Critical metrics on connected sums of Einstein four-manifolds

M. Gursky (Notre Dame)

Joint with J. Viaclovsky (Wisconsin)

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$$g \mapsto \int |\text{Riem}(g)|^2 dv$$

Can replace the first by

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By the C-G-B formula,

\[ 32\pi^2 \chi(X^4) = \int |W|^2 dv - 2 \int |\text{Ric}|^2 dv + \frac{2}{3} \int R^2 dv. \]
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\((t = \infty \text{ corresponds to } g \mapsto \int R^2 dv).\)
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• \( \mathcal{B}_0 \) is conformally invariant:

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- $B_0$ is conformally invariant:

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- Its gradient is the Bach tensor:

$$B_{ij} = -4(\nabla^k \nabla^\ell W_{ikj\ell} + \frac{1}{2} R^{k\ell} W_{ikj\ell}).$$

Critical metrics (i.e., metrics for which $B_{ij} = 0$) are called Bach-flat.
• In fact, $B$ is conformally invariant:

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Example 1: Einstein metrics

The $2^{nd}$ Bianchi identity implies

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• It follows that any metric which is locally conformal to an Einstein metric is also Bach-flat.
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$$\Lambda_2 = \Lambda_2^+ \oplus \Lambda_2^-$$

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**Definition.**

$(X^4, g)$ is called *self-dual* (resp., *anti-self-dual*) if $W^- \equiv 0$ (resp., $W^+ \equiv 0$.)
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**Examples.** Locally conformally flat manifolds, $(\mathbb{C}P^2, g_{FS})$. 

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- There are a number of gluing constructions for SD/ASD metrics: Floer, Donaldson-Friedman, Kovalev-Singer, LeBrun, Poon, Taubes, etc.
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• All known examples of (compact) Bach-flat manifolds are either SD/ASD or Einstein.
\(B^t\)-flat metrics

- Recall

\[
B^t[g] = \int |W(g)|^2 dv + t \int R(g)^2 dv.
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The gradient of \( g \mapsto \int R^2 \, dv \) is

\[ C_{ij} = 2 \left\{ \nabla_i \nabla_j R - (\Delta R)g_{ij} - R(R_{ij} - \frac{1}{4} Rg_{ij}) \right\}. \]
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**Lemma**
Critical points ($C_{ij} = 0$) are either scalar-flat or Einstein.

**Proof.** Taking the trace gives $\Delta R = 0$, hence CSC (constant scalar curvature). If $R \neq 0$, then $R_{ij} - \frac{1}{4} Rg_{ij} = 0$. QED.
$B^t$-flat metrics, cont.

Definition

$(X^4, g)$ is $B^t$-flat if it is a critical point of $B^t$. 

Lemma

$(X^4, g)$ is $B^t$-flat $\iff$ it has CSC and $B_{ij} = 2tR_{ij}$, where $E_{ij} = R_{ij} - \frac{1}{4}Rg_{ij}$ is the trace-free Ricci tensor.
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Glue two Einstein 4-manifolds to produce new examples of $B^t$-flat metrics.
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Glue two Einstein 4-manifolds to produce new examples of $B^t$-flat metrics.

The Catch: We cannot specify a priori the value of $t$. 
An overview of the gluing procedure

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**The Set-up:** Let $(Y, g_Y)$ and $(Z, g_Z)$ be Einstein (hence $B^t$-flat for all $t$), with positive scalar curvature.
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**The Set-up:** Let $(Y, g_Y)$ and $(Z, g_Z)$ be Einstein (hence $B^t$-flat for all $t$), with positive scalar curvature. Fix points $y_0 \in Y$ and $z_0 \in Z$.

We want to prove the existence of a $B^t$-flat metric on $X = Y \# Z$. 
Step 1: Blowing up

- Since $R_Y = R(g_Y) > 0$, the conformal laplacian $L = -\frac{1}{6} \Delta_Y + R_Y > 0$. Therefore, the Green’s function $G$ exists, with pole at $y_0$: 
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$$LG = 0 \text{ on } Y \setminus \{y_0\},$$

$$G(y, y_0) \sim \text{dist}_Y(y, y_0)^{-2} \text{ as } y \to y_0,$$

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• By the formulas for the Green’s function above, $g_N$ is scalar-flat:

$$R_N = G^{-3} LG = 0.$$
• Since \((N, g_N)\) is scalar-flat and Bach-flat, it is \(B^t\)-flat (for any \(t\)).
Step 1, cont.

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• It is also asymptotically flat: i.e., if we choose \(g_Y\)-normal coordinates \(\{y^i\}\) based at \(y_0\), then the inverted coordinates

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provide a coordinate system near infinity for \(N\). In this system,
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(g_N)_{ij} = \delta_{ij} + O(|x|^{-2}).
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Step 2: Gluing, cont.

• Since $g_N$ is flat near infinity, i.e., the gluing point, if we want the metrics to match (at least roughly) on the gluing region, we need the metric on $Z$ to be ‘almost’ flat near $p_2$. 
Step 2: Gluing, cont.

