Analytic Combinatorics in Several Variables

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Stony Book, 31 October, 2019

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Analytic Combinatorics in Several Variables

The aim of this project is to systematize (asymptotic) coefficient extraction from a wide class of naturally occurring multivariate generating functions. We aim to take a genuinely multivariate approach, and to improve over previous work in the areas of generality, ease of use, and suitability for effective computation. The topic is inherently interdisciplinary and uses complex analysis in one and several variables, asymptotics of integrals, topology, algebraic geometry, and symbolic computation. The application areas are many: we are particularly motivated by specific naturally occurring problems arising from areas such as multivariate recurrence relations, random tilings, queueing theory, and analysis of algorithms and data structures. If you are interested in working on these problems, please let us know.

Our methods use complex analysis and oscillatory integrals to analyse singularities of explicitly known generating functions. We deal with generating functions of the form $F(\mathbf{z}) = \sum a_{r_1} \ldots a_{r_d} z_1^{r_1} \cdots z_d^{r_d}$ which can locally be expressed in the form G/H with G and H analytic functions. The singular set of this function (zero-set of H) plays a large role.

People

The long-term organizers of this project are: **Robin Pemantle** | **Mark C. Wilson** | **Yuliy Baryshnikov** | **Stephen Melczer** | **Marni Mishna** .

Events related to this project

- Melczer, Mishna and Pemantle are organizing (and Baryshnikov and Wilson assisting with) an American Mathematical Society Mathematics Research Community for May-June 2020, with application deadline 2020-02-15. Participants should be between -2 and +5 years from award of PhD.

  Details and registration **HERE**

Outline

I Generating functions in combinatorics and probability

II Coefficients of series and complex variable methods

III Morse theoretic decompositions

IV Singularity theory (complex methods, part II)

Univariate generating functions

Let $\{a_n : n = 0, 1, 2, \ldots\}$ be a sequence of combinatorial interest. For example, suppose $a_n = p(n) =$ the number of partitions of the integer n.

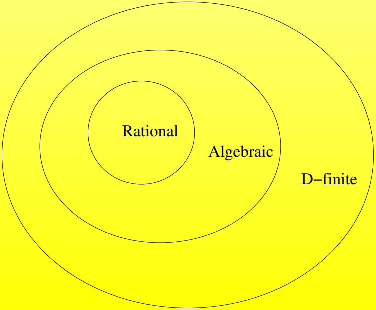
In order to study the sequence $\{a_n\}$, an particularly the asymptotics of a_n as $n \to \infty$, combinatorialists encode the sequence in a generating function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \,.$$

This sequence always exists as a **formal power series**, that is an element of $\mathbb{C}[[z]]$. When the series converges, analytic properties of the function f imply estimates on $\{a_n\}$.

Complexity classes

Recursions satisfied by $\{a_n\}$	$\longrightarrow$	niceness of f
linear recursion	$\longrightarrow$	rational function
convolution identity	$\longrightarrow$	algebraic function
polynomial recursion	$\longrightarrow$	D-finite function



Example: partitions

The generating function for integer partitions is given by

$$f(z) := \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1 - z^m} .$$

Hardy and Ramanujan famously used this to compute the leading asymptotic behavior

$$p(n) \sim \frac{1}{4\sqrt{3}\, n} e^{\pi\sqrt{2n/3}} .$$

This and all other such analyses via complex variable methods rely on Cauchy's integral formula

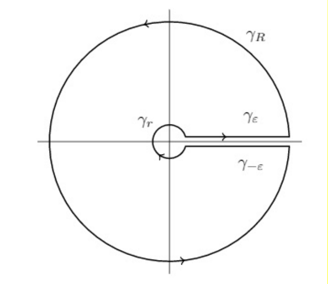
$$a_n = \frac{1}{2\pi i} \int_\gamma z^{-n-1} f(z)\, dz .$$

Singularity analysis

Over the decades from 1950 to 1990, the use of Cauchy's formula to determine asymptotics was methodized.

In 1956, Hayman showed how to apply the saddle point method to a wide class of generating functions.

In 1990, Flajolet and Odlyzko showed how the behavior of a generating function near an isolated or branch singularity auotmatically determines the coefficient asymptotics.



Multivariate generating functions

Multivariate generating functions are used

- to count combinatorial classes
- to compute recursively defined probabilities
- to encode "integrable" ensembles

Many applications: Queuing theory, lattice point enumeration, enumeration and analysis of search trees, transfer matrices, lattice paths, quantum walks, sequence alignment and matching, special functions and random tilings.

Scope of ACSV

ACSV extends the ideas of univariate singularity analysis to multivariate generating functions.

Much of the ACSV literature concentrates on rational functions.

