On positivity of a class of conformal covariant operators

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Background

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On \((M^d, g)\), \(k\) an integer, \((P_{2k}^d)_g\) is a class of differential operator of order \(2k\) with leading symbol \((-\Delta_g)^k\); with the conformal covariant property that under conformal change of metric \(g_w = e^{2w}g\), we have

\[
(P_{2k}^d)_{g_w}(\phi) = e^{-\frac{d+2}{2}w}(P_{2k}^d)_g(e^{\frac{d-2}{2}w}\phi)
\]

for all smooth functions \(\phi\) defined on \(M\).
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When \(k = 1\), \(P_2^d\) is the conformal Laplace or Yamabe operator:

\[
(P_2^d)_g = -\Delta_g + \frac{d - 2}{4(d - 1)}R_g
\]

where \(R_g\) is the scalar curvature of the metric \(g\).
When $d > 2k$, $(P_{2k}^d)_g(1) := \frac{d-2}{2}(Q_{2k}^d)_g$, e.g. when $k = 1$, $d > 2$, $Q_2^d = \frac{1}{2(d-1)}R$ while when $k = 1$, $d = 2$, $Q_2^2$ is defined to be the Gaussian curvature.
When $d > 2k$, $(P^d_{2k})_g(1) := \frac{d-2}{2}(Q^d_{2k})_g$, e.g. when $k = 1$, $d > 2$, $Q^d_2 = \frac{1}{2(d-1)}R$ while when $k = 1$, $d = 2$, $Q^2_2$ is defined to be the Gaussian curvature.

When $d = 2k$, Branson’s curvature $Q_{2k} = Q^{2k}_{2k}$ is also defined. When $(M^d, g)$ is locally conformally flat, $C_d \chi(M) = \int (Q_d)_g dv_g$; in general, $\int (Q_d)_g dv_g$ is a conformal invariant.
Graham-Zworski (’04) introduced fractional GJMS operators $P_{2\gamma}$ on the boundary of conformally compact Einstein or Asymptotic Hyperbolic manifolds of dimension $n + 1$ via Scattering matrix when $2\gamma \leq n$
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Conformal covariant property

$$(P_{2\gamma}^n)_{gw}(\phi) = e^{-\frac{n+2\gamma}{2}}w(P_{2k}^n)_{g}(e^{\frac{n-2\gamma}{2}}w\phi)$$
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$$\left( P_{2\gamma}^n \right)_g w (\phi) = e^{-\frac{n+2\gamma}{2}} w \left( P_{2k}^n \right)_g e^{\frac{n-2\gamma}{2}} w \phi$$

when $\gamma = \frac{1}{2}$,

$$P_1 = \frac{\partial}{\partial n} + \frac{n - 1}{2n} H$$

is the Robin boundary operator, where $H$ is the mean curvature and $Q_1 = \frac{1}{n} H$. 

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Positivity of Conformal Covariant Operators
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$$(P^n_{2\gamma})_{gw}(\phi) = e^{-\frac{n+2\gamma}{2} w}(P^n_{2k})_g(e^{\frac{n-2\gamma}{2} w} \phi)$$

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On the flat case, $\mathbb{R}^{n+1}_+, g_+ = \frac{dy^2 + dx^2}{y^2}$, where $y > 0$, $x \in \mathbb{R}^n$, for all $\gamma > 0$, we have

$$P_{2\gamma} = (-\Delta_x)^\gamma, \quad Q_{2\gamma} = 0.$$
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**Theorem (Gursky ('99))** On $(M^4, g)$ closed manifolds, $Y(M, g) > 0$ and $\int (Q_4)_g dv_g > 0$ implies that $(P_4)_g > 0$. In particular, when $R_g > 0$, $(Q_4)_g > 0$ then $(P_4)_g > 0$. 
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Formulas of $P_4^d$, $Q_4^d$ by Paneitz ('83) and GJMS,

$$(P_4^d)_g = (-\Delta_g)^2 + \text{div}(a_d \text{Ric}_g + b_d R_g^d)D + \frac{(d - 4)}{2} Q_4^d_g$$

$$Q_4^d_g = c_1(-\Delta_g R + c_2 R_g^2 - c_3 |\text{traceless Ric}_g|^2)$$

where $c_1, c_2, c_3$ are positive dimensional constants.
Gursky-Malchiodi ('13) \((d \geq 4)\),

On \((M^d, g)\) closed manifolds, if \(R_g > 0\), and \((Q^d_4)_g > 0\), then \((P^d_4)_g > 0\) when \(d \geq 4\).

They have made an in-depth study of \(P_4\) operators, proved strong maximal principle \((P_4u > 0 \text{ implies } u > 0)\).
Background

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  They have made an in-depth study of \( P_4 \) operators, proved strong maximal principle \((P_4 u > 0 \implies u > 0)\).

