Geometry and Algebra of Reflections

Oleg Viro

September 18, 2014

Plane Isometries

Theorem. Any plane isometry is a composition of at most three reflections in lines.

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Lemma. A plane isometry is determined by its restriction to any three non-collinear points. □

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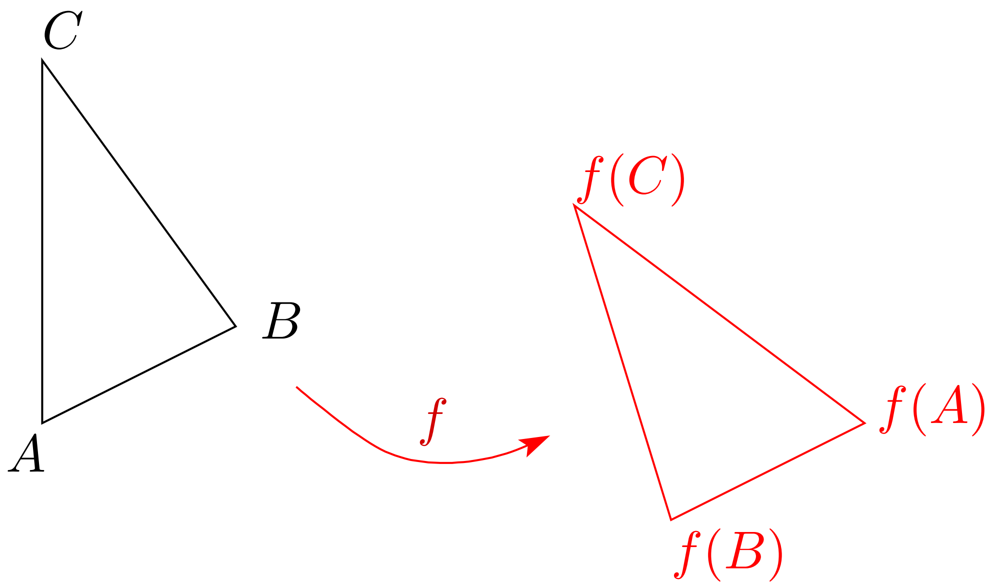
Proof of Theorem. Given an isometry:



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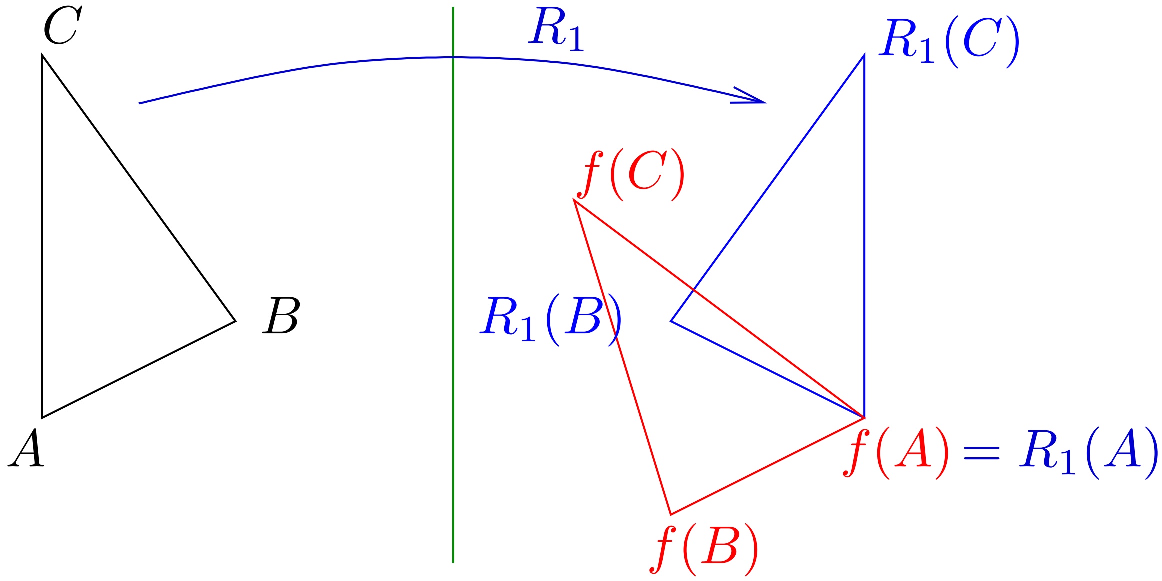

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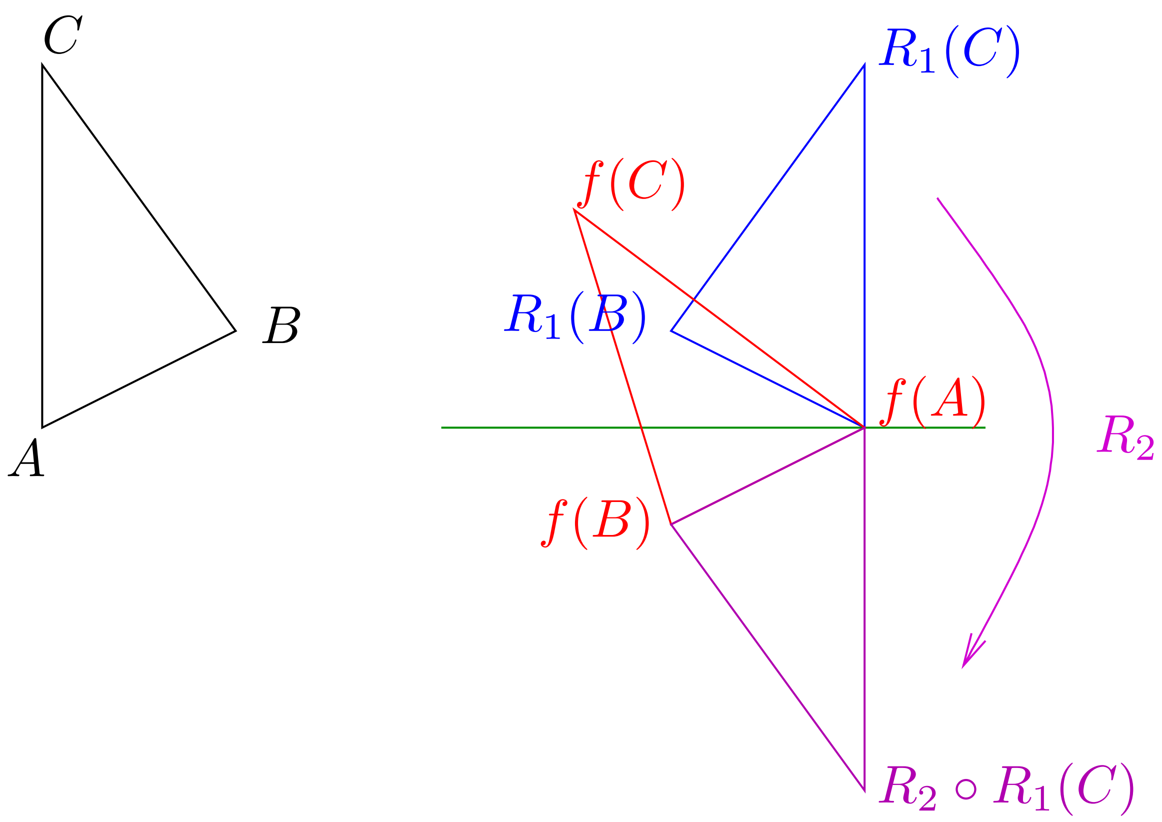

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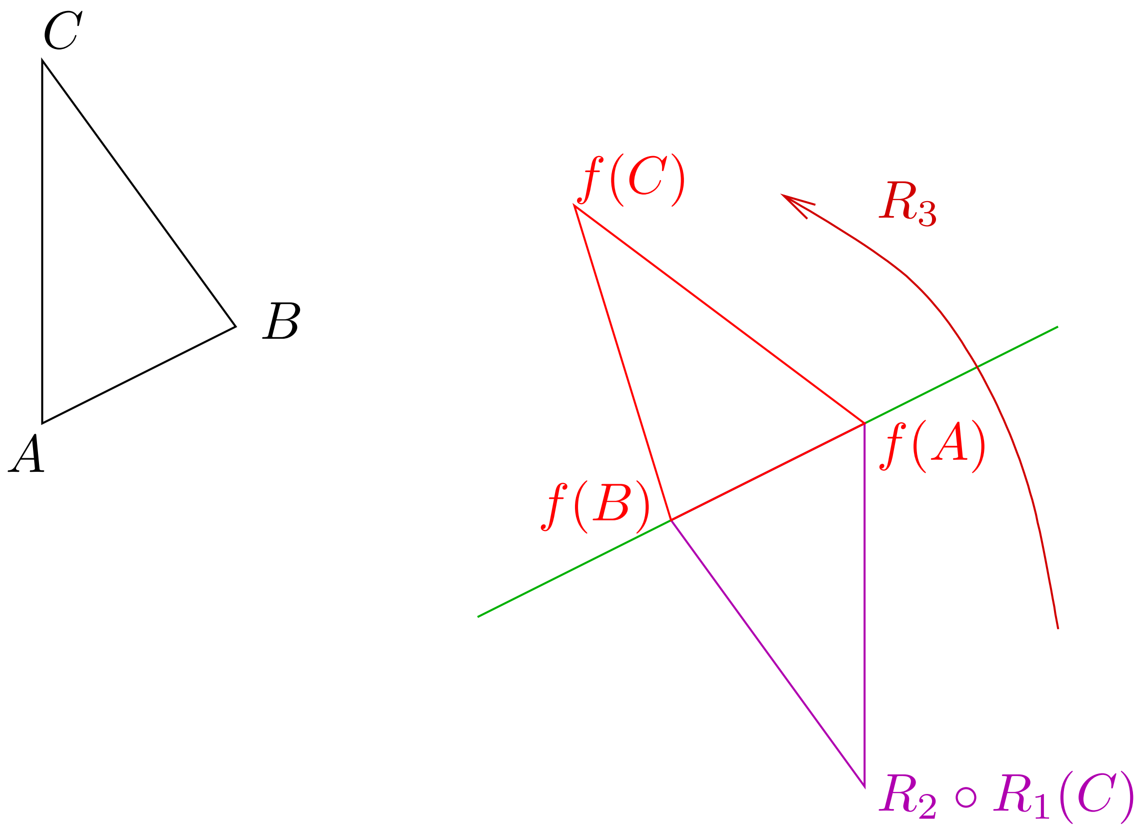

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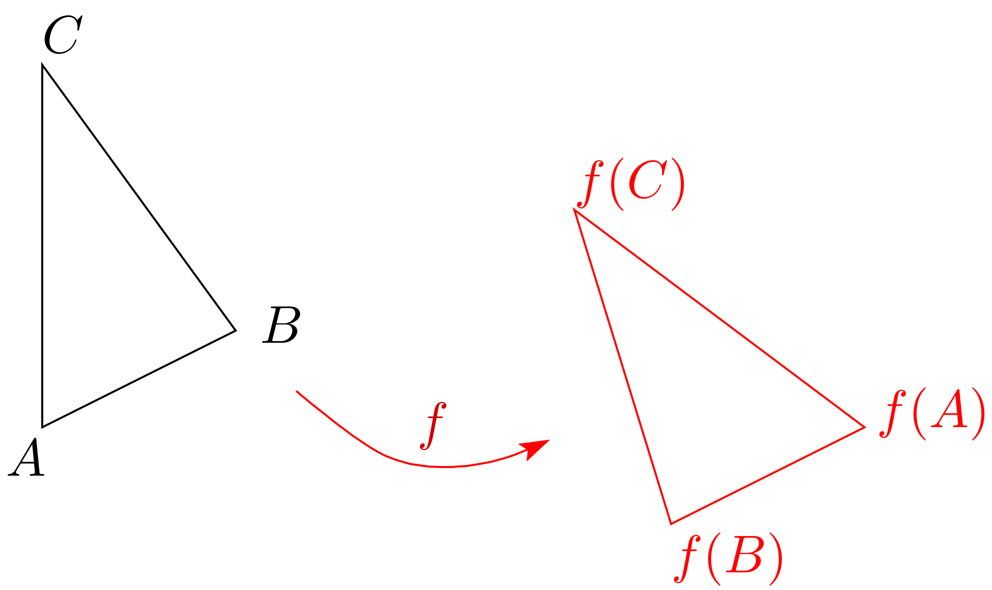


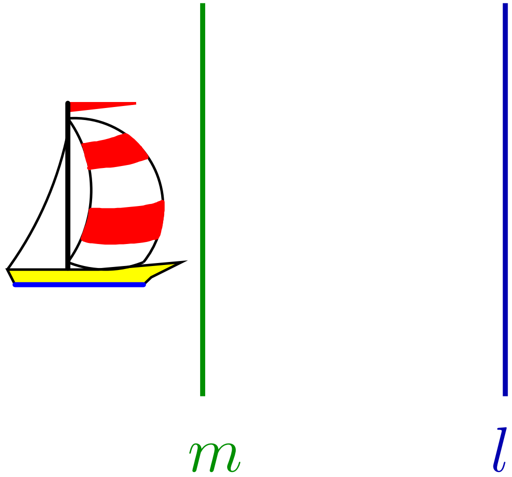
We are done. □

Plane Isometries

Theorem. Any plane isometry is a composition of at most three reflections in lines.

Generalization. Any isometry of a complete simply connected n-space of constant curvature is a composition of at most $n+1$ reflections in hyperplanes.

Compositions of reflections in parallel lines



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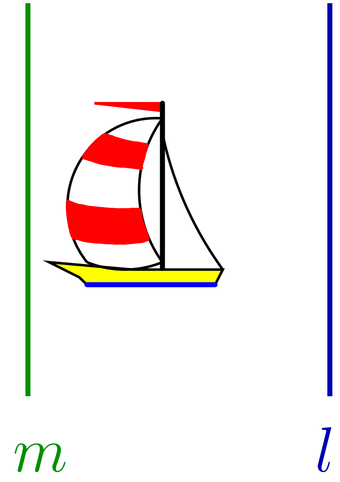

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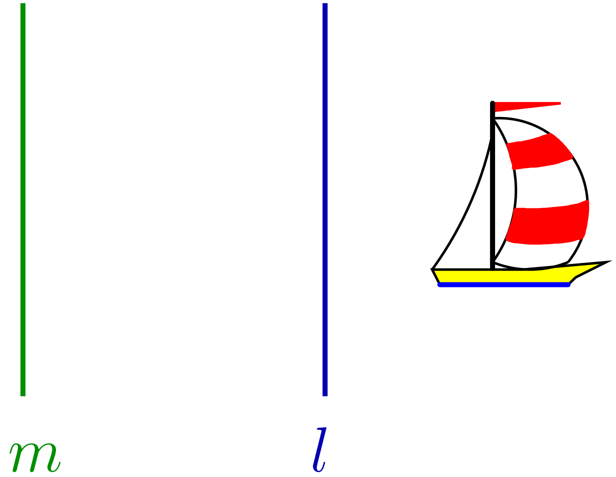

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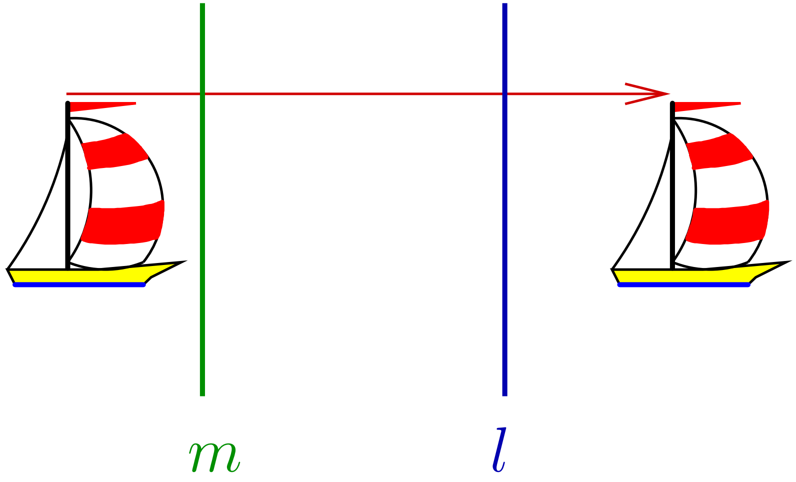

Compositions of reflections in parallel lines

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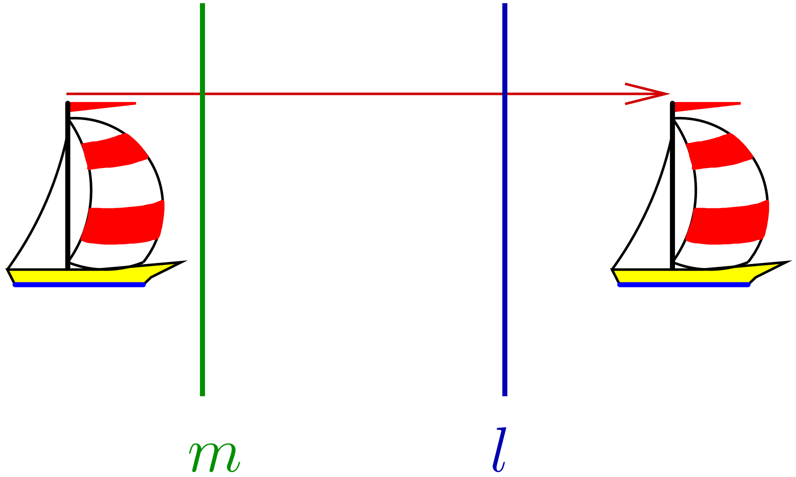

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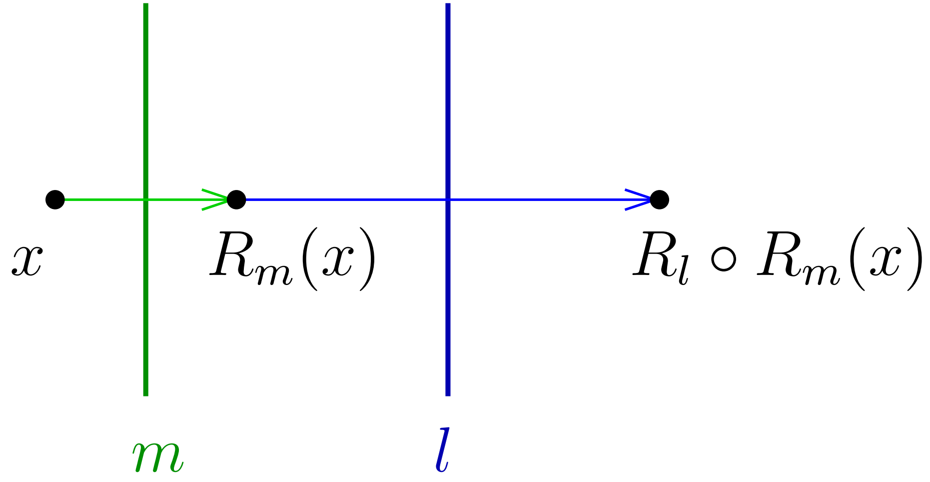

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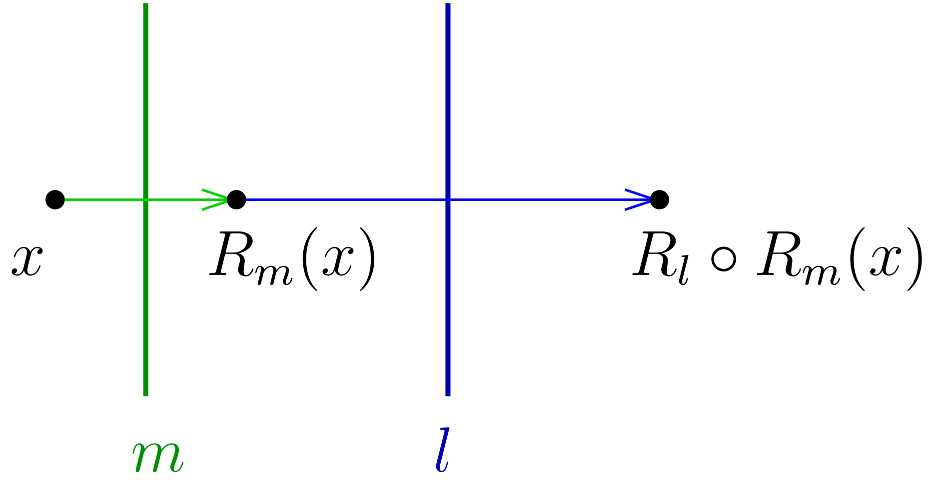

The decomposition is not unique:

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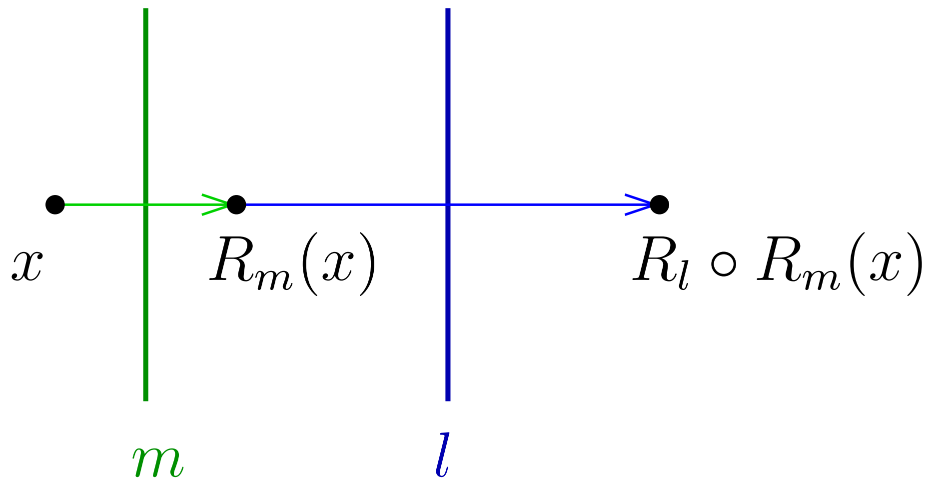

The decomposition is not unique:

$R_l \circ R_m = R_{l'} \circ R_{m'}$

iff l', m' can be obtained from l, m by a translation.

