Beyond Euclidean rectifiability

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Rectifiable spaces

Definition

A metric measure space (X, d, μ) is n-rectifiable if there exists a countable family of Lipschitz maps $\{f_i : A_i \to X\}_{i=1}^{\infty}$ where $A_i \subset \mathbb{R}^n$ is Borel such that

1. $\mu\left(X \setminus \bigcup_i f_i(A_i)\right) = 0$,
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Non-example: Consider $(\mathbb{R}^2, \mu)$ where $\mu(A) = \mathcal{L}^1(A \cap ([0, 1] \times \{0\}))$. This satisfies (1) for $n = 2$, but for every $x \in [0, 1] \times \{0\}$ we have

$$\frac{\mu(B(x, r))}{r^2} \geq \frac{r}{r^2} \xrightarrow{r \to 0} \infty.$$

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3. Subsets and countable unions

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4. Represent low dimensional structure in high dimensional space (n-rectifiable measures in $\mathbb{R}^m$).

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Theorem (Preiss, 1987)

Let μ be a Radon measure on $\mathbb{R}^m$. Then $(\mathbb{R}^m, |\cdot|, \mu)$ is n-rectifiable iff $\Theta^n(\mu; x)$ exists and is positive μ-a.e. $x \in \mathbb{R}^m$.

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The von Koch snowflake $(X, d, \mu) = (\mathbb{R}, \sqrt{|\cdot|}, \mathcal{L}^1)$ satisfies

$$\lim_{r \to 0^+} r^{-2} \mathcal{L}^1(B(x, r)) = 2, \qquad \forall x \in X,$$

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Idea: Lipschitz maps $f : ([a, b], |\cdot|) \to X$ correspond to 2-Hölder maps on $[a, b]$ and so are constant. Same then holds for $f : (Y, \rho) \to X$ any Lipchitz path connected space (Y, ρ).

Lipschitz differentiability spaces

Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}^n$ if there exists a unique $Df(x_0) \in L(\mathbb{R}^n, \mathbb{R})$ so that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(|x - x_0|).$$

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We then say that $Df(x_0)$ is the (Cheeger) derivative of f at x_0.

Lipschitz differentiability spaces (cont.)

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A metric measure space (X, d, μ) is a n-dimensional Lipschitz differentiability space (LDS) if there is a Lipschitz chart $\varphi : X \to \mathbb{R}^n$ so that every Lipschitz function $f : X \to \mathbb{R}$ is φ-differentiable at μ-a.e. $x \in X$.

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Examples include Euclidean spaces (Rademacher), Carnot groups (Pansu), doubling spaces with the Poincaré inequality (Cheeger).

Characterization of rectifiability

Theorem (Bate-L.)

A metric measure space (X, d, μ) is n-rectifiable iff there exists a countable number of Borel sets $U_i \subset X$ with $\mu(X \setminus \bigcup_i U_i) = 0$ so that

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Done if φ_i and f_i are biLipschitz! (That is f_i, φ_i are Lipschitz, injective, and f_i^{-1}, φ_i^{-1} are Lipschitz). But not every Lipschitz function is biLipschitz. Need more general notion of being biLipschitz.

BiLipschitz decomposition

Definition

We say Lipschitz $f : (X, d_X) \to (Y, d_Y, \nu)$ is biLipschitz decomposable if there exists a countable number of Borel subsets $E_i \subset X$ so that $f|_{E_i}$ are biLipschitz and $\nu(f(X \setminus \bigcup_j E_j)) = 0$.

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The Heisenberg group

The Heisenberg group $\mathbb{H}$ is the Lie group $(\mathbb{R}^3, \cdot)$ where

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Define the norm $N : \mathbb{H} \to [0, \infty)$ by $N(x, y, z) = \max\{|x|, |y|, |z|^{1/2}\}$. The left-invariant metric is $d(g, h) = N(g^{-1}h)$ and the measure is $\mathcal{L}^3$.

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For each $\lambda > 0$, define the scaling automorphism $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$.

The Heisenberg group (cont.)

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4. $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y)$ for all $x, y \in \mathbb{H}$, $\lambda > 0$,
5. Left translation is measure-preserving,
6. $\mathcal{L}(\delta_\lambda(A)) = \lambda^4 \mathcal{L}(A)$. In particular $\mathcal{L}(B(x,r)) = cr^4$.

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Properties of $\mathbb{H}$

1. Same topology as $\mathbb{R}^3$,
2. Geodesic*,
3. Left translation is isometry,
4. $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$ for all $x, y \in \mathbb{H}$, $\lambda > 0$,
5. Left translation is measure-preserving,
6. $\mathcal{L}(\delta_\lambda(A)) = \lambda^4 \mathcal{L}(A)$. In particular $\mathcal{L}(B(x, r)) = cr^4$.

Almost Euclidean, except that it is nonabelian.

The Heisenberg group (cont.)

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Heisenberg rectifiability

Definition

A metric measure space (X, d, μ) is $\mathbb{H}$-rectifiable if there exists a countable family of Lipschitz maps $\{f_i : A_i \to X\}_{i=1}^{\infty}$ where $A_i \subset \mathbb{H}$ is Borel such that

1. $\mu\left(X \setminus \bigcup_i f_i(A_i)\right) = 0$,
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What geometric properties do these spaces have? Are the f_i's biLipschitz decomposable? (The f_i's are biLipschitz decomposable for n-rectifiable spaces by Kirchheim).

BiLipschitz decomposition (redux)

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$$\mathcal{L}(f(Z)) < \delta.$$

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What about Lipschitz maps $f : \mathbb{H} \to X$ where X is a metric measure space?

BiLipschitz nondecomposition

Recall (X, d, μ) is Ahlfors s-regular if there exist $C > 1$ so that

$$\frac{1}{C} r^s \leq \mu(B(x,r)) \leq C r^s, \qquad \forall x \in X, r < \operatorname{diam}(X).$$

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Theorem (Le Donne-L.-Rajala)

There exists a 4-regular metric space (X, d, μ) and a Lipschitz surjection $f : \mathbb{H} \to X$ for which $f|_A$ is not biLipschitz for any Borel $A \subseteq \mathbb{H}$ of positive measure.

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We then manually push together vertically separated points q_1, q_2 at many locations and scales in a careful way to collapse the metric but not the measure.

Open problems

1. Mattila proved for any Borel $E \subseteq \mathbb{R}^m$ that $(\mathbb{R}^m, \mathcal{H}^n|_E)$ is n-rectifiable iff $\Theta^n(\mathcal{H}^n|_E; x) = 2^n$ for $\mathcal{H}^n|_E$-a.e. $x \in \mathbb{R}^m$. Does this hold when $(X, d, \mathcal{H}^n)$ is a n-rectifiable metric measure space?

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3. What "nice" Ahlfors n-regular spaces X (besides $\mathbb{R}^n$) have biLipschitz decomposition for all Lipschitz maps $f : X \to (Y, \mathcal{H}^n)$ into arbitrary metric spaces?

Thank you!