- **Hang-Yang ('14) \( d = 3 \)**
  When \( R_g > 0 \) and \((Q^3_4)_g > 0\), Green's function of \((P^3_4)_g\) is negative.
Main Theorems

- Positivity of $P_{2\gamma}$ as boundary operators of conformal compact Einstein manifolds on $(X^{n+1}, M^n, g_+)$ Poincare Einstein manifold, where $M^n = \partial X$, the conformal infinity boundary.

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Main Theorems

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- Theorem 1: When $0 < \gamma < 1$, $d \geq 2$, and $\Lambda_1(-\Delta g_+) > \frac{n^2}{4} - \gamma^2$, $Q_{2\gamma}^n > 0$ implies $P_{2\gamma}^n > 0$.

  **Remark:** By a work of J. Lee, if $R(M^n, g_0) > 0$, where $g_0 = g|_M$, then $\Lambda_1(-\Delta g_+) \geq \frac{n^2}{4}$. 
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  Remark: By a work of J. Lee, if $R_{(M^n, g_0)} > 0$, where $g_0 = g|_M$, then $\Lambda_1(-\Delta_{g_+}) \geq \frac{n^2}{4}$.

- Theorem 2: When $1 < \gamma < 2$, $n \geq 4$, $R_{(M, g_0)} > 0$ and 
  $$Q_{2\gamma}^n > 0$$ implies $P_{2\gamma}^n > 0$. When $n = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.
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- Theorem 1: When $0 < \gamma < 1$, $d \geq 2$, and
  \[ \Lambda_1(-\Delta_{g^+}) > \frac{n^2}{4} - \gamma^2, \quad Q^n_{2\gamma} > 0 \] implies $P^n_{2\gamma} > 0$.

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  $Q^n_{2\gamma} > 0$ implies $P^n_{2\gamma} > 0$. When $n = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

- Key step in the proof: The “right” choice of the conformal compactified Einstein metric in $X^{n+1}$.
Main tool, Extension Theorem

- Classical Setting: (work of Caffarelli-Silvestre)
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- Well-known result: \( f \) smooth on \( \mathbb{R}^n \)

\[
\Delta_{x,y} U(x, y) \text{on} \mathbb{R}^{n+1}_+ = (x, y | x \in \mathbb{R}^n, y > 0, \quad U\big|_{\mathbb{R}^n}(x) = f(x)
\]

then

\[
-U_y(x, 0) = (-\Delta)^{\frac{1}{2}} f(x).
\]

- Theorem (Caffarelli-Silvestre '06)

\[0 < \gamma < 1, \; a = 1 - 2\gamma,\]

\[
\left\{ \begin{array}{l}
\text{div}(y^a \nabla U) = 0 \text{ on } \mathbb{R}^{n+1}_+ \\
U\big|_{\mathbb{R}^n} = f.
\end{array} \right.
\]

Then
Actually

\[ f \in \dot{H}^{\gamma}(\mathbb{R}^n) = \left( \dot{W}^{\gamma,2}(\mathbb{R}^n) \right). \]
Actually \( 0 < \gamma < 1 \)

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\]

Then

\[
\int_{\mathbb{R}^n} \int_{y>0} |\nabla U|^2 y^a \, dx \, dy = \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} (-\Delta)^\gamma f \cdot f \, dx
\]

which implies \((a = 1 - 2\gamma)\)
Actually \( 0 < \gamma < 1 \)

\[
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\[
(-\Delta)^\gamma f = C_{n, \gamma} \lim_{y \to 0} y^a \frac{\partial U}{\partial n} \bigg|_{y=0}.
\]
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\[ f \in \dot{H}^\gamma(\mathbb{R}^n) = \left( \dot{W}^{\gamma,2}(\mathbb{R}^n) \right) . \]

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Applications to free-boundary problems, study of non-local minimal surface etc.
Work of Graham-Zworski

- On Conformal Compact Einstein Setting, a class of conformal covariant operators $P_{2\gamma}$ exists for

\[
0 < 2\gamma \leq n \quad (n \text{ even})
\]
\[
\text{all } \gamma > 0 \quad (n \text{ odd})
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$$0 < 2\gamma \leq n \quad (n \text{ even})$$

all $\gamma > 0 \quad (n \text{ odd})$

$P_{2\gamma} = (-\Delta)^\gamma$ in special setting of $(\mathbb{R}^{n+1}, \mathbb{R}^n, g_+)$, where

$$g_+ = \frac{dy^2 + dx^2}{y^2}$$

while

$$\bar{g} = y^2 g_+.$$

is the compactified metric.
Definition \((X^{n+1}, M^n, g_+)\) is **Conformally Compact Einstein** (or **C.C.E.**), where \(M = \partial X\), if
Definition $(X^{n+1}, M^n, g_+)$ is Conformally Compact Einstein (or \textit{C.C.E.}), where $M = \partial X$, if