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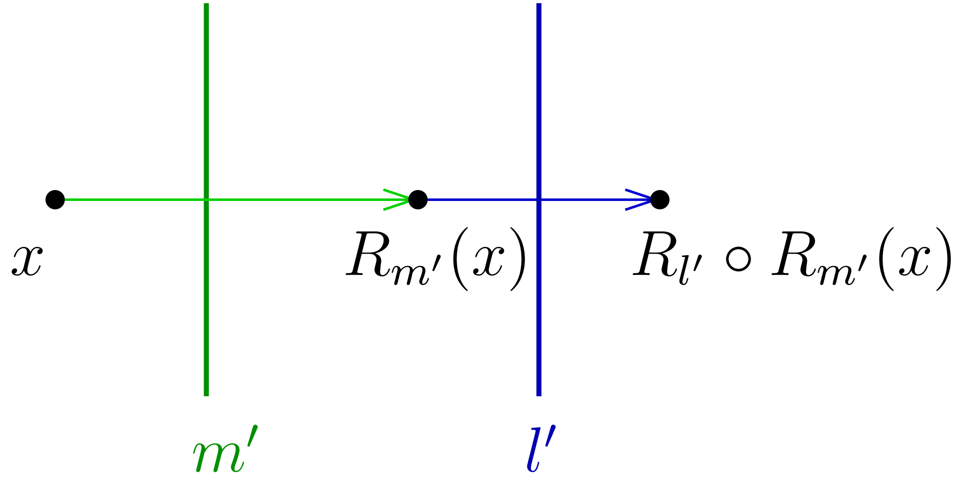

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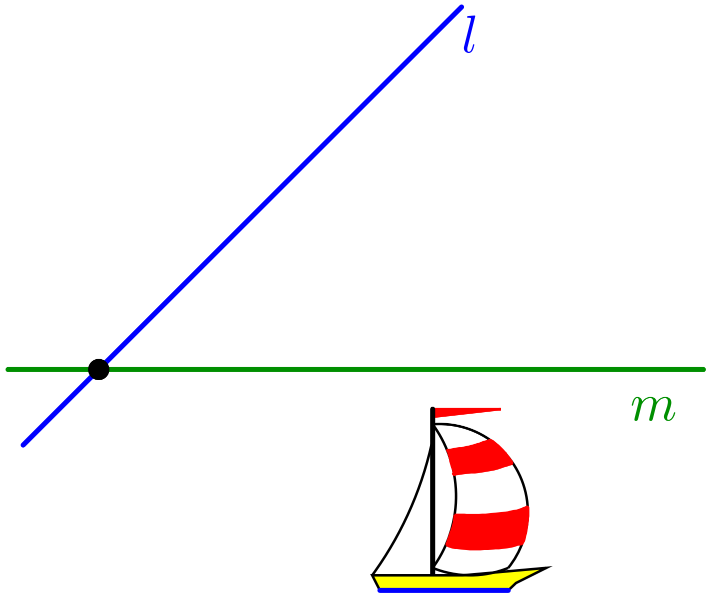

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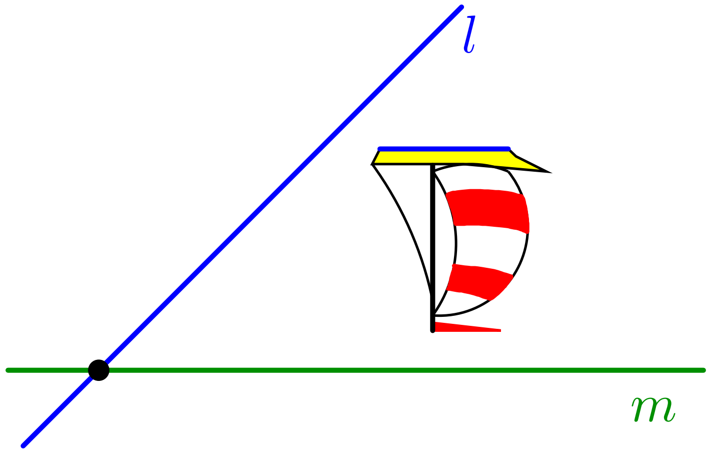

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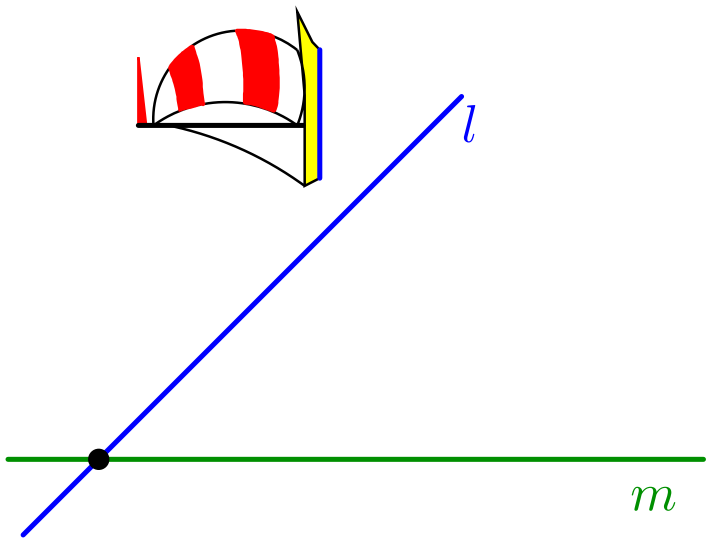

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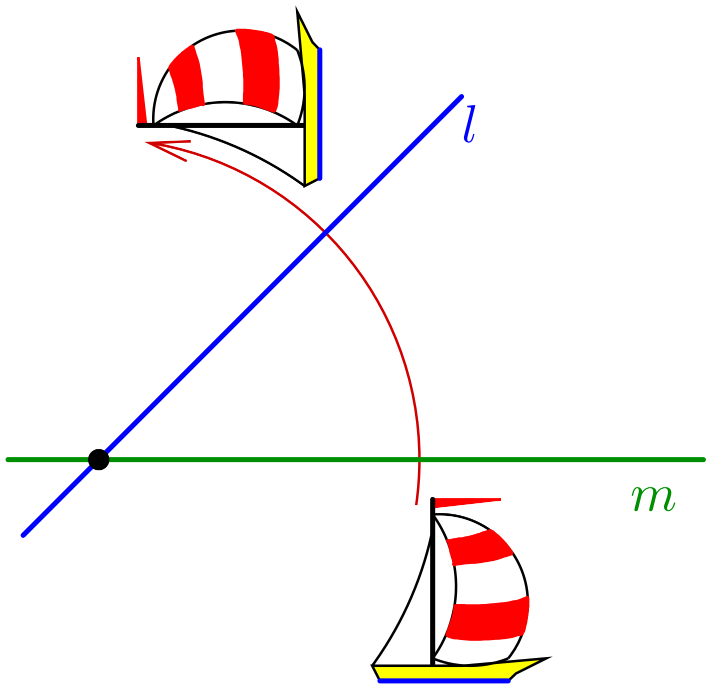

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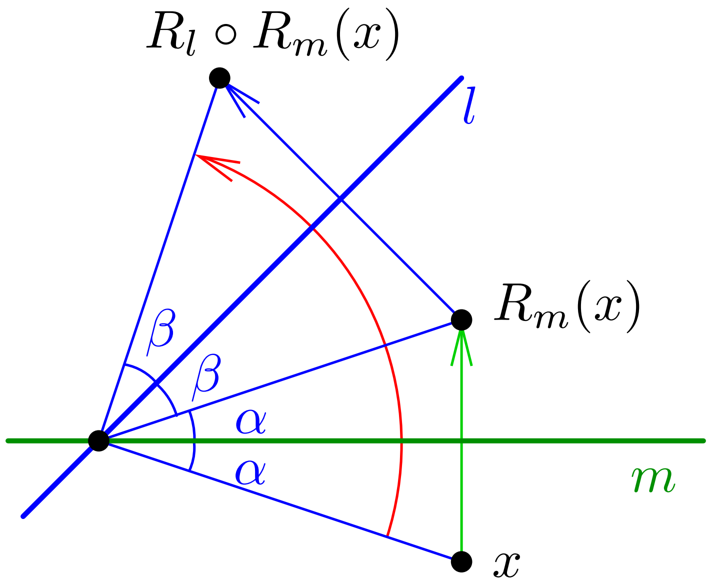

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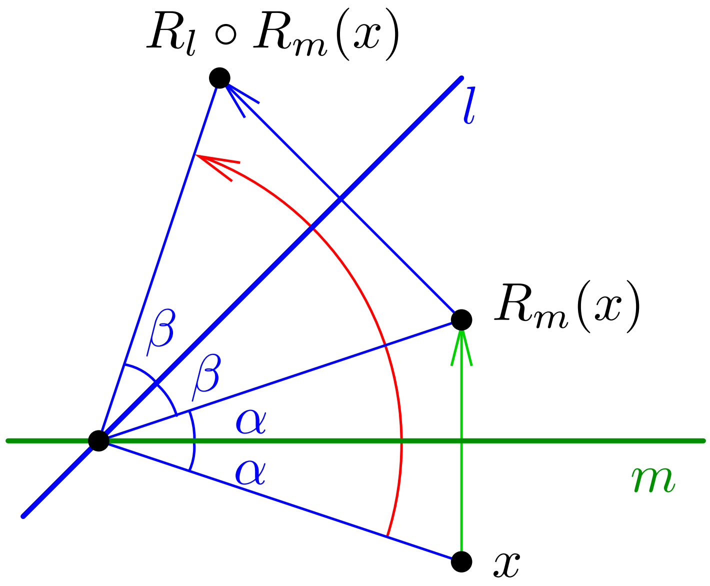

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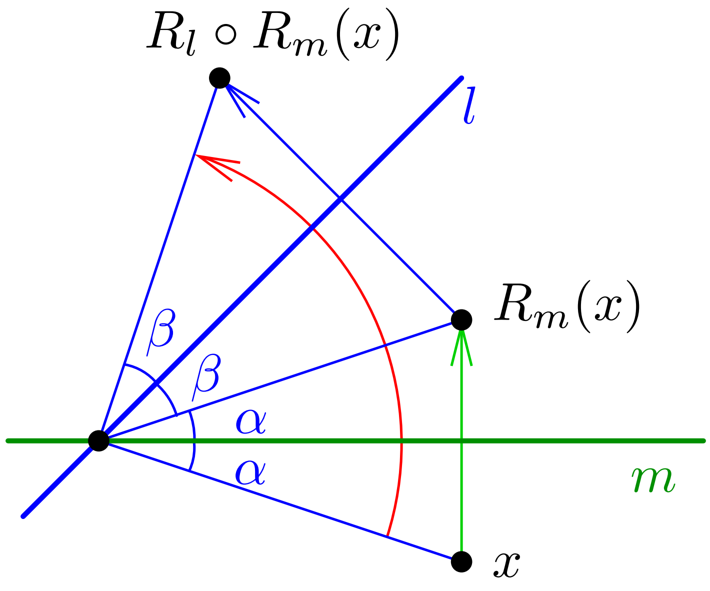

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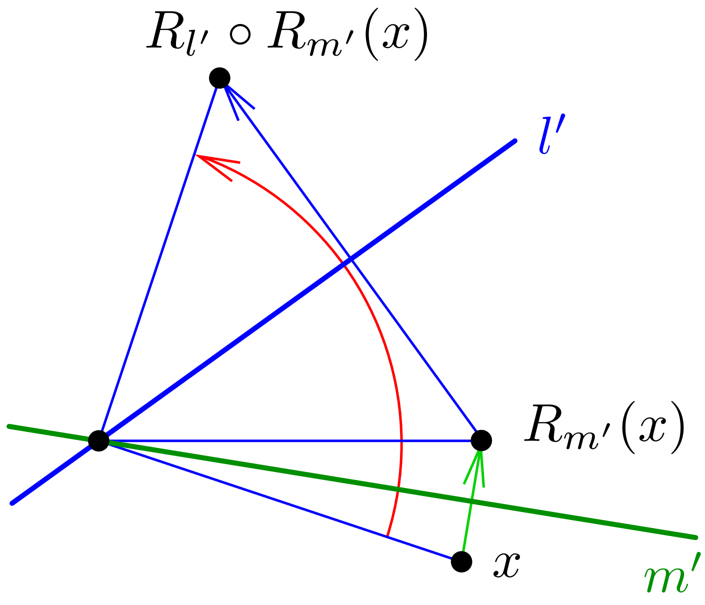

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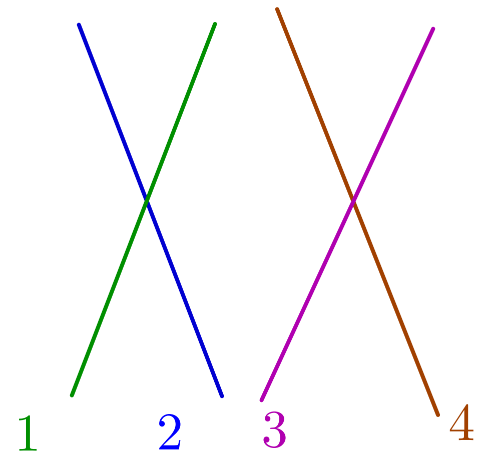

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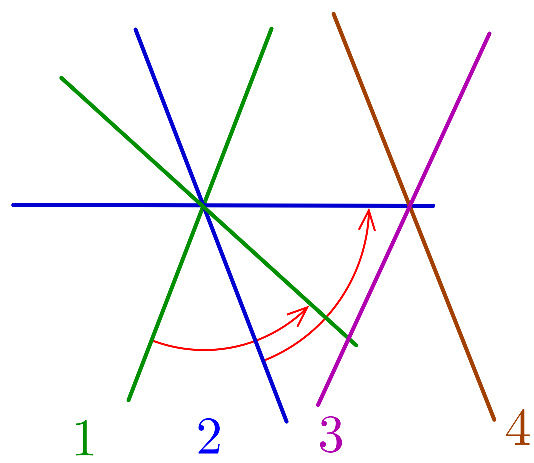

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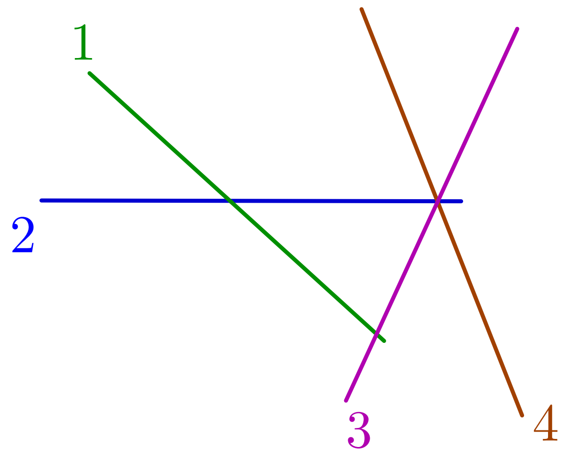

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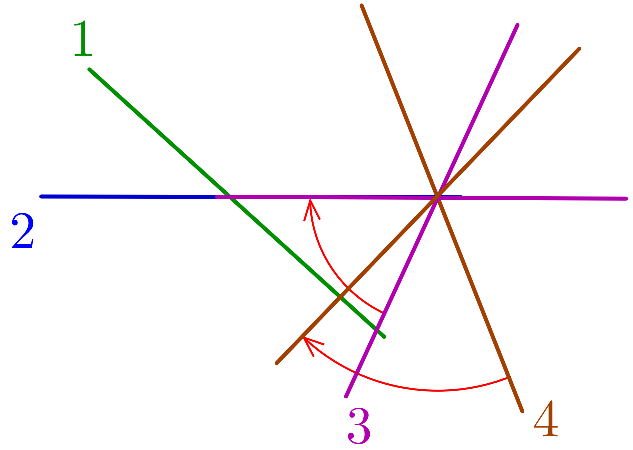

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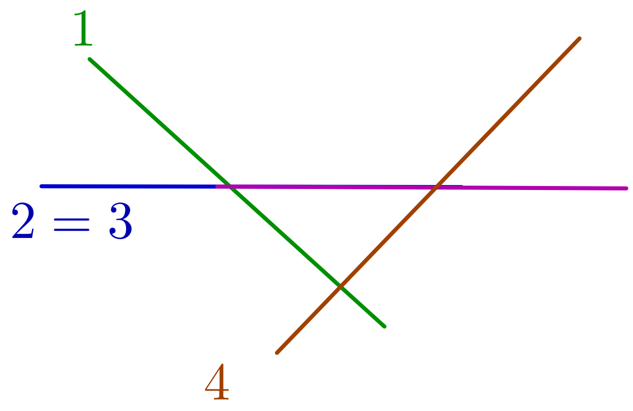

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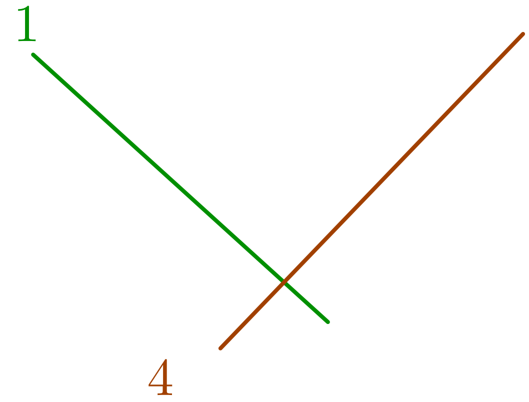

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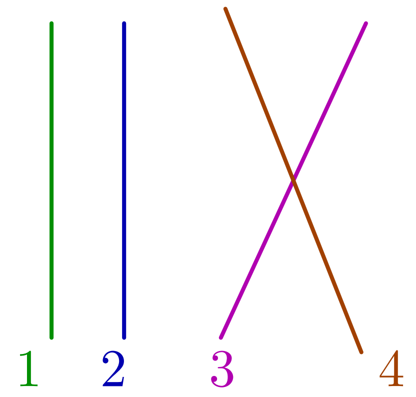

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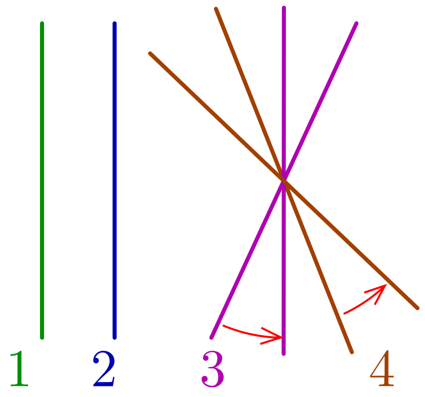

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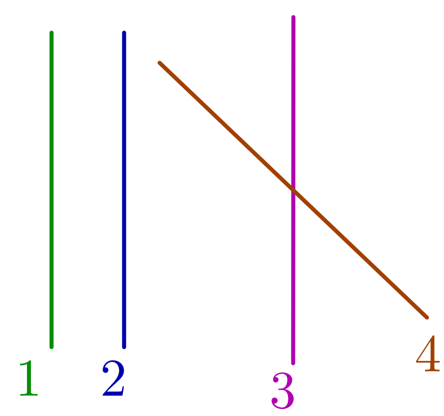

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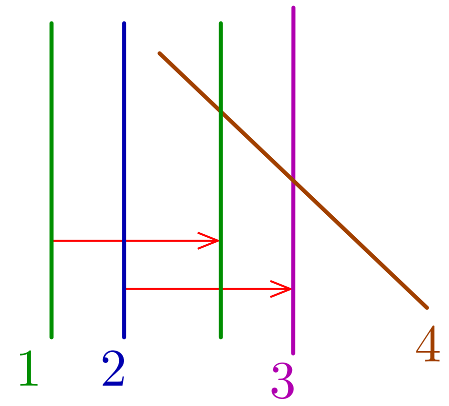

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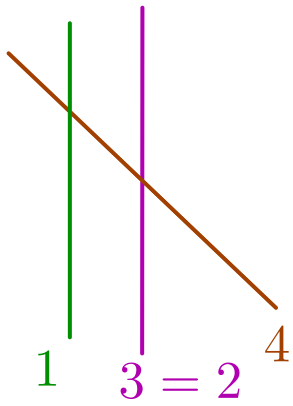

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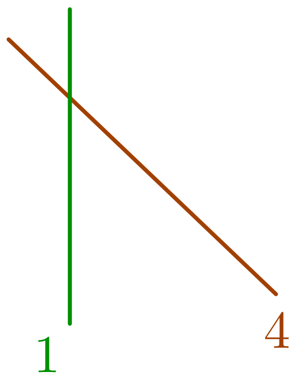

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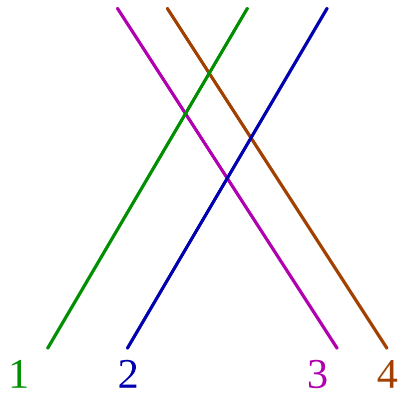

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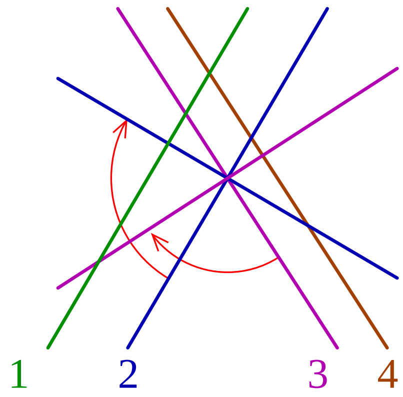

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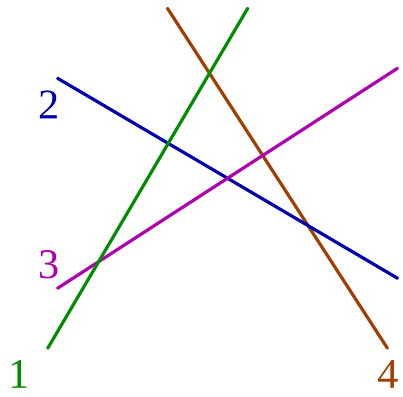

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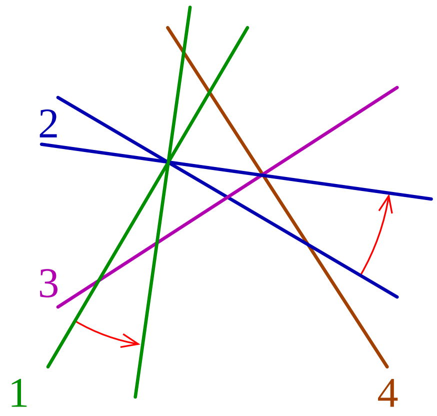

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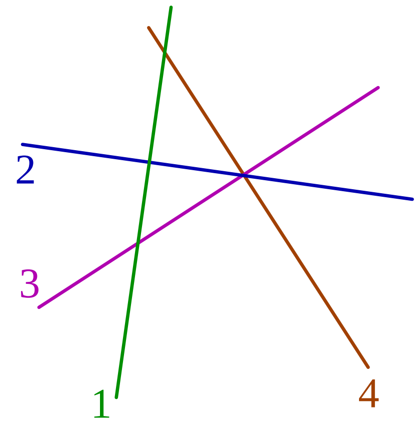

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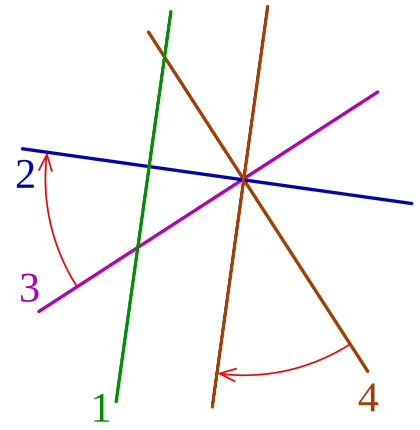

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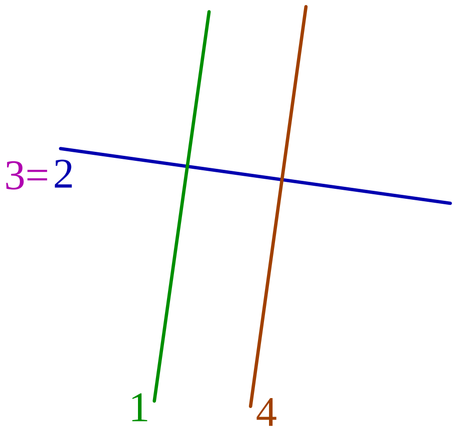

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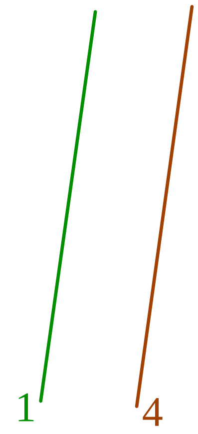

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Lemma. A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines. $\square$

Generalization of Lemma. In $\mathbb{R}^n$,
a composition of any $n+2$ reflections in hyperplanes is a composition of n reflections in hyperplanes.

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A composition of odd number of reflections reverses orientation and cannot be id.
A composition of two different reflections is not identity. $\square$

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Theorem. Any relation among reflections in lines follow from relations $R_l^2 = 1$ and $R_l \circ R_m = R_{l'} \circ R_{m'}$, where $l, m, l'm'$ are as on the preceding two slides. $\square$

Lemma. A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines. $\square$

Generalization of Lemma. In $\mathbb{R}^n$,
a composition of any $n+2$ reflections in hyperplanes is a composition of n reflections in hyperplanes.

Generalization of Theorem. Any relation among reflections in hyperplanes of $\mathbb{R}^n$ follow
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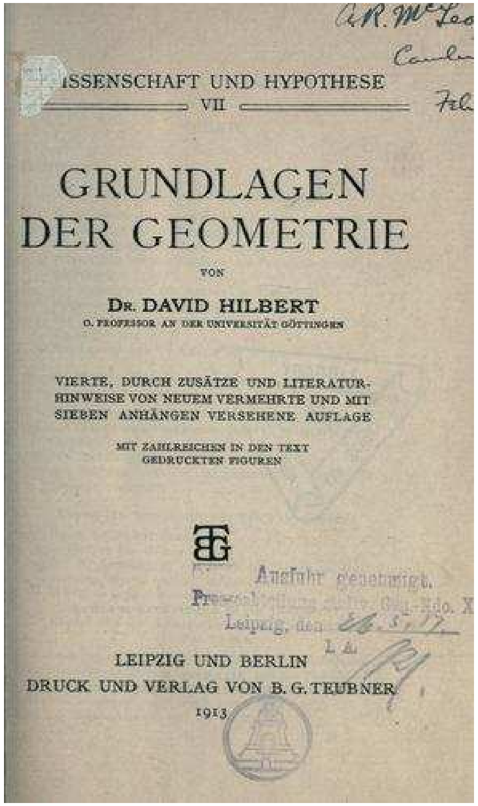
Mathematically satisfactory results were achieved in the XXth century.

Three major systems:

Bachmann’s foundations of geometry

1. David Hilbert’s *Foundations of Geometry*



WISSENSCHAFT UND HYPOTHESE

VII

GRUNDLAGEN
DER GEOMETRIE

VON

DR. DAVID HILBERT

O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN

VIERTE, DURCH ZUSÄTZE UND LITERATUR-
HINWEISE VON NEUEM VERMEHRTE UND MIT
SIEBEN ANHÄNGEN VERSEHENE AUFLAGE

MIT ZAHLREICHEN IN DEN TEXT
GEDRUCKTEN FIGUREN

Ausfuhr genehmigt.

LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
1913

Bachmann's foundations of geometry

2. Hermann Weil's *Space, Time, Matter*

RAUM · ZEIT · MATERIE

VORLESUNGEN ÜBER
ALLGEMEINE RELATIVITÄTSTHEORIE

VON

HERMANN WEYL

DRITTE, UMGEARBEITETE AUFLAGE

BERLIN
VERLAG VON JULIUS SPRINGER
1919

Bachmann's foundations of geometry

3. Friedrich Bachmann's

Construction of Geometry on the notion of reflections

Friedrich Bachmann

Aufbau der Geometrie
aus dem
Spiegelungsbegriff

Mit 160 Abbildungen

Zweite ergänzte Auflage

Springer-Verlag
Berlin Heidelberg New York 1973

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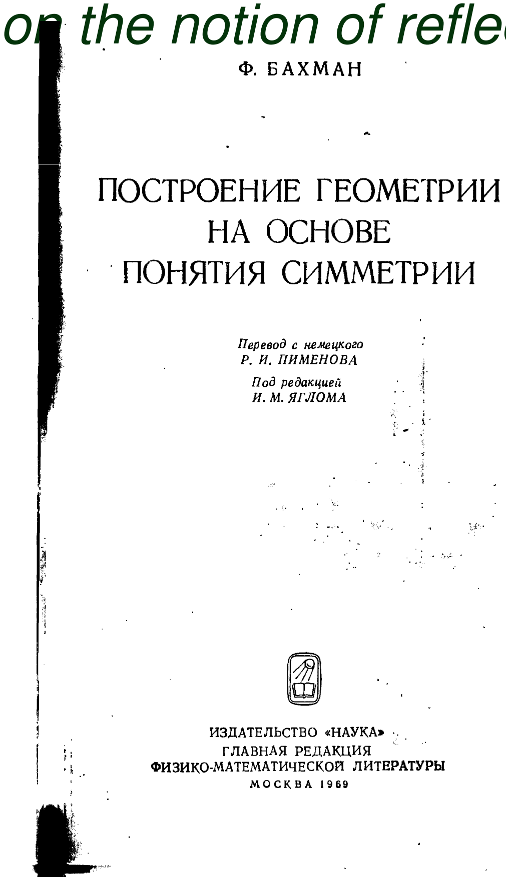

Ф. БАХМАН

ПОСТРОЕНИЕ ГЕОМЕТРИИ
НА ОСНОВЕ
ПОНЯТИЯ СИММЕТРИИ

Перевод с немецкого
Р. И. ПИМЕНОВА

Под редакцией
И. М. ЯГЛОМА

ИЗДАТЕЛЬСТВО «НАУКА»
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Four axioms for Absolute Plane Geometry:

1. Through any two points, one can draw a line.
2. If each of two points lies on two lines,
then either points or lines coincide.
3. If three lines have a common point,
then the composition of the reflections in them is a reflection in a line.
4. If three lines are perpendicular to a line,
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Higher dimensions, order and betweenness were out of consideration.