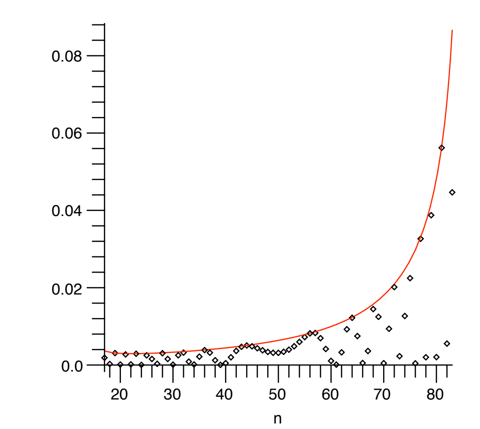
This loses less generality than you would think because all algebraic functions and many (conjecturally: all globally bounded) D-finite functions are representable as generalized diagonals of rational functions (Wilson and Raichev 2007, 2012; Christol 1990).

This implies that, unlike in the univariate case, coefficient asymptotics of multivariate rational functions exhibit a broad range of phenomena.

A PICTURE GALLERY OF PHENOMENA

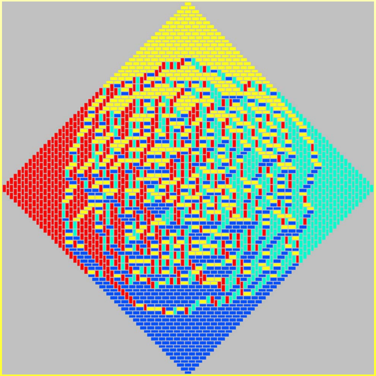
A one-dimensional example

This figure (dots) shows the amplitude squared of a one-dimensional quantum walk to be at any of N sites at time $N = 100$. The upper envelope (red) is a smooth algebraic curve. The phase factor moves at a well defined rate locally, changing with macroscopic location.



Random tilings

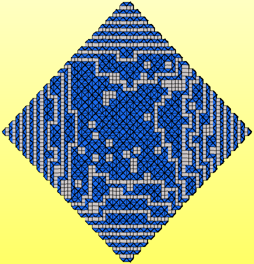
This figure shows a random tiling with four types of tiles. The 3-variable generating function for the probability of a red tile in position (i, j) in a tiling of size n is the relatively simple rational function below.



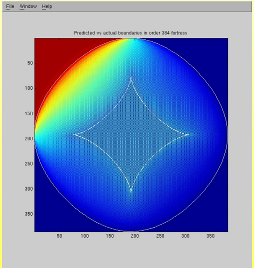
$$F(x, y, z) = \frac{y}{2(1 - yz)} \frac{1}{1 - (x + x^{-1} + y + y^{-1})\frac{z}{2} + z^2}$$

Another random tiling

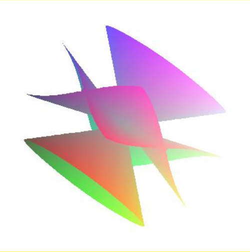
sample, size 47



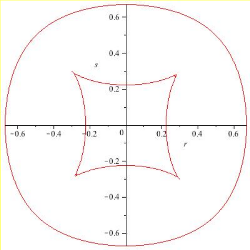
limit probs



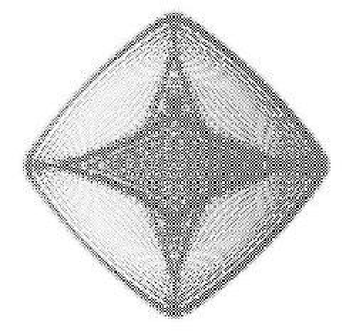
real zero set of denom. of GF



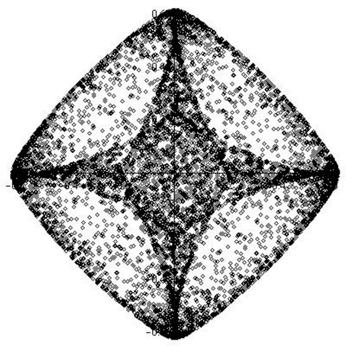
Dual curve to tangent cone of denominator:



2-D quantum walk

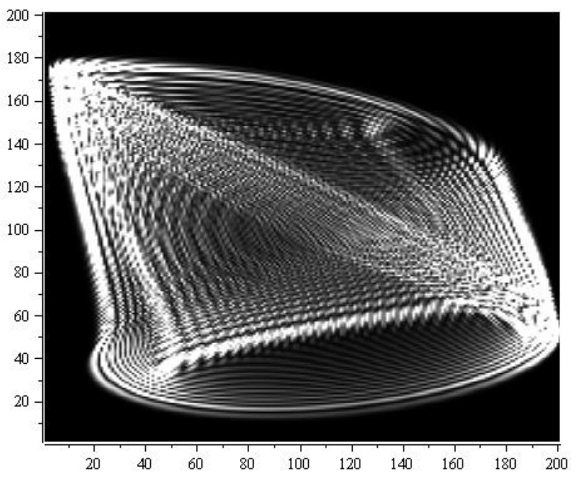


Amplitude of a 2-D quantum walk run to time 200



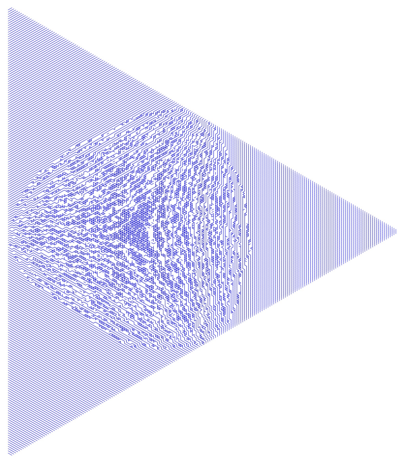
Sampled image of the log-Gauss map on the zero-set of the denominator on the unit 3-torus

