T$) and prove it can be corrected to torsion-free.

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**Kovalev** (2003) carried out Donaldson's proposal for AC CY 3-folds arising from Fano 3-folds.

# Twisted connect sum and hyperkahler rotation

Product $G_2$ structure on $M_{\pm} = \mathbb{S}^1 \times V_{\pm}$ asymptotic to

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To get a well-defined $G_2$ structure using

$$F : [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_- \to [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_+$$

given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T - 1 - t, \theta_2, \theta_1, f(y))$$

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- Some problems in Kovalev's original paper here.

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$\Rightarrow$ have reduced solving nonlinear PDEs for $G_2$-metric to two problems about complex projective 3-folds.

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Recently Hein-Haskins-Nordström gave simpler direct proof using ideas in Hein's thesis (and showed all "asymptotically split" ACyl CY 3-folds arise from such a construction).

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  - A holomorphic line bundle $L$ on $X$ is *nef* if

$$c_1(L).C = \int_C c_1(L) \geq 0$$

    for every irreducible holomorphic curve $C \subset X$.
  - A holomorphic line bundle $L$ on $X$ is *big* if

$$h^0(L^{\otimes m}) \geq Cm^n, \quad \text{for } m \gg 1, \quad n = \dim_{\mathbb{C}} X.$$

i.e. we replace condition $K_X^{-1}$ is positive with sufficiently "semi-positive".

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Kovalev used ACyl Calabi-Yau 3-folds of *Fano type* for his twisted connect sum $G_2$-manifolds; we generalise to (certain classes of) **weak Fano** 3-folds.

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$\Rightarrow$ can use them to construct compact twisted connect sum $G_2$ manifolds.

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**Theorem (CHNP)**

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**Remarks**

- Shows semi-Fano 3-folds can have larger Picard rank than Fanos. $\Rightarrow$ can get $G_2$ manifolds with larger Betti numbers.
- Classification results $\Rightarrow$ any Fano 3-fold has Picard rank $\leq 10$. In fact, Picard rank $\geq 6$ forces $X$ to be $\mathbb{P}^1 \times dP$ for some del Pezzo surface.

# Toric semi-Fano 3-folds

**Theorem (Coates-Haskins-Kasprzyk)**

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  - Most admit *many* nonisomorphic projective small resolutions. Can enumerate those completely in terms of geometry of the polytopes.
- Not every toric semi-Fano is rigid; rigidity is determined by polytope.

# $G_2$-manifolds and toric semi-Fano $3$-folds

**Theorem (CHNP+CHK)**

*There exist over 50 million matching pairs of ACyl CY 3-folds of semi-Fano type for which the resulting* $G_2$*-manifold is* 2*-connected.*

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$\Rightarrow$ only one diffeo type in almost-diffeo class for any 2-connected twisted connect sum with one side satisfying (**)

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Easy to find many examples satisfying i–iii from *toric* semi-Fanos
(but lots of other ways of doing this too..)