Geometry and Algebra of Reflections

Oleg Viro

September 18, 2014
Theorem. Any plane isometry is a composition of at most three reflections in lines.
Plane Isometries

**Theorem.** Any plane isometry is a composition of at most three reflections in lines.

**Lemma.** A plane isometry is determined by its restriction to any three non-collinear points.
Plane Isometries

Theorem. Any plane isometry is a composition of at most three reflections in lines.

Proof of Theorem. Given an isometry:

$\begin{align*}
A & \xrightarrow{f} B \\
B & \xrightarrow{f} C \\
C & \xrightarrow{f} A
\end{align*}$

Diagram:

$\begin{align*}
A & \quad B \\
\quad & \quad \\
\quad & \quad
\end{align*}$

$\begin{align*}
f(A) & \quad f(B) \\
\quad & \quad f(C)
\end{align*}$
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We are done. □
Theorem. Any plane isometry is a composition of at most three reflections in lines.

Generalization. Any isometry of a complete simply connected $n$-space of constant curvature is a composition of at most $n + 1$ reflections in hyperplanes.
Compositions of reflections in parallel lines
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\[ m \quad l \]
Compositions of reflections in parallel lines

\[ m \quad l \]
Compositions of reflections in parallel lines
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is a translation

\[ m \quad l \]
Compositions of reflections in parallel lines

is a translation

\[ x \rightarrow R_m(x) \rightarrow R_l \circ R_m(x) \]
Compositions of reflections in parallel lines

is a translation

\[ x \quad R_m(x) \quad R_l \circ R_m(x) \]

\[ \begin{array}{c|c|c}
    m & l & \\
\end{array} \]

The decomposition is not unique:
Compositions of reflections in parallel lines

is a translation

\[ R_l \circ R_m = R_{l'} \circ R_{m'} \]

iff \( l', m' \) can be obtained from \( l, m \) by a translation.
Compositions of reflections in parallel lines

is a translation

\[ x \quad R_{m'}(x) \quad R_{l'} \circ R_{m'}(x) \]

\[ m' \quad l' \]

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Compositions of reflections in intersecting lines
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is a rotation

\[ R_l \circ R_m(x) \]
Compositions of reflections in intersecting lines

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Decomposition of rotation is not unique:
Compositions of reflections in intersecting lines

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iff \( l', m' \) can be obtained from \( l, m \) by a rotation about the intersection point \( m \cap l \).
Compositions of reflections in intersecting lines

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Decomposition of rotation is not unique:

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iff \( l', m' \) can be obtained from \( l, m \) by a rotation about the intersection point \( m \cap l \).
Theorem. Any relation among reflections in lines follow from relations $R_l^2 = 1$ and $R_l \circ R_m = R_{l'} \circ R_{m'}$, where $l, m, l'm'$ are as on the preceding two slides.
Relations

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**Lemma.** A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines.
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![Diagram showing reflections in lines](image)
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![Diagram showing relations among lines and reflections](image-url)
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1

4
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\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
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![Diagram showing reflections and parallel lines](attachment://diagram.png)
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![Diagram of reflections in lines]
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![Diagram of reflections in lines with lines 1, 2, 3, and 4 and angles between them.]
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**Lemma.** A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines. □

**Generalization of Lemma.** In $\mathbb{R}^n$, a composition of any $n + 2$ reflections in hyperplanes is a composition of $n$ reflections in hyperplanes.
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**Proof of Theorem.** By Lemma, any relation can be reduced to a relation of length $\leq 3$. 
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A composition of odd number of reflections reverses orientation and cannot be $\text{id}$. □

A composition of two different reflections is not identity.
Relations

**Theorem.** Any relation among reflections in lines follow from relations $R^2_l = 1$ and $R_l \circ R_m = R_{l'} \circ R_{m'}$, where $l, m, l', m'$ are as on the preceding two slides. $\square$

**Lemma.** A composition of any 4 reflections in lines can be transformed by these relations to a composition of 2 reflections in lines. $\square$

**Generalization of Lemma.** In $\mathbb{R}^n$, a composition of any $n + 2$ reflections in hyperplanes is a composition of $n$ reflections in hyperplanes.