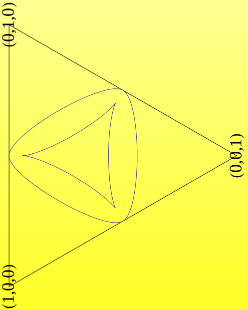
Another quantum walk



A double-dimer configuration

Double dimer configuratons are a combinatorial gadget obeying the hexahedron recurrence. The right-hand figure is the zero set of the algebraic dual of the denominator.





COMPLEX VARIABLE METHODS

Part I

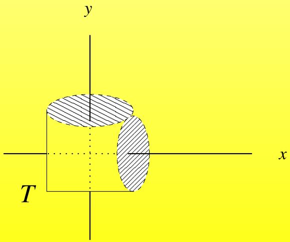
When $\mathcal{V} := \{Q = 0\}$ is smooth.

Getting from $F(X)$ to asymptotics for a_R

We begin with Cauchy's multivariate integral formula:

$$a_R = \left(\frac{1}{2\pi i}\right)^d \int_T Z^{-R} F(Z) \, \frac{dZ}{Z} \, .$$

Here T is a small torus, a product of circles winding once about the origin in each coordinate direction.

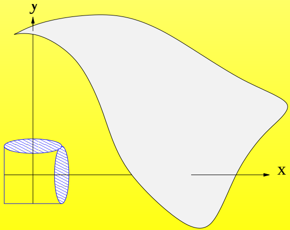


Domain of analyticity

If the generating function $F(Z) = P(Z)/Q(Z)$ is rational then F is analytic away from the singular variety

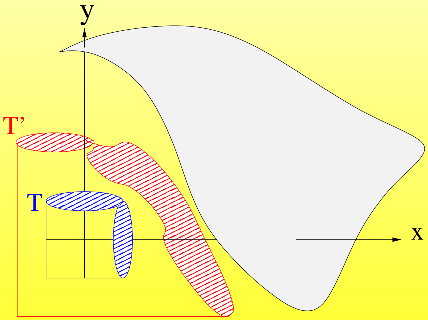
$$\mathcal{V} := \{Z : Q(Z) = 0\}\,.$$

More generally, one might have $F(Z) = G(Q(Z)^\alpha)$ or $F(Z) = G(\log Q(Z))$, where G is analytic but there is a branch singularity on $\mathcal{V}$.



Moving the chain of integration

We can move the contour of integration freely within the manifold $\mathcal{M} := (\mathcal{V} \cup \{\prod_{i=1}^{d} z_i = 0\})^c$.

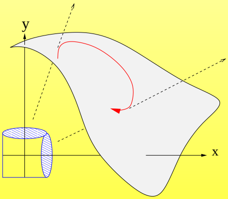


Once we cross $\mathcal{V}$, we pick up a residue.

Residue identity

The topological intersection of the d-cycle T with the variety $\mathcal{V}$ is a $(d-1)$-cycle $\mathcal{C}$, well defined at the level of homology.

The residue of the integrand $\omega := \mathbf{z}^{-\mathbf{r}}\mathbf{F}(\mathbf{z})\mathbf{dz}/\mathbf{z}$ is a $(\mathbf{d}-1)$-form $\mathrm{Res}(\omega)$.



The Cauchy integral reduces to an integral over $\mathcal{C}$:

$$(2\pi\mathbf{i})^{\mathbf{d}}\mathbf{a}_{\mathbf{R}} = \int_{\mathbf{T}} \omega = \int_{\mathcal{C}} \mathrm{Res}(\omega) \ .$$

Height function

To evaluate an integral containing the monomial $\mathbf{z}^{-\mathbf{r}}$ asymptotically as $\mathbf{r} \to \infty$, we need to push the contour "down" in the sense of making the term $\mathbf{z}^{-\mathbf{r}}$ as small as possible.

Equivalently, we want a contour minimizing the maximum value of

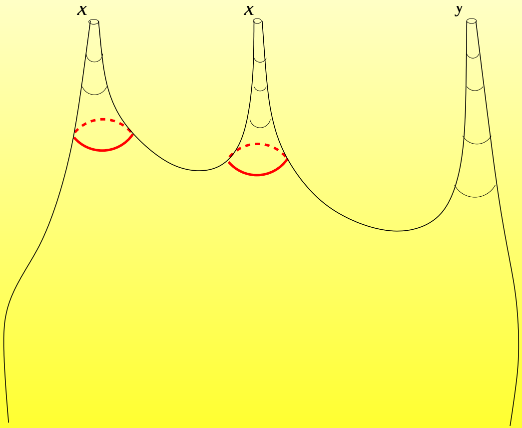
$$\mathbf{h}(\mathbf{z}) := -\mathbf{r} \cdot (\log |\mathbf{z}_1|, \ldots, \log |\mathbf{z}_\mathbf{d}|) .$$

In the next few pictures, $\mathcal{V}$ is drawn so that the height $\mathbf{h}$ decreases as you move down the screen.

