Another one-dimensional model for the 3D Euler equation

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Motivation: 3D Euler

Consider an ideal (no viscosity, incompressible) fluid filling up a domain M with boundary ∂M.

Velocity field U satisfies the Euler equation

$$U_t + U \cdot \nabla U = -\nabla P, \qquad \operatorname{div} U = 0.$$

Take divergence of both sides to get $\Delta P = -\operatorname{div}(U \cdot \nabla U)$.

Boundary condition: no flow through boundary, so $\langle U, n \rangle = 0$ where n is the unit normal field. Hence $\langle \nabla P, n \rangle = -\langle U \cdot \nabla U, n \rangle$.

1. Begin with initial divergence-free velocity U_0.
2. Solve Neumann problem to find pressure P_0.
3. Use ∇P_0 as acceleration to get velocity U_1.
4. etc.

Difficult because pressure P is nonlocal: change velocity anywhere and pressure changes everywhere!

Local well-posedness: If U_0 is smooth enough (e.g., $C^{1+\alpha}$ for $\alpha > 0$ or H^s for $s > \dim M/2$) then $U(t)$ exists on a short time interval $|t| < \varepsilon$, $U(t)$ is as smooth as U_0, and $U(t)$ is a continuous function of U_0. (Wolibner, Ebin-Marsden, Kato, etc.)

Global well-posedness: if U_0 is smooth, does $U(t)$ remain smooth for all time t? True if $\dim M = 2$, unknown for $\dim M \geq 3$. For positive viscosity, this is a Millennium problem.

Even for axisymmetric fluids, it's open (believed to be false due to numerical results (Luo-Hou, 2014).

Vorticity formulation

Take curl of the Euler equation to get

$$\begin{cases} \omega_t + U(\omega) = 0 & \dim M = 2 \\ \omega_t + [U, \omega] = 0 & \dim M = 3. \end{cases}$$

We can reconstruct U from ω using the Biot-Savart law since $\operatorname{div} U = 0$ and $\operatorname{curl} U = \omega$.

- If $\dim M = 2$ then ω is a function transported by the flow.
- If $\dim M = 3$ then ω is a vector field which may be stretched.

Theorem (Beale-Kato-Majda, 1984): If $\int_0^T \|\omega(t)\|_{L^\infty}\, dt < \infty$ then the solution can be continued past time T.

Lagrangian analysis

Let η denote the flow of U, satisfying $\dot{\eta} = U \circ \eta$, $\eta(0) = \mathrm{id}$. The Euler equation becomes

$$\ddot{\eta}(t) = -\nabla P \circ \eta, \qquad \det D\eta \equiv 1.$$

Vorticity:

- If $\dim M = 2$ then $\omega(t, \eta(t,x)) = \omega_0(x)$.
- If $\dim M = 3$ then $\omega(t, \eta(t,x)) = D\eta(t,x)\omega_0(x)$.

Vorticity growth comes from $D\eta \to \infty$ (or $D\eta \to 0$). And $D\eta$ satisfies

$$\frac{d^2}{dt^2} D\eta = -\nabla^2 P \circ \eta D\eta,$$

a linear ODE along a particle path.

Special paths

$r = 0$ and $r = 1$ are preserved (axis of symmetry and boundary). Write the flow as

$$\eta(t, r, \theta, z) = \big(\alpha(t, r, z), \theta + \beta(t, r, z), \gamma(t, r, z)\big).$$

Then $\alpha(t, 0, z) = 0$ and $\alpha(t, 1, z) = 1$.

If U_0 is odd through $z = 0$, then U will remain odd and $z = 0$ is a fixed point of the flow.

For example on the axis we have $\rho(t) = \alpha_r(t, 0, 0)$ satisfying

$$\ddot{\rho}(t) = \frac{b_0^2}{\rho(t)^3} - P_{rr}(t, 0, 0)\rho(t),$$

where b_0 is the initial "swirl" at the origin. This is the Ermakov-Pinney equation. Describes the radial motion of a harmonic oscillator in the plane with angular momentum b_0 and radial force $-P_{rr}$. If $\rho(t) \to 0$ then we have blowup.

