Some Homology and Cohomology Theories for a Metric Space

Robert Hardt
(Rice University)

BLAINEFEST

October 27, 2012

Coauthors

Thierry De Pauw (Paris VII) -H.,
Rectifiable and Flat G Chains in a Metric Space Amer.J.Math.2011

De Pauw, -H., Washek Pfeffer (UC Davis, Emeritus)
Homology of Normal Chains and Cohomology of Charges In preparation.

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* For M triangulated, use simplicial theory.
* For M a smooth manifold and for real coefficients, use differential forms and De Rham theory.
* For M semi-algebraic, use semi-algebraic chains, etc.

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Rectifiable Chains

Theorem. (H.Federer - W.Fleming, 1959) *Integer-multiplicity rectifiable chains give the ordinary integral homology for pairs of compact Euclidean Lipschitz neighborhood retracts (ELNR). Homology classes of such pairs contain mass-minimizing rectifiable chains.*

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What are rectifiable chains?

Rectifiable Sets

A subset M of a metric space X is $\mathcal{H}^m$ *rectifiable* if $\mathcal{H}^m(X \setminus f(E)) = 0$ for some Lebesgue measurable $E \subset \mathbb{R}^m$ and Lipschitz $f : E \to M$.
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Parameterization Theorem. *There exist disjoint compact $A_i \subset \mathbb{R}^m$ and an injective map $\alpha : A = \cup_{i=1}^{\infty} A_i \to M$ such that $\mathcal{H}^m[M \setminus \alpha(A)] = 0$, $\mathrm{Lip}\,\alpha \leq 1$, and $\mathrm{Lip}(\alpha \restriction A_i)^{-1} \leq 2\sqrt{m}$.*

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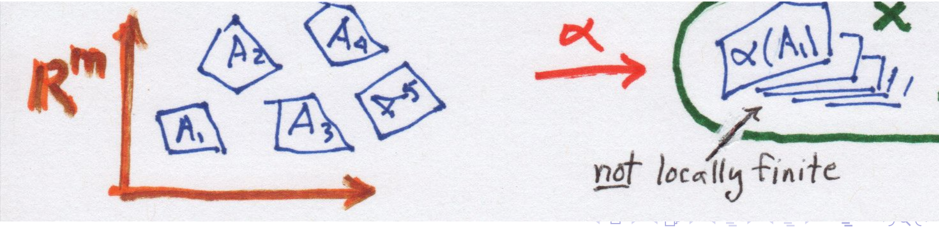

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and, $\mathcal{H}^m$ a.e. on $\alpha^{-1}[\alpha(A) \cap \beta(B)]$,

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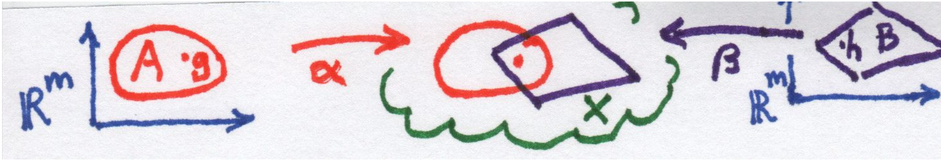

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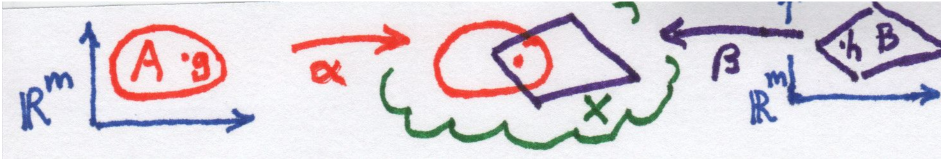

$\mathcal{R}_m(X; G) = \{m \text{ dimensional rectifiable } G \text{ chains } T \text{ in } X\}$.

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In particular, the standard space ℓ^∞ of bounded sequences essentially *contains* any separable metric metric space.

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A *polyhedral G chain* in Y is simply a finite sum $P = \sum_{i=1}^{I} [\![\gamma_i, \Delta_i, g_i]\!]$ where $\gamma_i : \mathbb{R}^m \to Y$ is affine, Δ_i is an m simplex, and g_i is *constant* on Δ_i.

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As the Koch snowflake in the plane shows, the boundary of a rectifiable chain is not expected to be rectifiable in general. So defining it requires completion of Lipschitz chains with respect to a weaker norm.

Flat Norm and Flat Chains

Note that in the space $\mathbb{R}$ the points $1/i$ approach the point 0, but the corresponding 0 dimensional chains $[\![1/i]\!]$ do not approach $[\![0]\!]$ *in mass norm* because $\mathbb{M}([\![1/i]\!] - [\![0]\!]) = 2$.

