What are the Equations Defining Algebraic Varieties?

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Defining Equations of Projective Varieties

Chow's Theorem. Consider

\[ X \subseteq \mathbb{P}^r = \mathbb{P}^r(C) \]

a complex submanifold. Then \( X \) is an algebraic variety, i.e. \( \exists \) homogeneous polynomials

\[ F_\alpha = F_\alpha(X_0, \ldots, X_r) \]

such that

\[ X = \{ F_1 = \ldots = F_N = 0 \} . \]

Question: What can one say about the defining equations \( \{ F_\alpha \} \), e.g. their degrees?

Answer: In this generality, nothing.

(Choose any \( \{ F_\alpha \} \), use to define \( X \subseteq \mathbb{P}^r \).)

Better Question: Study “nice” embeddings of given \( X \).

Today: “nice” = “very positive”
• Embedding $X \subseteq \mathbb{P}^r$ defined by choosing holomorphic line bundle $L$, and basis
  
  \[ s_0, \ldots, s_r \in \Gamma(X, L). \]

• Define
  
  \[ X \hookrightarrow \mathbb{P}^r \quad \text{via} \quad x \mapsto [s_0(x), \ldots s_r(x)]. \]

• We will be interested in $L$ where $c_1(L)$ is very positive.

Example. Take $C = \mathbb{P}^1$, $L = O_{\mathbb{P}^1}(3)$, giving

\[ \mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \quad [s,t] \mapsto [s^3, s^2t, st^2, t^3]. \]

Image is set

\[ C = \left\{ \text{rank} \begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{bmatrix} \leq 1 \right\} \subseteq \mathbb{P}^3. \]

So $C$ cut out by three quadratic polynomials

\[ \Delta_{01} = T_0T_2 - T_1^2 \]
\[ \Delta_{02} = T_0T_3 - T_1T_2 \]
\[ \Delta_{12} = T_1T_3 - T_2^2. \]
Example. Say $E = \mathbb{C}/\Lambda$ an elliptic curve.

- If $\deg L = 3$, get
  
  $$E \subseteq \mathbb{P}^2, \quad E = \{G = 0\}$$
  
  with $\deg(G) = 3$.

- If $\deg L = 4$, get
  
  $$E \subseteq \mathbb{P}^3, \quad E = \{Q_1 = Q_2 = 0\},$$
  
  with $\deg(Q_1) = \deg(Q_2) = 2$.

Theorem [Castelnuovo, Mumford, Kempf, ...]

If $X$ is smooth variety, and

$$X \subseteq \mathbb{P}^r$$

is defined by $L$ with

$$c_1(L) \gg 0,$$

then $X$ cut out by polynomials of degree 2.

Example. (Castelnuovo) When $X$ is curve of genus $g$, then conclusion holds when

$$\deg(L) \geq 2g + 2.$$
Sidman–Smith: When \( c_1(L) \gg 0 \), \( X \) is cut out in \( \mathbb{P}^r \) by the \( 2 \times 2 \) minors of matrix of linear forms.

**Two Issues.**

(I). The theorem guarantees that \( X \subseteq \mathbb{P}^r \) is cut out by quadrics if \( c_1(L) \) is sufficiently positive. What happens if we let \( L \) become even more positive?

**Example.** If \( g(X) = g \), what can we say when

\[
\deg(L) \geq 2g + 3?
\]

(II). Can’t easily read off invariants of \( X \) from number or form of quadrics defining it.

**Green:** Should study higher syzygies among defining equations.
Syzygies

Consider polynomial ring:
\[ S = \mathbb{C}[T_0, \ldots, T_r], \]
ideal
\[ I = (Q_1, \ldots, Q_N) \subseteq S \]
and to fix ideas say \( \deg(Q_\alpha) = 2 \).

Hilbert: Consider syzygies among the \( Q_\alpha \), i.e. relations of the form
\[ \sum R_\alpha \cdot Q_\alpha \equiv 0 \quad (*) \]
where the \( R_\alpha \) are polynomials with \( \deg R_\alpha = q \).
Say that (*) is a second syzygy of weight \( q \).