$\mathcal{V}$ pictured by height

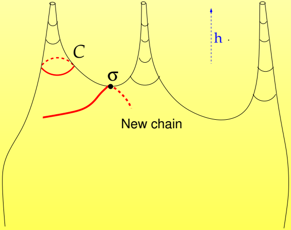
Where $\mathcal{V}$ intersects the **x**- or **y**-axis, height is infinite.

Expanding the 2-torus along the **x**-axis creates a ring (a 1-torus) around each such infinite height peak.

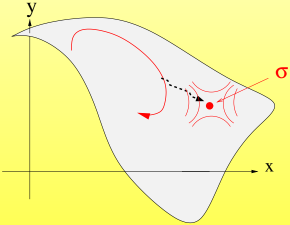


Pushing down = stationary phase

Pushing the contour down as far as it will go produces a stationary phase integral.



height view



regular view

The minimax is always located at critical point σ of h on $\mathcal{V}$.
The phase of the integrand is h, hence is stationary at σ.

Computer algebra

The critical points solve polynomial equations. Simplest case:

$$\begin{aligned} Q(z) &= 0 \\ r_d z_1 \frac{\partial Q}{\partial z_1}(z) &= r_1 z_d \frac{\partial Q}{\partial z_d}(z) \\ &\vdots \\ r_d z_{d-1} \frac{\partial Q}{\partial z_{d-1}}(z) &= r_{d-1} z_d \frac{\partial Q}{\partial z_d}(z)\,. \end{aligned}$$

Computer algebra can easily:

- Find the critical points.
- Compute the expansions of the residue form there.
- In most cases, read off asymptotic expressions such as

$$\int_{\text{near } \sigma} \sim \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \cdot \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \cdot \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{\sqrt{r^2+s^2}(r+s-\sqrt{r^2+s^2})^2}}$$

TOPOLOGICAL DECOMPOSITION

What's left to do

There may be many critical points.

1. How can we be certain the contour can be deformed to hang from a critical point?
2. To which such point(s) can the contour be deformed?
3. Once it gets into this position, what does the contour look like?

These questions are the province of Morse Theory.

Morse lemma: *Let $h : \mathcal{V} \to \mathbb{R}$ be a proper (stratified) Morse function*[1]*. Then the (stratified) downward gradient flow pushes any contour down to an attachment cycle at a critical point.*

[1] Plus some other technical assumptions.

Smooth Morse theory

In the smooth case, intersection cycle is a $(d-1)$-cycle in the $(2d-2)$-dimensional manifold $\mathcal{V}$ (thus, middle-dimensional).

The critical points of h on $\mathcal{V}$ all have middle dimension.

Morse theory tells us there is a set of cycles $\{\gamma_\sigma\}$ indexed by the critical points σ for h on $\mathcal{V}$, whose classes form a basis for $H_{d-1}(\mathcal{V})$, and such that h is maximized on γ_σ at σ.

[Locally, take the unstable manifolds for downward gradient flows.]

Using this basis

In the smooth case, integrals over these cycles are easy[2] to compute. One obtains an asymptotic formula $a_r \sim \phi_\sigma(r)$ looking something like $Cz^{-r}|r|^{-(d-1)/2}$ where C depends on r only through its direction $\hat{r} := r/|r|$.

All that remains is to resolve the original cycle into the basis

$$[T] = \sum_\sigma n_\sigma \gamma_\sigma$$

and the asymptotics can be read off. $\boxed{a_r \sim \sum_\sigma n_\sigma \phi_\sigma(r)}$

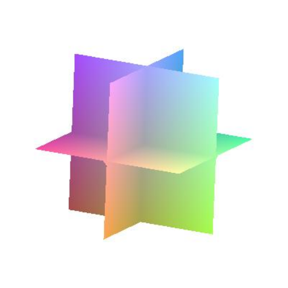
[2] Unless the "Morse" function fails to be quadratically nondegenerate at σ, or two or more critical points coalesce, or one seeks asymptotics approching one of these degenerate cases, etc.

When $\mathcal{V}$ is not smooth

$\mathcal{V}$ is a Whitney stratified space.

Critical points can appear on any stratum.

For example, if $d = 3$ and $\mathcal{V}$ looks locally like a union of three hyperplanes, it will have three one-dimensional strata and one zero-dimensional stratum.



Stratified Morse theory

Without going into the details, the stratified version of Morse theory tells you how to associate one or more homology generators with each critical point, giving again a homology basis for $H_{d-1}(\mathcal{V})$.

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OK if you really want details, there is a Künneth-like formula identifying the part of H_{d-1} local to σ via the attachment map with a product $H_{d-1-k}(N) \times H_k(S)$, where S is a stratum of co-dimension k and N is its normal slice.