**Generalization of Theorem.** Any relation among reflections in hyperplanes of $\mathbb{R}^n$ follow from relations $R^2_l = 1$ and $R_l \circ R_m = R_{l'} \circ R_{m'}$. 
Other planes

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How can this be? The groups are not isomorphic?
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In the isometry group of the Lobachevsky plane the same is true. **Theorem.** Any relation among reflections in lines follow from $R_l^2 = 1$ and $R_m \circ R_l = R_{m'} \circ R_{l'}$. How can this be? The groups are not isomorphic? How does curvature work? On sphere everything holds true. On the projective plane a reflection in line has extra fixed point.
Reflections

A reflection is an isometry (of a metric space) which is an involution (i.e., has period 2) and can be determined by its fixed point set.
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**Key example:** $\mathbb{R} \rightarrow \mathbb{R} : x \mapsto 2a - x$, the reflection of $\mathbb{R}$ in a point $a$. 
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Generalization: an orthogonal sum of $n - k$ copies of this reflection and $k$ copies of $\text{id} : \mathbb{R} \to \mathbb{R}$ is a reflection of $\mathbb{R}^n$ in a $k$-subspace.
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Correspondence: Subspace \( S \) \( \longleftrightarrow \) Reflection in \( S \) is the shortest connection between simple static geometric objects and isometries.
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Why involutions?
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Why involutions? \( x^2 = 1 \), and hence

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x^{-1} = x, \quad (xy)^{-1} = yx, \quad (xyz)^{-1} = zyx, \quad [x, y] = (xy)^2.
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Bachmann’s foundations of geometry

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Bachmann’s foundations of geometry

1. David Hilbert’s *Foundations of Geometry*
Bachmann’s foundations of geometry

2. Hermann Weil’s *Space, Time, Matter*
3. Friedrich Bachmann’s

*Construction of Geometry on the notion of reflections*

Friedrich Bachmann

Aufbau der Geometrie
aus dem
Spiegelungsbegriff

Mit 160 Abbildungen

Zweite ergänzte Auflage

Springer-Verlag
Berlin Heidelberg New York 1973
Bachmann’s foundations of geometry

3. Friedrich Bachmann’s
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Objects: a group $G$ generated by a set $S$ of involutions.
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Involutions from $S = \text{reflections in lines} = \text{lines}$. 
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Lines are perpendicular iff the reflections commute.
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A point $= g \circ h$, $g, h \in S$ if $g \circ h = h \circ g$. Denote $\{\text{points}\}$ by $\mathcal{P}$. 
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A point belongs to a line iff the reflections commute.
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Three lines are concurrent or parallel iff the composition of the reflections is a reflection.
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Three lines are concurrent or parallel

\[ \text{iff the composition of the reflections is a reflection.} \]

Four axioms for Absolute Plane Geometry:

1. Through any two points, one can draw a line.
2. If each of two points lies on two lines, then either points or lines coincide.
3. If three lines have a common point, then the composition of the reflections in them is a reflection in a line.
4. If three lines are perpendicular to a line, then the composition of the reflections in them is a reflection in a line.
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if the composition of the reflections is a reflection.

Four axioms for Absolute Plane Geometry.

Higher dimensions, order and betweenness were out of consideration.
Composition of reflections through points
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Composition of reflections through points

is a translation:

\[ R_B(R_A(X)) \]

\[ R_A(X) \]
Composition of reflections through points

is a translation:

\[ \overrightarrow{AB} = \frac{1}{2} \overrightarrow{X R_B(R_A(X))} \]
Composition of reflections through points

is a translation:

\[ X \xrightarrow{R_B(R_A(X))} \]

\[ \overrightarrow{AB} = \frac{1}{2} \overrightarrow{X R_B(R_A(X))} \]

\( \overrightarrow{AB} \) is half the arrow representing \( R_B \circ R_A \).
Composition of reflections through points

is a translation:

$$\overrightarrow{AB} = \frac{1}{2} \overrightarrow{XR_B(R_A(X))}$$

$$\overrightarrow{AB}$$ is half the arrow representing $$R_B \circ R_A$$.

Compare the head to tail addition $$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$.
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is a translation:

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\overrightarrow{AB} = \frac{1}{2} X \overrightarrow{R_B(R_A(X))}
\]

\(\overrightarrow{AB}\) is half the arrow representing \(R_B \circ R_A\).

Compare the head to tail addition

\[
\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}
\]

to

\[
(R_C \circ R_B) \circ (R_B \circ R_A) = R_C \circ R_B^2 \circ R_A = R_C \circ R_A.
\]
Head to tail

If $T = R_B \circ R_A$, $S = R_C \circ R_B$ and $R_B^2 = \text{id}$,
then $S \circ T = R_C \circ R_A$. 
If $T = R_B \circ R_A$, $S = R_C \circ R_B$ and $R_B^2 = \text{id}$, then $S \circ T = R_C \circ R_A$.

Which isometries are compositions of two reflections?
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Any isometry of $\mathbb{R}^n$. 