1. There exists some distance function $r$ so that $r^2 g_+$ is compact. $r > 0$ on $X$, $r = 0$ on $M$, and $dr \neq 0$ on $M$. $M$ is called the conformal infinity of $X^{n+1}$.
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\((X^{n+1}, M^n, g_+)\) is **Poincaré-Einstein**, if \(\text{Ric} \ g_+ = -ng_+\).
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- \((X^{n+1}, M^n, g_+)\) is Poincaré-Einstein, if \(\text{Ric } g_+ = -ng_+\).

- If \((X^{n+1}, M^n, g_+)\) is conformally compact Einstein, then there exists some special defining function \(y\) so that

\[
\begin{cases}
y > 0 \text{ on } X, \ y = 0 \text{ on } M \\
|\nabla y^2g_+y| = 1 \text{ on } M \times (0, \epsilon)
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\]

Thus

\[g_+ = \frac{dy^2 + g_y}{y^2} \text{ in } M \times (0, \epsilon).
\]

Denote \(\bar{g} = y^2 g_+\), then \(\bar{g} = dy^2 + g_y\) on \(X\), and 
\[g_y \big|_{y=0} = g_0 \text{ on } M.\]
On \((X^{n+1}, M^n, g_+)\), given \(f \in C^\infty(M)\)

Consider

\[
(\ast)_s - \Delta_{g_+} u - s(n - s) u = 0 \text{ on } X.
\]
On $(X^{n+1}, M^n, g_+)$, given $f \in C^\infty(M)$
Consider

\[(\ast)_s \quad \lambda \Delta g_+ u - s(n - s) u = 0 \text{ on } X.\]

**Mazzeo-Melrose** Except for finite number of points of
\(\lambda \in (0, \frac{n^2}{4})\), \(\lambda\) is not a (point) spectrum of \(-\Delta g_+\),
\(\lambda \in \left(\frac{n^2}{4}, \infty\right)\) are essential spectrum.
On $(X^{n+1}, M^n, g_+)$, given $f \in C^\infty(M)$

Consider

\[
(*)_s \quad -\Delta_{g_+} u - s(n-s) u = 0 \text{ on } X.
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**So for** \( \text{Re } s > \frac{n}{2} \), except finite no. of \( s \), \( y^{n-s} \) and \( y^s \) are asymptotic solution of equation \((*)_s\).
On \((X^{n+1}, M^n, g_+)\), given \(f \in C^\infty(M)\)

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So for \(\Re s > \frac{n}{2}\), except finite no. of \(s\), \(y^{n-s}\) and \(y^s\) are asymptotic solution of equation \(\text{(⋆)}_s\).

\[u = Fy^{n-s} + Hy^s, \quad F, H \in C^\infty(X)\]
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\[
F \big|_M = F \big|_{y=0} = f,
\]

\[
F = f + f_2 y^2 + f_4 y^4 \cdots, \text{ where } f_{2i} \in C^\infty(M)
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\(u\) is the solution of the Possion equation with Dirichlet data \(f\).
Define \textit{Scattering matrix}

\[
S(s) : C^\infty(M) \rightarrow C^\infty(M), \quad f \rightarrow H\big|_M.
\]
Define **Scattering matrix**

\[ S(s) : C^\infty(M) \to C^\infty(M), \quad f \to H\big|_M. \]

**Graham-Zworski**

When \( s = \frac{n}{2} + \gamma, \quad \gamma \notin \mathbb{Z}^+, \)

define \( P_{2\gamma} = S\left(\frac{n}{2} + \gamma\right) \) is a non-local pseudo-differential operator with leading symbol \( |\xi|^{2\gamma}. \)
Define *Scattering matrix*

\[ S(s) : C^\infty(M) \to C^\infty(M), \ f \to H|_M. \]

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define \( P_{2\gamma} = S\left( \frac{n}{2} + \gamma \right) \) is a non-local pseudo-differential operator with leading symbol \(|\xi|^{2\gamma}\).