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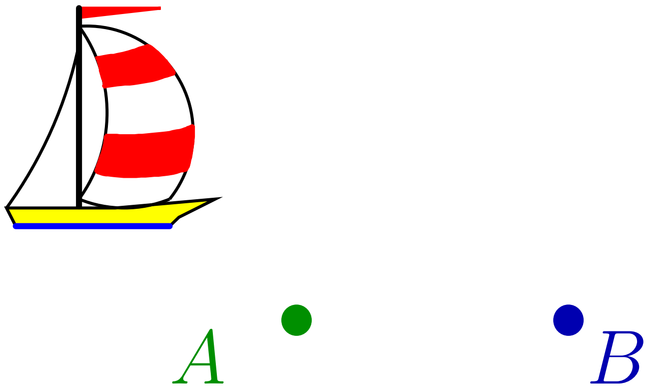

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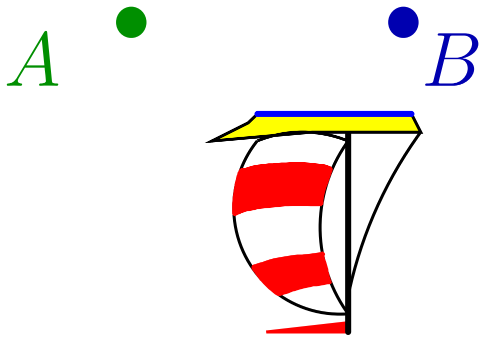

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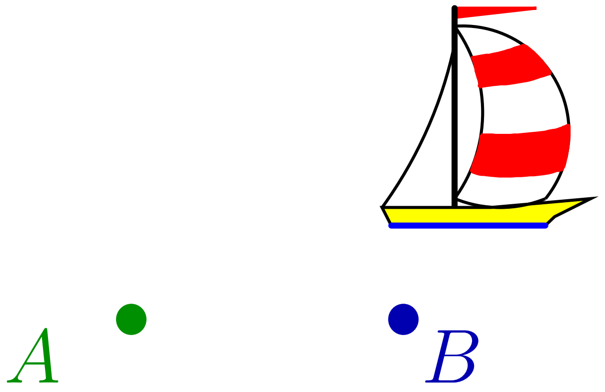

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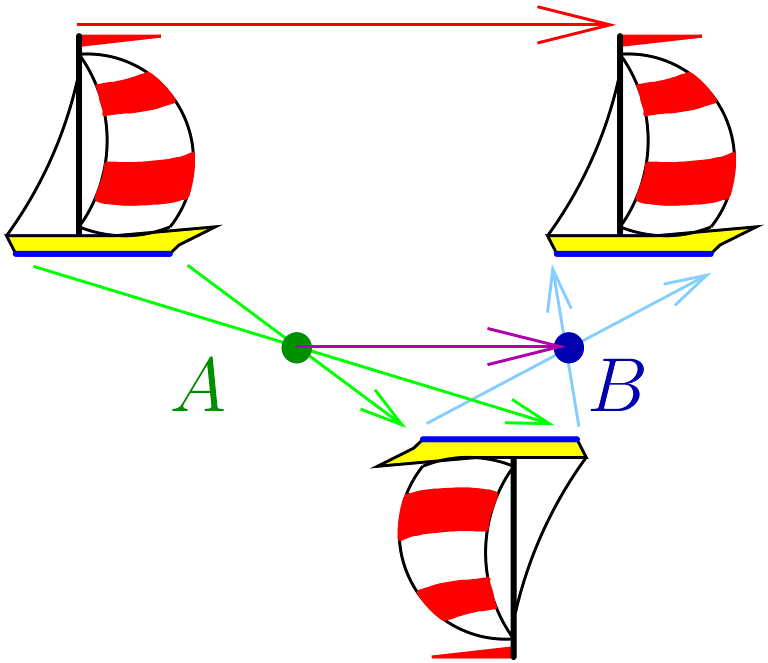

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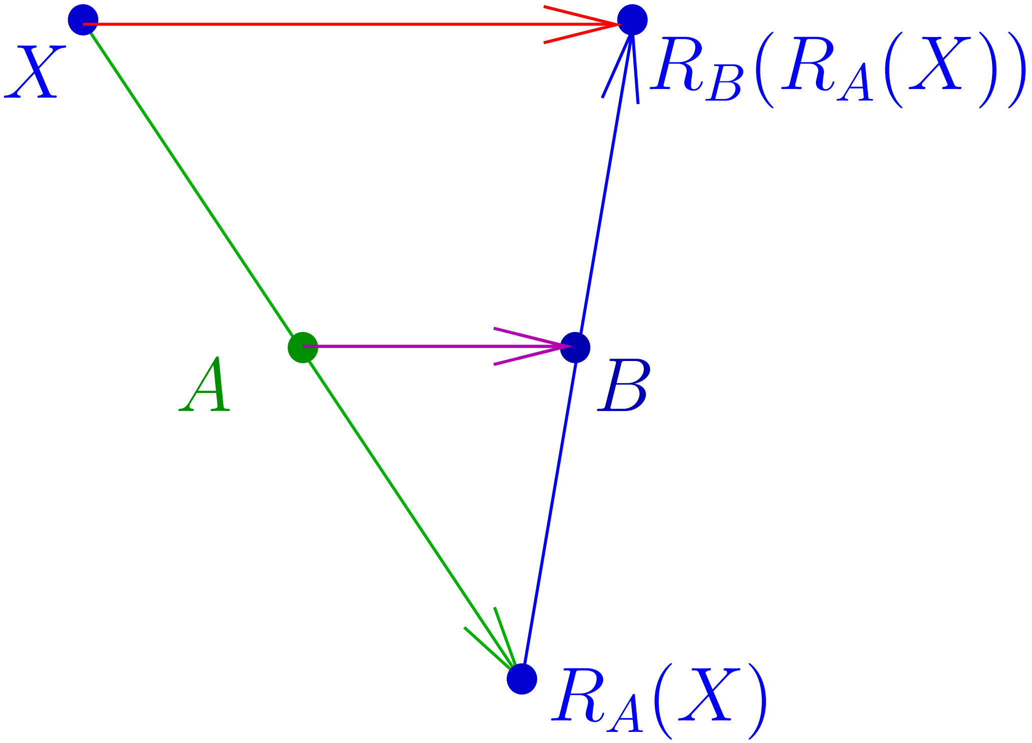

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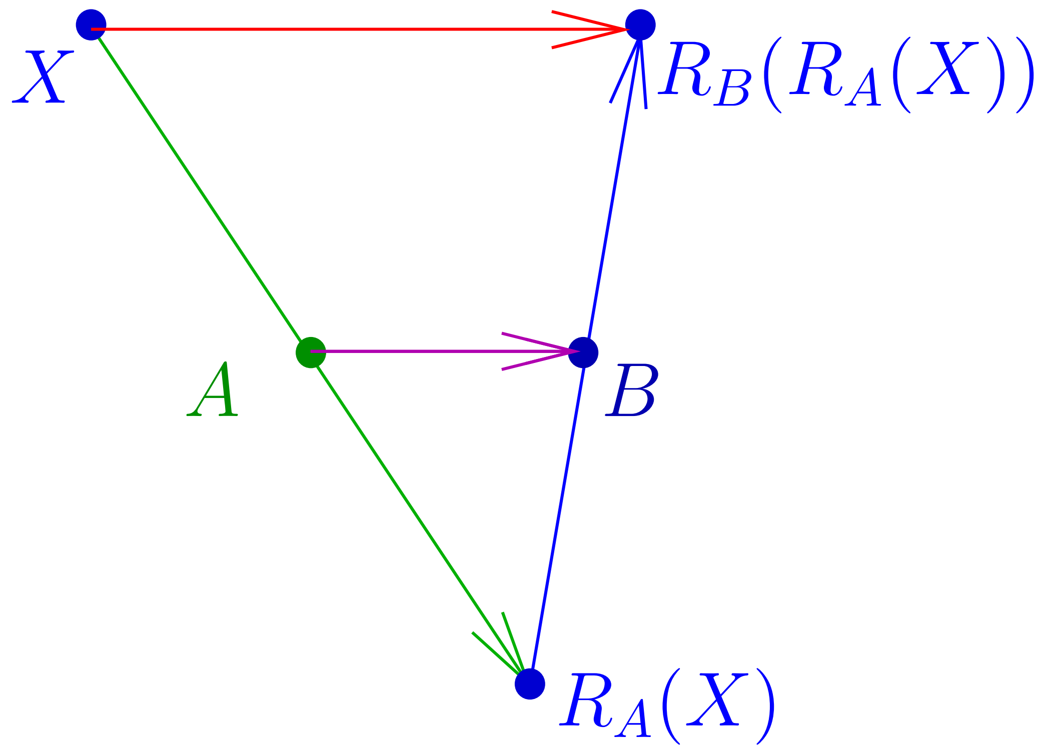

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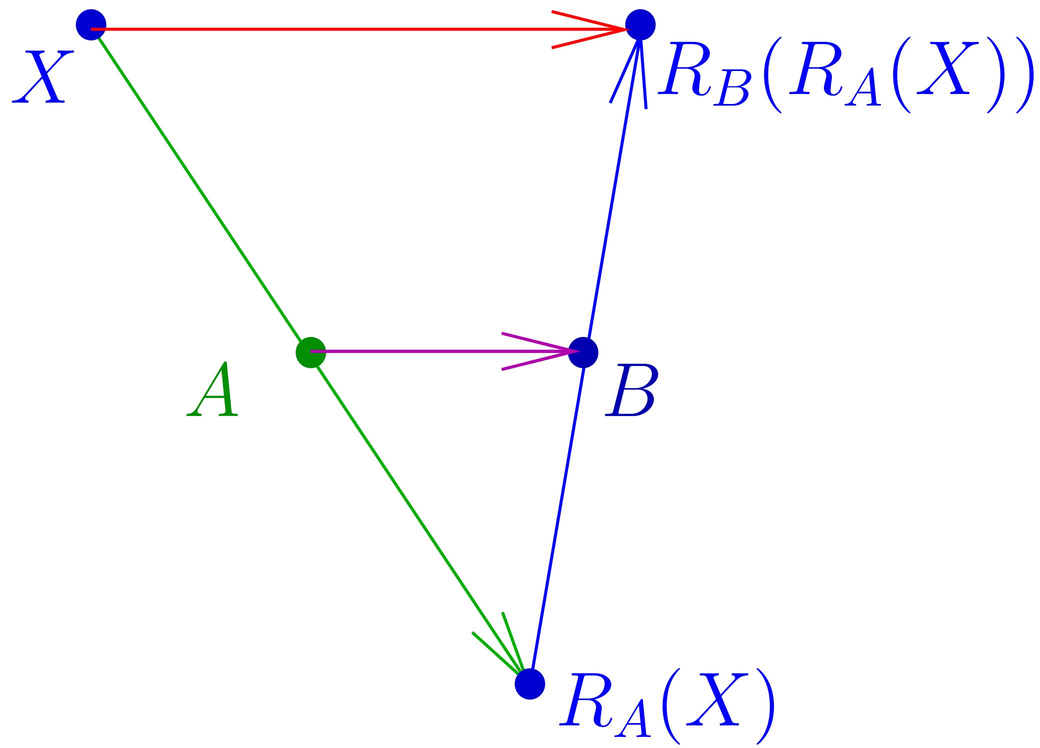

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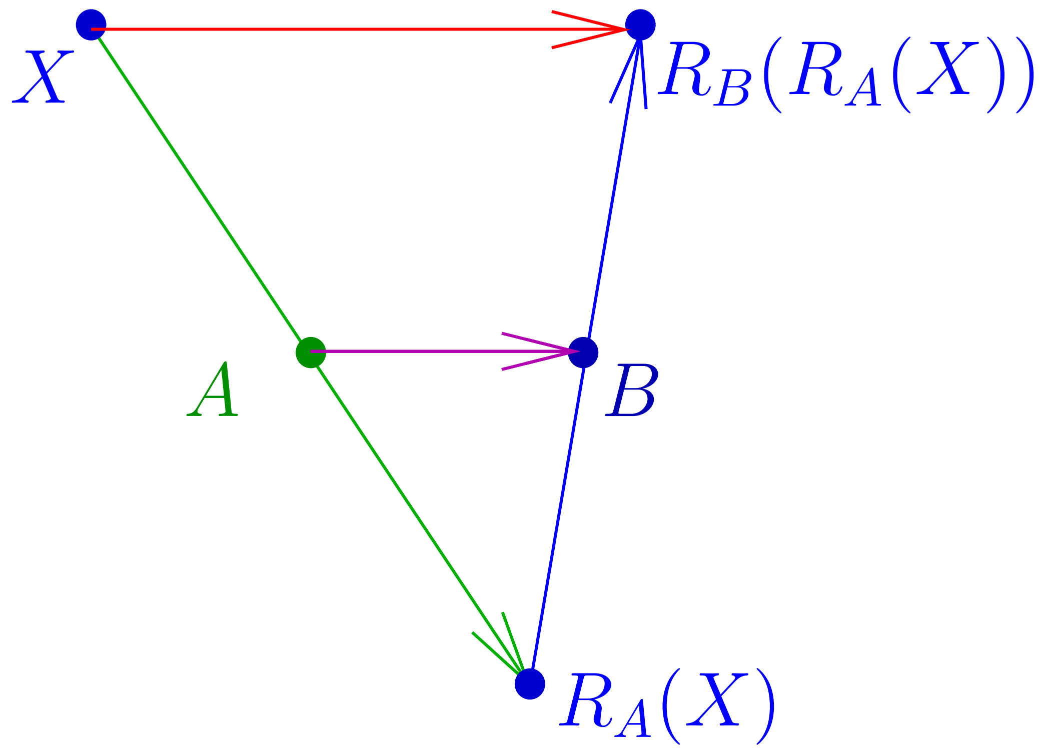

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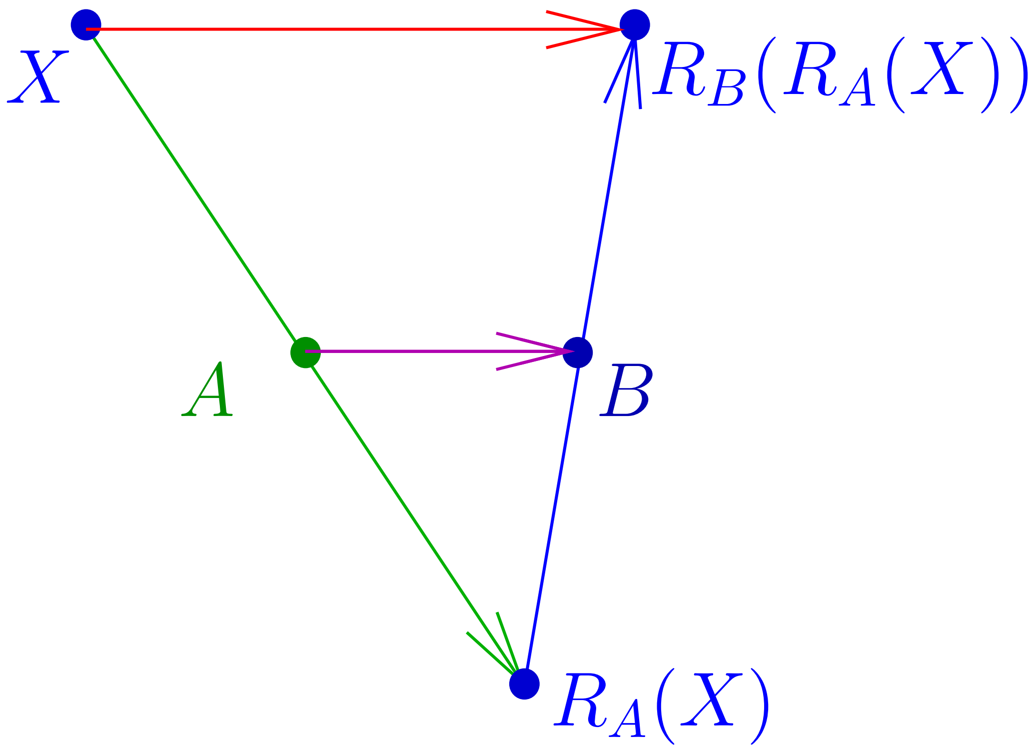

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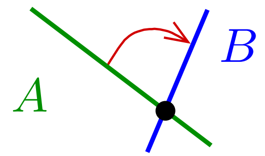

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Problem. Find an explicit description for the equivalence.

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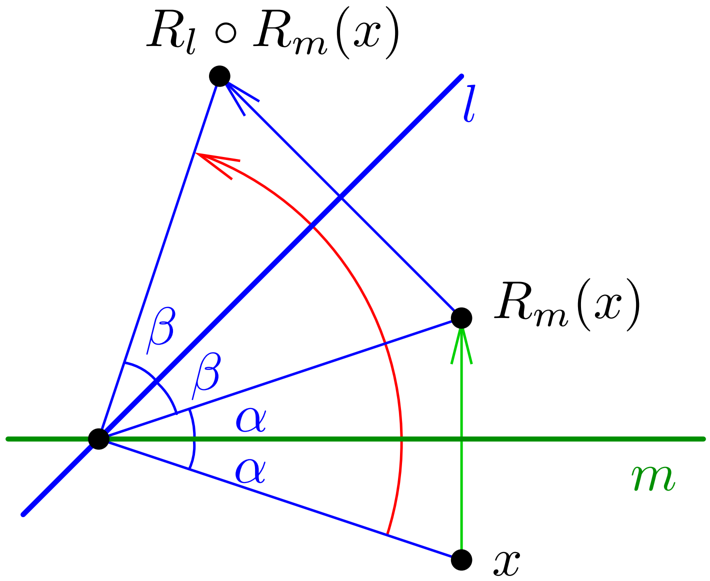

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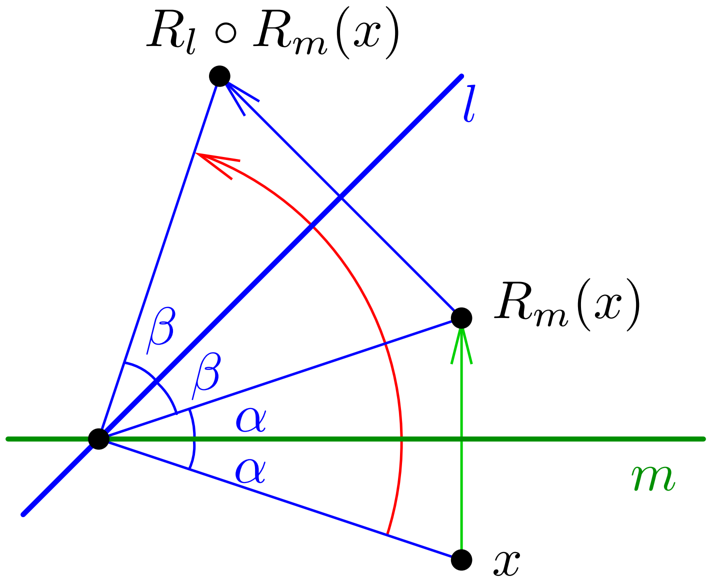

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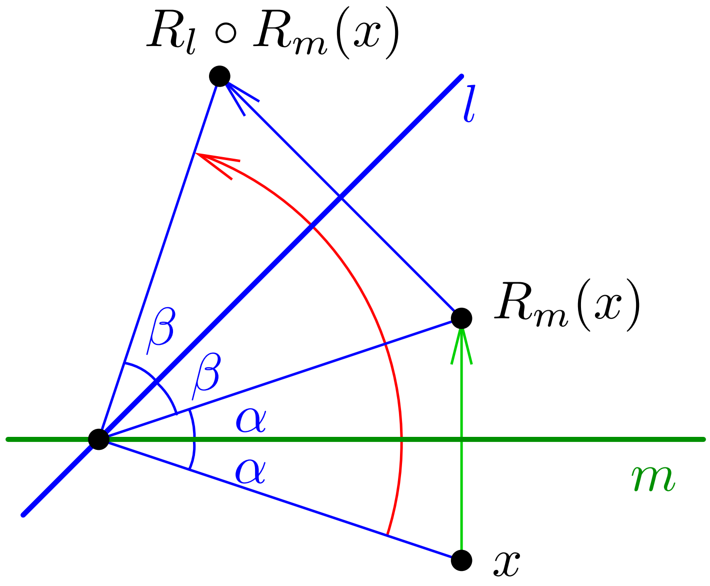

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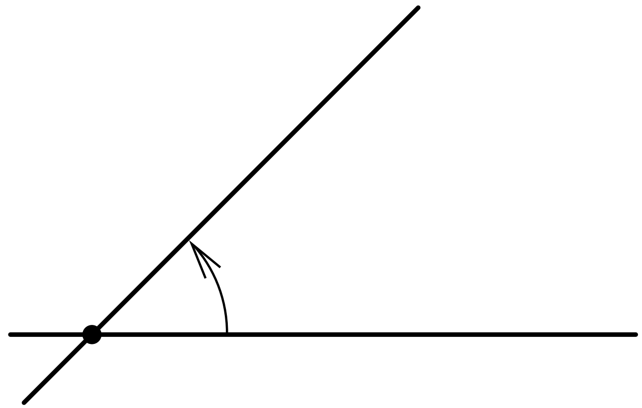
On a picture the order of lines is shown by an oriented arc.



Angle-arrows for a rotation

A rotation of a plane is encoded by an ordered pair of lines.

The lines intersect at the center of rotation.

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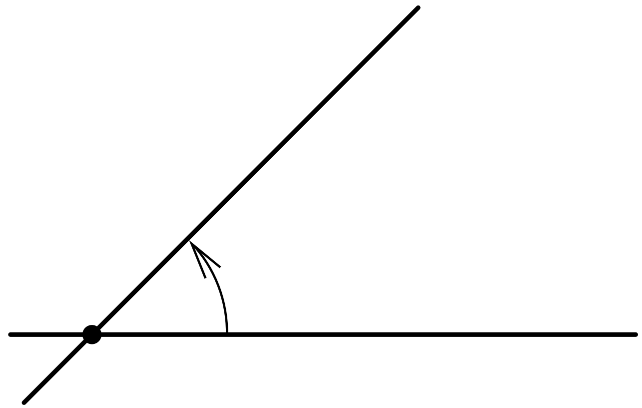
Equivalent angle-arrows are obtained by rotations about the vertex.