Example: Return to
\[ C = \mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \ [s, t] \mapsto [s^3, s^2 t, st^2, t^3]. \]
Recall that
\[ C = \left\{ \text{rank} \begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \end{bmatrix} \leq 1 \right\} \subseteq \mathbb{P}^3, \]
so \(C\) defined by

\[
\begin{align*}
\Delta_{01} &= T_0T_2 - T_1^2 \\
\Delta_{02} &= T_0T_3 - T_1T_2 \\
\Delta_{12} &= T_1T_3 - T_2^2.
\end{align*}
\]

Repeat row of matrix and expand resulting determinant:

\[
\det \begin{bmatrix}
T_0 & T_1 & T_2 \\
T_1 & T_2 & T_3 \\
T_0 & T_1 & T_2
\end{bmatrix} \equiv 0,
\]

so

\[
T_2 \cdot \Delta_{01} - T_1 \cdot \Delta_{02} + T_0 \cdot \Delta_{12} = 0.
\]

Similarly,

\[
T_3 \cdot \Delta_{01} - T_2 \cdot \Delta_{02} + T_1 \cdot \Delta_{12} = 0.
\]

No other syzygies.

So here all syzygies have minimal weight \(q = 1\).
Example: Consider degree four elliptic curve

\[ E = \mathbb{C}/\Lambda \subseteq \mathbb{P}^3. \]

Recall

\[ E = \{ Q_1 = Q_2 = 0 \}. \]

Here only syzygy is

\[ Q_2 \cdot Q_1 - Q_1 \cdot Q_2 = 0, \]

which has weight \( q = 2 \).

Returning to ideal

\[ I = (Q_1, \ldots, Q_N) \subseteq S, \]

one considers next

\[ \{ \text{Third syzygies} \} = \{ \text{Relations among coefficients of second syzygies} \}, \]

and so on.

(Constructing minimal free resolution of \( I \).)

Hibert’s Theorem on Syzygies: Process stops after at most \( r \) steps.
**Definition.** Given $L$ defining $X \subseteq \mathbb{P}^r$ one says that $L$ satisfies Property $(N_p)$ if:

- $X$ cut out by quadrics ($p = 1$);
- First $p$ modules of syzygies of $X$ generated by relations with minimal possible weight $q = 1$.

"Green’s Principle": On any smooth $X$, Property $(N_p)$ holds linearly in positivity of embedding line bundle.

Fix reference Kähler form $\omega_0$, and suppose that $L_d$ is line bundle such that

$$c_1(L_d) = d \cdot \omega_0 + \eta,$$

$\eta = \text{fixed} (1,1)$-form.

**Theorem.** (Many people...) There exist constants $A, B > 0$ (depending on $X, \omega_0, \eta$) such that $L_d$ satisfies $(N_p)$ when

$$d \geq A \cdot p + B.$$
Example (Green). Consider $X$ a curve with $g(X) = g$, and suppose $\deg(L_d) = d$. Then $(N_p)$ holds when

$$d \geq 2g + 1 + p.$$ 

Philosophy: As positivity of embedding grows, the algebraic properties of

$$X \subseteq \mathbb{P}^r$$

become simpler.

Note: Assume as above $c_1(L_d) = d \cdot \omega_0 + \eta$, and say

$$\dim X = n.$$ 

- Number of syzygy modules that occur is approximately

$$r(L_d) = C \cdot d^n + \text{LOT}$$

- Number of syzygy modules governed by results just stated grows linearly in $d$.

So: When $n = \dim X \geq 2$, Green’s principle ignores most of syzygies that occur!
Ottaviani-Paoletti: For

\[ X = \mathbb{P}^n, \quad L_d = \mathcal{O}_{\mathbb{P}^n}(d), \]

\((N_p)\) fails when \(p = 3d - 2\).

**Question:** When \(n \geq 2\), what can one say about the asymptotic shape of syzygies of embedding

\[ X \subseteq \mathbb{P}^{rd} \]

defined by \(L_d\) as \(d \to \infty\)?