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An ordered pair of subspaces $(A, B)$ such that $T = R_B \circ R_A$ is an analogue for an arrow representing a translation.
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If $A \cap B = \emptyset$, then we can connect them with the shortest arrow and consider $A$ and $B$ as decorations at the end points.
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\[ A \quad \rightarrow \quad B \]
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To what extent are the representations non-unique?
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**Equivalence relation:**

\[(A, B) \sim (A', B') \quad \text{if} \quad R_B \circ R_A = R_{B'} \circ R_{A'} .\]
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**Equivalence relation:**

\((A, B) \sim (A', B')\) if \( R_B \circ R_A = R_{B'} \circ R_{A'} \).

**Problem.** Find an explicit description for the equivalence.
Angle-arrows for a rotation
Angle-arrows for a rotation

A rotation of a plane is encoded by an ordered pair of lines.
Angle-arrows for a rotation

A rotation of a plane is encoded by an ordered pair of lines. The lines intersect at the center of rotation.
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Head to tail for rotations
Head to tail for rotations

Given two rotations, present them by angle-arrows.
Head to tail for rotations

Given two rotations, present them by angle-arrows.

By rotating the angle-arrows, make the second line in the first angle coinciding with the first line in the second, so that the angle-arrows are \((l, m)\) and \((m, n)\).
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Erase \(m\) and draw an oriented arc from \(l\) to \(n\), i.e., form the ordered angle \((l, n)\).
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\[ C = A + B \]
Reflections in points

A reflection of the plane in a point is the rotation by $\pi$ about the point.
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Therefore it is a composition of reflections in any two orthogonal lines passing through the point.
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Relations involving three reflections, $R_l$, $R_m$ and $R_O$:

$R_O = R_m \circ R_l$ and hence $R_O \circ R_l \circ R_m = \text{id}$ and $R_O \circ R_l = R_m$. 
**Reflections in points**

A reflection of the plane in a point is the rotation by $\pi$ about the point.

Therefore it is a composition of reflections in any two orthogonal lines passing through the point.

Furthermore, $R_l$, $R_m$ and $R_O$, together with $\text{id}$, form the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$. 
Composing reflections in line and point
Composing reflections in line and point
Composing reflections in line and point
Composing reflections in line and point

\[ m \]

\[ O \]
Composing reflections in line and point

This is a glide reflection!
Composing reflections in line and point

Indeed!

This is a glide reflection!
Composing reflections in line and point

Indeed!

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Composing reflections in line and point

Indeed!

This is a glide reflection!
Decorated arrows for a glide reflection

\[ x \rightarrow R_m(x) \rightarrow RO(R_m(x)) \]
Decorated arrows for a glide reflection

\[ x \rightarrow R_m(x) \rightarrow O \rightarrow R_O(R_m(x)) \]

\[ m \]
Decorated arrows for a glide reflection
Decorated arrows for a glide reflection
Decorated arrows for a glide reflection
Decorated arrows for a glide reflection

\[
R_{O'}(x) \xrightarrow{m} R_m(x) \xrightarrow{l} R_l(R_m(x)) = R_l(R_{O'}(x))
\]
Decorated arrows for a glide reflection

![Diagram showing a glide reflection with decorated arrows]

- $x$
- $O'$
- $R_{O'}(x)$
- $R_l(R_{O'}(x))$
- $l$
Decorated arrows for a glide reflection

\[ O' \]

\[ l \]
Decorated arrows for a glide reflection
Decorated arrows for a glide reflection

A decorated arrow for a glide reflection may glide along itself.
Decorated arrows for a glide reflection

A decorated arrow for a glide reflection may glide along itself.
Decorated arrows for a glide reflection

A decorated arrow for a glide reflection may glide along itself.
Head to tail for glide reflections
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows.
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should NOT be decorated with lines.
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are \( \overrightarrow{IO} \) and \( \overrightarrow{On} \).
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines. By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are $\overrightarrow{IO}$ and $\overrightarrow{On}$.
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By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second.

so that the decorated arrows are $\vec{lO}$ and $\vec{On}$.

Draw an oriented arc from $l$ to $n$.

![Diagram showing the head to tail for glide reflections](image-url)
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are $\overrightarrow{lo}$ and $\overrightarrow{On}$. Draw an oriented arc from $l$ to $n$. 
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are $\overrightarrow{lo}$ and $\overrightarrow{On}$. Draw an oriented arc from $l$ to $n$ and erase $O$. 
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are $\vec{lO}$ and $\vec{On}$. Draw an oriented arc from $l$ to $n$ and erase $O$. 