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$$\mathcal{F}(T) \; = \; \inf\{\mathbb{M}(S) + \mathbb{M}(T - \partial S) \; : \; S \in \mathcal{L}_{m+1}(Y, G)\} \; .$$

Then the flat norm $\mathcal{F}([\![1/i]\!] - [\![0]\!]) \; \leq \; 1/i \; \to \; 0$ because $[\![1/i]\!] - [\![0]\!] = \partial[\![0, 1/i]\!]$.

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Since $\mathcal{F} \leq \mathbb{M}$, a rectifiable chain $T \in \mathcal{R}_m(Y; G)$ is flat and so now has a well-defined boundary $\partial T \in \mathcal{F}_{m-1}(Y; G)$.

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$$\mathbf{N}_m(X;G) = \{T \in \mathcal{F}_m(Y;G) \,:\, \operatorname{spt} T \subset X,\ \mathbb{M}(T) + \mathbb{M}(\partial T) < \infty\}$$

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denote the subgroup of *normal chains*. Then working with either rectifiable chains having rectifiable boundaries or with normal chains, one can similarly define *rectifiable chains homology*

$$\mathcal{H}_m^{\mathcal{R}}(X;G)$$

and *normal chains homology*

$$\mathbf{H}_m(X;G)\,.$$

An Example

For X being the standard fractal boundary of the Koch snowflake in $\mathbb{R}^2$,

$$\mathbf{H}_1(X;\mathbb{Z}) \;=\; 0\;, \mathcal{H}_1^{\mathcal{R}}(X;\mathbb{Z}) \;=\; 0\;,\; \text{and}\; \mathcal{H}_1^{\mathcal{F}}(X;\mathbb{Z}) \;=\; \mathbb{Z}$$

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Let's look at a few examples relevant to *mass-minimizing* G chains.

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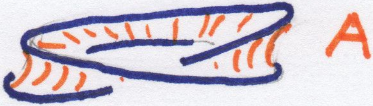

Example 2. B is three (similarly-oriented) semi-circles bounding A which is three half-disks. Here $\partial B = 0$ and $\partial A = B$ as $\mathbb{Z}/3\mathbb{Z}$ chains.

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An important property relevant to the *existence* of mass-minimizing chains is a suitable compactness theorem. Our version is the following:

Compactness Theorem

Theorem. [DHP] *Suppose X is a compact metric space and G is a complete normed group with closed balls being compact. For $R > 0$,*

(I) $K_R = \{T \in \mathcal{F}_m(X; G) \ : \ \mathbb{M}(T) + \mathbb{M}(\partial T) \leq R\}$ *is $\mathcal{F}$ compact.*

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The *rectifiability* in (B) give the desired geometric character to the Plateau problem solutions. While this rectifiabiity is *not true* for $G = \mathbb{R}$ with the usual absolute value norm $|\,|$, it is true for another norm on $\mathbb{R}$:

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For $0 < \alpha < 1$, we define the norm $\|r\|_\alpha = |r|^\alpha$ for $r \in \mathbb{R}$. Then $(\mathbb{R}, \|\cdot\|_\alpha)$ *does* satisfy condition * . Also "merging" paths in T may reduce the corresponding mass $\mathbb{M}_\alpha(T)$.

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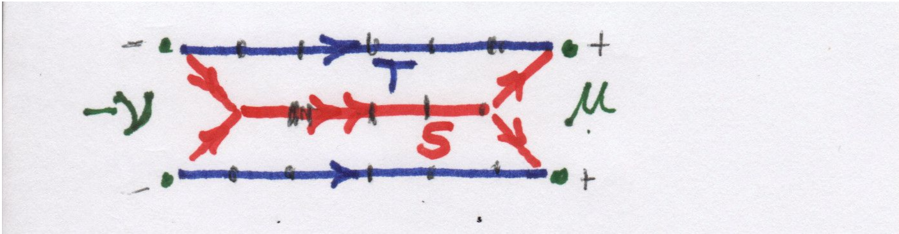
Example.



$$\mathbb{M}_{\frac{1}{2}}(T) \;=\; 1\cdot(6+6) \;>\; 1\cdot 4\sqrt{2}+\sqrt{2}\cdot 4 \;=\; \mathbb{M}_{\frac{1}{2}}(S).$$

$\mathbb{M}_\alpha$ Minimizers

Corollary.(Q. Xia) *There exists a* $\mathbb{M}_\alpha$ *minimizing* $T \in \mathcal{R}_1(\mathbb{R}^n, \mathbb{R})$ *with* $\partial T = \mu - \nu$.