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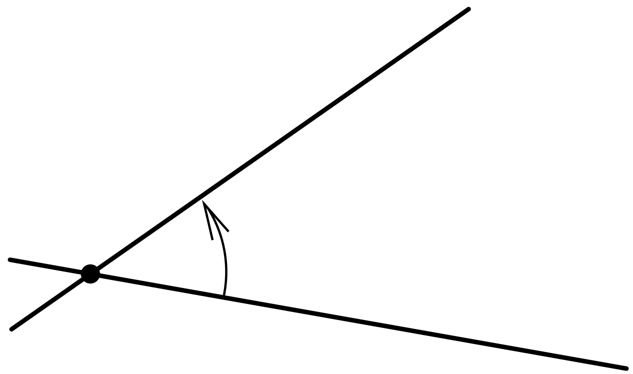

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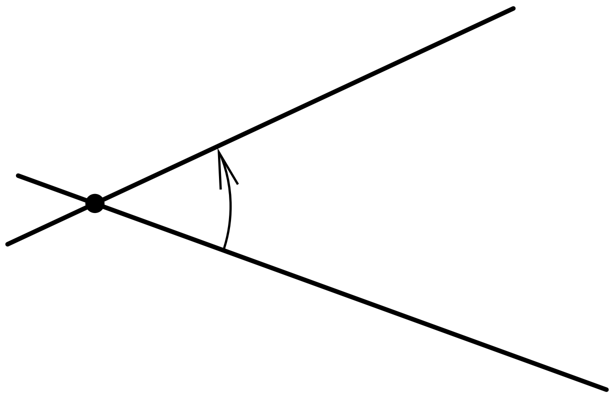

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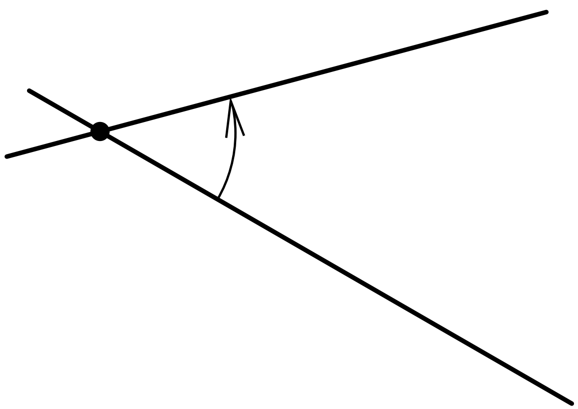

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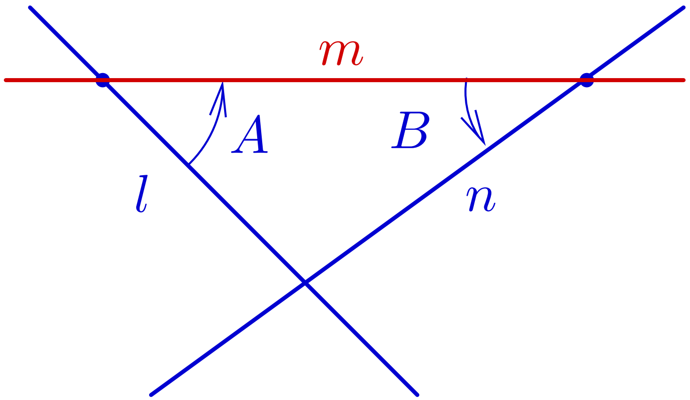

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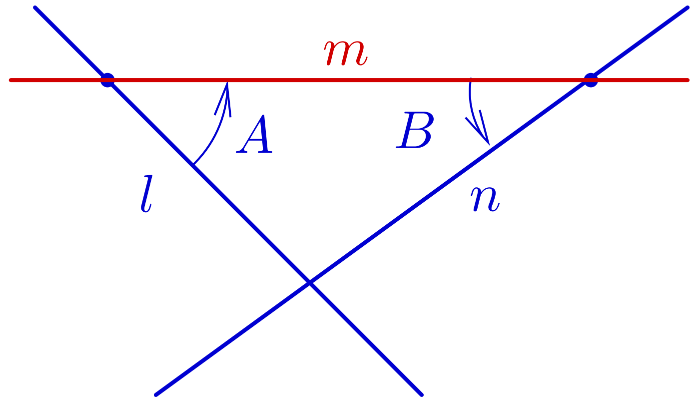

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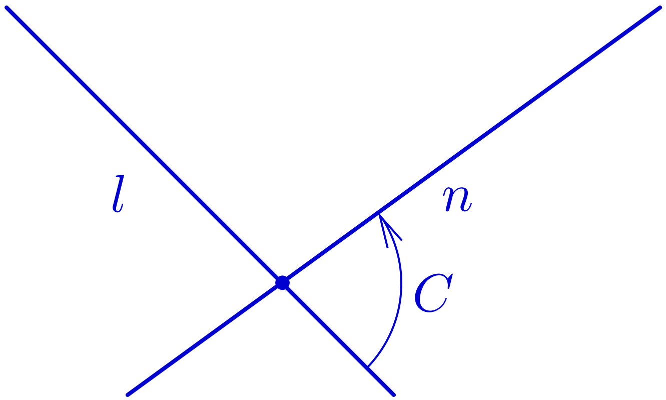

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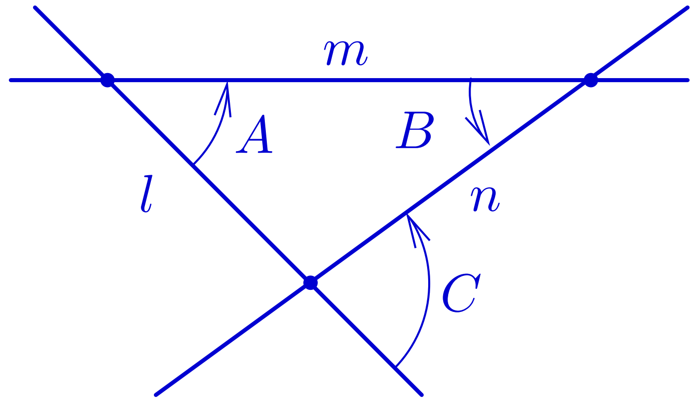

$$C = A + B$$

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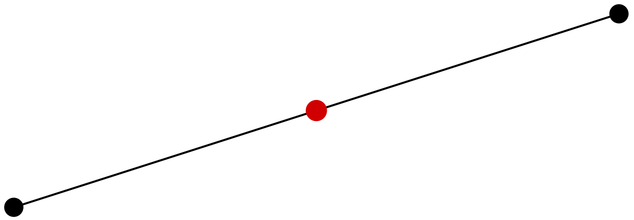

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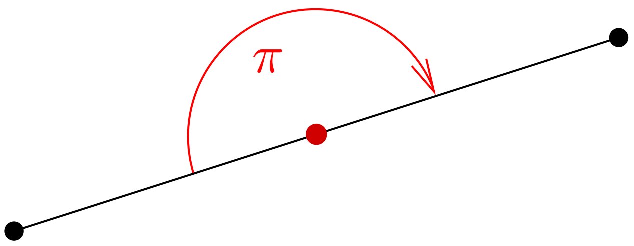

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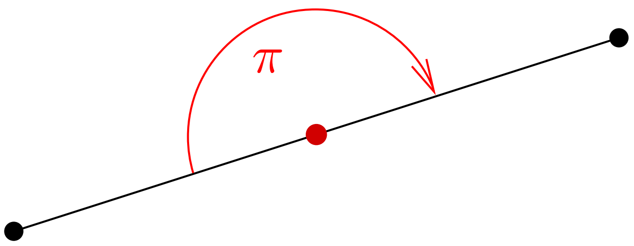

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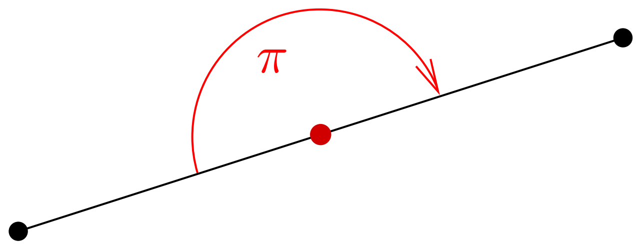

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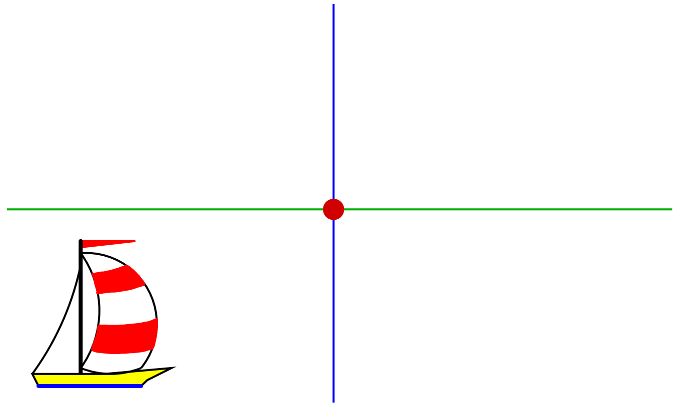

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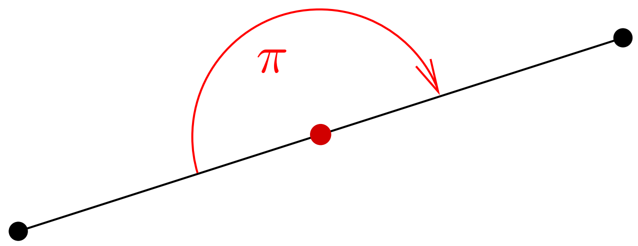

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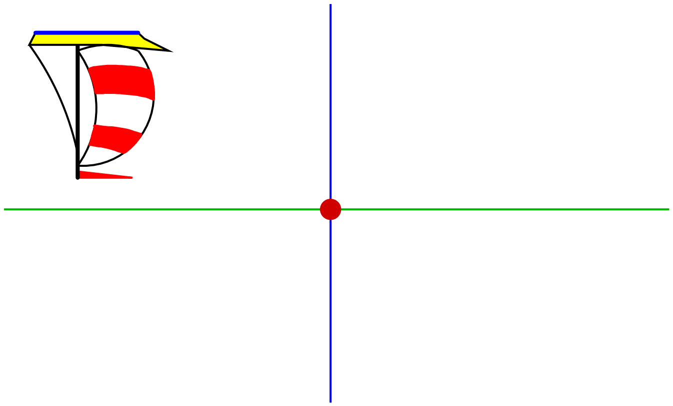

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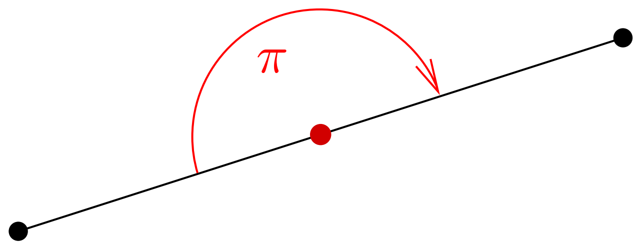

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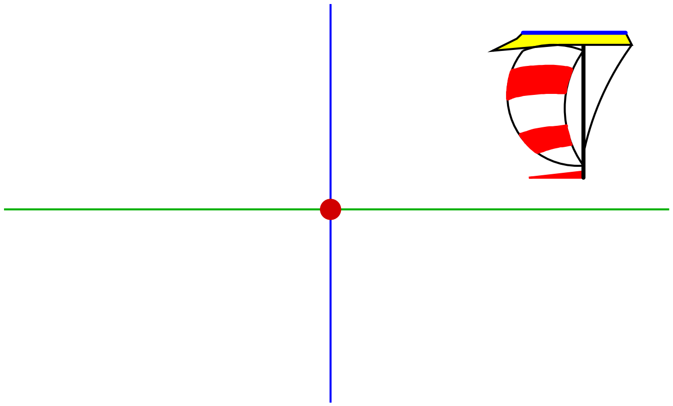

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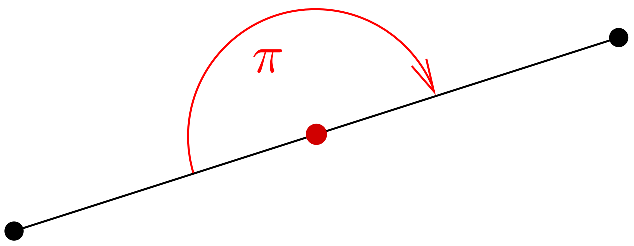

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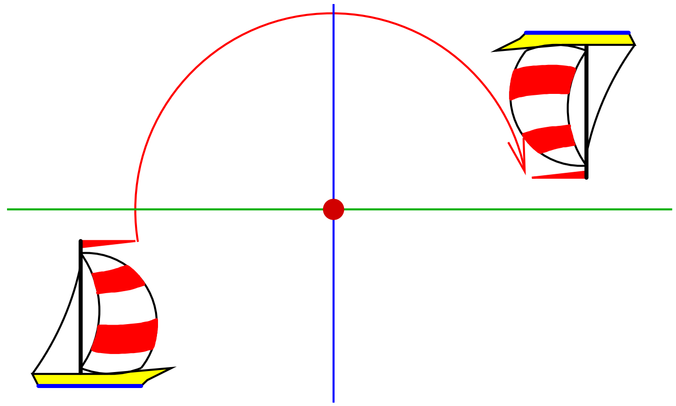

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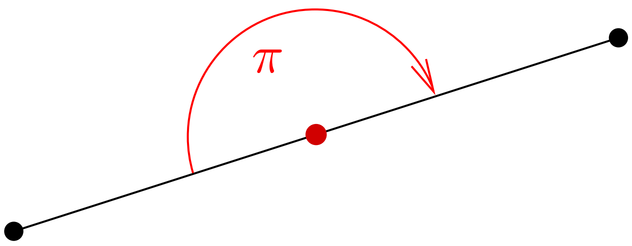

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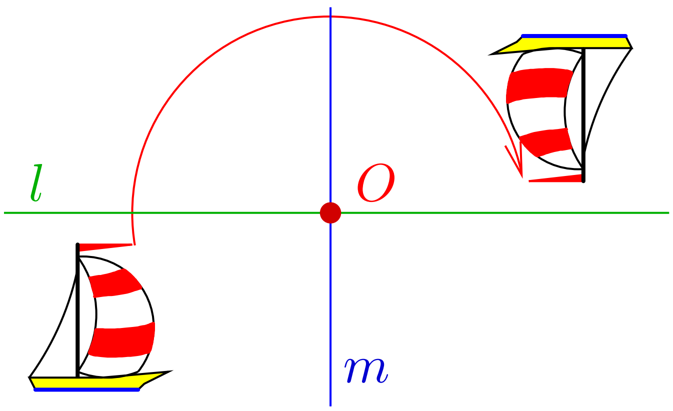

Relations involving three reflections, R_l, R_m and R_O:

$R_O = R_m \circ R_l$ and hence $R_O \circ R_l \circ R_m = \mathrm{id}$ and $R_O \circ R_l = R_m$.

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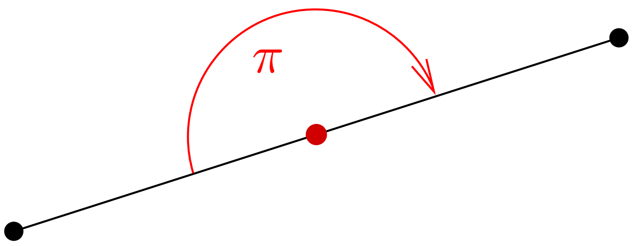

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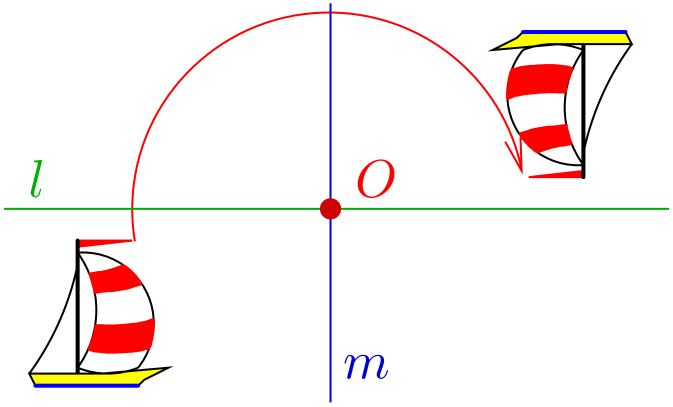
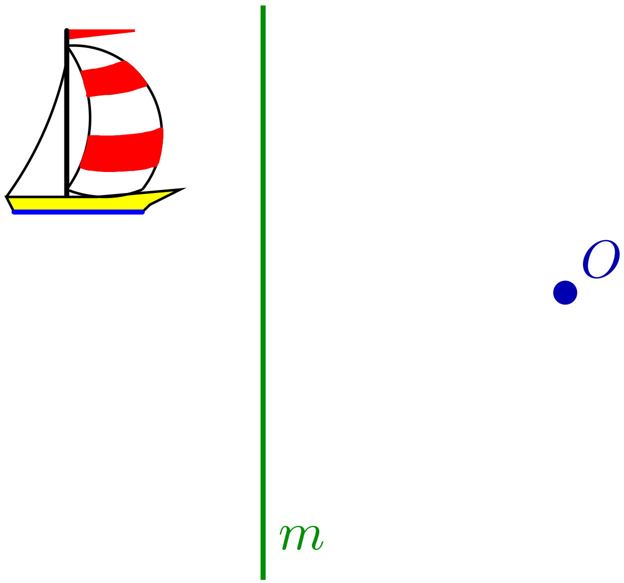


Furthermore, R_l, R_m and R_O, together with id, form the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Composing reflections in line and point



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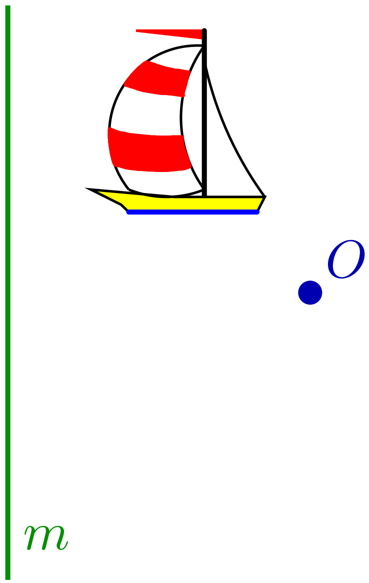

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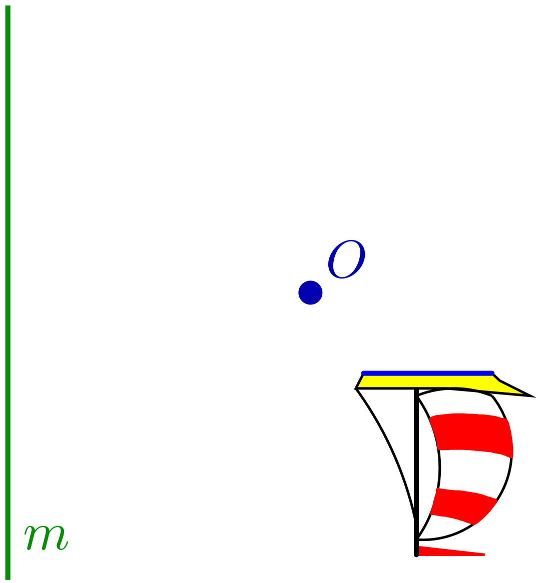

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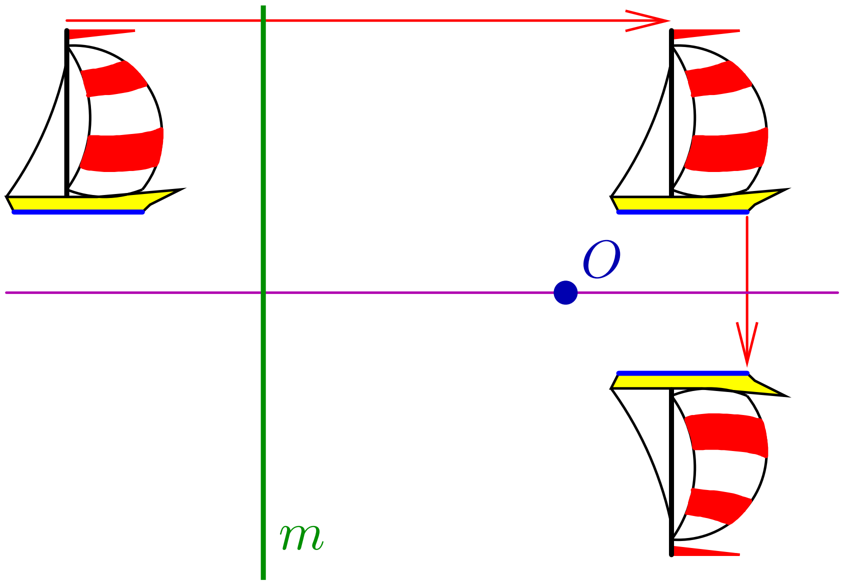

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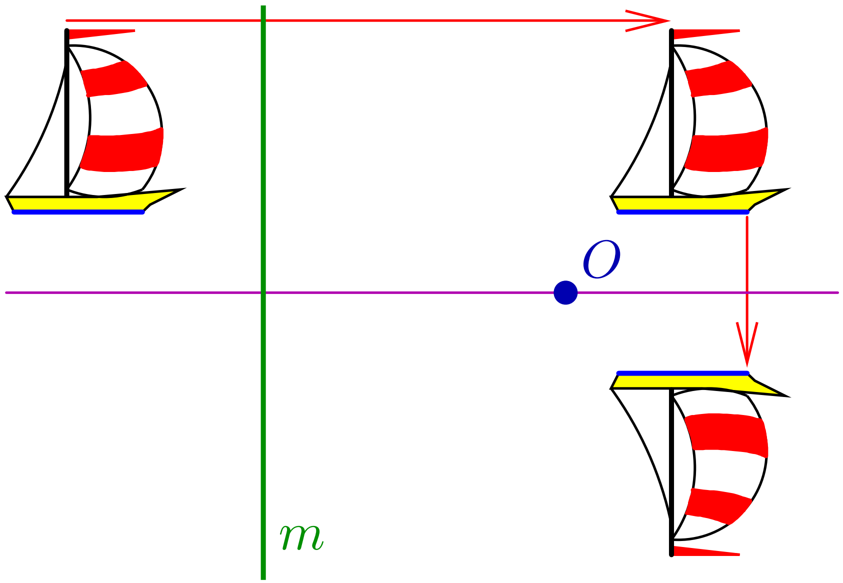

This is a glide reflection!

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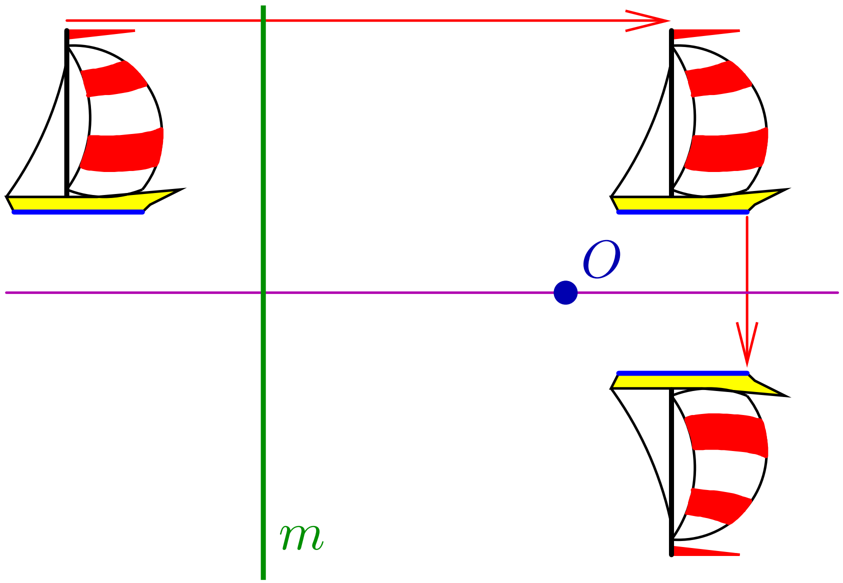

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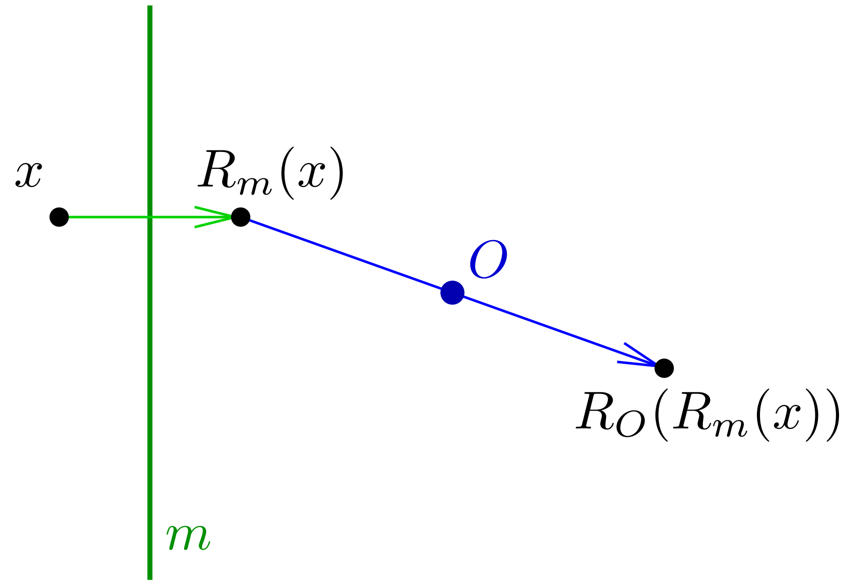

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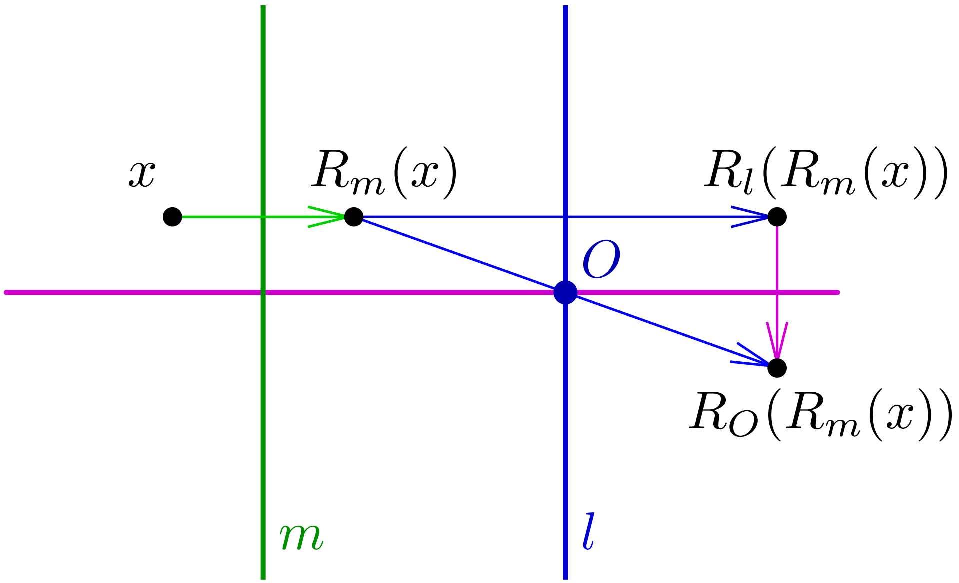