- Since $g_N$ is flat near infinity, i.e., the gluing point, if we want the metrics to match (at least roughly) on the gluing region, we need the metric on $Z$ to be ‘almost’ flat near $p_2$. Therefore, we scale the metric $g_Z$: given $a, b > 0$ small, let $\tilde{g} = a^{-2} b^{-2} g_Z$: 
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- On $Z$, choose $g_Z$-normal coordinates centered at $z_0$, and consider the annulus $A_Z = \{ z : b < |z| < 2b \}$. 
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- On $N$, use the coordinates $\{x^i\}$ near infinity to define the annulus $A_N = \{x : a^{-1} < |x| < 2a^{-1}\}$.
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\[(N, g_N) \quad (Z \setminus \{z_0\}, \tilde{g})\]
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\[ X_{a,b} = Y \# Z \]
Step 3: Gluing the metrics

The next step is gluing the metrics to obtain a metric $g_{a,b}$ on $X_{a,b}$. 

Let $g_{a,b} = \begin{cases} g_N & \text{on } N \{ |x| > a - 1 \} \\ g_Z & \text{on } Z \{ |z| < 2b \} \end{cases}$.

- On the overlapping region, we can use cut-off functions to glue $g_N$ and $\tilde{g}$ to obtain a new metric $g_{a,b}$.
- The tensor $B + tC$ of $g_{a,b}$ will satisfy $| (B + tC)(g_{a,b}) | = O(a^4b^2) + O(a^6)$, which unfortunately is too crude. (We need the metrics $g_N$ and $\tilde{g}$ to 'match' to higher order.)
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$$P(\theta) = (B + tC)(g_{a,b} + \theta) + \mathcal{K}_{g_{a,b}+\theta}[\Gamma(\theta)],$$

where

$\mathcal{K}_{g_{a,b}+\theta}[\Gamma(\theta)]$ is the conformal Killing operator, and $\Gamma : \mathbb{S}^2(T^*X) \to T^*X$ is a third order linear differential operator.
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- The operator $\Gamma$ (for “gauge-fixing”) is chosen so that the linearized operator is elliptic.
Step 4: The nonlinear map, cont.

• Let

\[ S(h) = \frac{d}{ds} P(sh) \bigg|_{s=0} \]

denote the linearized operator.

\[ (i) \quad \text{For } \theta \text{ sufficiently small, if } P(\theta) = 0 \text{ then } \hat{g} = g + \theta \text{ is a } Bt \text{-flat metric.} \]

\[ (ii) \quad \text{The linearized operator } S \text{ is elliptic.} \]
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**Proposition**

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**Proposition**

(i) For \( \theta \) sufficiently small, if \( P(\theta) = 0 \) then \( \hat{g} = g_{a,b} + \theta \) is a \( B^t \)-flat metric.

(ii) The linearized operator \( S \) is elliptic.
To find a zero of the nonlinear map, we apply an implicit function theorem-type argument. The key to making this work is the \textit{surjectivity} of the linearized operator $S$ when $a, b$ are chosen sufficiently small.

By a standard limiting argument, surjectivity of $S$ can be reduced to the surjectivity of $S_N$ and $S_Z$; i.e., the linearized operator on the 'neck' and the punctured manifold $Z \{ z_0 \}$.

As in other gluing results, we work in weighted function spaces, where the weight is (roughly) the distance function from a fixed point. In general, however, the cokernel will be non-trivial.
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• Each has a torus action,
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- From now on we assume these symmetries for all metrics, functions, etc.
The linearized operator on $(\mathbb{Z} \setminus \{z_0\}, g_z)$

**Theorem**

(G- Viaclovsky, '12) For $t < 0$, the cokernel of $S = S_z$ is

$$c \cdot g_z$$

(which comes from scaling).
Splitting Proposition

Assume

$$S_N h = 0,$$

and write $h = z + fg_N$, where $z$ is trace-free.
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Furthermore, in AF coordinates

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• In fact, \( h \) decays quadratically.
Using cut-off functions, we have two globally defined “approximate” cokernel elements:

\[ \kappa_1 \] (corresponding to cokernel on \( N \)),

\[ \kappa_2 \] (cut-off) \( g_{a, b} \) (corresponding to cokernel on \( \mathbb{Z} \)).
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For \( a, b \) sufficiently small we can then solve

\[ P(\theta) = \lambda_1 \kappa_1 + \lambda_2 \kappa_2. \]
Key Proposition

Take $a = b$. Then

$$\lambda_1 = \mu a^4 + O(a^5),$$

where

$$\mu = \left\{ \frac{2}{3} W(y_0) \ast W(z_0) + 4tR(z_0)\text{mass}(g_N) \right\} |S^3|.$$
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Solving modulo the cokernel(s), cont.

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• Then an easy argument shows \( \lambda_2 = 0 \); i.e.,

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P(\theta) = 0.
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Statement of the Main Result

Theorem

(G - Viaclovsky) The following 4-manifolds admit a (toric-invariant) $B^t$-flat metric:

\[ \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \mathbb{CP}^2 \# 2 \overline{\mathbb{CP}^2}, 2 \# S^2 \times S^2. \]
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<thead>
<tr>
<th>Topology of connected sum</th>
<th>Value(s) of $t_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}P^2 # \overline{\mathbb{C}P^2}$</td>
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</tr>
<tr>
<td>$S^2 \times S^2 # \overline{\mathbb{C}P^2} = \mathbb{C}P^2 # 2 \overline{\mathbb{C}P^2}$</td>
<td>$-1/3, -1.1892...$</td>
</tr>
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• Trying to determine which happens is ongoing work.