Will initially focus on which weights of syzygies appear.
Asymptotic Non-Vanishing Thms (with L. Ein)

As before, consider $X$ with $\dim X = n$, and $L_d$ on $X$ with

$$c_1(L_d) = d \cdot \omega_0 + \eta.$$ 

$L_d$ defines

$$X \subseteq \mathbb{P}^{r_d} \text{ with } r_d = O(d^n).$$

Are interested in $p^{\text{th}}$ syzygies of $X$ for

$$1 \leq p \leq r_d$$

when $d \gg 0$.

General Facts. For $d \gg 0$:

(I). All syzygies of $X$ have weights

$$1 \leq q \leq n + 1.$$
(II). [Green et al] $L_d$ has syzygies of maximal weight $q = n + 1$ if and only if

$$\Gamma(X, \Omega^n_X) \neq 0,$$

in which case such syzygies appear only for a few large values of $p$.

Rmk. Follows from (I) and (II), that for curves, essentially only syzygies that appear are those of weight $q = 1$ (⇒ Green’s theorem.)

Problem: Fix $q \in [1, n]$. For which

$$p \in [1, r_d]$$

does $L_d$ give rise to a $p^{th}$ syzygy of weight $q$ when $d \gg 0$?
Theorem A  Fix $q \in [1,n]$. There exist constants $C_1, C_2 > 0$ with the property that if

$$d \gg 0,$$

then $L_d$ determines $p^{th}$ syzygy of weight $q$ for every $p$ with

$$C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{m-1}.$$

Rmk. For fixed $q \in [1,n]$, consider the ratio

$$\frac{\# \{ p \in [1,r_d] \mid \exists p^{th} \text{ syz. of weight } q \}}{\# \{ p \in [1,r_d] \}}$$

Since $r_d = O(d^m)$, Theorem implies:

$$\text{Ratio} \to 1 \text{ as } d \to \infty.$$ 

(I.e. asymptotically in $d$, “essentially all” the syzygy modules that could have generator in weight $q$ actually do have such generators.)
Conjecture. Fix $1 \leq q \leq n$. Then

$$K_{p,q}(L_d) = 0$$

for $p \leq O(d^{q-1})$.

Veronese Varieties

Take $X = \mathbb{P}^n$ and $L_d = \mathcal{O}_{\mathbb{P}^n}(d)$. Use all monomials of degree $d$ to define embedding

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^{r_d}, \quad r_d = \binom{n+d}{d} - 1.$$

Image is $d^{th}$ Veronese embedding of $\mathbb{P}^n$.

Syzygies of Veronese varieties studied eg by Ottaviani-Paoletti, Rubei, Bruns-Conca-Römer.
Theorem B. Fix $q \in [1, n]$. If $d \gg 0$ then the Veronese variety carries $p^{\text{th}}$ syzygies of weight $q$ for

$$\binom{d+q}{q} - \binom{d-1}{q} - q \leq p$$

$$p \leq \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n}{n-q} - q - 1.$$ 

Ex. Take $q = 2, n = 2$. Then $\exists$ syzygies of weight $q = 2$ for

$$3d - 2 \leq p \leq r_d - 2.$$ 

(Thm of Ottaviani-Paoletti.)

Conjecture. Bound is optimal for all $q \in [1, n], d \geq q + 1$.

Vector space of $p^{\text{th}}$ syzygies of weight $q$ are a representation of $\text{SL}(n+1)$.  

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Ask: How many different irreducible representations appear?

Fulger-Zhou: For fixed $p$, as $d \to \infty$:

$$\begin{pmatrix}
\text{# of irreps in space of $p^{th}$ syzygies of weight $q = 1$}
\end{pmatrix} = O(d^p).$$

Intuition for Proof of Thm A.