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Higher Dimensions.(H.–De Pauw, In progress) *For* $m \geq 1$,
dim $(\text{spt}\, T \setminus \text{spt}\, \partial T) \;\leq\; m-1$ *for any* $\mathbb{M}_\alpha$ *minimizing* $T \in \mathcal{R}_m(X, \mathbb{Z})$.

Proof of (I)

K_R is $\mathcal{F}$ *complete* by the lower semicontinuity of $\hat{\mathbb{M}}$. So we need only show that K_R is also *totally bounded*. For this, it suffices to find, for each $\varepsilon > 0$, a compact subset C_ε of $\mathcal{F}_m(Y; G)$ so that

$$K_R \subset \{T \in \mathcal{F}_m(Y; G) \; : \; \mathrm{dist}_{\mathcal{F}}(T, C_\varepsilon) < 2\varepsilon R\} \; .$$

Continuation of Proof of (I)

By the MAP (Metric Approximation Property) of $Y = \ell^\infty(D)$ there is a Lipschitz 1 linear projection p of Y onto some finite n dimensional $W \subset Y$ so that $\|p(x) - x\| < \varepsilon$ for all x in the compact set X. W is equivalent to $\mathbb{R}^n$ (with bounds only depending on X and ε). So we assume $W = \mathbb{R}^n$ and use the Deformation Theorem of B. White.

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First, note that

$$C_\varepsilon = \{ \sum_{i=1}^{I} g_i Q_i \ : \ Q_i = m \text{ cube of a size } \varepsilon \text{ subdivision,}$$

$$Q_i \cap p(X) \neq \emptyset, \text{ and } \sum_{i=1}^{I} \|g_i\| \varepsilon^m \leq cR \}$$

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is $\mathcal{F}$ compact. Now, for any $T \in K_R$, an affine homotopy shows that $\mathcal{F}(p_\# T - T) \leq \varepsilon R$. Next the Deformation Theorem implies that $\mathcal{F}(p_\# T - Q) \leq \varepsilon R$ for some $Q \in C_\varepsilon$. So $\mathrm{dist}_{\mathcal{F}}(T, C_\varepsilon) < 2\varepsilon R$.

Proof of (II), $m = 0$

For the case $m = 0$, we follow the argument of White and note that T is a G valued Borel measure, which we wish to show is purely atomic. First we verify the general

Lemma. *For any positive Borel measure* μ *without atoms on* X*, there exists a* μ *measurable function* $f : X \to [0,1]$ *so that* $\mu[f^{-1}\{t\}] = 0$ *for every* $t \in [0,1]$.

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Then we apply the lemma with $\mu = |\nu_T|$ where ν_T is the nonatomic part of T. Fot nonzero μ, we get, *in the group* G, the nonconstant continuous curve $\gamma(t) = \nu_T\left[f^{-1}[0,t)\right]$ of finite length $\leq \mathbb{M}(T)$, contradicting (*).

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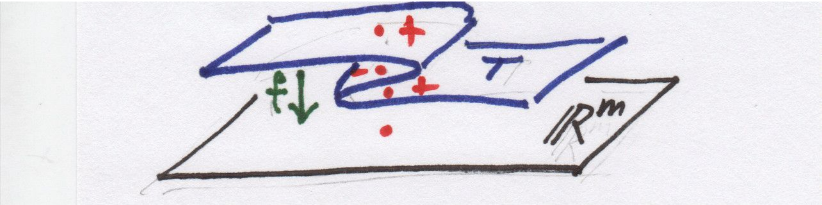
For the case $m > 0$ we generalize Jerrard's observation showing that, for any Lipschitz map $f : X \to \mathbb{R}^m$, the slice function

$$\langle T, f, \cdot \rangle \in \mathrm{BV}\left(\mathbb{R}^m, \mathcal{R}_0(X; G)\right) .$$



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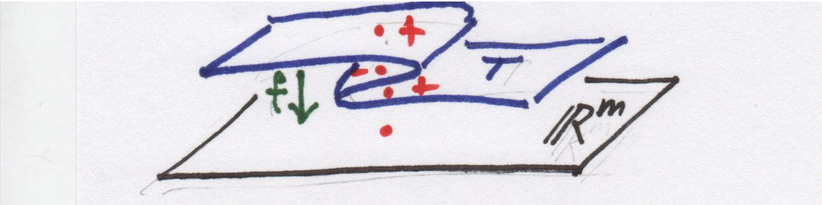

By an argument of Ambrosio-Kirchheim, this may be approximated by a Lipschitz function, leading to the rectifiability of T.

Real Normal Chains and Dual Cochains

For simplicity, we will, for the rest of the lecture, assume that

X is a compact metric space, G is the coefficeint group $\mathbb{R}$ with the standard norm $|\,|$, and drop the $\mathbb{R}$ symbol.