When \( s = \frac{n}{2} + k \) \( S \) has a simple pole,

define

$$P_{2k} = C_{n,k} \text{ Res}_{s=\frac{n}{2}+k} S\left( \frac{n}{2} + k \right)$$
Define Scattering matrix

\[ S(s) : C^\infty(M) \to C^\infty(M), \ f \to H|_M. \]

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\[ P_{2k} = C_{n,k} \text{Res}_{s=\frac{n}{2}+k} S \left(\frac{n}{2} + k\right) \]

\( P_{2k} \) is the GJMS operators.
Examples

- Flat model \((\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_H)\). where

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g_H = \frac{dy^2 + dx^2}{y^2}.
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- Another model example is \((H^{n+1}, S^n, g_h)\). In this case, denote 

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g_h = \frac{|dy|^2}{(1 - |y|^2)^2}.
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  \]

- Choose
  \[
r = 2 \frac{1 - |y|}{1 + |y|},
  \]
  then
  \[
g_h = r^{-2}(|dr|^2 + (1 - \frac{r^2}{4})^2 g_{S^n}).
  \]
Examples

◮ Flat model \((\mathbb{R}^{n+1}_{+}, \mathbb{R}^n, g_H)\). where

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g_H = \frac{dy^2 + dx^2}{y^2}.
\]

◮ Another model example is \((H^{n+1}, S^n, g_h)\). In this case, denote

\[
g_h = \frac{|dy|^2}{(1 - |y|^2)^2}.
\]

◮ Choose

\[
r = 2 \frac{1 - |y|}{1 + |y|},
\]

then

\[
g_h = r^{-2}(|dr|^2 + (1 - \frac{r^2}{4}) g_{S^n}).
\]

◮ \((H^{n+1}/\Gamma, \Omega(\Gamma)/\Gamma, g_h)\), where \(\Gamma\) a Kleinian group.
Examples

- Flat model \((\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_H)\). where
  \[
g_H = \frac{dy^2 + dx^2}{y^2}.
  \]

- Another model example is \((H^{n+1}, S^n, g_h)\). In this case, denote
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- \((H^{n+1}/\Gamma, \Omega(\Gamma)/\Gamma, g_h)\), where \(\Gamma\) a Kleinian group.

- Schwarzschild space.
Theorem (C- Gonzalez ’11)

On \((X^{n+1}, M^n, g_+)\) conformal compact Einstein setting, given a function \(f \in C^\infty(M)\);

\[(\ast)_s\]

\[-\Delta_{g_+} u - s(n - s)u = 0 \text{ on } X\]

\[
\uparrow
\]

\[
s = \frac{n}{2} + \gamma
\]

\[(\ast)'_s\]

\[-\text{div}_{\bar{g}}(\rho^a \nabla_{\bar{g}} U) + E(\rho, a)U = 0 \text{ on } X\]

\[
U = \rho^{s-n}u \quad U| = f
\]

where \(\bar{g} = \rho^2 g_+\), \(0 < \gamma \leq \frac{n}{2}\), \(\rho\): any totally geodesic defining function.

We can express \(P_{2\gamma} f\) in terms of boundary behavior of \(U\).
In general, the expression of $E(\rho, a)$ is complicated, but in the special case when $\gamma = \frac{1}{2}$, $a = 0$, $E(\rho, a) = C_n R_{\bar{g}}$, the equation becomes

$$(P_2)_{\bar{g}} U = 0 \text{ on } X.$$
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It turns out a good choice of the $\rho$ is $\rho^\ast$ defined as follows: Suppose

$$-\Delta_{g^+} v - s(n - s)v = 0 \quad (\ast)_s$$

and $v$ is Possion operator on data $f \equiv 1$. Note if $v > 0$ on $X^{n+1}$, one can define $\rho^\ast = \nu \frac{1}{n - s}$ then $E(\rho^\ast, a) = 0$, where $s = \frac{n}{2} + \gamma$ and $a = 1 - 2\gamma$. 
In general, the expression of $E(\rho, a)$ is complicated, but in the special case when $\gamma = \frac{1}{2}$, $a = 0$, $E(\rho, a) = C_n R_{\bar{g}}$, the equation becomes

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It turns out a good choice of the $\rho$ is $\rho^*$ defined as follows: Suppose

$$-\Delta_{g^+} v - s(n-s)v = 0 \quad (*)_s$$

and $v$ is Possion operator on data $f \equiv 1$. Note if $v > 0$ on $X^{n+1}$, one can define $\rho^* = \sqrt[n-s]{\frac{1}{v}}$ then $E(\rho^*, a) = 0$, where $s = \frac{n}{2} + \gamma$ and $a = 1 - 2\gamma$.

It turns out when $0 < \gamma < 1$, $v > 0$ if and only if $\lambda_1(-\Delta_+) > \frac{n^2}{4} - \gamma^2$. 

