‘Hodge Theory’ from Floer homology:

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Definition: FOOO = Fukaya-Oh-Ohta-Ono
Hodge Theory in complex geometry

$X$ \quad Kahler maniflds

$\Omega^{p,q}(X)$: \quad (p,q)-forms on $X$

$\partial : \Omega^{p,q}(X) \rightarrow \Omega^{p+1,q}(X)$

$\bar{\partial} : \Omega^{p,q}(X) \rightarrow \Omega^{p,q+1}(X)$
\[
\overline{\partial} + u\partial : \bigoplus \Omega^{p,q}(X)[[u]] \rightarrow \bigoplus \Omega^{p,q}(X)[[u]]
\]

\[
H(\overline{\partial} + u\partial) = \frac{\text{Ker}(\overline{\partial} + u\partial)}{\text{Im}(\overline{\partial} + u\partial)} \cong H(X) \otimes \mathbb{C}[[u]]
\]

\[
\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}
\]
\[ \int : \bigoplus_{p,q} \Omega^{p,q}(X) \to \mathbb{C} \]

\[ \int : H(\bar{\partial} + u\partial) \to \mathbb{C}[[u]] \]
\[ \int \partial u \wedge v \pm \int u \wedge \partial v = 0 \]
\[ \int \bar{\partial} u \wedge v \pm \int u \wedge \bar{\partial} v = 0 \]

\[ \langle \rangle : H(\bar{\partial} + u\partial)^\otimes 2 \to \mathbb{C}[[u]] \]
\[ \langle u, v \rangle = \int u \wedge v \]

There is a family version of this story: Variation of Hodge structure
Mirror symmetry

\[ X \]

\[ X^\vee \]

\[ H(\bar{\partial} + u\partial) \]

\[ \int \ : H(\bar{\partial} + u\partial) \rightarrow \mathbb{C}[[u]] \]

Yukawa coupling

(actually need to identify \((k,0)\)-form with \(n-k\) poly vector field. using holomorphic \(n\) form.)
Mirror symmetry

\( X \) \hspace{2cm} \( X^\vee \)

Hodge theory of \( X \)

Something similar to Hodge theory?

Derived category of coherent sheaves

Cyclic filtered A infinity category

Homological mirror symmetry (Kontsevitch)

Lagrangian Floer theory

FOOO
Something similar to Hodge theory?

Cyclic filtered A infinity category

There are proposals by

Kontsevitch-Soibelman-Kazarkov-Pantev-Auroux, etc.

K. Saito – A. Takahashi

e tc.
\( A \) differential graded algebra

(special case of A infinity algebra, the story works in the same way for filtered A infinity algebra)

\[
HH(A) = H(CH(A), \delta^H)
\]

Hochshild complex

\[
\delta^H + uB : CH(A) \otimes \mathbb{C}[[u]] \rightarrow CH(A) \otimes \mathbb{C}[[u]]
\]

\[
\bar{\partial} + u\partial : \bigoplus_{p,q} \Omega^{p,q}(X)[[u]] \rightarrow \bigoplus_{p,q} \Omega^{p,q}(X)[[u]]
\]

\( B \) Connes’ opearor
\[ H(\delta^H + uB) = HC(A) \]

Cyclic homology.
Example

$A = \Omega M$ \hspace{1cm} \text{de-Rham complex of a manifold } M$

$HH(A) = H(LM)$ \hspace{1cm} \text{homology of free loop space of } M$

$HC(A) = H_{S^1}(LM)$ \hspace{1cm} $S^1$ equivariant homology of free loop space of $M$

$B : H(LM) \rightarrow H(LM)$

$P \xrightarrow{\sigma} LM \hspace{1cm} S^1 \times P \rightarrow LM$

$t, x \rightarrow t\sigma(x)$
Remark

\[ A = \Omega M \]

\[ HH(A) = H(LM) \]

\[ HC(A) = H_{S^1}(LM) \]

In this example

\[ HC(A) \neq HH(A) \otimes \mathbb{C}[[u]] \]

No reasonable non-degenerate pairing on

\[ HH(A) = H(LM) \]

Reason

This corresponds to the case when symplectic manifold is

\[ T^*M \]

non-compact.
$\delta^H + uB : CH(A) \otimes \Lambda[[u]] \to CH(A) \otimes \Lambda[[u]]$

$\bar{\partial} + u\partial : \bigoplus \Omega^{p,q}(X)[[u]] \to \bigoplus \Omega^{p,q}(X)[[u]]$

This talk:

There are cases that this proposal actually works and gives Mirror.