Unfortunately the closest that (stratified) Morse theory comes to telling us how to decompose $[T]$ in the given basis is this:

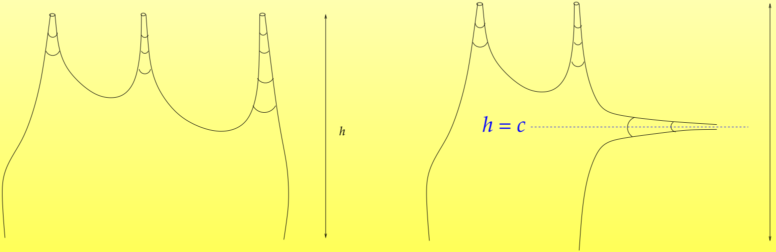
> *Use a (stratified) downward gradient flow to deform T until it lies below the critical height except near σ, then apply some more deformations and you will find the local component of T, at least in relative homology, up to arbitrary lower stuff (Stokes' phenomenon).*

And, what is worse, sometimes the Morse decomposition FAILS COMPLETELY and the cycles at critical points do not form a basis!

How can this happen? $\longrightarrow$

Properness

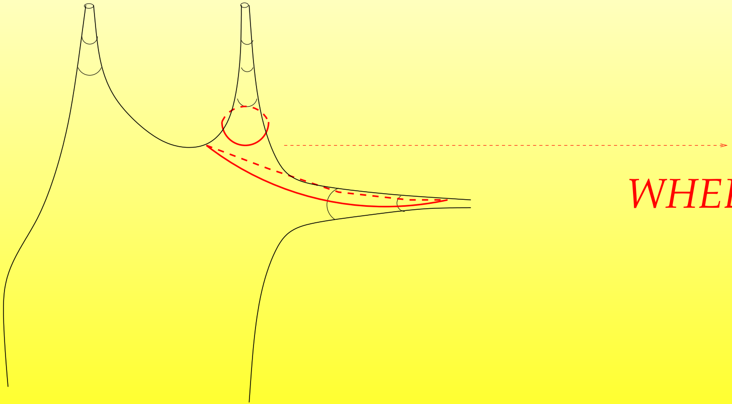
There are some technical assumptions, but the worst one is that h is proper. In the picture on the left, h is proper.



But in higher dimensions, the picture on the right is more typical. When $c \in (a, b)$, the inverse image of $h[a, b]$ is no longer compact. This means that the downward gradient flow could experience...

Shooting off to infinity

To infinity, and beyond!



WHEE !

The cycle goes to infinity before ever getting down to height c, and it never reaches a stationary phase point.

When can this happen?

This can happen. In some contexts it happens a lot!

For example, the algebraic function $\tilde{F}(x, y) = x/\sqrt{1 - x - y}$, is a diagonal of P/Q where

$$Q(x, y, z) = 2 + x + xy - z + 2xz + 2xyz + z^2x + z^2xy \, .$$

All downward flows on $\mathcal{V}_Q$ get forced to infinity.

This sort of obstruction can only occur when there is a critical point at infinity.

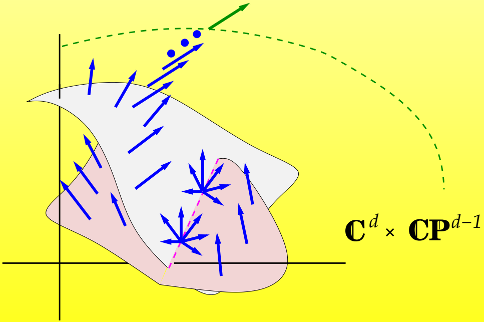
Critical points at infinity

The classical definition of a critical point at infinity is to solve the critical point equations (remember them?) projectively.

Unfortunately, this will inundate you with spurious solutions.

Disturbances at infinity

A disturbance at infinity in direction r is a point at infinity of the projective variety $\tilde{\mathcal{V}}$ which is the limit of points $z^{(n)}$ that are critical points for the height function $-r^{(n)} \cdot z$ where $r^{(n)} \to r$ as $n \to \infty$.



A disturbance at ∞ is a point at infinity on the closure of the relation over $\mathcal{V}$:

z is critical in dir. r

Computability of disturbances at infinity

Fortunately, computer algebra can detect disturbances at infinity. It is pretty straightforward to translate the definition into Gröbner basis computations, yielding the following steps.

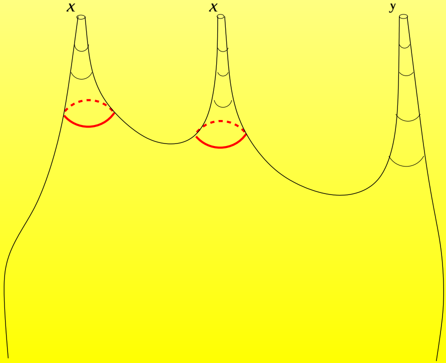
1. Projectivize Q
2. Let I be the projectivized ideal for the critical point equations as functions of both z and r
3. Saturate I by the projectivizing variable
4. Set the projectivizing variable to zero
5. Substitute to specify the r variables

Good news / bad news

The good news is that we can usually prove there are no critical points at infinity. The bad news is, now that we know the basis exists, we still have to decompose $[T]$.