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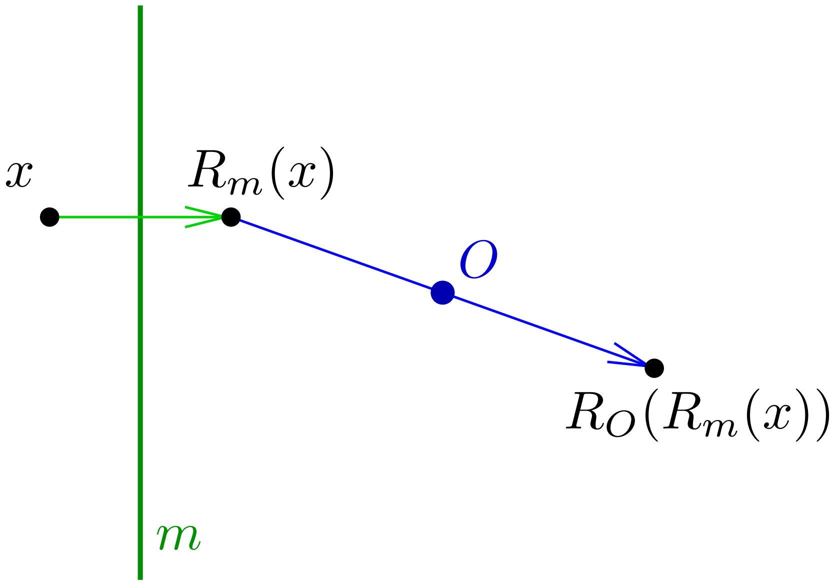
Decorated arrows for a glide reflection



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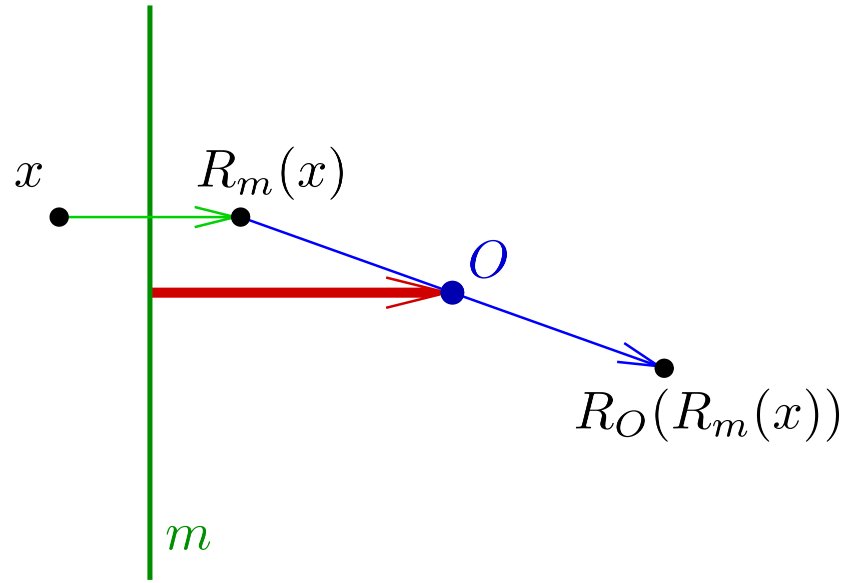

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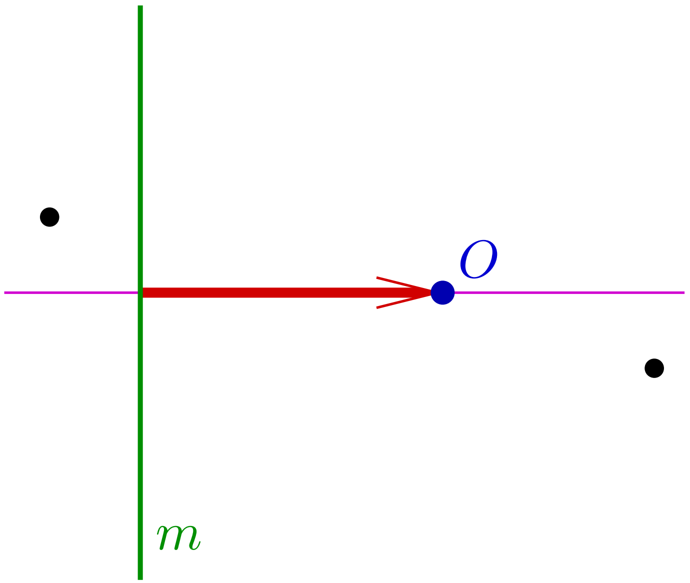

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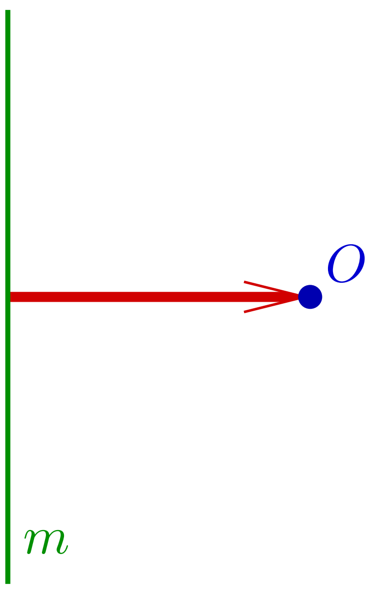

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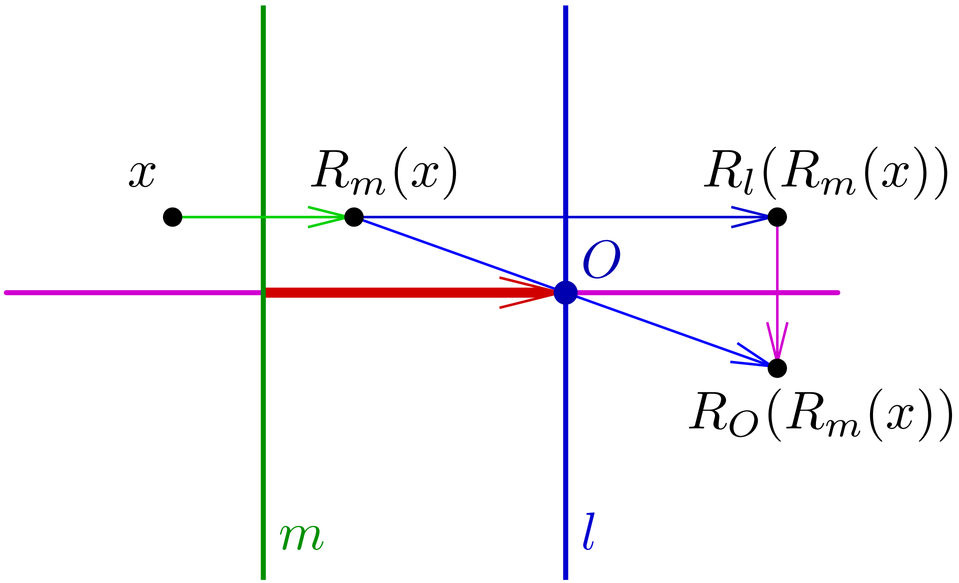

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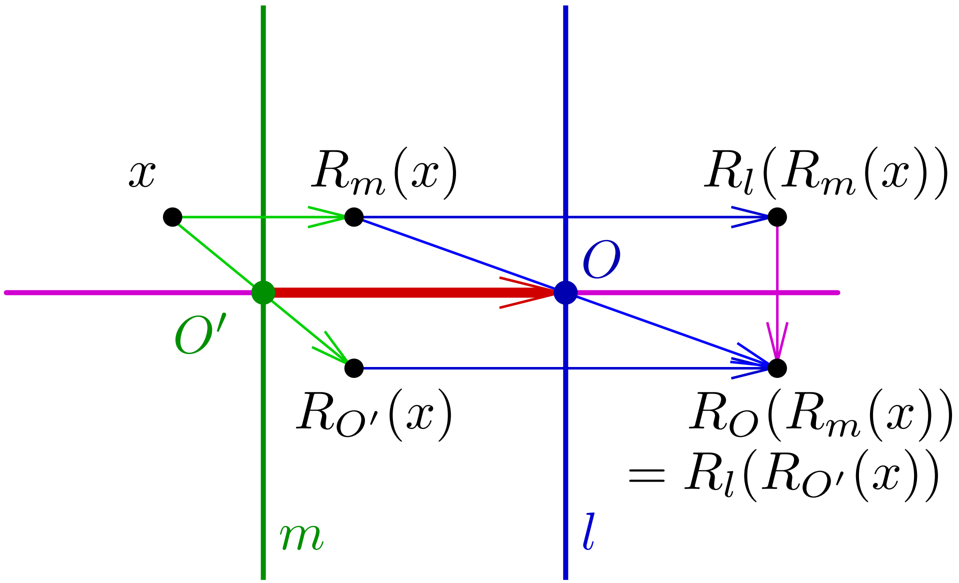

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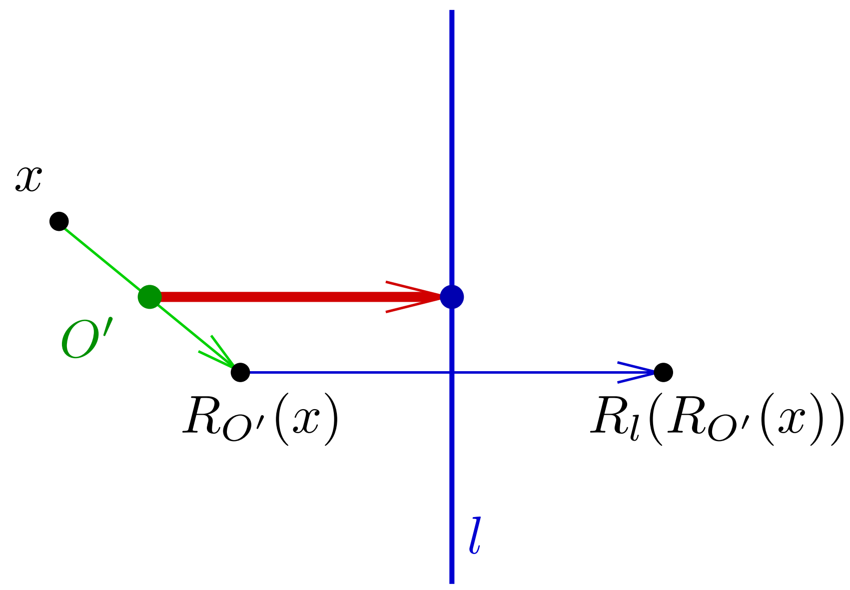

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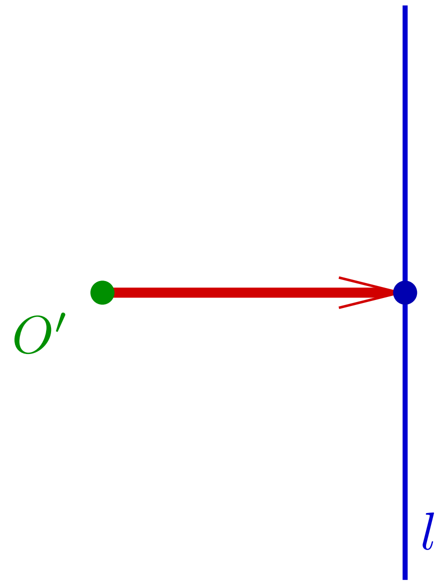

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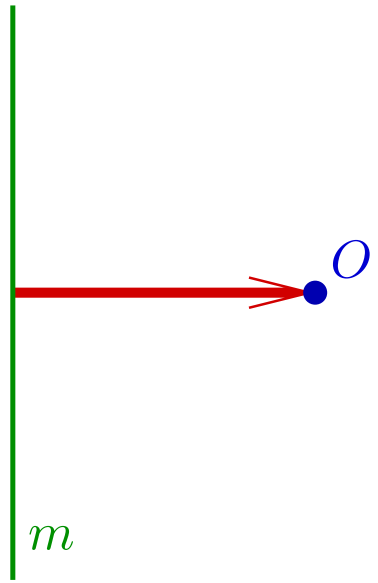

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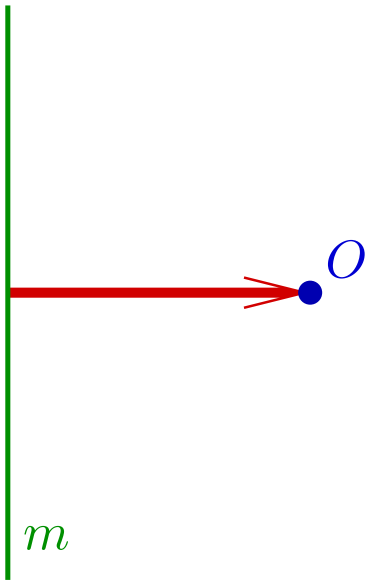

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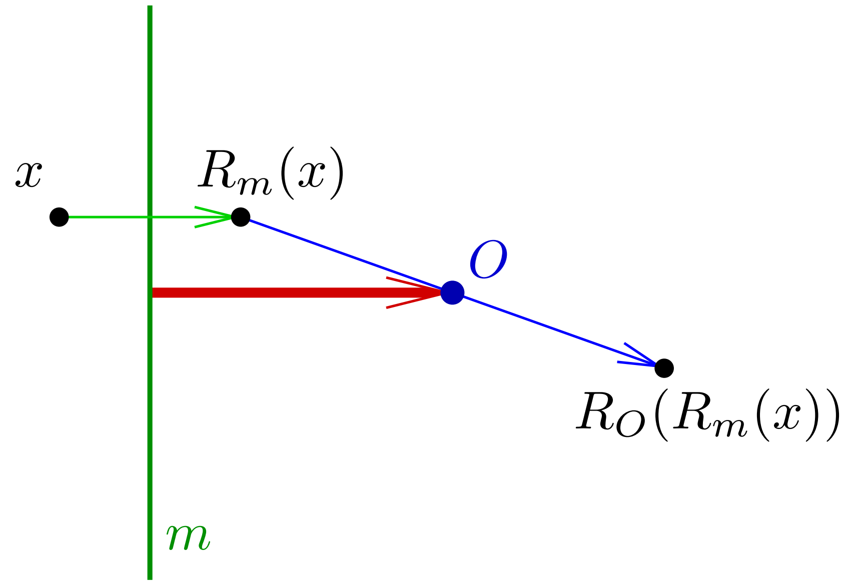

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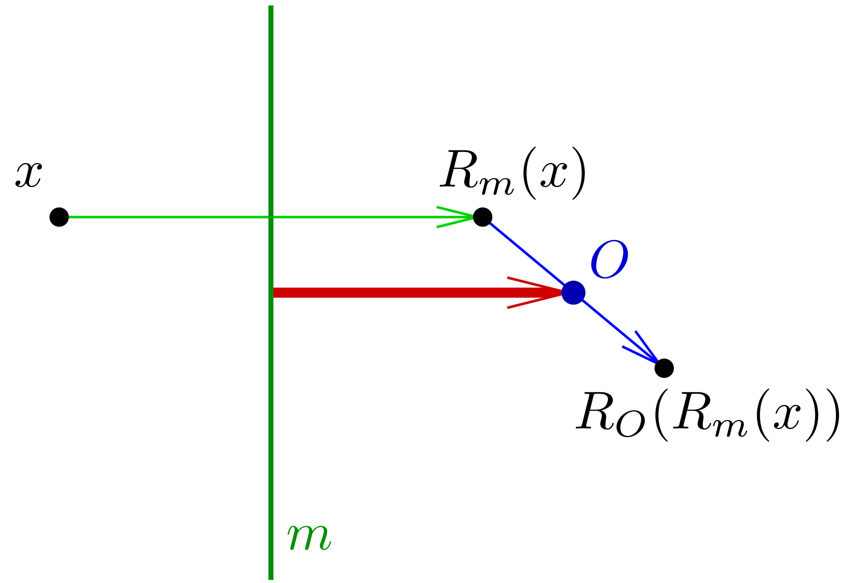

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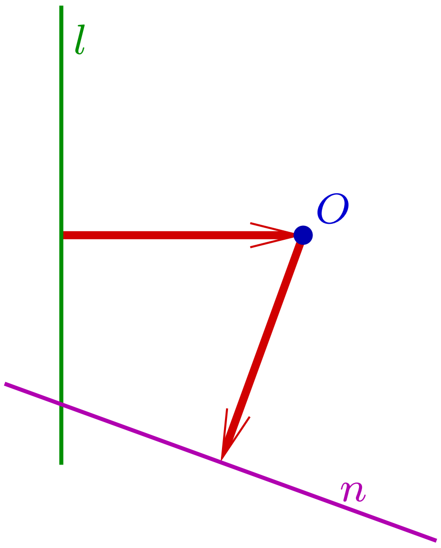

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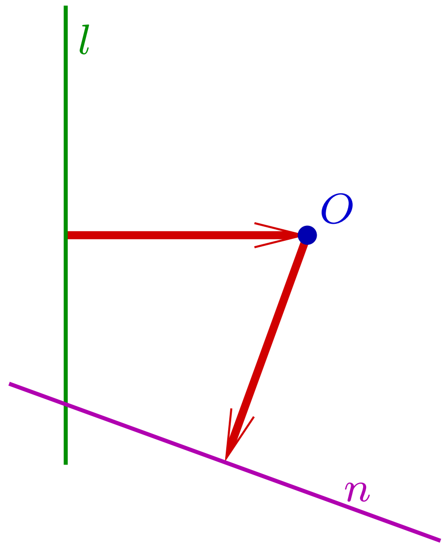

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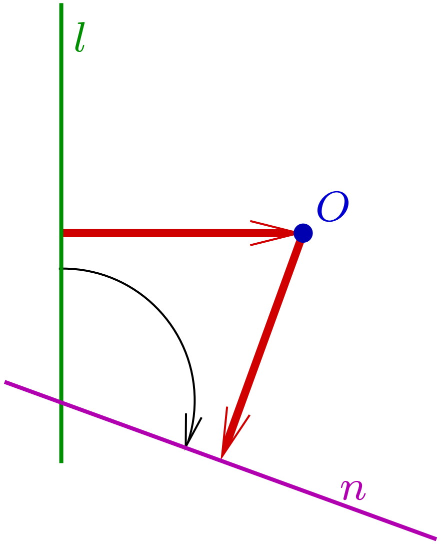

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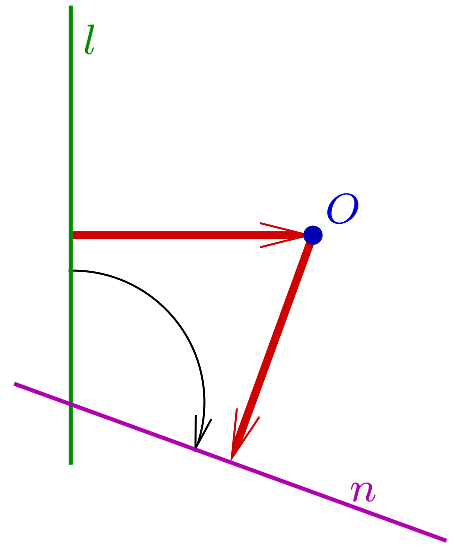
Draw an oriented arc from l to n and erase O.



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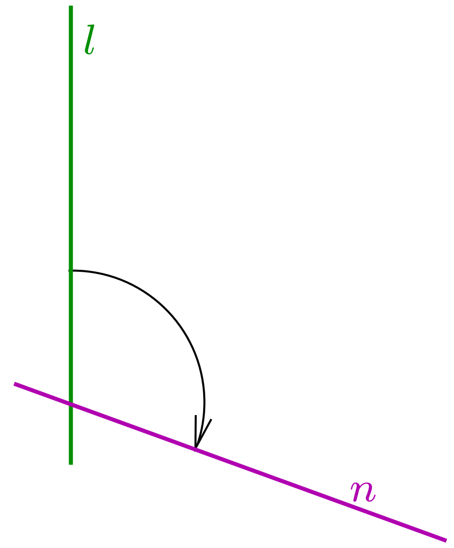

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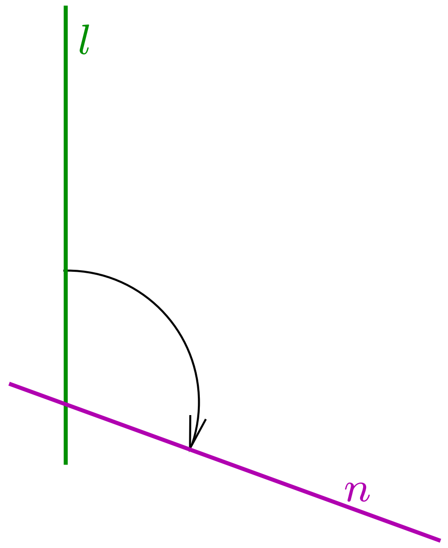

This is a rotation!

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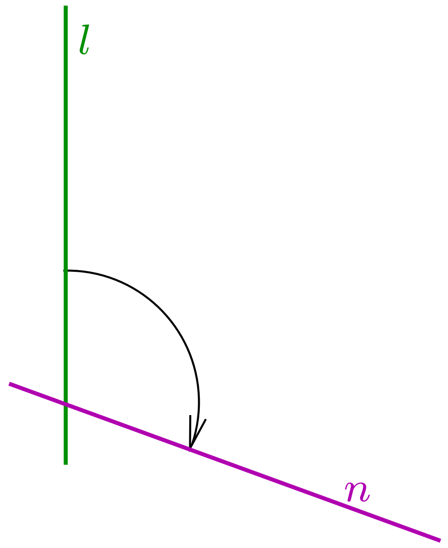
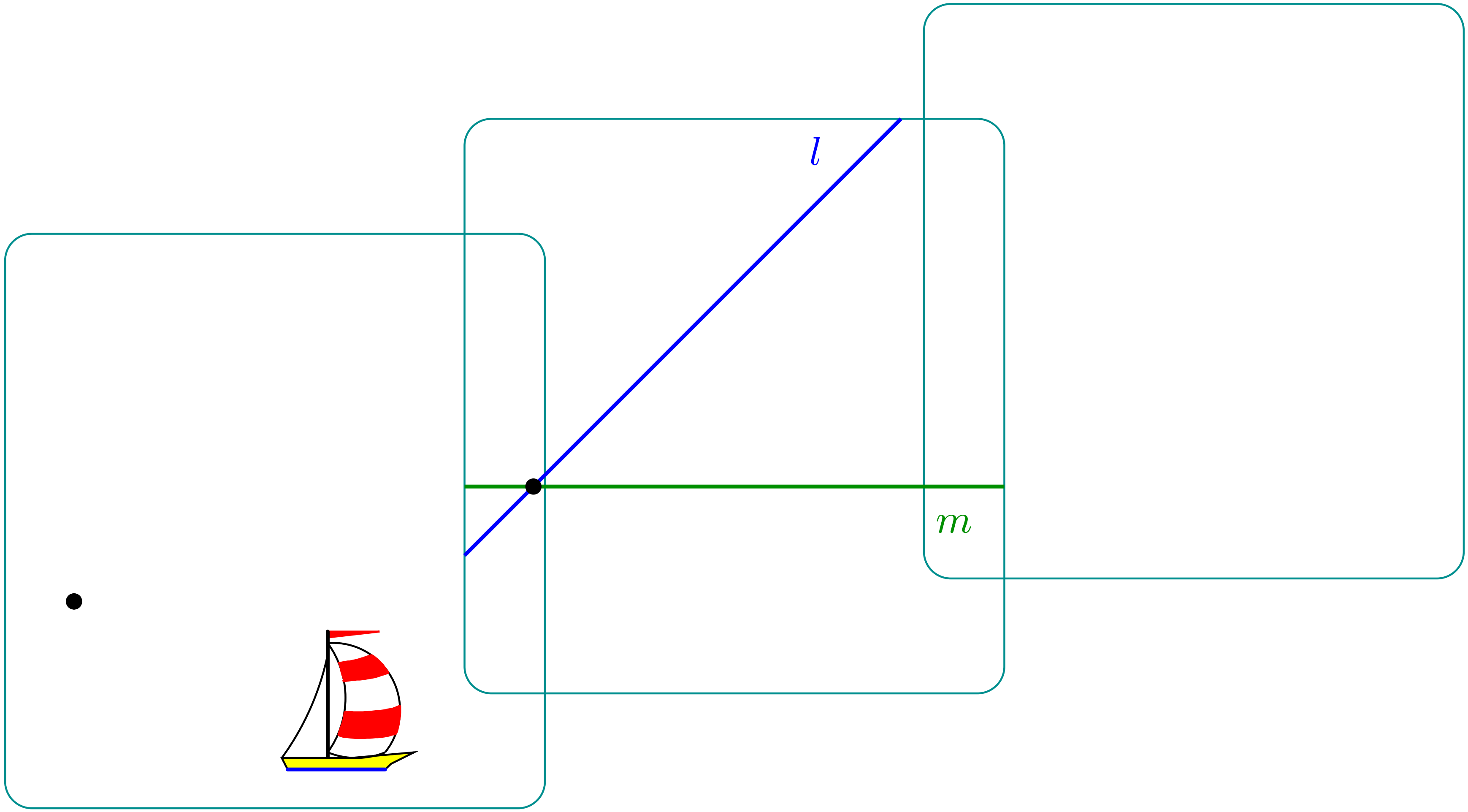