Riemannian geometry

Arnold (1966) noticed that the Euler equation $\eta_{tt} = -\nabla P \circ \eta$ and $\det D\eta \equiv 1$ is formally the geodesic equation on the group $\mathrm{Diff}_{\mathrm{vol}}(M)$, the group of volume-preserving diffeomorphisms. That is, fluids locally minimize the length determined by kinetic energy when constrained by volume.

Ebin-Marsden (1970) proved that in the context of Sobolev H^s diffeomorphisms, the geodesic equation is actually smooth. In other words, the Euler PDE is actually an infinite-dimensional ODE, which does not lose derivatives! (This almost never happens.)

Thus there is an exponential map which takes initial velocity U_0 to final position $\eta(1)$, and this map is *smooth*. (The data-to-solution map $U_0 \mapsto U(1)$ is not smooth, or even uniformly continuous; Himonas-Misiołek, 2010.)

The derivative of the exponential map is a linear map from one Hilbert space to another.

- Is it invertible? (If not, conjugate points.)
- If not, is the kernel finite-dimensional?
- Is the cokernel finite-dimensional?
- How can one find the singular points? What do they mean?

If the kernel and cokernel are always finite-dimensional, the exponential map is called Fredholm.

Theorem: (Ebin, Misiołek, P.) If $\dim M = 2$ and $\partial M = \emptyset$, then the exponential map is Fredholm. If $\dim M = 3$ it is not.

Failure of Fredholmness relates conjugate points to blowup via BKM (P. 2010).

A good model of 3D Euler should have:

- smooth exponential map
- non-Fredholm exponential map
- energy conservation
- vorticity stretching
- velocity determined nonlocally from vorticity
- a BKM criterion for blowup via vorticity

The only known model in one dimension having all of these is the Wunsch equation, which is the topic of this talk (finally!).

The Wunsch equation

Vorticity form:

$$\omega_t + u\omega_\theta + 2u_\theta\omega = 0, \qquad \omega = Hu_\theta.$$

Here H is the Hilbert transform. Intuitively: H is a (nonlocal) rotation, and Hu_x is like a curl.

Operator form:

$$Hf(\theta) = \frac{1}{\pi} P.V. \int_0^{2\pi} \cot \tfrac{\theta-\psi}{2} f(\psi)\, d\psi = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|\theta-\psi|>\varepsilon} \cot \tfrac{\theta-\psi}{2} f(\psi)\, d\psi.$$

(Like the Biot-Savart operator that recovers U from curl U.)

Fourier coefficient form: if $f(\theta) = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}$ then

$$Hf(\theta) = \sum_{n\in\mathbb{Z}} -i \operatorname{sgn} n f_n e^{in\theta}.$$

Hilbert transform

- If $f\colon S^1 \to \mathbb{R}$, then there is a unique $\phi(x, y)$ such that $\phi_{xx} + \phi_{yy} = 0$ for $y > 0$ and $\phi(e^{i\theta}) = f(\theta)$.
- This ϕ is harmonic and has a harmonic conjugate $\psi(x, y)$ satisfying the Cauchy-Riemann equations $\phi_x = \psi_y$ and $\phi_y = -\psi_x$.
- Then $g(\theta) = \psi(e^{i\theta})$ is the Hilbert transform of f. (Determined uniquely by condition that $\int_{S^1} \psi = 0$.)

In other words, $g = Hf$ iff there is a complex analytic function F on the unit disc such that $F(e^{i\theta}) = f(\theta) + ig(\theta)$.

For example $H(\cos n\theta) = \sin n\theta$ using $F(z) = z^n$ and $H(\sin n\theta) = -\cos n\theta$ using $F(z) = -iz^n$, assuming $n > 0$. Briefly $H(e^{in\theta}) = -i \operatorname{sgn} n e^{in\theta}$.

Consequences if $\int_{S^1} f = 0$:

- $H^2 f = -f$ since iF is also analytic if F is.
- $fHg + gHf = H(fg - HfHg)$ since FG is also analytic if F and G are.

Define $\Lambda = H\partial_\theta$.

Λ is symmetric since

$$\int_{S^1} (fHg' - gHf')\, d\theta - \int_{S^1} (fHg' + g'Hf)\, d\theta = \int_{S^1} H(fg' - HfHg')\, d\theta = 0.$$

Λ is positive-definite since $e^{in\theta}$ is an orthogonal basis of eigenvectors with $H(e^{in\theta}) = |n|e^{in\theta}$. (Question: how to prove this directly?)