We have effective algorithms for this only in some cases.

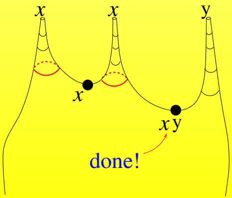
In this picture, for example, can you tell?



Two variables, $\mathcal{V}$ is smooth

Algorithm 1 (Find dominant critical point in 2D)

1. *Order the saddles by height.*
2. *Beginning with the highest, follow the two ascent paths. Each path must end at a pole or another saddle.*
3. *If both paths to a point marked x, then mark the saddle as x and continue; do similarly with a double y.*
4. *If one goes to x and one goes to y, you are done: output that saddle, as well as any of equal height that also go to x and y.*



More cases we can do

There are three more cases in which a decomposition algorithm is known.

- Product linear case (when $\mathcal{V}$ is a hyperplane arrangement)
- When the coefficients are now to be nonnegative (use Pringsheim's theorem)
- Certain symmetric functions: the Grace-Walsh-Szegö Theorem provides an analogue of Pringsheim's theorem

The general case is wide open.

Problem:

Find a general algorithm for the smooth tri-variate case.

COMPLEX VARIABLE METHODS

Part II

Integrating near singularities

[sorry, this part is going to go by very fast]

Techniques for computing the Cauchy integral when $\mathcal{V}$ has singularities and a residue does not yield a simple saddle point integral were developed in the context of PDE's by Atiyah, Bott and Gårding.

1. Transfer to logarithmic coordinates, e.g., $(x, y, z) = (x_0 e^X, y_0 e^Y, z_0 e^Z)$.
2. Homgenize at the origin.
3. Integrate over an imaginary fiber approaching the imaginary d-plane.
4. Use hyperbolicity to find semi-continuous families of cones.
5. Use cones to create deformations

6a. In some cases, use known generalized Fourier transforms.

6b. In other cases, take a residue and projectively integrate over the Leray or Petrovsky cycle.

First two steps

Step 1 (log): Transform to log coordinates $z = z_0 \exp(x + iy)$. The torus of integration turns into $x + i[-\pi, \pi]^d$.

To simplify, suppose $z_0 = (1, 1, \ldots, 1)$.

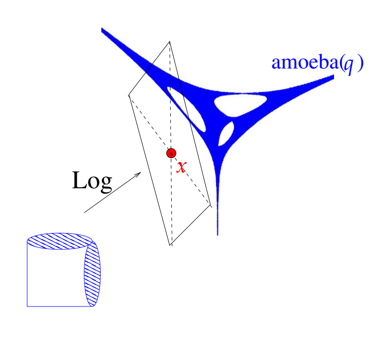
Step 2 (homogenize): Replace $q := Q \circ \exp$ by its homogenization at the origin. An N-term expansion in negative powers of $\tilde{q}$ with remainder shows that the integral of y^m/q is well approximated by the integral of $y^m/\tilde{q}$, up to a term whose order is $O(|x|^{m-N})$ for some m. We will be taking $x \to 0$, therefore, with $F \circ \exp = \tilde{p}/\tilde{q}$, replacing $\tilde{p}$ by $\tilde{\tilde{p}}$ obtained by subtacting off the finitely many monomials of degree less than m shows that

$$a_r \approx \int_{x+i\,[-\pi,\pi]^d} \frac{\tilde{\tilde{p}}}{\tilde{q}}\, dy \, .$$

Step 3: Imaginary fiber

The integral over $[-\pi, \pi]^d$ may be replaced by the integral over the imaginary fiber $x + i\mathbb{R}^d$, provided we are careful.

The integral is always defined when x is a point in the complement of the **amoeba** of Q, in the component corresponding to the domain of convergence of the series $\sum a_r z^r$.



Generalized inverse Fourier transform

The integral converges at infinity provided we define it via integration by parts (this is generalized function theory).

Now take $x \to 0$ to make sense of the *a priori* nonconvergent integral

$$\int_{\mathbb{R}^d} e^{ir \cdot y} \frac{\tilde{\tilde{p}}}{\tilde{q}} \, dy \, .$$

Adding back the low degree monomials, we see that the Cauchy integral that computes a_r is a generalized Fourier transform of $F \circ \exp = \tilde{p}/\tilde{q}$.

How to make this a local computation

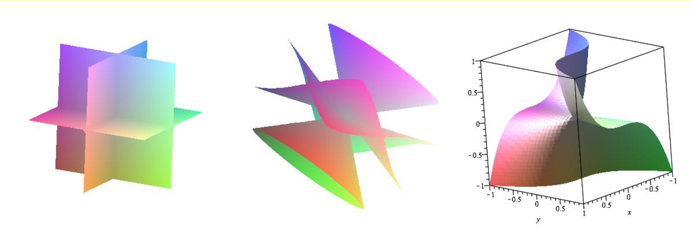
The next constructions show how to move the imaginary fiber **past** the origin, except in a small neighborhood. In terms of the Morse height function h, the contour goes **below** the critical height, except in a small neighborhood of σ, as Morse theory says should be possible.