![Diagram of glide reflections with arrows and arc]
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second. so that the decorated arrows are $\overrightarrow{lo}$ and $\overrightarrow{On}$.

Draw an oriented arc from $l$ to $n$ and erase $O$.

This is a rotation!
Head to tail for glide reflections

Given two glide reflections, present them by decorated arrows. The head in the first arrow and tail in the second one should **NOT** be decorated with lines.

By gliding the arrows, make the arrow head of the first arrow coinciding with the arrow tail of the second.

so that the decorated arrows are $\vec{O}l$ and $\vec{On}$.

Draw an oriented arc from $l$ to $n$ and erase $O$.

**Exercise.** Find head to tail rules for composing rotation and glide reflection.
In the 3-space. Rotation
In the 3-space. Rotation
In the 3-space. Rotation
In the 3-space. Rotation
In the 3-space. Rotation
In the 3-space. Rotation
In the 3-space. Rotation

Everything like on the plane.
A decorated angle-arrow formed by two intersecting lines defines a rotation of the 3-space about the axis \( l \) to the plane of the lines.
A decorated angle-arrow formed by two intersecting lines defines a rotation of the 3-space about the axis \( \perp \) to the plane of the lines.
Rotation of a sphere
Rotation of a sphere

Great circle arrow versus angular displacement vector.
Screw displacement
Screw displacement
Screw displacement
Screw displacement
Screw displacement
Screw displacement

$R_{l'} \circ R_m(x)$

$R_m(x)$

$l$

$m$

$x$

$\alpha$

$\beta$

$\beta$

$\beta$

$\alpha$

$\alpha$
A decorated arrow presenting a screw displacement is an arrow with two perpendicular lines at the end points skew to each other.
Screw displacement

A decorated arrow presenting a screw displacement is an arrow with two perpendicular lines at the end points skew to each other.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.

By gliding the arrows along their lines and rotating the decorations, make the head decorations of the first arrow coinciding with the tail decoration of the second arrow.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.

By gliding the arrows along their lines and rotating the decorations, make the head decorations of the first arrow coinciding with the tail decoration of the second arrow.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.

Find common perpendicular for the tail decoration of the first arrow and head decoration of the second. Draw an arrow along it connecting the decorations.

\[ m \]

\[ n \]
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.

Find common perpendicular for the tail decoration of the first arrow and head decoration of the second. Draw an arrow along it connecting the decorations.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

Find the common perpendicular for the lines of the arrows.

Find common perpendicular for the tail decoration of the first arrow and head decoration of the second. Draw an arrow along it connecting the decorations. Erase old arrows and their common decoration line.
Head to tail for screws

Given two screw displacement, present them by decorated arrows.

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Find common perpendicular for the tail decoration of the first arrow and head decoration of the second. Draw an arrow along it connecting the decorations. Erase old arrows and their common decoration line.
Spin

Two-fold covering $Spin(n) \rightarrow SO(n)$. 
Spin

Two-fold covering $\text{Spin}(n) \to \text{SO}(n)$.

An element of $\text{Spin}(n)$ is an element $T$ of $\text{SO}(n)$ together with a homotopy class of a path connecting $\text{id}$ to $T$. 
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An element of $\text{Spin}(n)$ is an element $T$ of $\text{SO}(n)$ together with a homotopy class of a path connecting $\text{id}$ to $T$.

If $T = R_A$, where $A$ has codimension two, then the homotopy class of the path can be encoded in orientation of $A$. 
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An element of $\text{Spin}(n)$ is an element $T$ of $\text{SO}(n)$ together with a homotopy class of a path connecting $\text{id}$ to $T$.

If $T = R_A$, where $A$ has codimension two, then the homotopy class of the path can be encoded in orientation of $A$.

$\text{SO}(n)$ is generated by reflections in codimension two subspaces.
In the 3-space

Types of decorated arrows = types of isometries.
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Types of decorated arrows = types of isometries.

A rotation is presented by an ordered pair of planes or intersecting lines.
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Types of decorated arrows = types of isometries.

A rotation is presented by an ordered pair of planes or intersecting lines.

A general fact: an orthogonal sum of two isometries is presented by orthogonal sum of decorated arrows.
In the 3-space

Types of decorated arrows = types of isometries.

A rotation is presented by an ordered pair of planes or intersecting lines.

A general fact: an orthogonal sum of two isometries is presented by orthogonal sum of decorated arrows.

The identity map is presented by a decorated arrow consisting of two coinciding spaces with coinciding decorations.
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Thank you for your attention!
Thank you for your attention!
Thank you for your attention!
Thank you for your attention!
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