New results on the global geometry of scalar curvature

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Curvature

Fundamental theme in Riemannian geometry:

Question (Local to global)

How does the curvature of a Riemannian manifold influence the global geometry & topology of the manifold?

Scalar curvature

Today:

Question (Local to global for scalar curvature)

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Question (Local to global for scalar curvature)

How does the scalar curvature of a Riemannian manifold influence the global geometry & topology of the manifold?

Scalar curvature is the weakest of the classical curvature invariants. It is hard for it to "transmit information" between different points of the manifold (e.g., compared to Ricci curvature).

Scalar curvature is trace of Ricci curvature, so think of tr $M \geq 0$ vs $M \geq 0$ for an $n \times n$ matrix M.

Scalar curvature & volume

Scalar curvature R measures the volume of *small* geodesic balls:

$$|B_r(p)| = \omega_n r^n \left(1 - \frac{R(p)}{6(n+1)} r^2 + O(r^3)\right)$$

as $r \to 0$.

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as $r \to 0$.

By contrast, Ricci curvature gives control on geodesic balls of all size. For example, $\mathrm{Ric} \geq 0$ implies that

$$|B_r(p)| \leq \omega_n r^n$$

for *all* $r > 0$.

The isoperimetric problem

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The list of situations in which the isoperimetric regions are "well" understood is *very short*, e.g.: simply connected space forms, some quotients and products of space forms, certain rotationally symmetric manifolds.

Small isoperimetric regions

We do understand the isoperimetric problem for small volumes in any manifold, and how it relates to scalar curvature. (Druet, Nardulli, Morgan–Johnson, Ye, Pacard–Xu, and others.)

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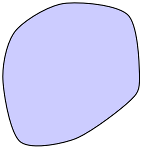
The above result is one of the only such result on isoperimetric regions in manifolds which are not “special,” i.e., it applies to manifolds with no symmetries.

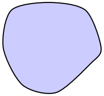
Small isoperimetric regions

small isoperimetric region



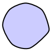
Small isoperimetric regions

smaller isoperimetric region



Small isoperimetric regions

tiny isoperimetric region



Small isoperimetric regions

minuscule isoperimetric region

Small isoperimetric regions

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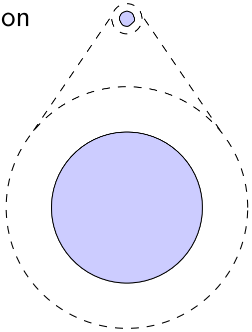
rescaled picture

metric and region
close to Euclidean



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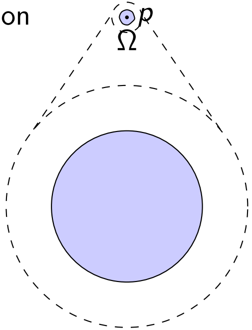

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$$|\partial\Omega| = n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}}(1 - c_n R_g(p)|\Omega|^{\frac{2}{n}} + \dots)$$

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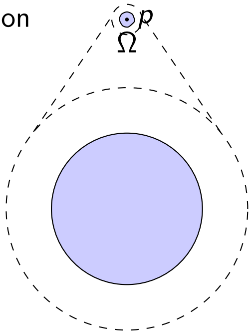

$$|\partial\Omega| = \underbrace{n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}}}_{\text{“}\mathbb{R}^n\text{”}}\underbrace{\left(1 - c_n R_g(p)|\Omega|^{\frac{2}{n}} + \dots\right)}_{\text{“}g,p\text{”}}$$

Scalar curvature & the isoperimetric problem

The expansion,

$$|\partial\Omega| = n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}}(1 - c_n R_g(p)|\Omega|^{\frac{2}{n}} + \dots)$$

implies:

Theorem (Druet, Nardulli, Morgan–Johnson)

Small isoperimetric regions are close to small geodesic balls, centered at points of maximal scalar curvature.

Scalar curvature & the isoperimetric problem in the large

What about the isoperimetric problem for other (large) volumes?

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In general, scalar curvature gives no information about the isoperimetric problem other than $V \to 0$ regime.