Thus

$$\langle\langle u, v \rangle\rangle_{\dot{H}^{1/2}} = \int_{S^1} u\Lambda v\, d\theta$$

defines a metric on the space of mean-zero vector fields on S^1.

Consider the group $\mathrm{Diff}(S^1)$ of smooth diffeomorphisms of the circle, under composition.

It is a Fréchet manifold: tangent space at $\eta \in \mathrm{Diff}(S^1)$ is

$$T_\eta \mathrm{Diff}(S^1) = \{U \colon S^1 \to TS^1 \mid U(\theta) \in T_{\eta(\theta)} S^1 \forall \theta \in S^1\}.$$

In particular $T_{\mathrm{id}} \mathrm{Diff}(S^1)$ is the space of vector fields on S^1.

Left-translation is $(DL_\eta)(U) = D\eta(U)$ and right-translation is $(DR_\eta)(U) = U \circ \eta$. Define right-invariant metric on $\mathrm{Diff}(S^1)$ by

$$\langle\!\langle U, V \rangle\!\rangle_{\dot{H}^{1/2},\eta} = \int_M (U \circ \eta^{-1}) \Lambda (V \circ \eta^{-1}) \, d\theta$$

for $U, V \in T_\eta \mathrm{Diff}(S^1)$.

Geodesic equation for $\eta(t) \in \mathrm{Diff}(S^1)$ is (with $\omega = \Lambda u$)

$$\eta_t(t,\theta) = u\big(t, \eta(t,\theta)\big)$$

$$\omega_t(t,\theta) + u(t,\theta)\omega_\theta(t,\theta) + 2u_\theta(t,\theta)\omega(t,\theta) = 0$$

This is called the Euler-Arnold equation. The Wunsch equation comes from $\Lambda = H\partial_\theta$.

Other Euler-Arnold equations:

- If $\Lambda = 1$ on $\mathrm{Diff}(S^1)$ we get $u_t + 3uu_\theta = 0$. (Burgers')
- If $\Lambda = 1 - \partial_\theta^2$ on $\mathrm{Diff}(S^1)$ we get $u_t - u_{t\theta\theta} + 3uu_\theta - 2u_\theta u_{\theta\theta} - uu_{\theta\theta\theta} = 0$. (Camassa-Holm) (Kouranbaeva, Misiołek)
- If $\Lambda = -\partial_\theta^2$ on $\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$ we get $u_{t\theta\theta} + 2u_\theta u_{\theta\theta} + uu_{\theta\theta\theta} = 0$. (Hunter-Saxton) (Lenells)
- If $\Lambda = 1$ on Bott-Virasoro group we get $u_t + 3uu_\theta + u_{\theta\theta\theta} = 0$. (Korteweg-DeVries) (Ovsienko-Khesin)
- If $\Lambda = 1$ on $\mathrm{Diff}_{\mathrm{vol}}(M)$ we get $u_t + u \cdot \nabla u = -\nabla p$. (Ideal Euler) (Arnold)

Studying PDEs as geodesic equations allows us to:

- study stability using sectional curvature (Arnold);
- prove well-posedness using FTODE rather than PDE methods (Ebin-Marsden);
- understand blowup and weak solutions geometrically (Lenells).

Well-posedness:

- Note that $\omega_t + u\omega_\theta + 2\omega u_\theta = 0$, $u = (H\partial_\theta)^{-1}\omega$ is *not* an ODE for ω.
- However if $\eta_t(t,\theta) = u(t,\eta(t,\theta))$, then

$$\frac{\partial}{\partial t}\omega(t,\eta(t,\theta)) = -2\omega(t,\eta(t,\theta))u_\theta(t,\eta(t,\theta)).$$