Sun-Yung Alice Chang, joint with Jeffrey Case  Princeton University  Geometry Festival  Stony Brook, NY  Positivity of Conformal Covariant Operators
Proof of Theorem 1

Lemma: When $0 < \gamma < 1$, $(a = 1 - 2\gamma)$ and

$$\rho^* = y + d_\gamma Q_{2\gamma}y^{1+2\gamma} + O(y^3) > 0.$$  

Given $f$, $u$ = solution of Poission equation with data $f$, $U := (\rho^*)^{-\frac{n}{2}}u$. Then for some positive $c_\gamma$, we have

$$P_{2\gamma}f(x) = c_\gamma \lim_{y \to 0} (\rho^*)^{a} \frac{\partial U}{\partial n}(x) + \frac{n - 2\gamma}{2} Q_{2\gamma}f(x),$$  

for $x \in \partial X$. 

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Positivity of Conformal Covariant Operators
Proof of Theorem 1

- Theorem 1: when $\rho^* > 0$ on $X^{d+1}$, $Q_{2\gamma} > 0$ implies $P_{2\gamma} > 0$
Proof of Theorem 1

- Theorem 1: when $\rho^* > 0$ on $\mathcal{X}^{d+1}$, $Q_{2\gamma} > 0$ implies $P_{2\gamma} > 0$

- Proof:

With the choice of $\rho^*$, $g^* = (\rho^*)^2 g_+$, we have

$U := (\rho^*)^{\gamma - \frac{n}{2}} u$ and $U|_M = f$, $U$ satisfies the PDE

$$-\text{div}_{g^*}((\rho^*)^a \nabla_{g^*} U) = 0.$$ 

Hence

$$\int_X (-\text{div}_{g^*}((\rho^*)^a \nabla_{g^*} U) \, U = 0$$

and

$$\int_M (\rho^*)^a \frac{\partial U}{\partial n} U = \int_X (\rho^*)^a |\nabla U|^2 \geq 0.$$ 

We then apply the Lemma to finish the proof.
Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R_{(M^d, g_0)} > 0$ and $Q_{2\gamma}^d > 0$ implies $P_{2\gamma}^d > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$. 
Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R(M^d,g_0) > 0$ and $Q^d_{2\gamma} > 0$ implies $P^d_{2\gamma} > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

Outline of Proof:
Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R(M^d, g_0) > 0$ and $Q_{2\gamma}^d > 0$ implies $P_{2\gamma}^d > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

Outline of Proof:

Step 1: Extend Caffarelli-Silvestre’s Extension Theorem to $1 < \gamma$. On flat setting, work by R. Yang.
Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R(M^d, g_0) > 0$ and $Q_{2\gamma}^d > 0$ implies $P_{2\gamma}^d > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

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Theorem 2: When $1 < \gamma < 2$, $d \geq 4$, $R(\mathcal{M}_{d}, g_{0}) > 0$ and $Q_{2\gamma}^{d} > 0$ implies $P_{2\gamma}^{d} > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

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Step 3: When $R(\partial X, g_{0}) > 0$, prove that $R(X, g^{*}) > 0$, where $g^{*} = (\rho^{*})^{2}g_{+}$ and $\rho^{*}$ the special defining function.
Outline of the Proof of Theorem 2

- **Theorem 2**: When $1 < \gamma < 2$, $d \geq 4$, $R(M^d, g_0) > 0$ and $Q_2^d > 0$ implies $P_2^d > 0$. When $d = 3$, the same result holds when $1 < \gamma \leq \frac{3}{2}$.

- **Outline of Proof**:
  - **Step 1**: Extend Caffarelli-Silvestre’s Extension Theorem to $1 < \gamma$. On flat setting, work by R. Yang.
  - **Step 2**: On compactified Poincare Einstein metric, using the notion of metric space with measures to express the fractional GJMS operators and its curvature terms, thus generalize the extension theorem to such manifolds.
  - **Step 3**: When $R(\partial X, g_0) > 0$, prove that $R(X, g^*) > 0$, where $g^* = (\rho^*)^2 g_+$ and $\rho^*$ the special defining function.
  - **Step 4**: Apply the extension theorem and some proof similar to that of Gursky-Malchiodi to establish the theorem.
Extension Theorem in flat case when $\gamma > 1$

- Recent work of R. Yang, here for the special case when $1 < \gamma < 2$. 
Recent work of R. Yang, here for the special case when $1 < \gamma < 2$.

On $\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$, Denote

$$\Delta_a U = y^{-a} \text{div}(y^a \nabla U) = \Delta U + \frac{a}{y} \frac{\partial U}{\partial y}$$

Then

$$\Delta_a U = 0, \text{ with } U|_{\mathbb{R}^n} = f$$

where $a = 1 - 2\gamma$, iff

$$(\Delta_b)^2 U = 0, \text{ with } U|_{\mathbb{R}^n} = f, \text{ and } \lim_{y \to 0} y^b \frac{\partial U}{\partial y} = 0$$

with $b = 3 - 2\gamma$. 