We can define paring $(CH(A) \otimes \Lambda[[u]])^{\otimes 2} \to \Lambda[[u]]$

which has all the required properties.
$(X, \omega)$  Symplectic manifold

$L \subset X$  Lagrangian submanifold

$(\Omega L, d, \wedge)$  differential graded algebra (de Rham complex)

deformation by pseudo-holomorphic discs [FOOO]

$(\Omega L, \{m_k\})$  A infinity algebra

Consider many Lagrangian submanifolds [FOOO]

A infinity category  $F(X)$
Suppose \( X \) compact

**Conjecture**  (Kontsevitch, Seidel .........)

\[
HH(F(X)) \cong H(X)
\]

in certain cases.

More precisely

\[
p_* : CH(F(X)) \rightarrow S_*(X)
\]

chain map, always.  Called open closed map.  ([FOOO] general case.)

\[
p_* : CH(F(X)) \rightarrow S_*(X)
\]

is expected to induce an isomorphism in cohomology in a good case.
\[ \langle p^*(P, \ldots, P), Q \rangle_{PD_X} \]
Proposition

\[ p_* \circ B = 0 \]

Corollary 1

\[ p_* \text{ induces } \]

\[ p_{c*} : HC(F(X)) \to H_{S^1}(X) = H(X) \otimes \Lambda[[u]] \]

Corollary 2

If \( p_* : HH(F(X)) \to H(X) \)

is isomorphism then

\[ p_{c*} : HC(F(X)) \to H_{S^1}(X) \]

is an isomorphism.

and

\[ HC(F(X)) \cong HH(F(X)) \otimes \Lambda[[u]] \]
Remark

\[ HC(A) \cong HH(A) \otimes \Lambda[[u]] \]

is proved for A infinity category \( A \) which is `smooth and compact'.

by Kaledlin (in purely algebraic method.)
Pairing (FOOO and Abouzaid-FOOO)

\[ \langle \quad \rangle_{\text{res}} : CH(F(X))^{\otimes 2} \to \Lambda \]

\[ CH(H(L)) = \bigoplus_{k} H(L)^{\otimes k} \]

\[ a, b \in H(L) \subset CH(H(L)) \]

\[ \langle a, b \rangle_{\text{res}} = \sum_{i(1),i(2),j(1),j(2)} g^{i(1)j(2)} \langle a, e_{i(1)} \cup e_{i(2)} \rangle \langle b, e_{j(1)} \cup e_{j(2)} \rangle_{PD_L} \]

\[ e_i \quad \text{basis of } H(L) \quad (g^{ij}) = (g_{ij})^{-1} \quad g_{ij} = \langle e_i, e_j \rangle_{PD_L} \]

\[ \cup \quad \text{Product structure in Floer homology} \quad (\text{cup product} \quad \text{plus deformation by holomorphic disc.}) \]
\[ \langle P_0 \otimes \cdots \otimes P_n, Q_0 \otimes \cdots \otimes Q_m \rangle_{\text{res}} \]
Proposition

\[
\langle \delta_H x, y \rangle_{\text{res}} = \pm \langle x, \delta_H y \rangle_{\text{res}}
\]

\[
\langle Bx, y \rangle_{\text{res}} = \pm \langle x, By \rangle_{\text{res}}
\]

Note

\[
\int \bar{\partial}u \wedge v = \pm \int u \wedge \bar{\partial}v
\]

\[
\int \partial u \wedge v = \pm \int u \wedge \partial v
\]

Corollary 1

\[
\langle \ \rangle_{\text{res}} \text{ extends to `higher residue pairing'}
\]

\[
\langle \ \rangle_{\text{res}} : HC(F(X)) \otimes^2 \rightarrow \Lambda[[u]]
\]
\[ p_* : HH(F(X)) \to H(X) \]

**Proposition** (FOOO-AFOOOO)

\[
\langle x, y \rangle_{\text{res}} = \langle p_* x, p_* y \rangle_{PD_X}
\]

extends to `higher residue pairing`

\[ p_{c*} : HC(F(X)) \to H_{S^1}(X) \]

**Corollary**

\[
\langle x, y \rangle_{\text{res}} = \langle p_{c*} x, p_{c*} y \rangle_{PD_X}
\]

right hand side is \[ H_{S^1}(X) \otimes H_{S^1}(X) \to H_{S^1}(pt) = \Lambda[[u]] \]
How they work in the case of Toric – Landau-Ginzburg Mirror

A variant of Hodge Theory in complex geometry (K. Saitoh’s theory of isolated singularity)

$$F : \mathbb{C}^n \rightarrow \mathbb{C}$$

holomorphic function with isolated critical point at 0

$$\Omega(\mathbb{C}^n)$$

differential forms on \( \mathbb{C}^n \)
$F : \mathbb{C}^n \rightarrow \mathbb{C}$ holomorphic function with isolated critical point at 0