The method for doing this relies on the notion of **hyperbolicity** of the homogeneous polynomial $\tilde{q}$.

Fun Fact: The notion of hyperbolicity was developed by Gårding in the 1950's for the study of PDE's and has recently been used to prove the Kadison-Singer conjecture.

Hyperbolicity

It would not make sense to try to give the definition at this late hour. Roughly, it has to do with not twisting in a way that make it impossible to define tangent cones consistently around the origin.



The first two pictures are hyperbolic at the origin; the third is not.

Consequences of hyperbolicity

In order to use the methods of [ABG70], we require:

Proposition 2

1. *For every* $Z := \exp(X + iY)$ *with* X *on the boundary of the amoeba of* Q*, the polynomial* p_Z *is hyperbolic.*
2. p_Z *has a cone of hyperbolicity contaning the support cone* B *to the amoeba complement at* X*.*

Family of cones

Step 4 (cones): The key construction for evaluating the Cauchy integral is a set of real cones.

Theorem 3 (semi-continuous family of cones)

Let p be any hyperbolic homogeneous polynomial and let B be a cone of hyperbolicity for p. There is a family of cones $K(x)$ indexed by the points x at which p vanishes, such that the following hold.

(i) *Each $K(x)$ is a cone of hyperbolicity for the tangent cone p_x.*

(ii) *All of the cones $K(x)$ contain B.*

(iii) *$K(x)$ is* **semi-continuous** *in x, meaning that if $x_n \to x$, then $K(x) \subseteq \liminf K(x_n)$.*

Vector field and deformations

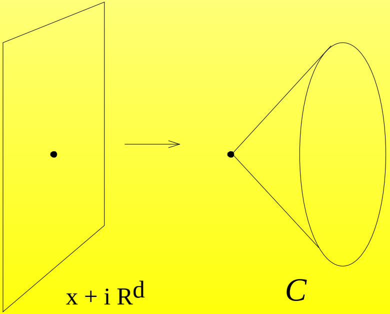
Step 5a (Vector field):

Theorem 4 (Morse deformation; BP2011, after ABG1970)

If $\{K(x)\}$ is a semi-continuous family of cones, then a continuous vector field Ψ may be constucted with $\Psi(x) \subseteq K(x)$ for each x.

Step 5b (deformation): This allows the chain of integration for the Cauchy integral to be deformed so that the integrand is very small except in a neighborhood of Z.

The deformation is locally projective.



The Cauchy integral and the Riesz kernel

Note to self: Skip this slide.

Recall the integral in logarithmic coordinates

$$a_R = (2\pi)^{(1-d)/2} \exp(-R \cdot X) \int \exp -(iR \cdot Y) f(Y)\, dY \,.$$

Pushing the chain of integration from the imaginary fiber outward in a conical manner produces a homogeneous inverse Fourier transform.

Leading asymptotic behavior only depends on leading behavior of the homogenization $1/p_Z$. We recognize the IFT

$$\int_\gamma p_z^{-1} \exp(iR \cdot Y)$$

as the Riesz kernel for the homogeneous polynoimal p_z.

Computing inverse Fourier transforms

The estimates needed to establish the existence of the integral rely on the projective deformation.

Inverse Fourier transforms (IFT's) may be manipulated via formal rules, e.g., $\widehat{x^m f} = (d/dr)^m \hat{f}$, $\widehat{fg} = \hat{f} * \hat{g}$, etc.

The infrastructure for this is laid out in the theory of **boundaries of holomorphic functions**, for example in Hörmander (1990).

Examples of computed IFT's

Step 6a (known cases): The simple cases are well known.

Example 5 (hyperplane)

For example, the IFT of a linear function $ax + by + cz$ is a delta function on the ray $\lambda\langle a, b, c\rangle$ in the dual space. (Here the index space $\mathbb{Z}^d$ is the dual space and the real/complex space in which the generating function variables live is the primal space.)

This corresponds to the fact that the coefficients a_r of $G := 1/((a + b + c) - ax - by - cz)$ decrease exponentially except along the ray $r = \lambda\langle a, b, c\rangle$.

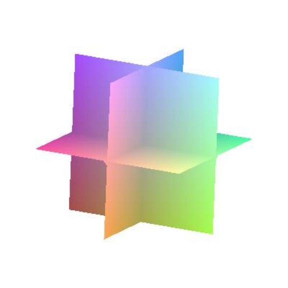
Multiplying a generating function by G corresponds to convolving the IFT with the ray $\lambda\langle a, b, c\rangle$.