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Positivity of Conformal Covariant Operators
Extension Theorem in flat case when $\gamma > 1$

In this case

\[
\int_{\mathbb{R}^n} (-\Delta_x f)^\gamma f dx = c_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} (\Delta_b U)^2 y^b dxdy.
\]

and
Extension Theorem in flat case when $\gamma > 1$

In this case

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\]

and

\[
(-\Delta_x f)^\gamma(x) = c_{n,\gamma} \lim_{y \to 0} y^b \frac{\partial}{\partial y} \Delta_b U(x, y).
\]

We have the “renormalized energy”, e.g. when $\gamma = \frac{3}{2}$,

\[
a = 1 - 2\gamma = -2, \quad b = 3 - 2\gamma = 0,
\]

then

\[
\lim_{\epsilon \to 0} \left( -\int_{\mathbb{R}^n} \int_{y \geq \epsilon} |\nabla U|^2 y^{-2} dxdy + \frac{1}{\epsilon} \int_{y=\epsilon} |\nabla_x f|^2 dx \right) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{y \geq \epsilon} |\nabla U|^2 y^{-2} dxdy.
\]
Notation: Given a number $m \in \mathbb{R}$, $\phi$ a function defined on $(X, g)$, $(F, h)$ a metric space of dimension $m$; on the metric measure space $(X, g, e^{-\phi} dv)$, denote $P_{2k, \phi}^m$ the GJMS operators on the warped product space

$$(X \times_{e^{-\phi}} F^m, g \oplus e^{-\frac{2\phi}{m}} h)$$

restricted to functions on $X$. 
Notation: Given a number $m \in \mathbb{R}$, $\phi$ a function defined on $(X, g)$, $(F, h)$ a metric space of dimension $m$; on the metric measure space $(X, g, e^{-\phi} dv_g)$, denote $P_{2k, \phi}^m$ the GJMS operators on the warped product space 

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In this notion, when $m = \infty$, $Ric_{\phi}^m = Ric + \nabla^2 \phi$ the Bakry-Emery Ricci tensor, $\Delta$ operator is replaced by $\Delta_{\phi} := \Delta - \nabla\phi \nabla$. 

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Extension Theorem on Poincare Einstein setting

Two key observations
Two key observations

1. On \((X^{n+1}, \partial X, g_+),\) C.C.E. with \(Ric_{g_+} = -n,\) When \(s = \frac{n}{2} + \gamma,\) \(g = \rho^2 g_+,\) the \((*)_s\) equation

\[-\Delta_{g_+} u - s(n - s)u = 0, \text{ on } X\]

can be re-written as \((*)''\)

\[P_{2,\phi}^m U = 0 \text{ on } X\]

where \((F^m, h)\) is chosen to be the (sphere) with \(Ric_h = (m - 1)h,\) \(U = \rho^{s-n} u\) and \(g = \rho^2 g_+,\) \(m = 1 - 2\gamma\) and \(e^{-\phi} = \rho^m.\)
Extension Theorem on Poincare Einstein setting

- Second Observation:
Extension Theorem on Poincare Einstein setting

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(2) When $1 < \gamma < 2$, then $u$ satisfies $(\ast)_s$ implies it satisfies

$$(-\Delta_g + -(s-2)(n-(s-2))) \circ (-\Delta_g + -s(n-s)) u = 0, \text{ on } X \ (\ast\ast)_s$$
Second Observation:

(2) When $1 < \gamma < 2$, then $u$ satisfies $(\ast)_s$ implies it satisfies

$$\left(-\Delta_{g^+} - (s-2)(n-(s-2))\right) \circ \left(-\Delta_{g^+} - s(n-s)\right) u = 0, \text{ on } X \ (\ast\ast)_s$$

which turns out to be equivalent to

$$P_{4,\phi_2}^m U = 0 \text{ on } X \ (\ast\ast)'_s,$$

where $m_2 = 3 - 2\gamma$ and $e^{-\phi_2} = \rho^{m_2}$. 
Extension Theorem on Poincare Einstein setting

- Second Observation:

  (2) When $1 < \gamma < 2$, then $u$ satisfies $(\ast)_s$ implies it satisfies

  $$( -\Delta_{g^+} - (s-2)(n-(s-2))) \circ (-\Delta_{g^+} - s(n-s)) u = 0,$$

  on $X$ $(\ast\ast)_s$

  which turns out to be equivalent to

  $$P^{m_2}_{4, \phi_2} U = 0 \text{ on } X (\ast\ast)'_s,$$

  where $m_2 = 3 - 2\gamma$ and $e^{-\phi_2} = \rho^{m_2}$.