Exercise. *Find head to tail rules for composing rotation and glide reflection.*

In the 3-space. Rotation



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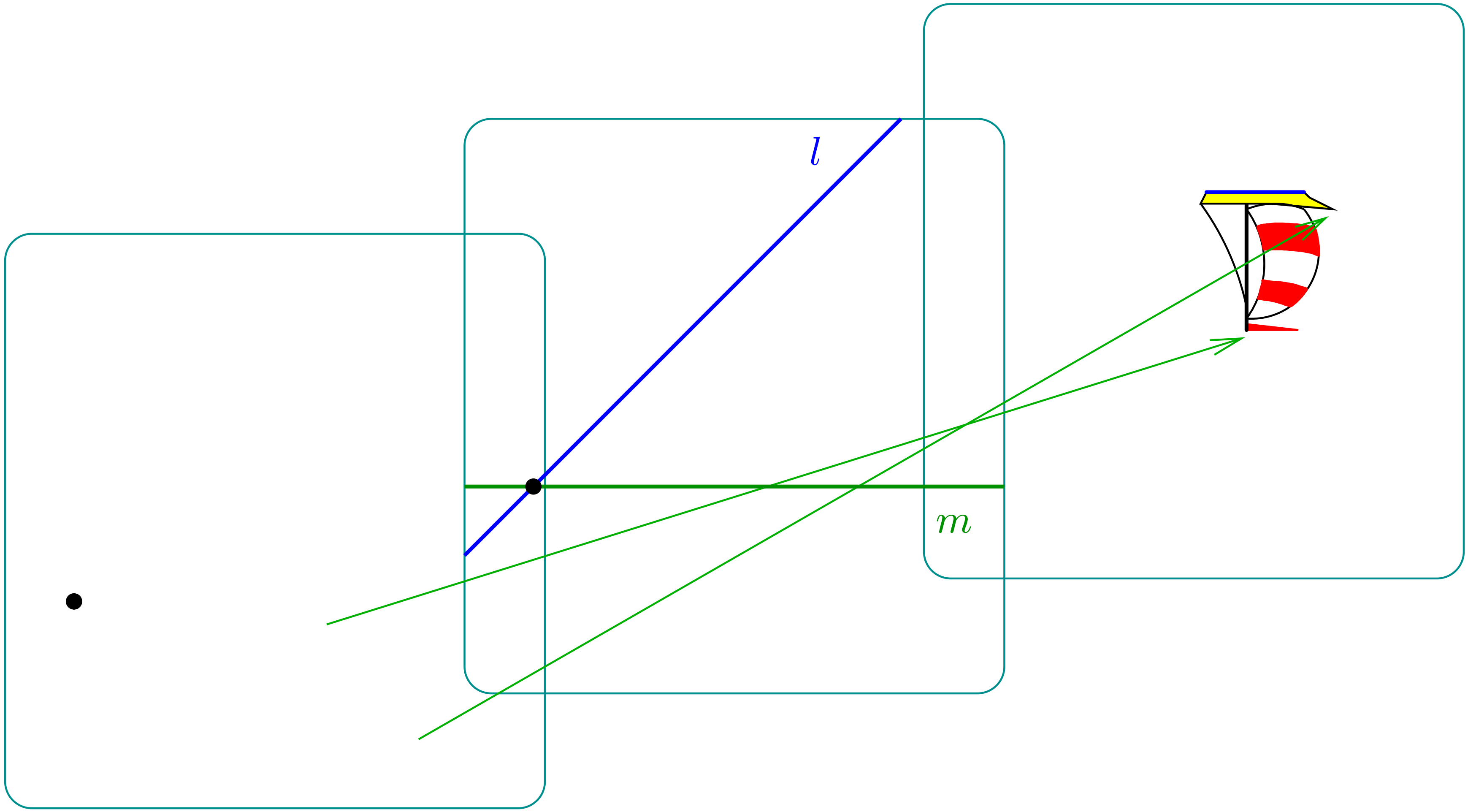
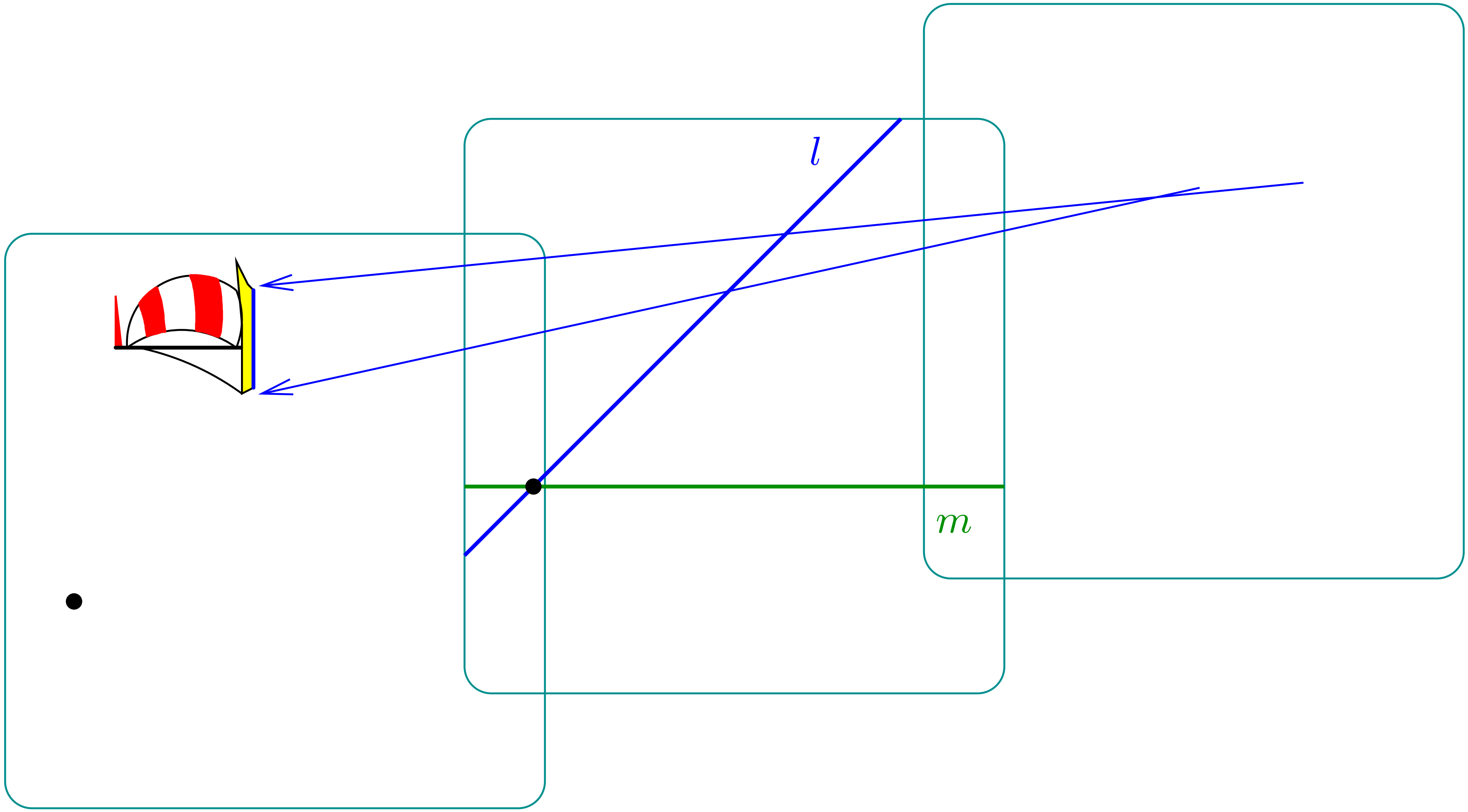


In the 3-space. Rotation



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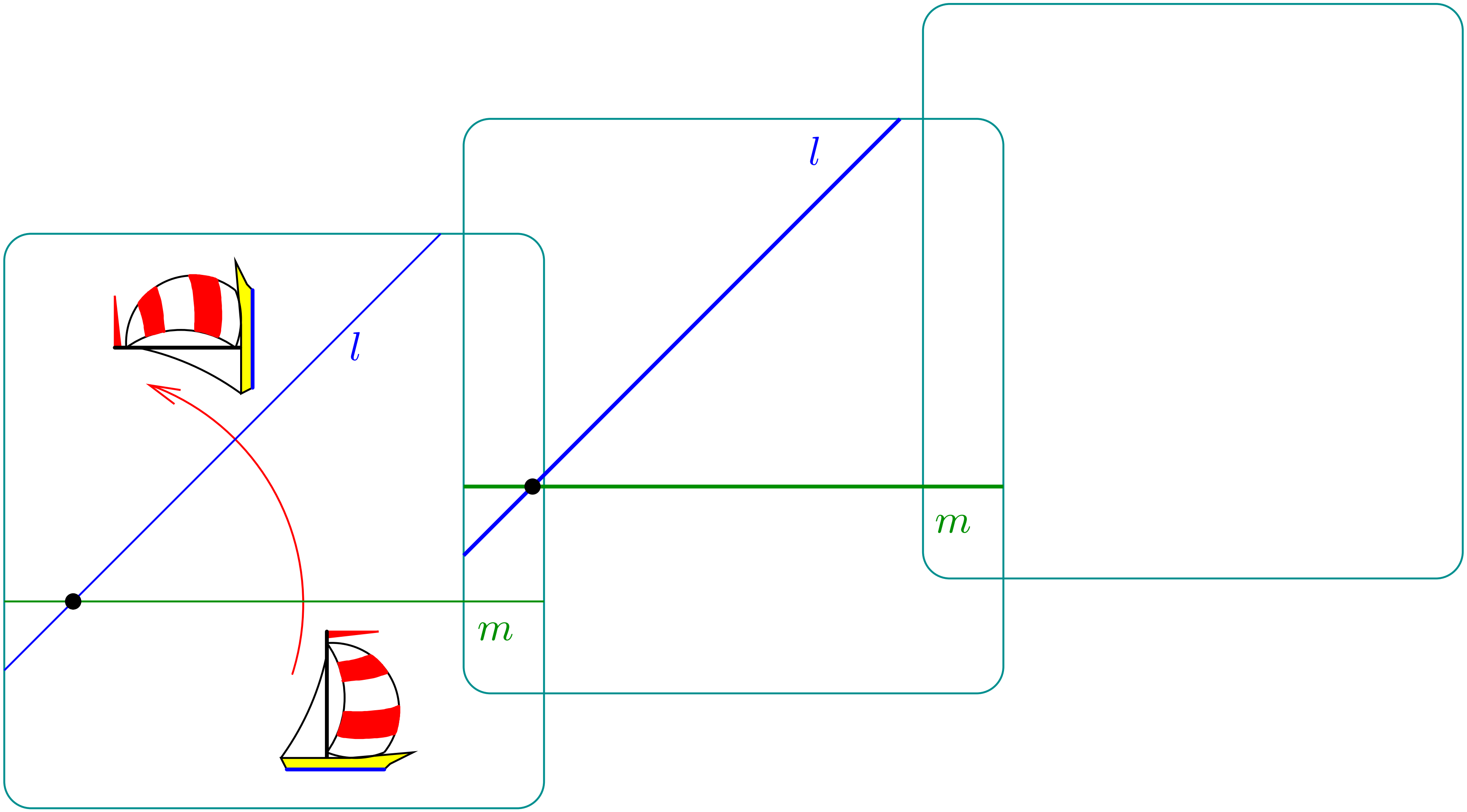

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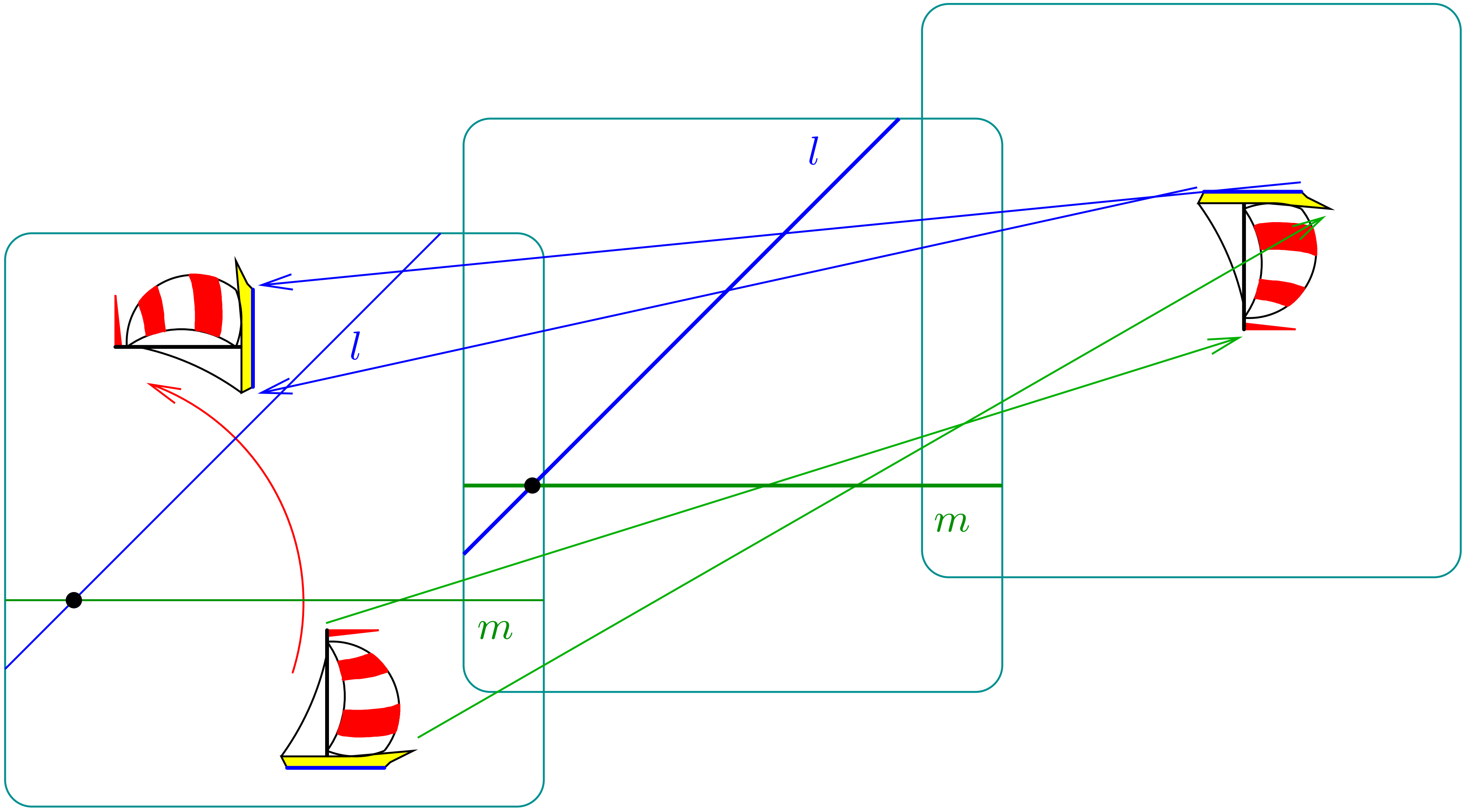

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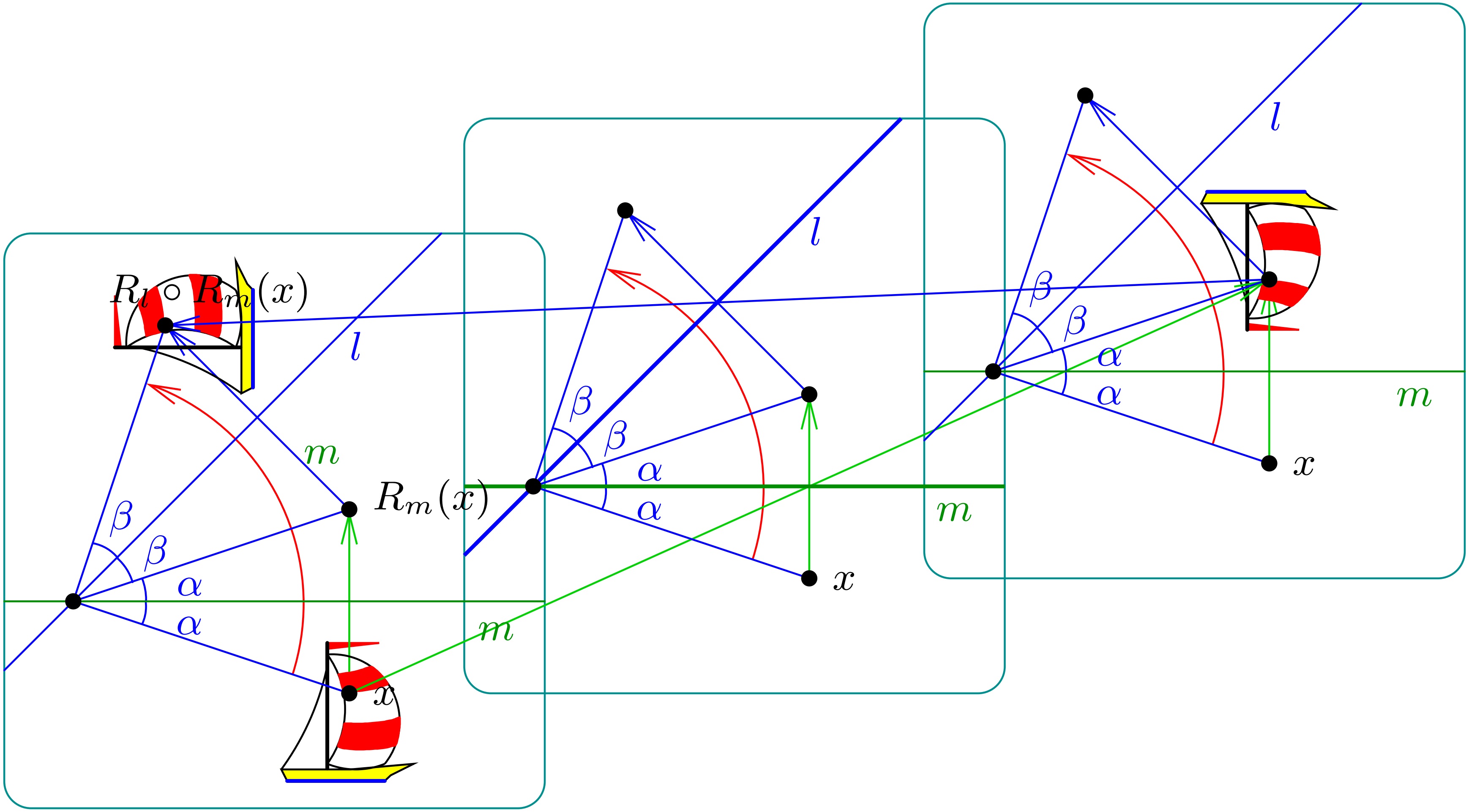

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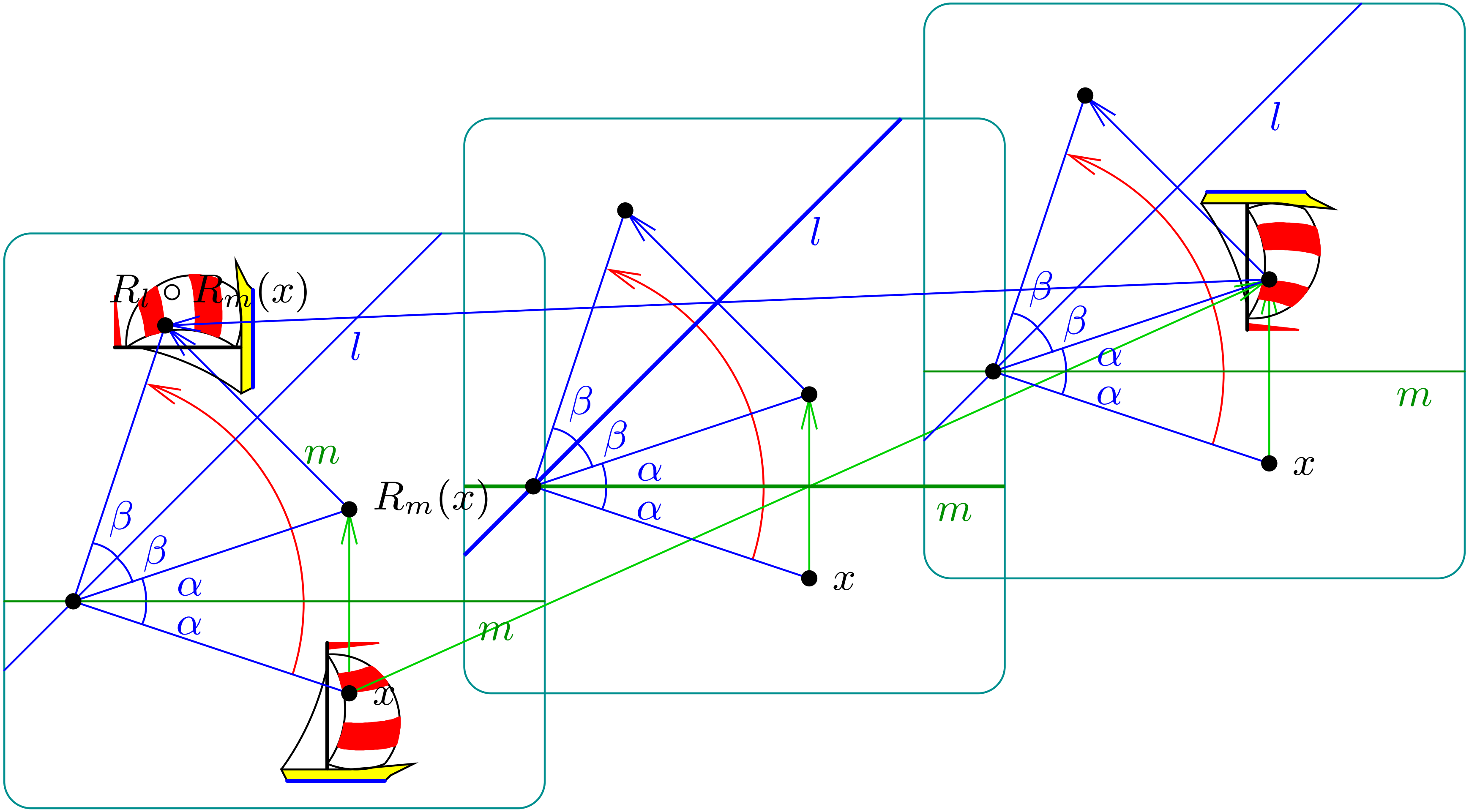
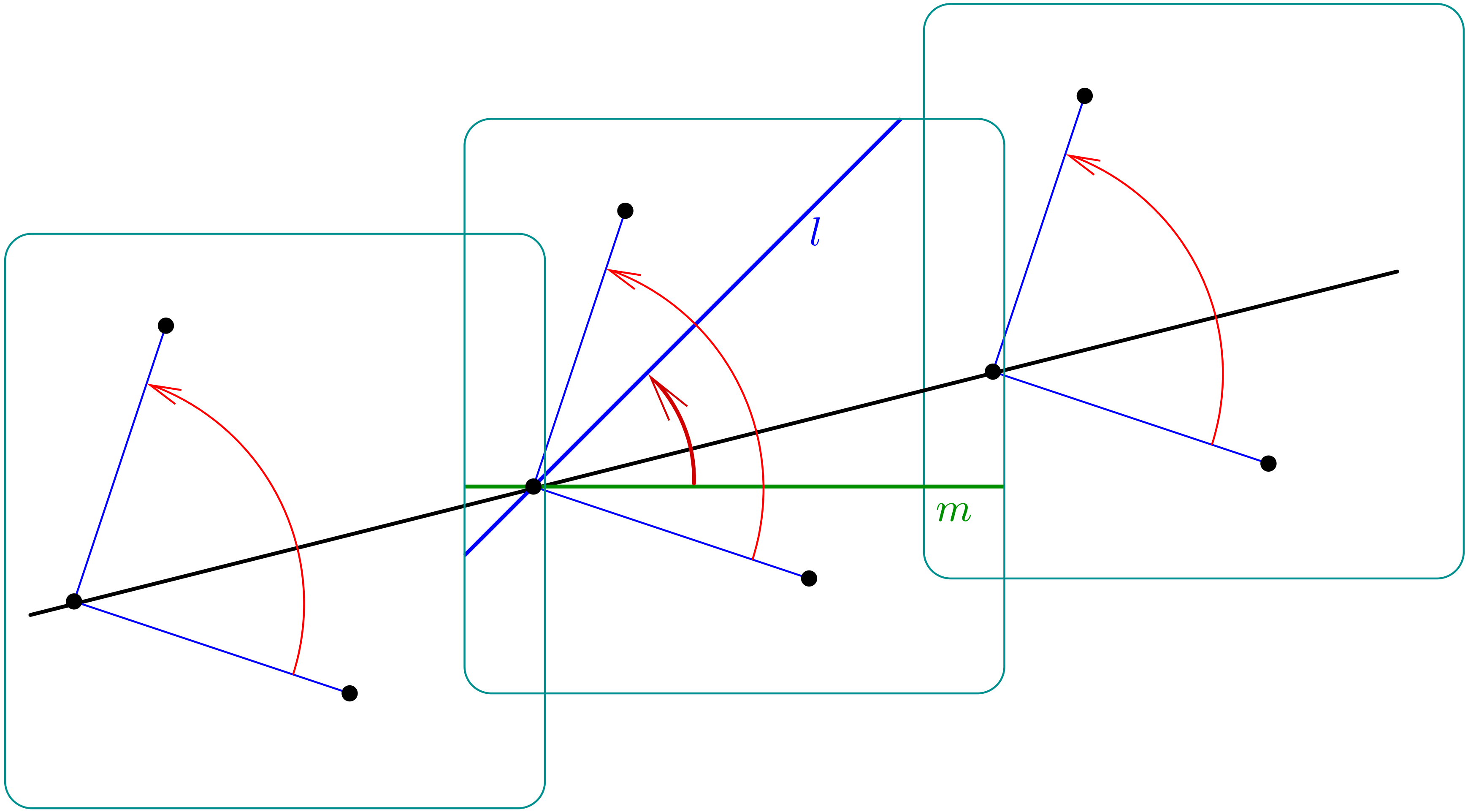


Everything like on the plane.