$\Omega(\mathbb{C}^n)$ differential forms on $\mathbb{C}^n$

$\bar{\partial} + dW \wedge \partial$ $\bar{\partial}$

Hodge structure in singularity theory

Hodge structure in Kahler manifold
\[ \overline{\partial} + dW \wedge \partial \]

(Higher) residue pairing

\[ H(u\partial + \overline{\partial} + dW \wedge) \]

\[ \text{Jac}(W) \otimes \Lambda[[u]] \]

\[ \alpha, \beta \mapsto \int \alpha \wedge \beta \]

\[ H(u\partial + \overline{\partial}) \]

\[ H(X) \otimes \Lambda[[u]] \]
$$Jac(W) = \frac{O(\mathbb{C}^n)}{\left( \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n} \right)}$$

Jacobian ring

(Higher) residue pairing is defined on

$$H(u \partial + \bar{\partial} + dW \wedge) \downarrow$$

$$Jac(W) \otimes \Lambda[[u]]$$
\[ \bar{\partial} + dW \wedge \partial \]

Hodge structure in singularity theory

(Higher) residue pairing

\[ H(u\partial + \bar{\partial} + dW \wedge) \]

\[ Jac(W) \otimes \Lambda[[u]] \]

Such a structure appears in Lagrangian Floer theory of Toric manifold
Toric symplectic manifold

$$(X, \omega) \quad \text{has symplectic} \quad T^n \quad \text{action}$$

moment maps

$$\pi : (X, \omega) \rightarrow P \subseteq T^n \quad \text{moment map}$$

$$\pi^{-1}(c) = T^*_c \cong T^n \quad \text{Lagrangian torus}$$
\[ \pi^{-1}(c) = T^c_c \cong T^n \]  \hspace{1cm} \text{Lagrangian torus}

\[ (\Omega T^n_c, \{m_k\}) \] \hspace{1cm} \text{Lagrangian Floer theory of} \quad T^n_c

\[ (\Omega T^n_c, d, \wedge) \quad \text{plus correction by} \quad \text{pseudo-holomorphic disk} \]

\[ HH(\Omega T^n_c, d, \wedge) \cong H(LT^n_c) \cong \bigoplus_{\gamma \in \mathbb{Z}^n = \pi_1(T^n)} H(T^n) \]
\[ HH(\Omega T^n_c, d, \wedge) \cong H(LT^n) \cong \bigoplus_{\gamma \in \mathbb{Z}^n = \pi_1(T^n)} H(T^n) \]

\[ HH(\Omega T^n_c, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \]

\[ H(T^n) \cong \text{exterior algebra with n generators} \]

\[ \sum_{\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n} a_{\gamma} \gamma \quad \leftrightarrow \quad \sum_{\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n} a_{\gamma} x_1^{\gamma_1} \cdots x_n^{\gamma_n} \]
\[ HH(\Omega T^n_c, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \]

\[ \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \rightarrow \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \]

\[ dW \wedge + \text{higher} \]

\[ HH(\Omega T^n_c, \{m_k\}) \cong \text{Jac}(W) \]

\[ W \quad \text{generating function of the counting holomorphic discs} \]
$W(b) = \sum_{u} \pm T \int_{\partial D^2} u^* b$

for $b \in H^1(T^n)$
We can calculate

\[ B : HH(\Omega \Gamma^n_c, d, \wedge) \cong \Gamma(\mathbb{C}^n; \Lambda^*, 0) \to \Gamma(\mathbb{C}^n; \Lambda^*, 0) \]

and obtain:

\[ B = \partial : \Gamma(\mathbb{C}^n; \Lambda^*, 0) \to \Gamma(\mathbb{C}^n; \Lambda^*, 0) \]
\[ B = \partial : \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \rightarrow \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \]

\[ \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \rightarrow \Gamma(\mathbb{C}^n; \Lambda^{*,0}) \]

\[ dW \wedge + \text{higher} \]

\[ HH(\Omega T^*_c, \{m_k\}) \congJac_c(W) \]

Hodge structure in singularity theory

Hodge structure in Lagrangian Floer theory of toric manifold
Moreover in the case:

\[(X, \omega)\]

is a compact toric manifold:

\[p_* : \bigsqcup_c HH(\Omega T^n_c, \{m_k\}) \cong \bigsqcup_c Jac_c(W) \rightarrow H(X)\]

is an isomorphism.

so the story works in this case.
Summary:

- Using Floer homology and its A infinity structure
- We can find the same kinds of structure as Hodge theory on Hochshild and cyclic homology.
- Especially when open closed map is an isomorphism:
  - It has a non-degenerate pairing (higher residue pairing) and Hodge to de Rham degeneration can be proved geometrically.
- In the case of toric manifold, the whole structure match with K. Saito’s theory of singularity.