- With this notion, we have the extension theorem on Poincare Einstein manifolds.
Recall if \( v = v_s \) satisfies the Possion equation with Dirichlet data \( f \equiv 1 \), under the condition \( R_{\partial X, g_0} > 0 \), we have \( v > 0 \) on \( X \). Denote \( \rho^* = v^{\frac{1}{n-s}} \), and \( g = g^* = (\rho^*)^2 g_+ \), \( s = \frac{n}{2} + \gamma \).
Recall if $v = v_s$ satisfies the Possion equation with Dirichlet data $f \equiv 1$, under the condition $R_{\partial X, g_0} > 0$, we have $v > 0$ on $X$. Denote $\rho^* = v^{\frac{1}{n-s}}$, and $g = g^* = (\rho^*)^2 g_+$, $s = \frac{n}{2} + \gamma$.

Good properties of $g^*$ metric:
Step 3, the right choice of compactified metric

- Recall if \( \nu = \nu_s \) satisfies the Possion equation with Dirichlet data \( f \equiv 1 \), under the condition \( R_{\partial X, g_0} > 0 \), we have \( \nu > 0 \) on \( X \). Denote \( \rho^* = \nu^{1/(n-s)} \), and \( g = g^* = (\rho^*)^2 g_+ \), \( s = \frac{n}{2} + \gamma \).

- Good properties of \( g^* \) metric:

- (a) \( R^m_{\phi} = 0 \), when \( m = 1 - 2\gamma \), \( e^{-\phi} = (\rho^*)^m \). (This corresponds to the fact \( E(\rho^*, a) = 0 \) in previous notations).
Step 3, the right choice of compactified metric

- Recall if $v = v_s$ satisfies the Poisson equation with Dirichlet data $f \equiv 1$, under the condition $R_{\partial X, g_0} > 0$, we have $v > 0$ on $X$. Denote $\rho^* = v^{\frac{1}{n-s}}$, and $g = g^* = (\rho^*)^2 g_+$, $s = \frac{n}{2} + \gamma$.

- Good properties of $g^*$ metric:
  
  - (a) $R^m_{\phi} = 0$, when $m = 1 - 2\gamma$, $e^{-\phi} = (\rho^*)^m$. (This corresponds to the fact $E(\rho^*, a) = 0$ in previous notations).
  
  - (b) When $1 < \gamma < 2$, $(Q_4)^{m_2}_{\phi_2} = 0$, where $m_2 = 1 - 2\gamma$, $e^{-\phi_2} = (\rho^*)^{m_2}$
Step 3, the right choice of compactified metric

- Recall if $v = v_s$ satisfies the Possion equation with Dirichlet data $f \equiv 1$, under the condition $R_{\partial X, g_0} > 0$, we have $v > 0$ on $X$. Denote $\rho_* = v^{\frac{1}{n-s}}$, and $g = g_* = (\rho_*)^2 g_+$, $s = \frac{n}{2} + \gamma$.

- Good properties of $g_*$ metric:
  - (a) $R^m_{\phi} = 0$, when $m = 1 - 2\gamma$, $e^{-\phi} = (\rho_*)^m$. (This corresponds to the fact $E(\rho_*, a) = 0$ in previous notations).
  - (b) When $1 < \gamma < 2$, $(Q_4)^{m_2}_{\phi_2} = 0$, where $m_2 = 1 - 2\gamma$, $e^{-\phi_2} = (\rho_*)^{m_2}$
  - (c) When $\gamma > 1$, $R_{g_*}|_{\partial X} = c_\gamma R(\partial X, g_0) > 0$, where $c_\gamma > 0$. 

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Positivity of Conformal Covariant Operators
Another crucial property:
Lemma: Under the assumption $R_{\partial X, g_0} > 0$, for all $s \geq \frac{n}{2} + 1$, $R_{g^*} > 0$ on $X$.
Proof: Due to property (b) above, we have the PDE for $R = R(g^*)$,
$$\Delta \phi_2 R = c_1 R^2 - c_2 |E|^2,$$
where $c_1 = c_1(s)$, $c_2 = c_2(s)$ are positive constants, and $E$ the traceless Ricci.
Step 3, the right choice of compactified metric

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  where $c_1 = c_1(s)$, $c_2 = c_2(s)$ are positive constants, and $E$ the traceless Ricci.

- It turns out when $s = n + 1$, $c_1 = 0$, the metric has been studied before by J. Lee ’95, where the equation by maximal principle together with property (c) gives $R_{g^*} > 0$.
Step 3, the right choice of compactified metric

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It turns out when $s = n + 1$, $c_1 = 0$, the metric has been studied before by J. Lee ’95, where the equation by maximal principle together with property (c) gives $R_{g^*} > 0$.