In the 3-space. Rotation



A decorated angle-arrow formed by two intersecting lines defines a rotation of the 3-space about the axis $\perp$ to the plane of the lines.

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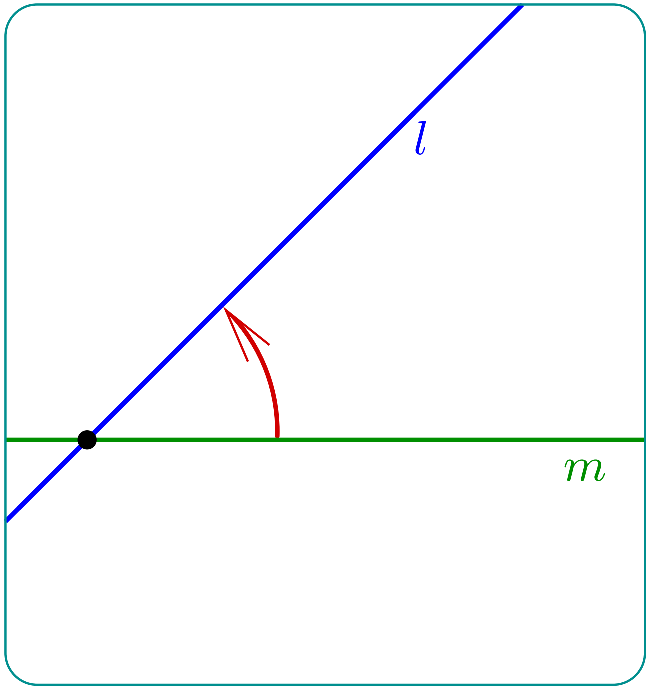

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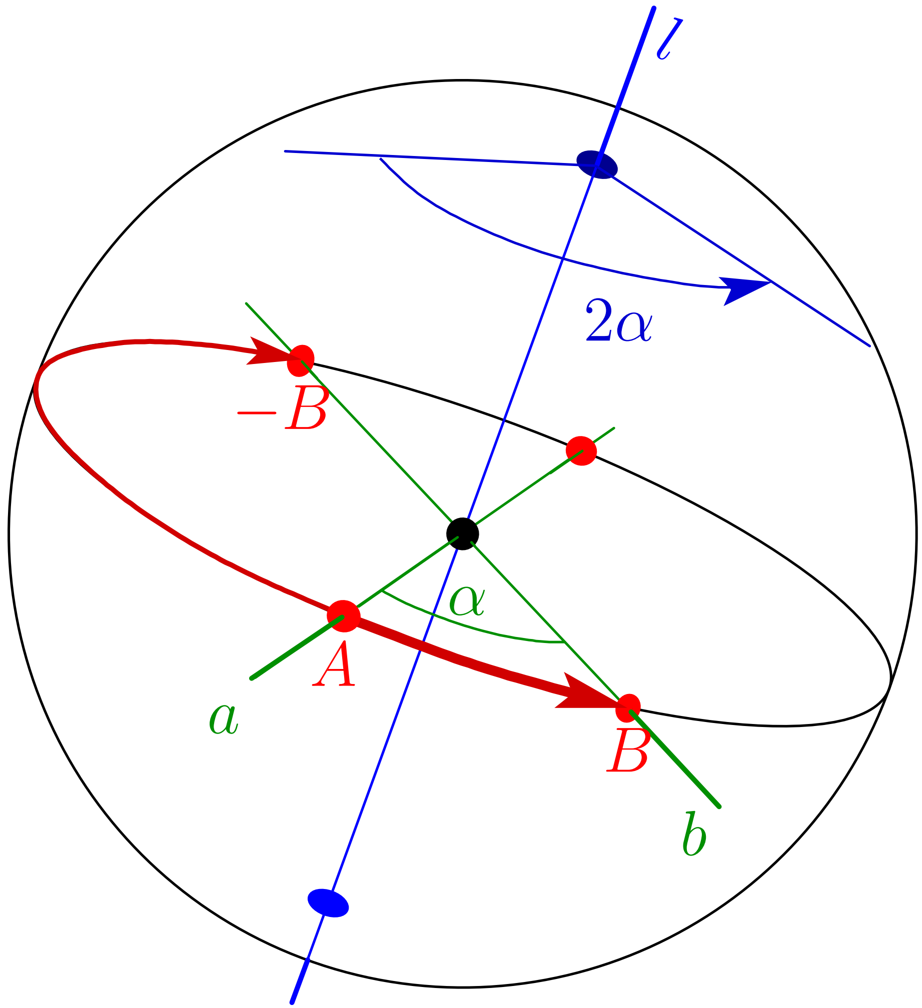

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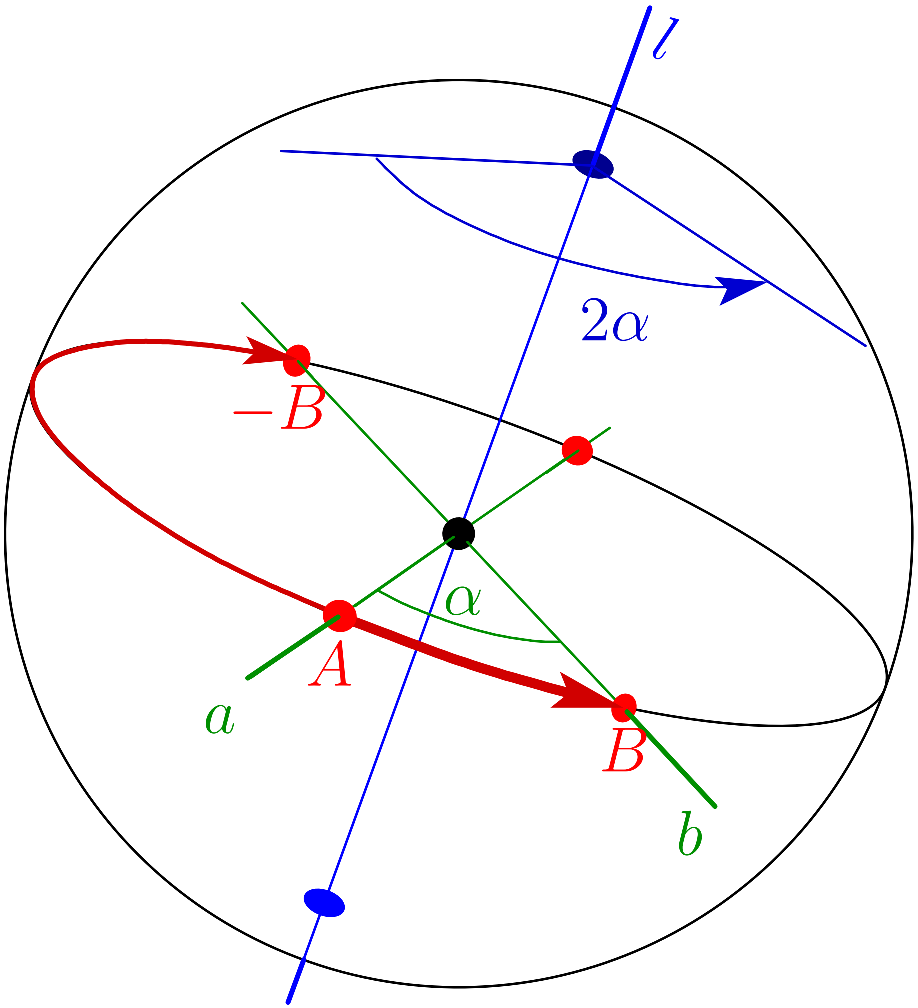
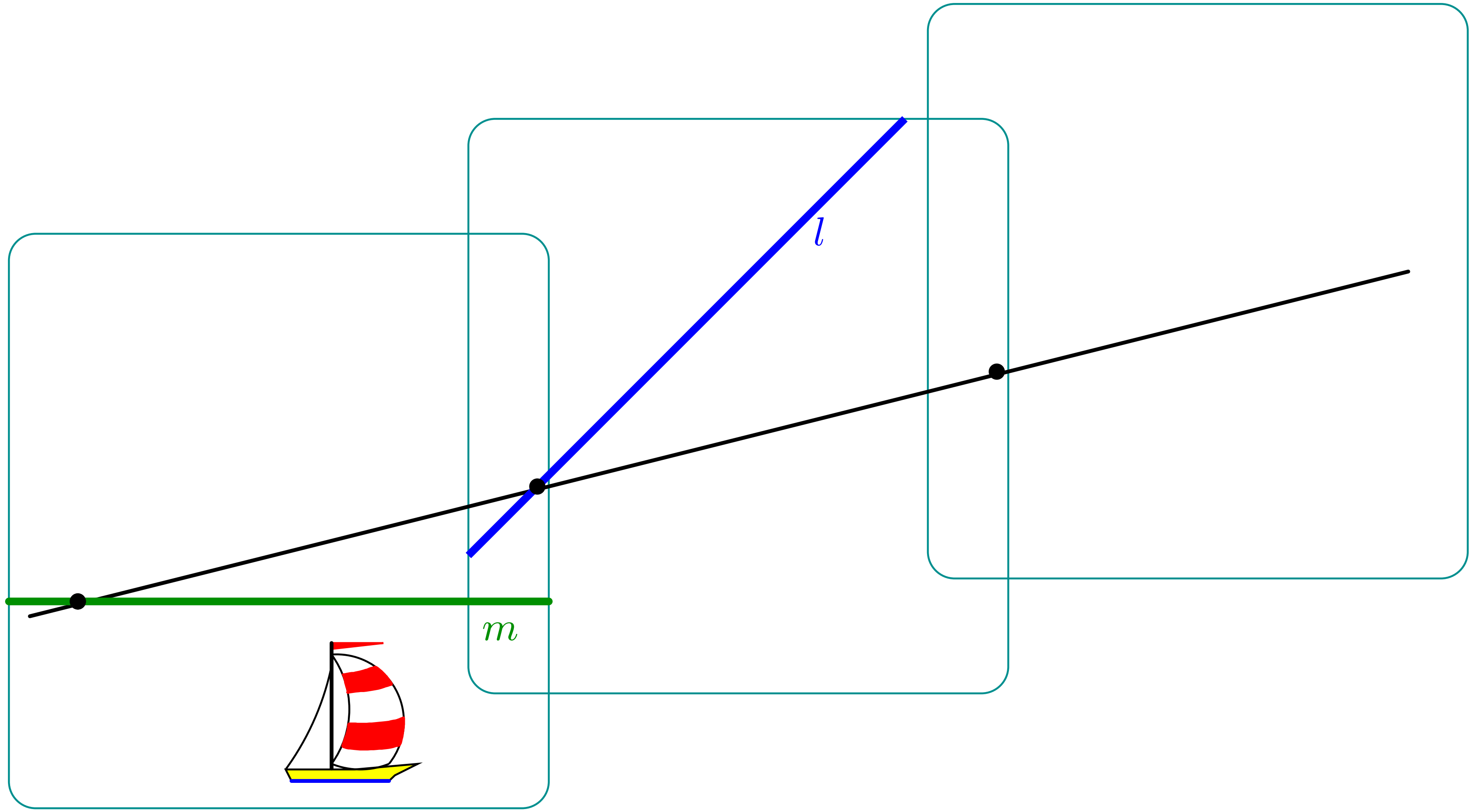


Great circle arrow versus angular displacement vector.

Screw displacement



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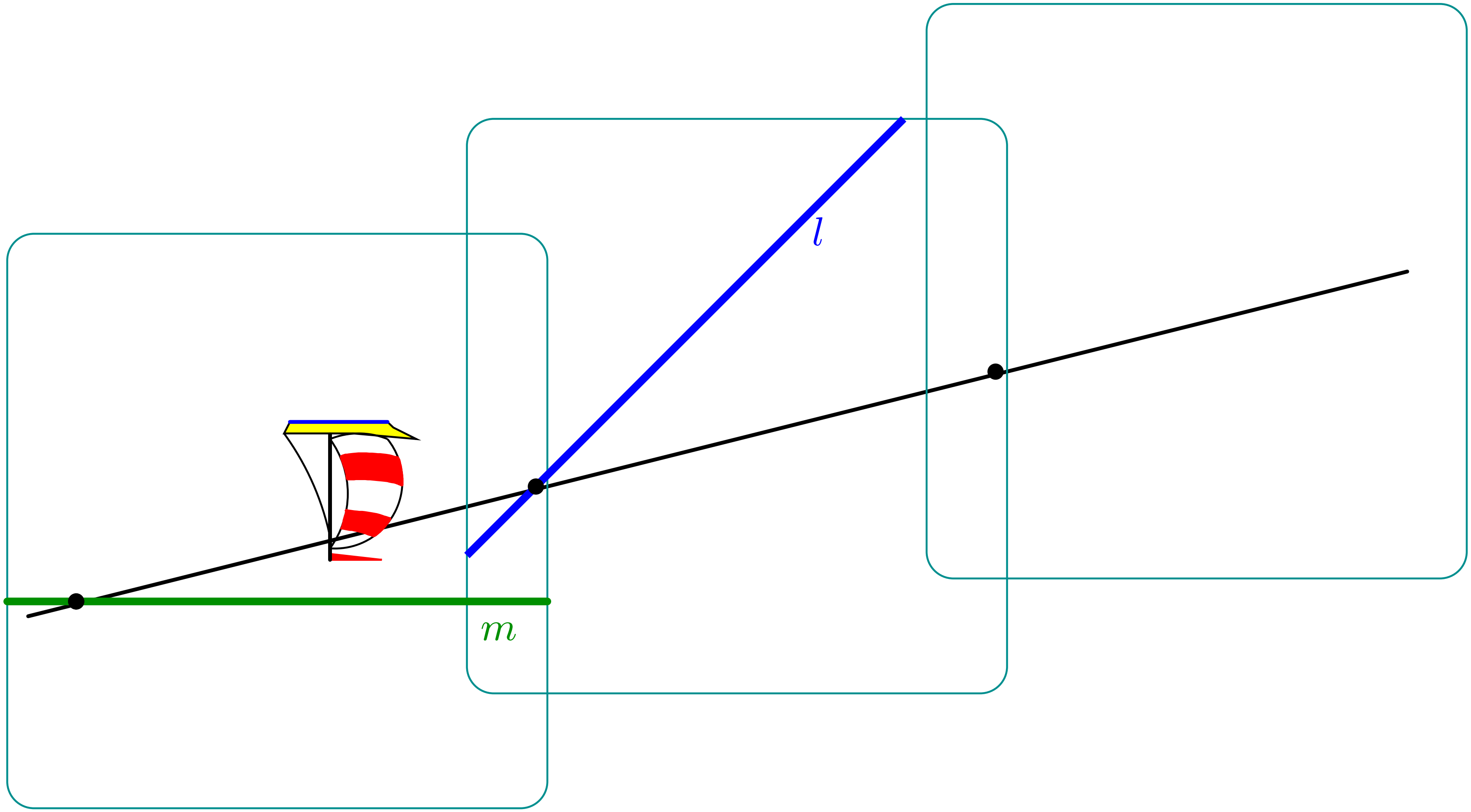

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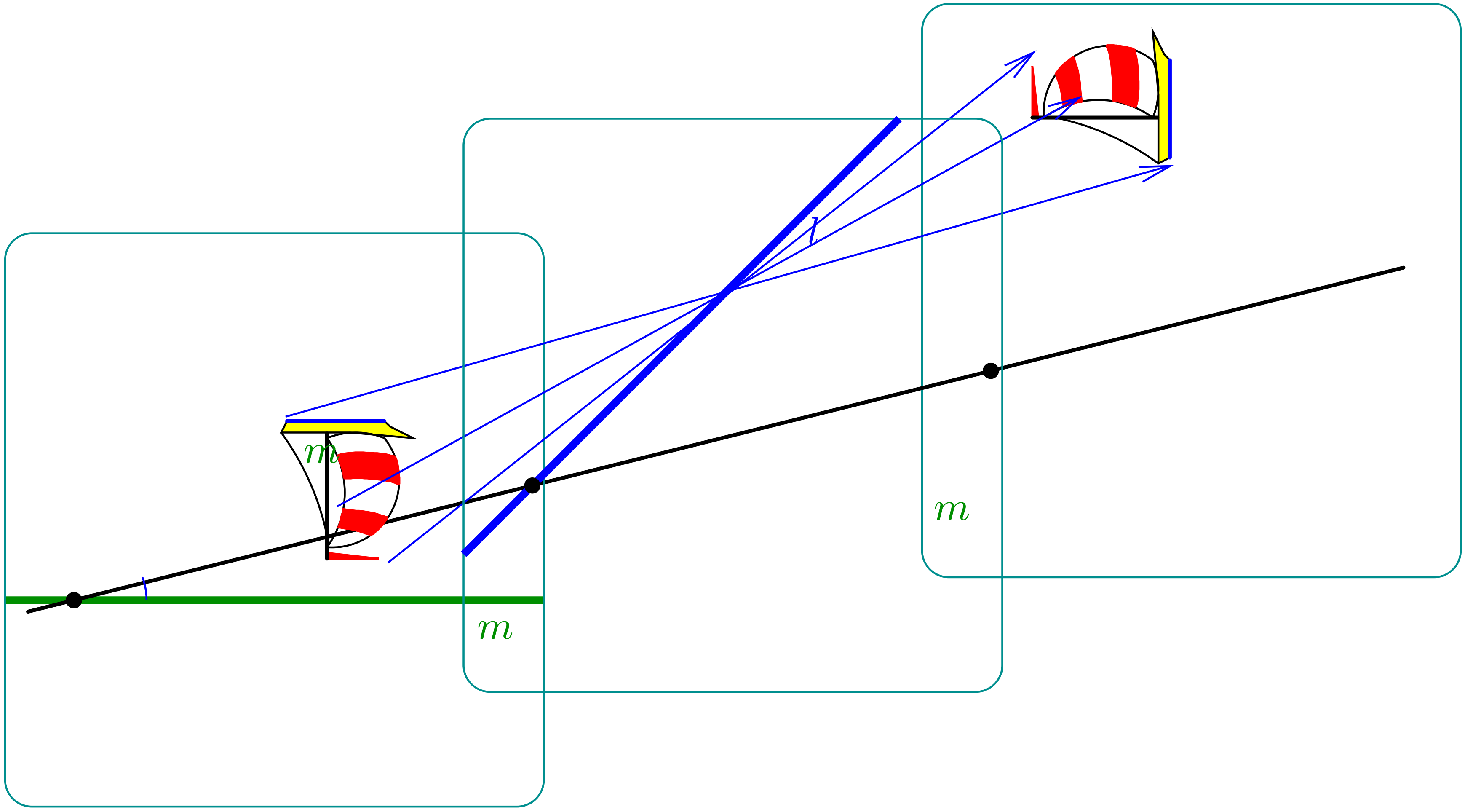

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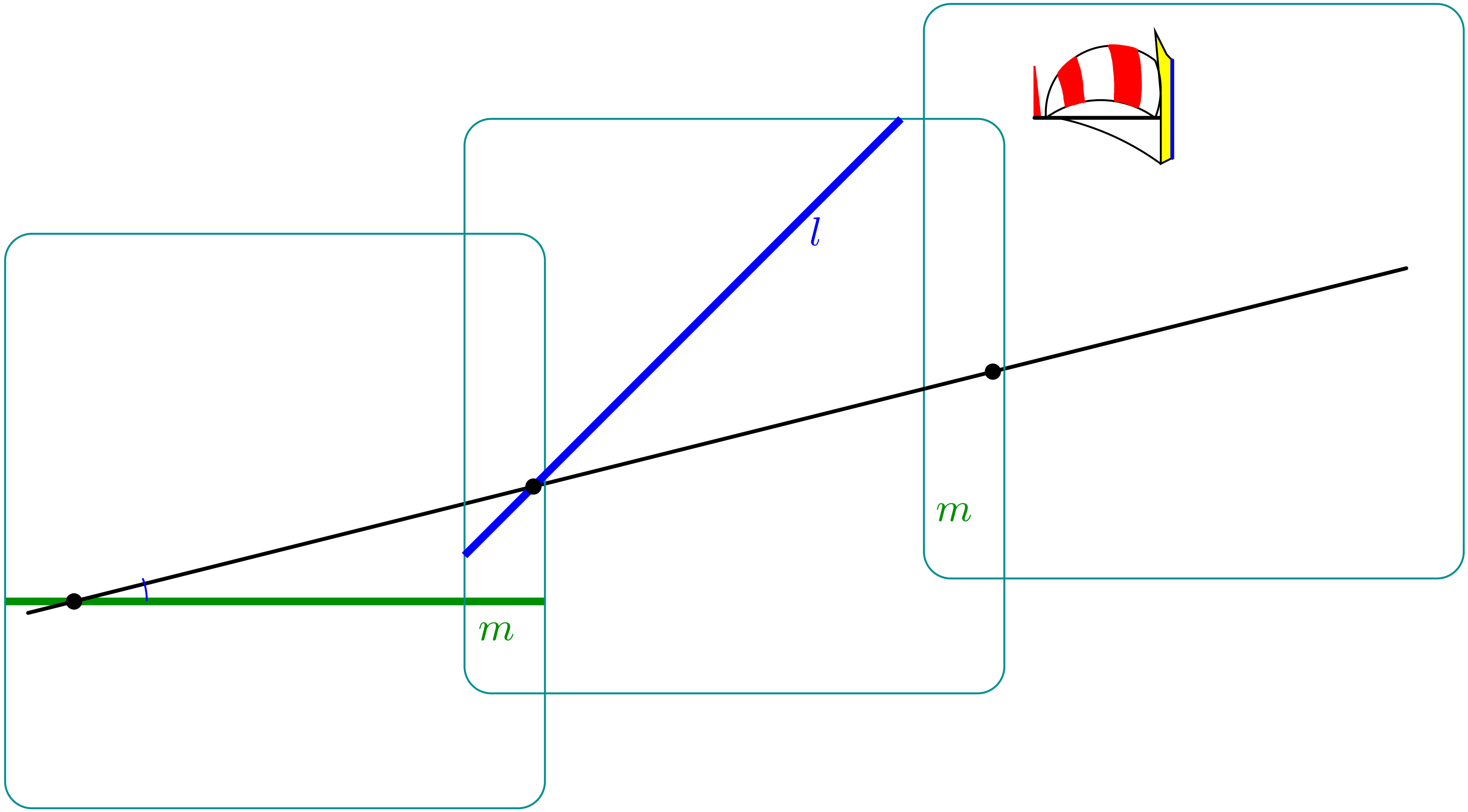

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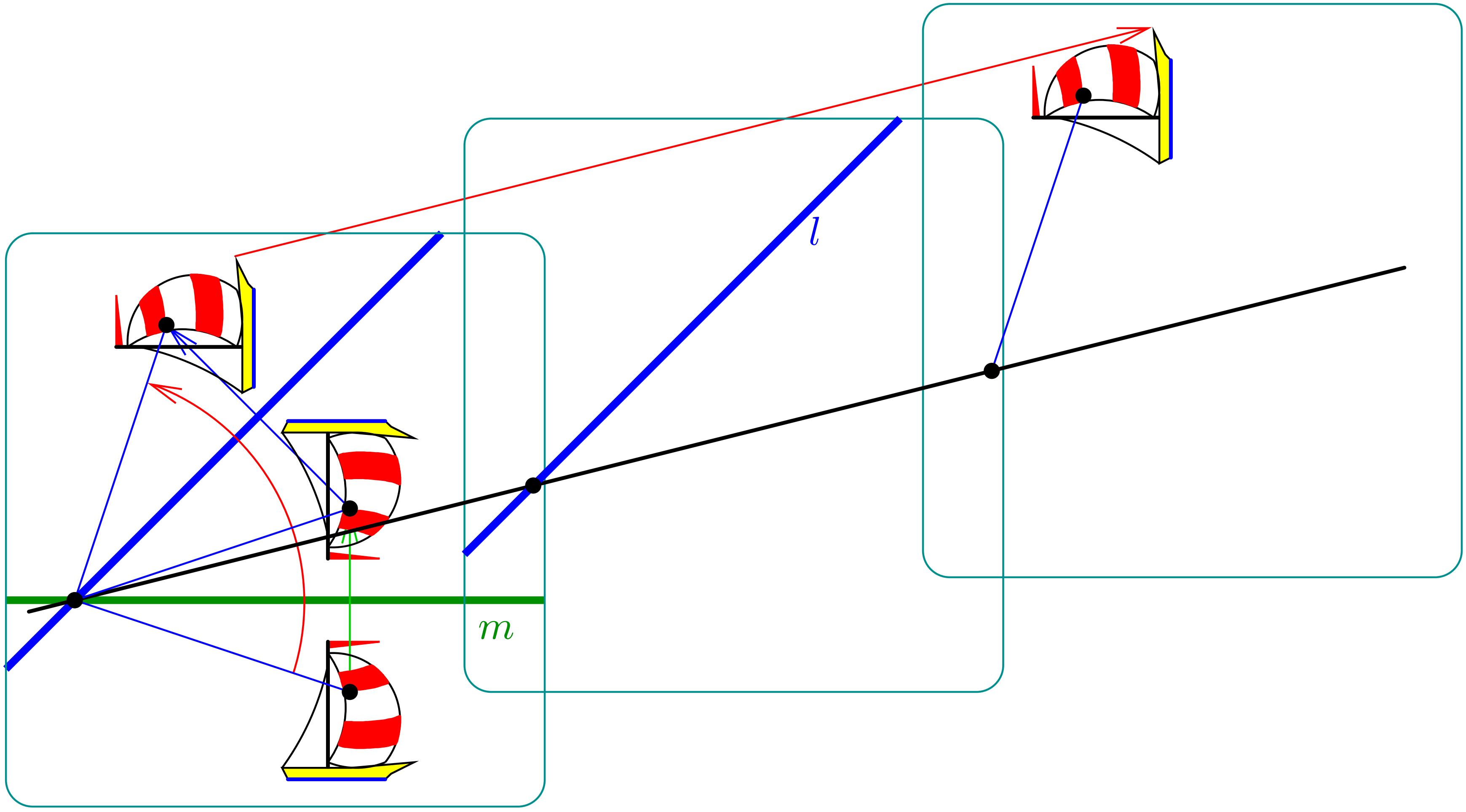