Several planes

Example 6 (orthant)

The IFT of $1/(xyz)$ *is the constant 1 on the orthant. This corresponds to the generating function*

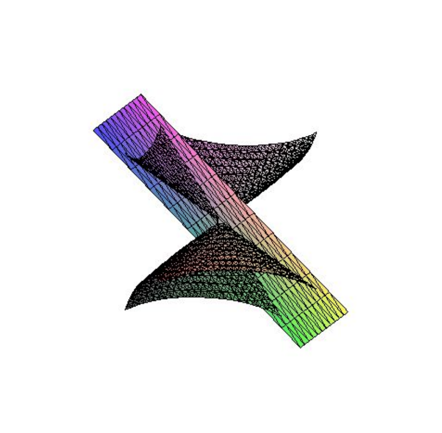
$$\frac{1}{(1-x)(1-y)(1-z)}\,.$$



Quadratic times linear

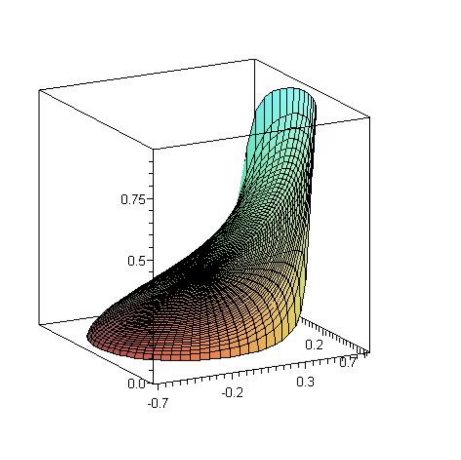
"Quadratic times linear" describes the homogeneous part of the Aztec Diamond probability generating function

$$\frac{z/2}{(1 - yz)\,[1 - (x + x^{-1} + y + y^{-1})z + z^2]}\,.$$

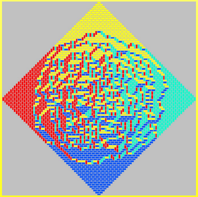


Example: Aztec diamond tilings

IFT is a convolution of a delta function on the ray $(0, \lambda, \lambda)$ with the IFT of a circular quadratic. The quadratic is self-dual, with IFT equal to $t^2 - r^2 - s^2$.



shown: plot of $(r, s) \mapsto \lim_{t \to \infty} a_{rt,st,t}$



General case

Step 6b (harder cases): There is no effective algorithm to read off inverse Fourier transforms for homogenous polynomials, say, from their monomial degrees and integer coefficients.

A roadmap is given by Atiyah-Bott-Gårding (Acta Math. 1970) "Lacunas for hyperbolic differential operators with constant coefficients, I and II." In particular, for degrees up to six a table of possible indecomposable singularity types is given (see Arnold, Gusein-Zade and Varchenko, *Singularities of Differentiable Maps*, volumes 1 and 2) and the first steps in the computation are taken.

Reductions

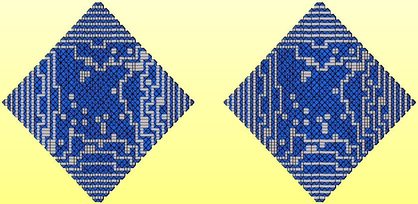
Because one is always computing IFT's of **homogenoeus** polynomials, the radial part of the integral may always be done first, reducing dimension by one.

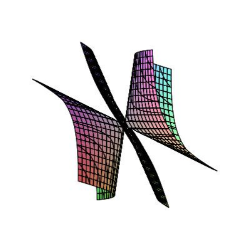
It is also always possible to take a single residue, resulting in a cyce of integration of dimension $d-2$ in $\mathbb{CP}^d$ with a known linking relation to the projectivization of $\mathcal{V}$. This will be either the **Leray** cycle or the **Petrovsky** cycle, depending on whether the degree of the denominator Q is respectively at least or less than the number of variables plus the degree of the numerator.

Details, adapted from [ABG70], are given in [BP11].

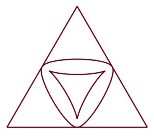
More to be done

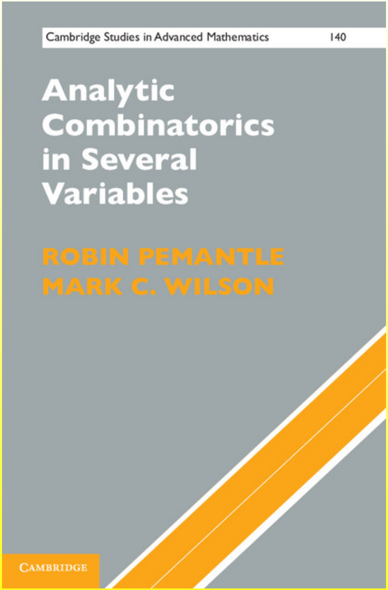
General cubic and quartic integrals are a little trickier to evaluate explicitly than are the quadratics and factored cubics. The so-called fortress tillings are an example of this (work in progress with Y. Baryshnikov).





This cubic arises in analysis of the hexahedron recurrence. Its IFT gives a limit shape theorem (work in progress). At present we can describe the feasible region but not the limit statistics within the region. For a particular parameter value, the central collar becomes a plane and the results for Quardatic times Linear apply.

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