We can now run a “continuity” argument on the parameter $s$ starting at $s = (n + 1)$, together property (c), apply strong maximal principle to conclude $R_{g^*} > 0$ on $X$ for all $s \geq \frac{n}{2} + 1$. 

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Positivity of Conformal Covariant Operators
Proof of Theorem 2

When $1 < \gamma < 2$, we will show that when $R(\partial X, g_0) > 0$ and $Q_{2\gamma} > 0$, implies $P_{2\gamma} > 0$.

Proof:

Given $f$ defined on $\partial X$, by Extension theorem

$$
\int_{\partial X} (P_{2\gamma} f) \, dv_{g_0} = \frac{n-2\gamma}{2} \int_{\partial X} (Q_{2\gamma} f) \, dv_{g_0} + c_\gamma \text{ Energy term of } (P_4)^{m_2}.
$$

We apply the fact $R_{g^*} > 0$, together with an argument similar to that of Grusky-Malchiodi to prove the 4-th order energy term is non-negative, and which together with $Q_{2\gamma} > 0$ establishes the result.
Some discussion

- Theorem

( J. Qing - Guillarmou '10 ) On $(X^{n+1}, M^n, g^+)$ C.C.E. manifolds with $n + 1 > 3$, $Y(M^n, g_0) > 0$ iff the first real scattering pole $\leq \frac{n}{2} - 1$. 
Theorem

(J. Qing - Guillarmou '10) On $(X^{n+1}, M^n, g^+)$ C.C.E. manifolds with $n + 1 > 3$, $Y(M^n, g_0) > 0$ iff the first real scattering pole $\leq \frac{n}{2} - 1$.

Equivalent Statement: Under the assumption $Y(M, g_0) > 0$, $P_{2\gamma} \geq 0$ for all $0 < \gamma < 1$. 
Some discussion

► Theorem

(J. Qing - Guillarmou '10) On \((X^{n+1}, M^n, g^+)\) C.C.E. manifolds with \(n + 1 > 3\), \(Y(M^n, g_0) > 0\) iff the first real scattering pole \(\leq \frac{n}{2} - 1\).

► Equivalent Statement: Under the assumption \(Y(M, g_0) > 0\), \(P_{2\gamma} \geq 0\) for all \(0 < \gamma < 1\).

► The result generalizes an earlier work of Schoen-Yau.

\(X = H^{n+1}/\Gamma\), \(\Gamma\) a Kleinian group
\(\Omega(\Gamma) \subset S^n\) domain of discontinuity of \(\Gamma\)
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Some discussion

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- Equivalent Statement: Under the assumption \( Y(M, g_0) > 0 \), \( P_{2\gamma} \geq 0 \) for all \( 0 < \gamma < 1 \).

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  \( X = H^{n+1}/\Gamma \), \( \Gamma \) a Kleinian group
  
  \( \Omega(\Gamma) \subset S^n \) domain of discontinuity of \( \Gamma \)
  
  \( M = \Omega(\Gamma)/\Gamma \) locally conformally compact

- **Schoen-Yau**: If \( M \) is of positive scalar curvature, then \( \delta(\Gamma) \backsimeq \) Hausdorff dim of \( S^n \setminus \Omega(\Gamma) \), then \( \delta(\Gamma) \leq \frac{n}{2} - 1 \).

- Work of **Sullivan — Patterson, P.Perry** etc.
Some discussion

Some open questions:
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When $0 < \gamma < 1$, does $Q_{2\gamma} > 0$ imply $P_{2\gamma'} > 0$ when $0 < \gamma' \leq \gamma$?
Some discussion

- Some open questions:
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  - On $\mathbb{R}^n$, $P_{2\gamma_1} \circ P_{2\gamma_2} = P_{2(\gamma_1+\gamma_2)}$, In general, under curvature conditions, do we expect semi-group property of the family $P_{2\gamma}$?
Some open questions:

- When $0 < \gamma < 1$, does $Q_{2\gamma} > 0$ imply $P_{2\gamma'} > 0$ when $0 < \gamma' \leq \gamma$?

- On $\mathbb{R}^n$, $P_{2\gamma_1} \circ P_{2\gamma_2} = P_{2(\gamma_1+\gamma_2)}$. In general, under curvature conditions, do we expect semi-group property of the family $P_{2\gamma}$?

- Work of Gonzalez-Qing ’12 studied the $Q_{2\gamma}$ equation and related positive mass problem when $0 < \gamma < 1$. When $\gamma = \frac{1}{2}$, $Q_1 = cH$, the mean curvature. In general, is there a geometric description of the fractional $Q$ curvature?