Fix a hypersurface $\overline{X} \subseteq X$, and consider composition

$$\overline{X} \subseteq X \subseteq \mathbb{P}^{r_d}.$$ 

Then $\overline{X}$ embeds in a linear space of very large codimension, and by induction on dim, one can see that syzygies of $\overline{X}$ in $\mathbb{P}^{r_d}$ have many different weights. Expect that these contribute to syzygies of $X$ in $\mathbb{P}^{r_d}$.
Betti Numbers. (with Ein, Erman)

Consider $X$, $L_d$ as before. Define

$$k_{p,q}(L_d) = \dim \left\{ p^{th} \text{ syzygies of weight } q \right\}.$$ 

Question. Fix $1 \leq q \leq n$. Can one say anything about the asymptotics of these betti numbers as $d \to \infty$?

Curves: Take

$$g(C) = g, \quad \deg(L_d) = d$$

$$r_d = d - g.$$ 

For large $d$, want to consider the behavior of the dimension

$$k_{p,1} = \text{def} \ k_{p,1}(L_d)$$ 

as a function of $p$. 
Ex. Plot of $k_{p,1}$ for $g = 0$, $d = 60$.

Ex. Plot of $k_{p,1}$ for $g = 10$, $d = 60$. 

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Prop. Fix $C, L_d$ as above, and let $\{p_d\}$ be a sequence of integers such that

$$p_d \to \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$$

for some fixed number $a$ (i.e. $\lim \frac{2p_d - r_d}{\sqrt{r_d}} = a$).

Then as $d \to \infty$,

$$\frac{1}{2r_d} \cdot \sqrt{\frac{2\pi}{r_d}} \cdot k_{p_d,1} \to e^{-a^2/2}.$$

What about general $X, L_d$?

One can hope that similar picture holds for $k_{p,q}(L_d)$ for every $q \in [1, n]$.

Conjecture: For each $q \in [1, n]$ there is a function $F(d)$ (depending on $X$ and geometric data) such that

$$F(d) \cdot k_{p_d,q}(L_d) \to e^{-a^2/2}$$

as $d \to \infty$ and $p_d \to \frac{r_d}{2} + a \cdot \frac{\sqrt{r_d}}{2}$.
Example. Betti numbers $k_{p,1}$ of 4-fold Veronese embedding of $\mathbb{P}^2$.

(Biggest example we could work out exactly on computer.)

Confession: Don’t know how to verify Conjecture for any $X$ of dimension $n \geq 2$ !

(Ex. What are asymptotics of betti numbers for Veronese embeddings of $\mathbb{P}^2$ ??)
Evidence for Conjecture comes from

**Probabilistic Picture**: For “random resolutions” having syzygies with fixed weights, betti numbers become normally distributed as length of resolution grows.

**Ask**: What does one mean by “random resolution?”

As model for syzygies of very positive embeddings of varieties of fixed dimension $n$, consider resolutions of modules $M$ over polynomial rings in $r + 1$ variables that have syzygies only in weights $1 \leq q \leq n$.

**Eisenbud–Schreyer**: Proved conjecture of Boij-Söderberg describing (up to scaling) all possible configurations of betti numbers $k_{p,q}(M)$ for $M$ as above.
Betti tables are (essentially) parametrized up to scaling by numerical parameters

\[ x = \{x_I\} \in [0, 1]^\binom{r}{n-1} \]

So get functions

\[ k_{p,q} : \Omega_r \overset{\text{def}}{=} [0, 1]^\binom{r}{n-1} \longrightarrow \mathbb{R} \]

describing Betti numbers of formal resolution described by Boij-Söderberg coefficient vector \( x \in \Omega_r \).

Plan: For fixed \( 1 \leq q \leq n \), choose

\[ x \in \Omega_r \] uniformly at random,

and study distribution in \( p \) of the formal betti numbers \( k_{p,q}(x) \).
Example. Plot of $k_{p,1}$ for random $x$ with $n = 2$ and $r = 14$.

Example. Plot of $k_{p,1}$ for random $x$ with $n = 2$ and $r = 60$. 
Theorem C. (Informal Statement). Fix $1 \leq q \leq n$. Then as $r \to \infty$, for “most” choices of $x \in \Omega_r$

the formal betti numbers

$$k_{p,q}(x) \in \mathbb{R}$$

display the sort of normal distribution (in $p$) that is predicted by the conjecture.

So at least the Conjecture predicts that “real-life” Betti numbers have typical, rather than exceptional, behavior.