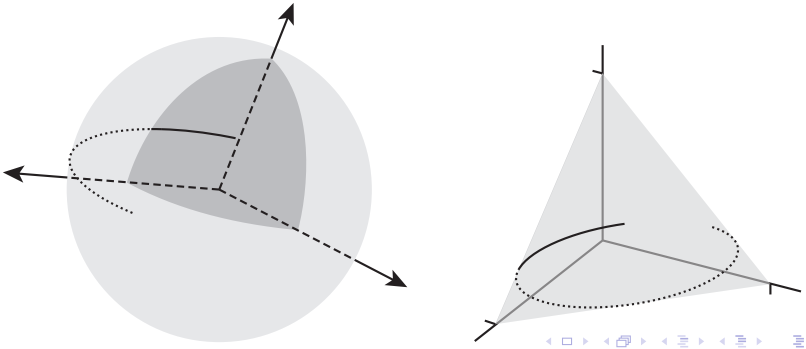
- In addition $\eta_{t\theta}(t,\theta) = u_\theta(t,\eta(t,\theta))\eta_\theta(t,\theta)$. We get vorticity conservation:

$$\eta_\theta(t,\theta)^2\omega(t,\eta(t,\theta)) = \omega_0(\theta).$$

- Solve for u_θ in terms of ω: we have $u_\theta = -H(\omega)$, and $u_\theta \circ \eta = -H_\eta(\omega_0/\eta_\theta^2)$ where $H_\eta f = H(f \circ \eta^{-1}) \circ \eta$.
- Thus $\eta_{t\theta}/\eta_\theta = -H_\eta(\omega_0/\eta_\theta^2)$, and this is an ODE for η_θ.

Curvature and blowup (Lenells, Hunter-Saxton):

- The sectional curvature for the $\dot{H}^1$ metric giving Hunter-Saxton is a *positive constant*.
- This implies it is isometric to a sphere. In fact the isometry is $\eta \mapsto \sqrt{\eta_\theta} = \rho$; note that $\int_{S^1} \rho^2 \, d\theta = 1$.
- The image of $\mathrm{Diff}(S^1)$ is the positive "octant." All geodesics leave it, but squaring a spherical geodesic gives a weak solution.



Recall Wunsch equation: $Hu_{t\theta} + uHu_{\theta\theta} + 2u_\theta Hu_\theta = 0$. Magic formula $2H(fHf) = (Hf)^2 - f^2$ implies

$$u_{t\theta} + u_\theta^2 - (Hu_\theta)^2 = H(uHu_{\theta\theta})$$

Notice:

$$\begin{aligned}
\eta_t(t,\theta) &= u(t,\eta(t,\theta)) \\
\eta_{t\theta}(t,\theta) &= u_\theta(t,\eta(t,\theta))\eta_\theta(t,\theta) \\
\eta_{tt\theta}(t,\theta) &= u_{t\theta}(t,\eta(t,\theta))\eta_\theta(t,\theta) + u_{\theta\theta}(t,\eta(t,\theta))\eta_\theta(t,\theta)\eta_t(t,\theta) \\
&\quad + u_\theta(t,\eta(t,\theta))\eta_{t\theta}(t,\theta) \\
&= [u_{t\theta} + uu_{\theta\theta} + u_\theta^2](t,\eta(t,\theta))\eta_\theta(t,\theta).
\end{aligned}$$

Thus

$$\eta_{tt\theta}(t,\theta) - \omega(t,\eta(t,\theta))^2\eta_\theta(t,\theta) = -F(t,\eta(t,\theta))\eta_\theta$$

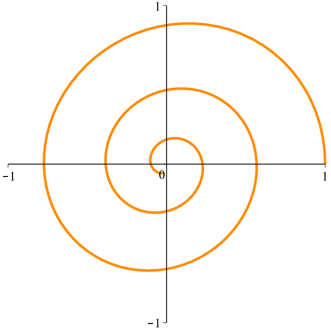
where $F = -H(uHu_{\theta\theta}) - uu_{\theta\theta}$.

Recall $\omega(t,\eta(t,\theta)) = \omega_0(\theta)/\eta_\theta(t,\theta)^2$, and thus we get

$$\ddot{\rho}(t) - \frac{\omega_0^2}{\rho(t)^3} = -f(t)\rho(t),$$

for $\rho(t) = \eta_{tt\theta}(t,\theta_0)$, with $f(t) = F(t,\eta(t,\theta_0))$.

This is the Ermakov-Pinney equation again! Planar harmonic oscillator, angular momentum ω_0, radial force $f(t)$. Heuristic example with $\omega_0 = 1$ and $f(t) = \frac{C}{(1-t)^2}$:



But what do we know about the mystery force $F = -H(uH_{u\theta\theta}) - uu_{\theta\theta}$?

Theorem (Bauer, Kolev, P.)