Thus we have the vector space

$$\mathbf{N}_m(X) \;=\; \{T \in \mathcal{F}_m(X,\mathbb{R}) \;:\; \mathbb{M}(T) + \mathbb{M}(\partial T) \;<\; \infty\}$$

of *normal* $\mathbb{R}$ chains in X as well as the closed subspaces of *cycles*

$$\mathbf{Z}_m(X) \;=\; \{T \in \mathbf{N}_m(X) \;:\; \partial T = 0\} \quad \text{for} \quad m \geq 1 \;,$$

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$$\mathbf{H}_m(X) \ = \ \mathbf{Z}_m(X)/\{\partial S \ : \ S \in \mathbf{N}_{m+1}(X)\} \ .$$

Cochains

Whitney also studied the dual space $\mathcal{F}_m(\mathbb{R}^n;\mathbb{R})^*$ of *flat cochains*, and his student J. Wolfe (1957) showed that any flat cochain comes from bounded Borel m form ω where $d\omega$ is a bounded Borel $m+1$ forms. This means $\alpha(T) = T(\omega)$ for $T \in \mathcal{F}_m(\mathbb{R}^n;\mathbb{R})$.

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Charges, which act on normal chains, were defined by De Pauw to study solutions of div $v = F$ by using the terms of $\int_{\partial\Omega} v\cdot\nu = \int_\Omega F$ as functionals of the set Ω of finite perimeter.

De Pauw, Moonens, Pfeffer (2009) showed that charges in $\mathbb{R}^n$ correspond to $\omega + d\eta$ for some *continuous* ω, η.

Charges

The *localized topology* $\mathcal{T}_{\mathbf{N}}$ on $\mathbf{N}_m(X)$ has the property that

$$T_j \to T \text{ in } \mathcal{T}_{\mathbf{N}} \iff \mathcal{F}(T_j - T) \to 0 \textit{ and } \sup_j \hat{\mathbb{M}}(T_j) + \hat{\mathbb{M}}(\partial T_j) < \infty \ .$$

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A *charge* is a continuous linear $\alpha : (\mathbf{N}_m(X), \mathcal{T}_{\mathbf{N}}) \to \mathbb{R}$. Let

$$\mathbf{CH}^m(X) \ = \ \{m \text{ dimensional charges in } X\} \ .$$

We have the continuous operators

$$\delta : \mathbf{CH}^m(X) \ \to \ \mathbf{CH}^{m+1}(X) \ , \quad (\delta\alpha)(S) = \alpha(\partial S)$$

$$\phi^{\#} : \mathbf{CH}^m(Y) \to \mathbf{CH}^m(X), \ (\phi^{\#}\alpha)(T) = \alpha(\phi_{\#} T)$$

for Lipschitz $\phi : X \to Y$.

Vanishing of $\mathbf{H}_0(X)$, $\mathbf{H}^0(X)$

Theorem. *Consider the following three conditions.*

(A) $\mathbf{H}_0(X) = 0$.

(B) X *is Lipschitz path connected.*

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Then (A) implies (B) and (B) implies (C).

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Example. $X = \cup_{i=1}^{\infty} X_i$ where X_i are embedded curves in $\mathbb{R}^3$ joining points a_i to 0, disjoint away from 0, and with length$(X_i) = 2^i$. Then X is Lipschitz path connected, but $T = [\![0]\!] - \sum_{i=1}^{\infty} 2^{-i} [\![a_i]\!]$ has $\chi(T) = 0$ although T bounds no one chain of finite mass in X. So (B) does not imply (A) in general.

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Theorem. *(C) implies (A) if X satisfies the linear isoperimetric condition*

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Definition. X *is m bounded* $\iff \mathbb{M}(S) \le c_m(X)\mathbb{M}(\partial S)$ for all $S \in \mathbf{N}_{m+1}(X)$. This *linearly isoperimetric* condition has been studied by many people (Gromov,. . . ,Wenger).

Duality

Theorem. *X is m bounded* $\iff$ $\{\partial S \ : \ S \in \mathbf{N}_{m+1}(X)\}$ *is* $\mathcal{T}_{\mathbf{N}}$ *closed.*
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In 1974 Federer proved a duality theory using real flat chains and flat cochains for the category of Euclidean Lipschitz neighborhood retracts. Our goal with normal chain homology and charge cohomology is to understand *metric properties* of more general spaces such as varieties, fractals, or Gromov-Hausdorff limits of manifolds.

Questions

(1) Determine when various specific spaces are m bounded.

(2) Interpret $\mathbf{H}_m(X)$ and $\mathbf{H}_m(X)$ (as Banach spaces) for $m \geq 1$.

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HAPPY BIRTHDAY BLAINE !