Scalar curvature & the isoperimetric problem in the large

Key for small volumes: metric is nearly flat $\Rightarrow$ isoperimetric region is nearly round.

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For large volumes: natural assumption is that the metric becomes Euclidean at large scales, i.e. is asymptotically flat.

Such metrics also arise naturally in General Relativity as initial data for the Einstein equations describing an isolated gravitating system.

Scalar curvature & the AF isoperimetric problem

Theorem (C.–Eichmair–Shi–Yu)

Suppose that (M^3, g) is asymptotically flat and has non-negative scalar curvature. Then, either (M^3, g) is flat space $(\mathbb{R}^3, g_{\mathbb{R}^3})$ or for V sufficiently large, there exists an unique isoperimetric region Ω_V containing volume V.

Note: no uniqueness for $\mathbb{R}^3$!

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Prior work by Bray, Eichmair–Metzger with more asymptotic symmetry.

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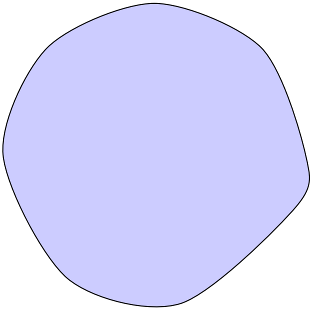

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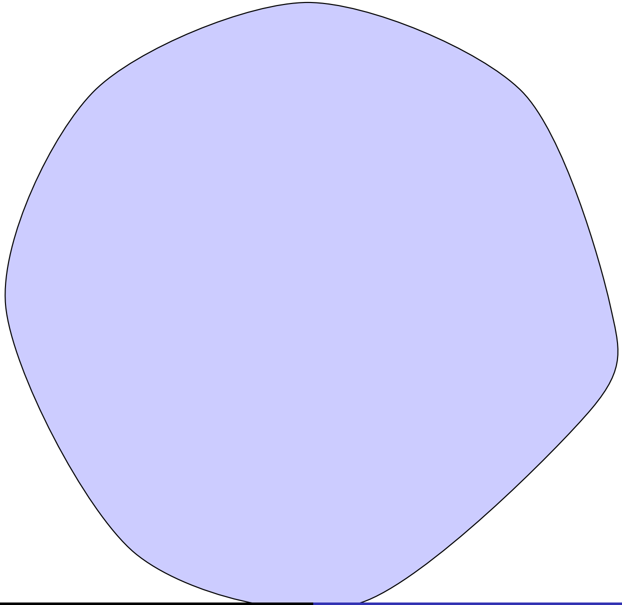

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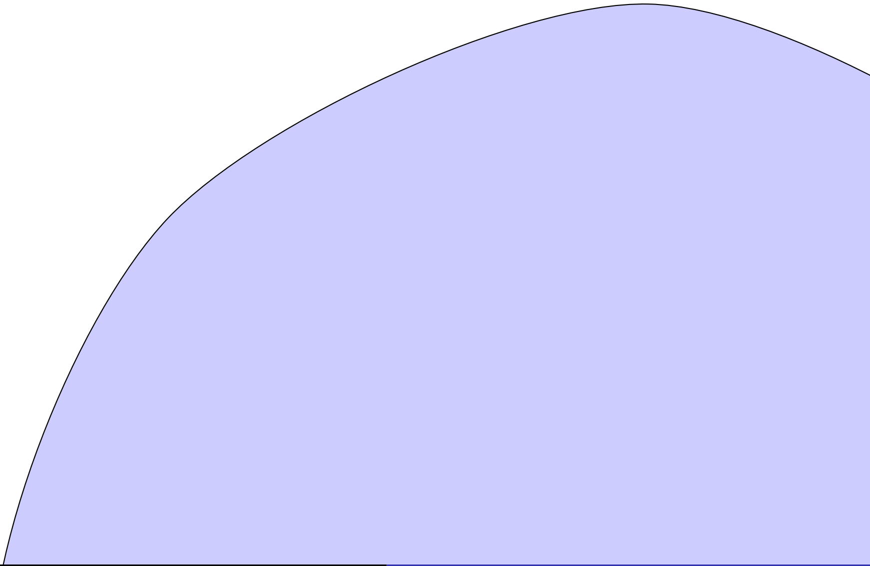

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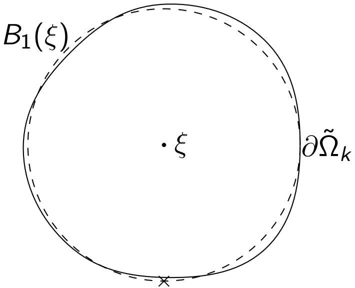

non-smooth convergence at center of rescaling!!