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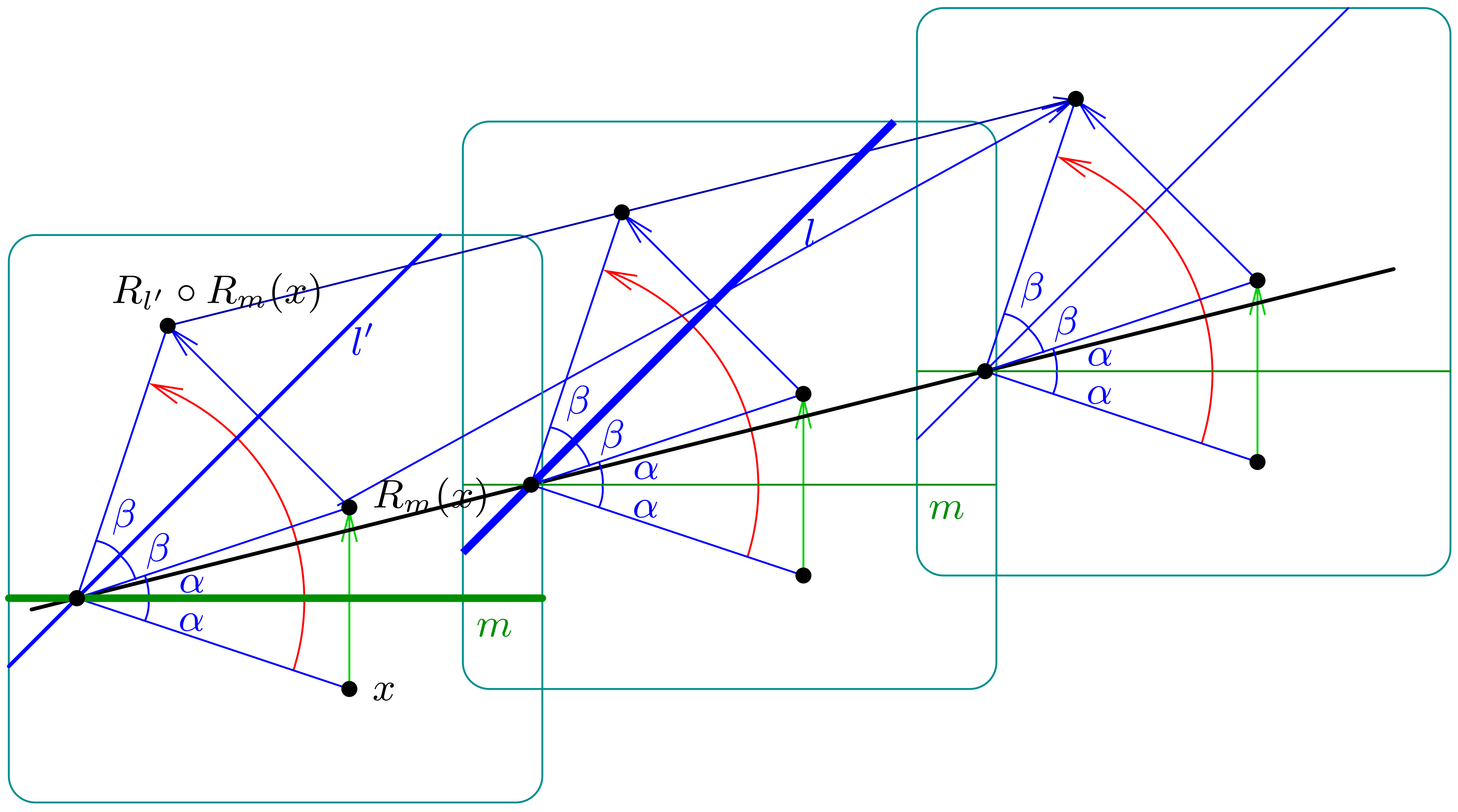

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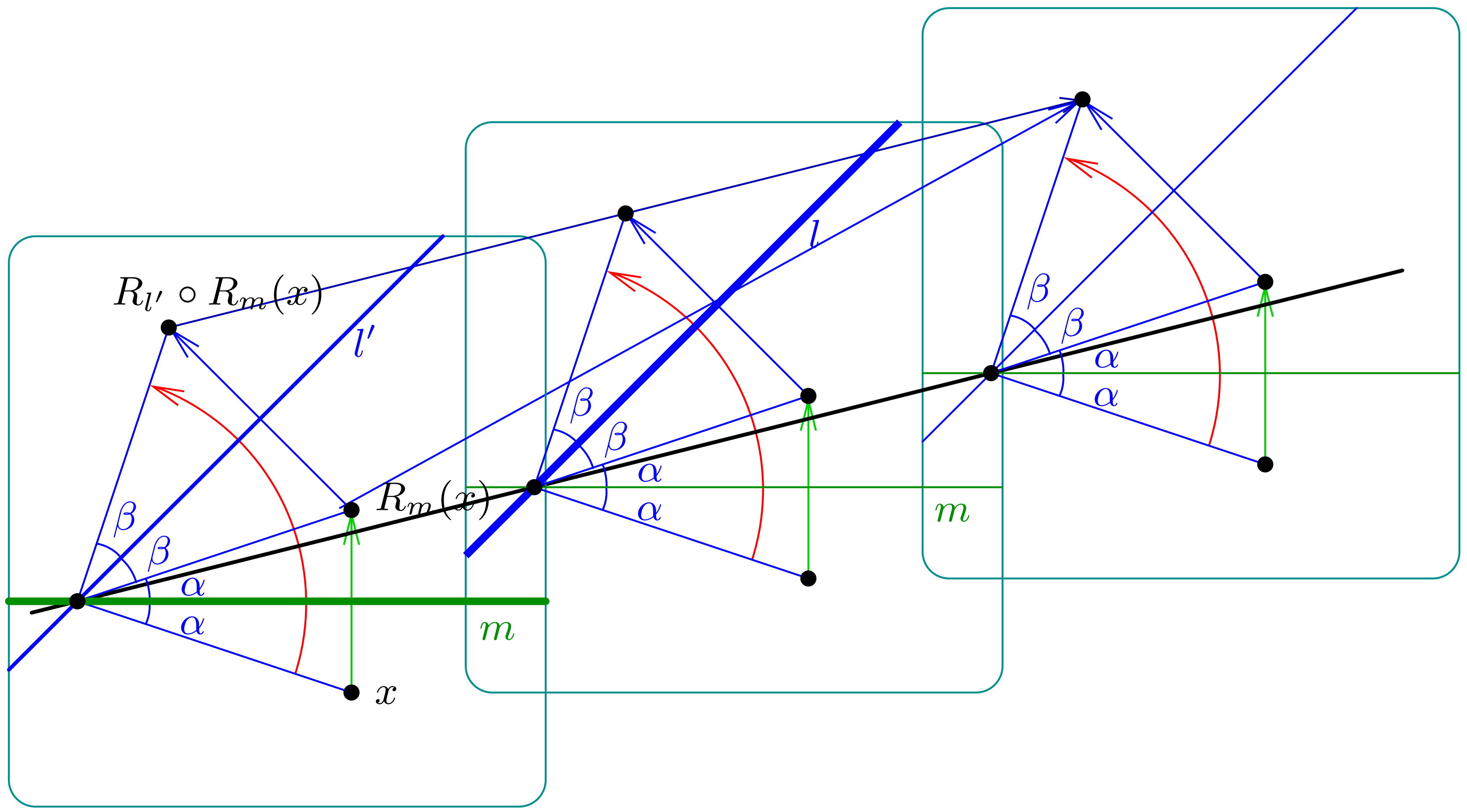

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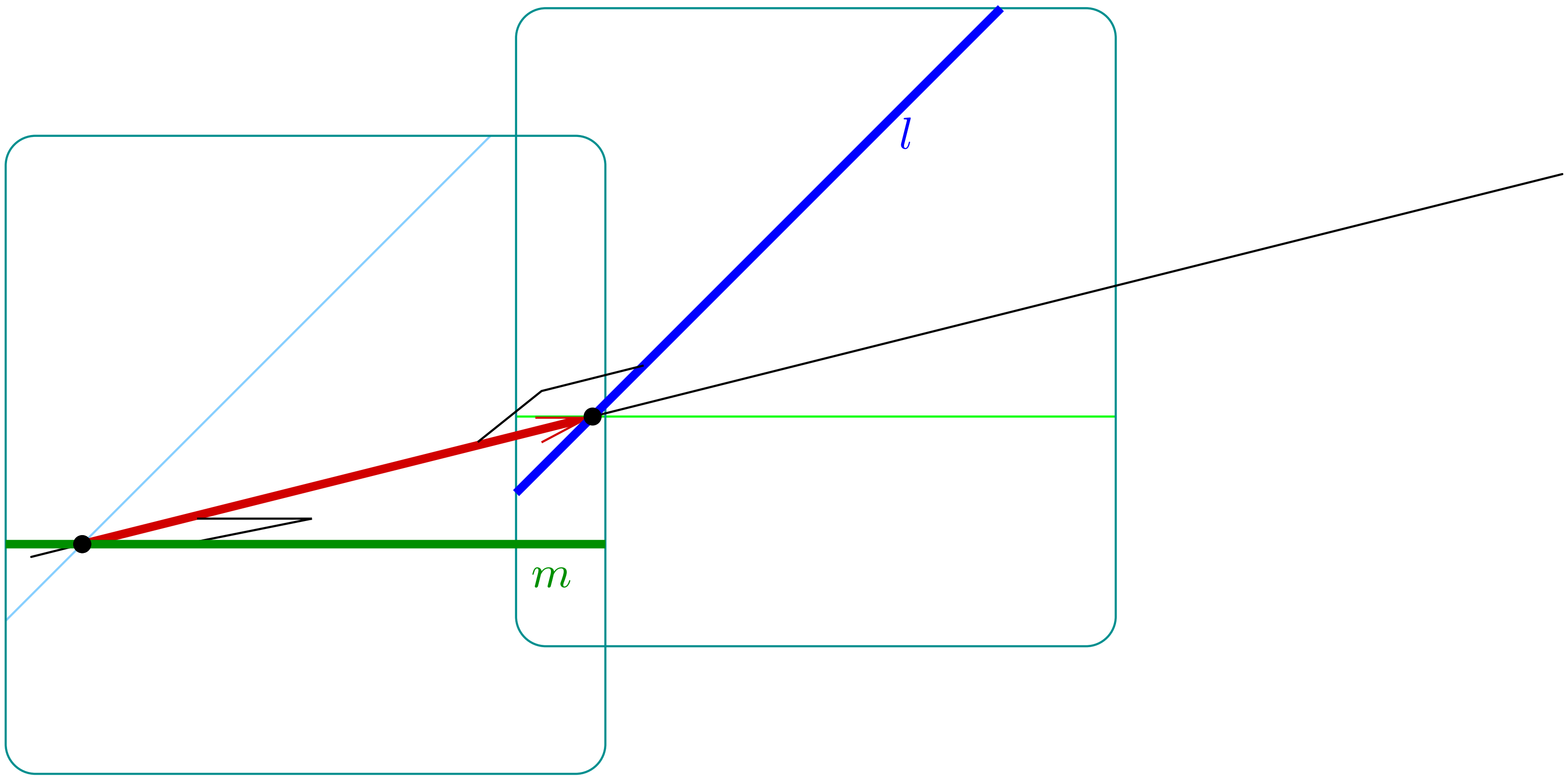

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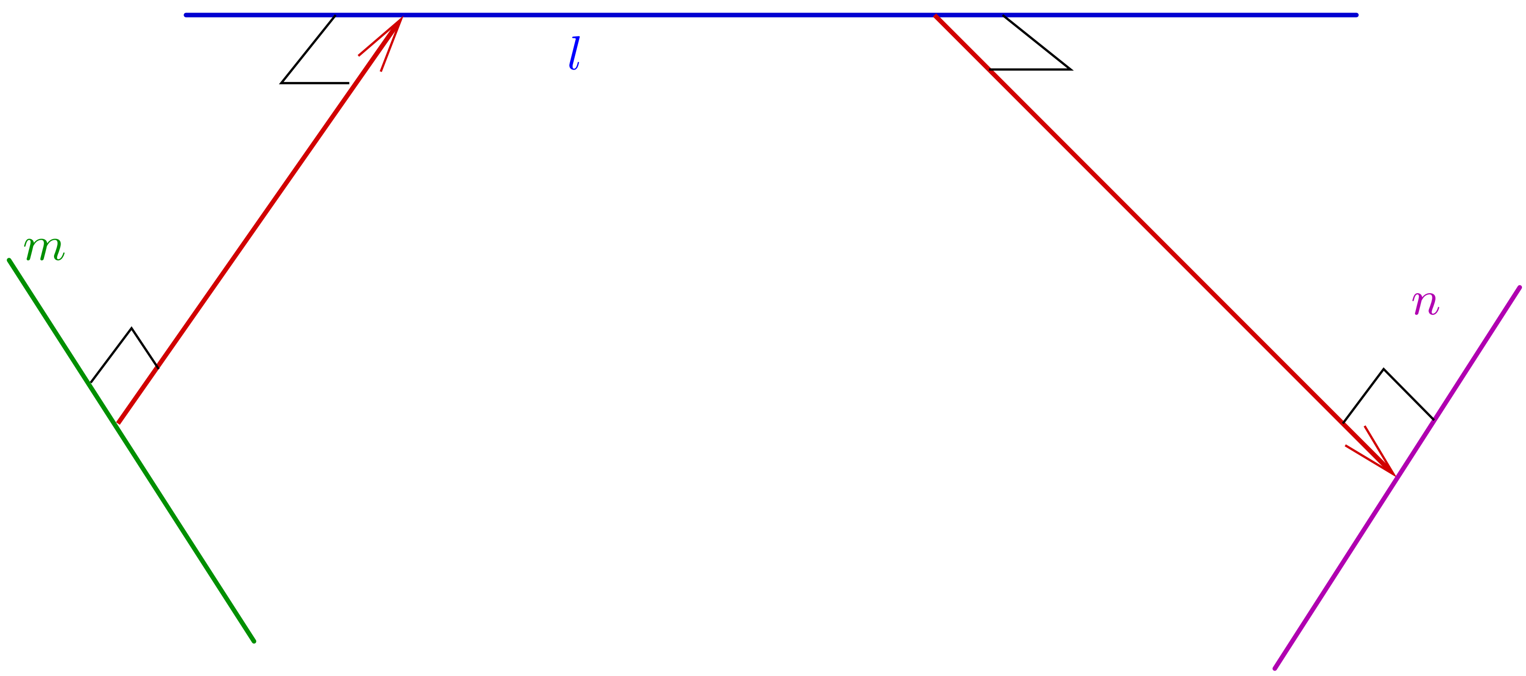

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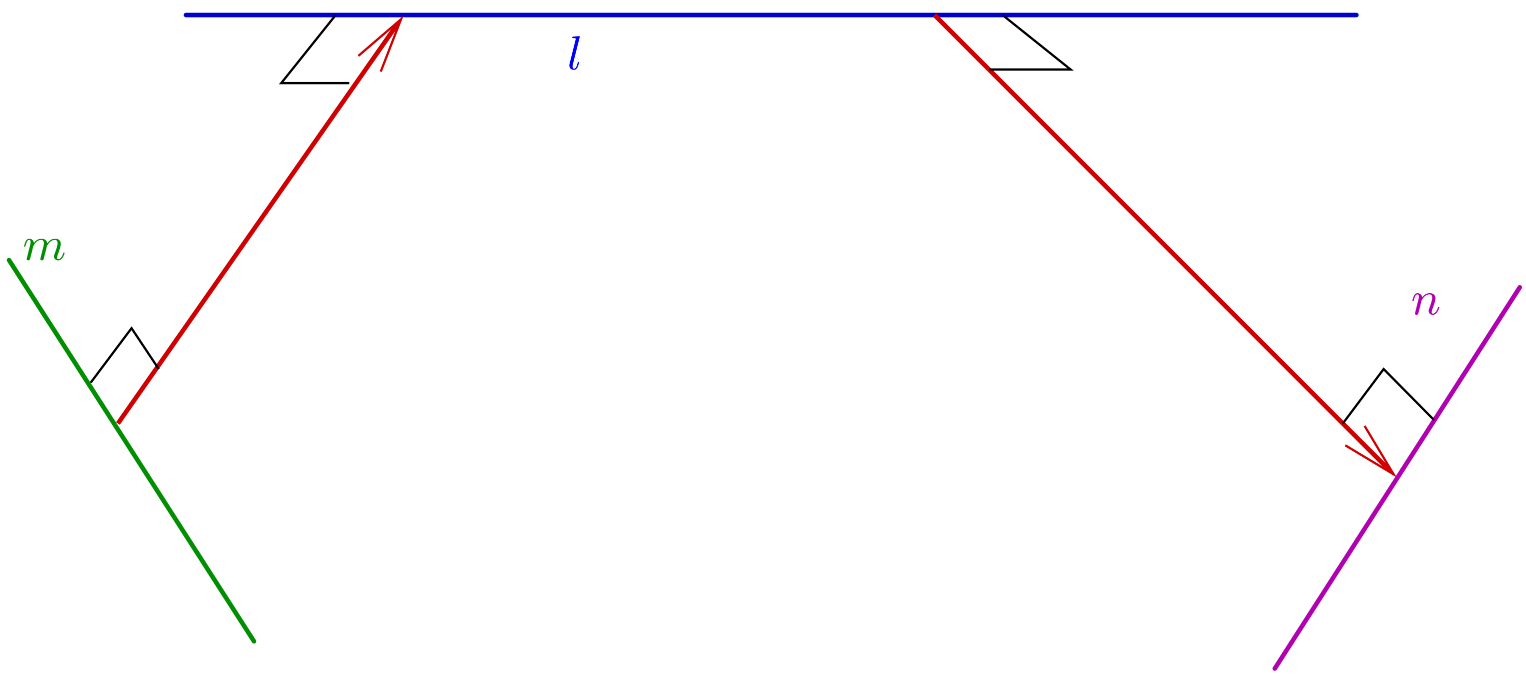

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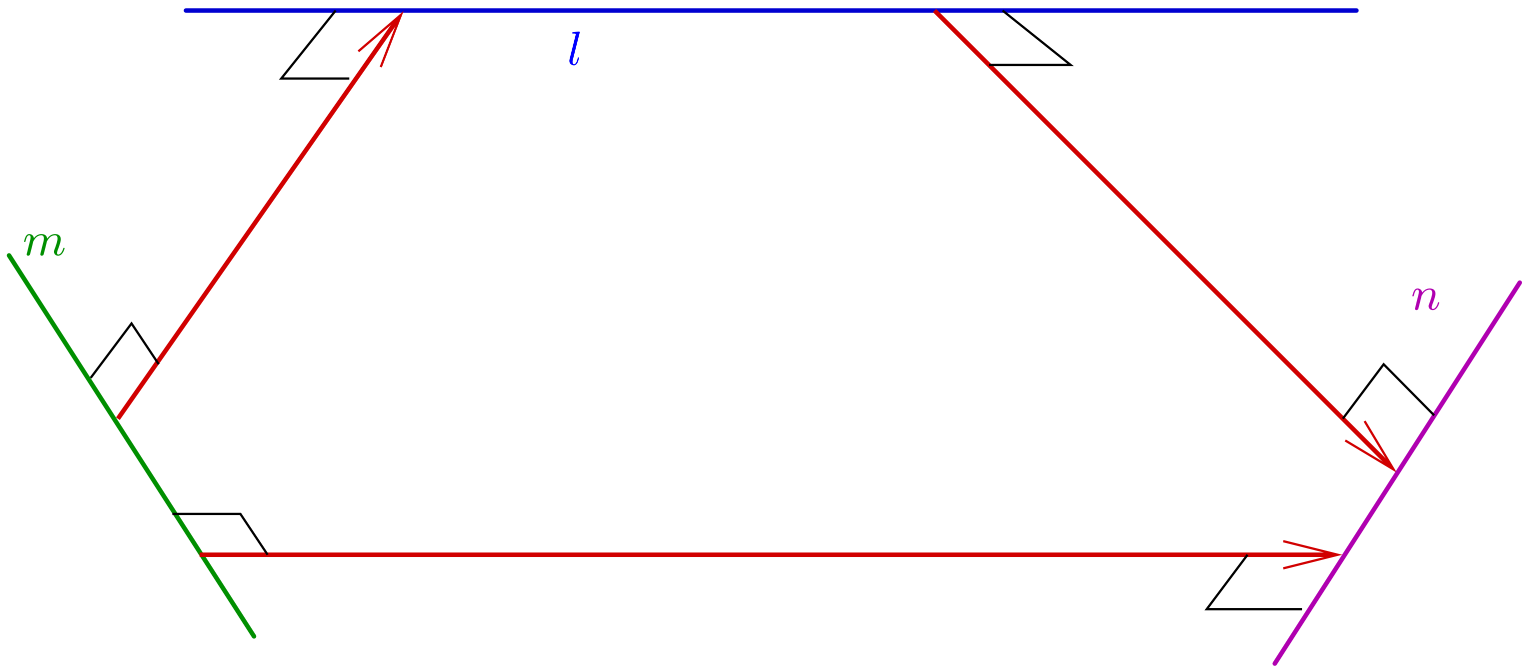

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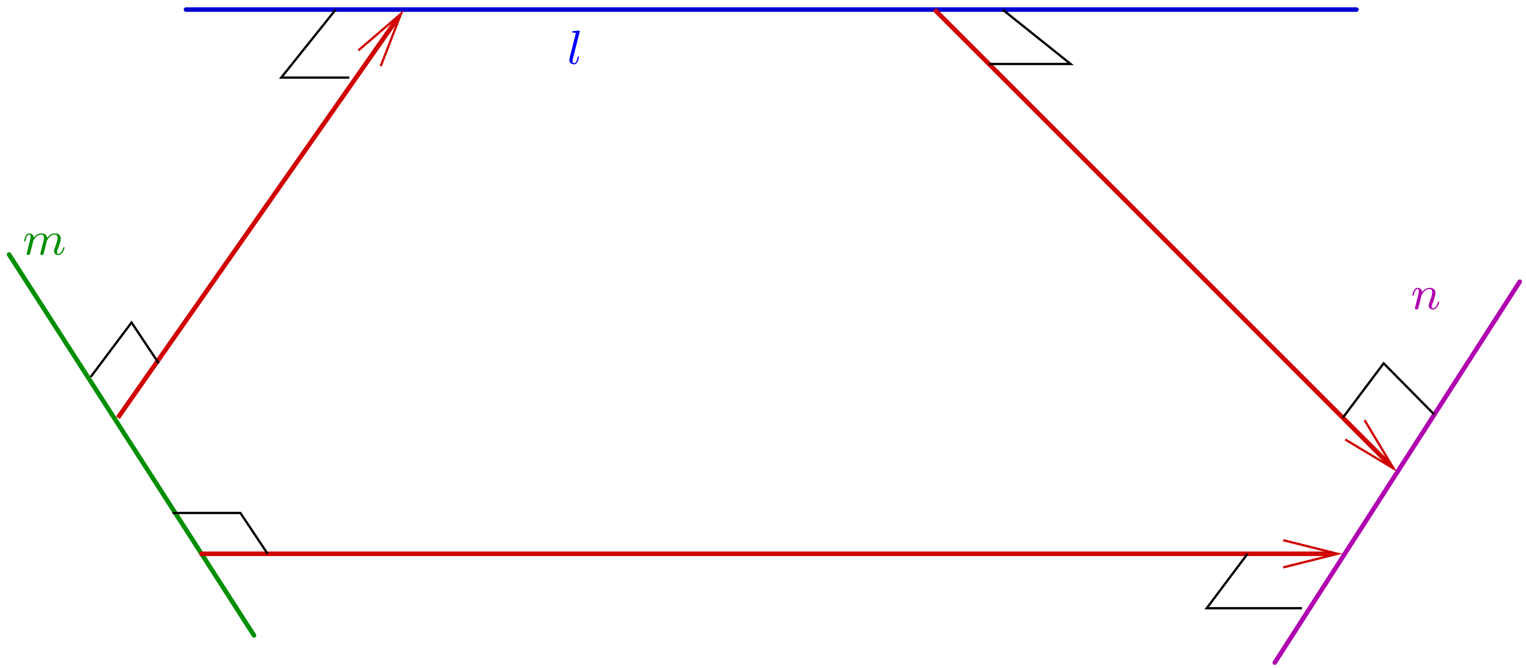

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$SO(n)$ is generated by reflections in codimension two subspaces.

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The identity map is presented by a decorated arrow consisting of two coinciding spaces with coinciding decorations.

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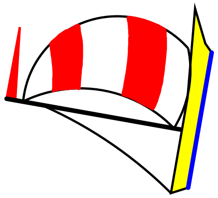

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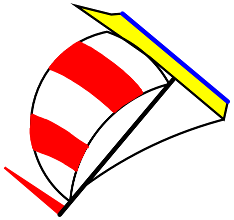

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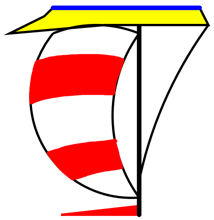

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