For any $f\colon S^1 \to \mathbb{R}$ *and any* $p > 0$ *we have*

$$H(fH\Lambda^p f) + f\Lambda^p f \geq 0$$

everywhere, where $\Lambda = H\partial_\theta$.

Proof: expand f in a Fourier series $f(\theta) = \sum_n f_n e^{in\theta}$. Manipulate series to get

$$H(fH\Lambda^p f) + f\Lambda^p f = 2\sum_{n=1}^{\infty}[n^p - (n-1)^p]|\phi_n|^2$$

where $\phi_n(x) = \sum_{m=n}^{\infty} f_m e^{im\theta}$.

Special case $p = 1$ discovered by Córdoba-Córdoba, special case $p = 2$ with f odd discovered by Castro-Córdoba.

In particular $H(fHf'') + ff'' \leq 0$ since $H^2 = -1$, so the mystery force is always positive!.

Intuition: blowup requires $\eta_x \to 0$. Angular momentum (initial vorticity) tries to prevent it. Mystery force tries to send "particle" to origin.

Beale-Kato-Majda criterion (Bauer-Kolev-P.): if $\int_0^T \|\omega(t)\|_L^\infty \, dt < \infty$ then existence up to time T. (Works the same way for Wunsch equation as for 3D Euler.)

Intuition: since $\omega \circ \eta = \omega_0/\eta_x^2$, blowup at T should require $\int_0^T dt/\rho(t)^2 = \infty$. But angular momentum means $\rho^2\dot{\theta}$ is constant, so we need $\theta(t) \to \infty$ as $t \to T$.

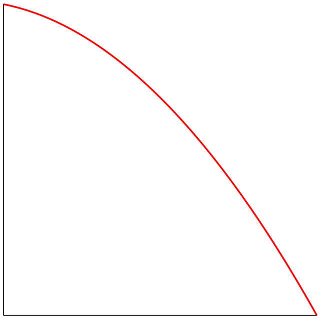
Still mysterious!

Special case: $\omega(x_0) = 0$. Then

$$\eta_{ttx}(t, x_0) = -F(t, \eta(t, x_0))\eta_x(t, x_0).$$

Now $\eta_x(0, x_0) = 1$. Since F is always positive, the function $t \mapsto \eta_x(t, x_0)$ is always concave down.

If $\eta_{tx}(0, x_0) = u_x(0, x_0) \leq 0$ then $\eta_x(T, x_0) = 0$ for some $T > 0$. (Set the controls for the heart of the sun.)

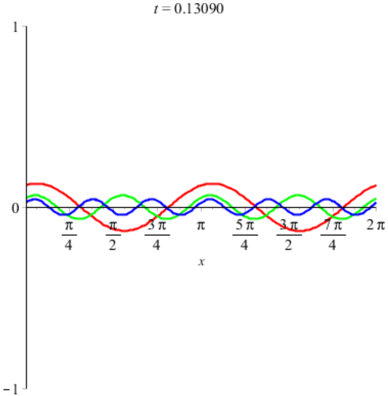


What 3D Euler and the Wunsch equation have in common:

- Smooth Riemannian exponential map on H^s Sobolev-class diffeomorphism groups.
- Exponential map is not Fredholm, due to too many conjugate points.
- Vorticity conservation law and Beale-Kato-Majda vorticity criterion for blowup.
- Flow map differential satisfies Ermakov-Pinney equation (sometimes).
- Intrinsic distance locally bounded (finite diameter for 3D fluids, zero distance for Wunsch equation).

Non-Fredholmness? Related to conjugate points. Geodesics locally minimize length between two points, but may not minimize globally. (E.g., on a sphere.)

Roughly, $\eta(a)$ and $\eta(b)$ are *conjugate* if some family of geodesics connects them with shorter length. Fredholmness implies there are at most a finite-dimensional family of length-shortening perturbations along any finite portion of a geodesic.



Open questions:

- Does every geodesic end in finite time?
- Are there infinitely many conjugate pairs along a geodesic that ends in finite time? (Probably yes.)
- Is failure of Fredholmness related to vanishing geodesic distance?
- What happens *near* the blowup location?
- Does the flow η remain smooth even if it fails to be a diffeomorphism (as happens for Camassa-Holm)?
- Does the "magic inequality" $H(fH\Lambda^p f) + f\Lambda^p f \geq 0$ generalize to higher dimensions, using e.g., Riesz transforms instead?