Scalar curvature & the AF isoperimetric problem

Analogy: Assume that $f(0) = f'(0) = 0$

$$\lambda^{-1} f(\lambda x) \xrightarrow{\lambda \to 0} 0$$

Get strong information about f near 0.

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On the other hand, assume that $f(x) = o(|x|^{-1})$ at infinity, so

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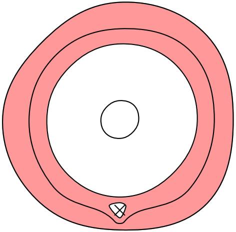
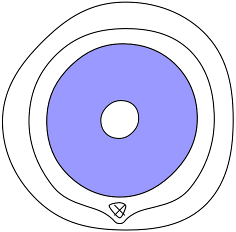
but the convergence is in, say, $C^0_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. Have no information about f in compact region!

Scalar curvature & the AF isoperimetric problem

New approach: combine Huisken's monotonicity of isoperimetric defect along mean curvature flow with analysis of Willmore energy/Hawking mass (flux integral, relating asymptotics of metric to "mass" and thus scalar curvature) of flowing surfaces.





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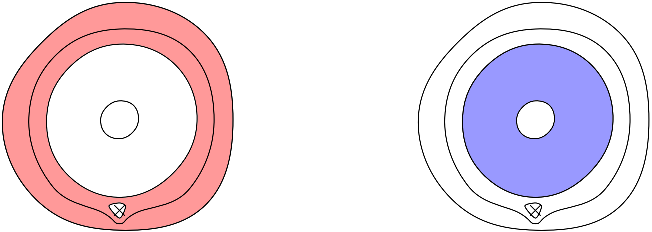

Red region: a priori bound on "Hawking mass" coming from inverse mean curvature flow (Huisken–Ilmanen). Blue region: off center, so "Hawking mass" is very small ⇒ behaves like region in $\mathbb{R}^3$.

Limits of large isoperimetric regions

Most extreme example of poor convergence of rescaled isoperimetric regions: Ω_j isoperimetric in asymptotically flat 3-manifold with $|\Omega_j| \to \infty$ but bounded inner radius, i.e.

$$\limsup_{j\to\infty} d(p, \Omega_j) < \infty.$$

Can show (Eichmair–Metzger) that a subsequence of $\partial\Omega_j$ limits to an area-minimizing surface. Thus: no area-minimizing surfaces $\Rightarrow$ inner radius tends to infinity.

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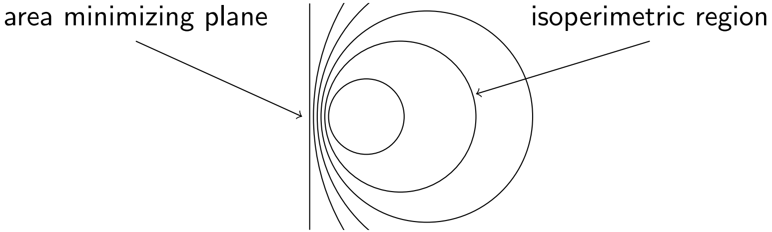

Area minimizing surfaces & scalar curvature

Theorem (C.–Eichmair)

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Builds on work of Anderson–Rodriguez and Liu.

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First difficulty: non-negative scalar curvature vs. the distance function. Compare to:

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Second difficulty: non-compactness of surface makes it hard to argue using its "surface area," which is a quantity that is "controlled" by scalar curvature in 3-manifolds (Schoen–Yau).