Math 683 - Collapsing Riemannian Manifolds

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AN ACCOUNT OF THE THEORY OF CRYSTALLOGRAPHIC GROUPS[1]

LOUIS AUSLANDER

Introduction. L. Bieberbach in two fundamental papers [2], [3] established the fundamental theorems for the crystallographic groups or Raumgruppen. We propose in this paper to give an almost completely self-contained account of these fundamental facts. We will use only the elementary theory of groups, matrices and polynomials from algebra, the basic geometry of euclidean space and the most elementary topological considerations. At one point we will need the exponential mapping for Lie matrix groups for which various elementary accounts are available.

I would like to thank M. Rosenlicht and P. Fong for useful conversations.

1. **Definition of crystallographic groups.** Let E^n denote the n-dimensional euclidean space and let $R(n)$ denote the group of rigid motions of E^n. Then let 0 be a point in E^n. The subgroup of $R(n)$ leaving 0 fixed is called the orthogonal group and we will denote it by $O(n)$. Let R^n be the subgroup of $R(n)$ consisting of pure translations. Then there are two facts which should be recalled: First, R^n may be identified with E^n under the map $r \in R^n$ goes into $r(0)$. Secondly, R^n is a normal subgroup of $R(n)$, $O(n) \cap R^n$ is empty and every element of $R(n)$ can be uniquely represented in the form gt, where $g \in O(n)$ and $t \in R^n$. These last three conditions are abbreviated by writing $R(n) = O(n) \cdot R^n$.

A subgroup $\Gamma \subset R(n)$ is called a crystallographic group if the following two conditions are satisfied:

1. If $\gamma_1, \cdots, \gamma_n, \cdots$ is a sequence of elements from Γ and $x \in E^n$, then the sequence $\gamma_i x$, $i = 1, 2, \cdots$, is Cauchy if and only if there exists $N > 0$ such that $\gamma_i = \gamma_N$ for all $i > N$.

2. There exists a compact subset of E^n, say F, such that for every $x \in E^n$ there exists $\gamma \in \Gamma$ with the property that $\gamma(x) \in F$.

These two conditions are slightly awkward to work with. The following theorem gives a more convenient formulation of the crystallographic groups.

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THEOREM 1. *A necessary and sufficient condition for a subgroup of $R(n)$ to be a crystallographic group is that Γ is a discrete subset of $R(n)$ and $R(n)/\Gamma$ is compact in the quotient topology.*

PROOF. We will prove this theorem by means of the following two propositions.

(A) A necessary and sufficient condition for a subgroup Γ of $R(n)$ to act on E^n without accumulation points (i.e., so as to satisfy condition (1) of the definition of crystallographic groups) is that Γ be a discrete subset of $R(n)$.

For let Γ be a discrete subgroup of $R(n)$ and assume there exist $x_0 \in E^n$ and $\gamma_i \in \Gamma$, $i=1, \cdots$, such that $\gamma_i(x_0)$ is Cauchy. Further, let $t_0 \in R^n$ be such that $t_0(0)=x_0$. Then consider $\gamma_i t_0 \in R(n)$. Now $\gamma_i=(g_i, t_i)$ and

$$\gamma_i t_0 = (g_i, \operatorname{ad}(g_i)t_0 + t_i).^2$$

Since $\gamma_i(x_0)$ is Cauchy and is exactly $(\operatorname{ad}(g_i)t_0+t_i)(0)$, we have that $\operatorname{ad}(g_i)t_0+t_i$ is a Cauchy sequence in R^n. Since $O(n)$ is compact, we can find a subsequence of the g_i which is Cauchy. Hence the sequence $\gamma_i t_0$ is Cauchy. But $R(n) \to R(n)$ obtained by right multiplication by t_0 is a homeomorphism and, hence, γ_i must be Cauchy. Hence it must be trivial from some point on and we have proven our first assertion.

Let $\Gamma \subset R(n)$ operate without accumulation points on E^n. Assume $\gamma_i \in \Gamma$ is a Cauchy sequence and $\gamma_i=(g_i, t_i)$. We must have, since $R(n)$ is topologically $O(n) \times R^n$, that g_i and t_i are both Cauchy sequences. Hence $t_i(0)=\gamma_i(0)$ must be Cauchy. This proves (A).

(B) A necessary and sufficient condition for a subgroup Γ of $R(n)$ to have a compact fundamental domain (i.e., satisfy (2) in the definition of crystallographic group) is that $R(n)/\Gamma$ be compact.

PROOF. It is trivial to verify that Γ has a compact fundamental domain is equivalent to E^n/Γ, in the quotient topology, shall be compact. Now there is a well-defined continuous mapping of $R(n)/\Gamma$ into E^n/Γ obtained by identifying two points of $R(n)$ that differ by an element of $O(n)$ acting to the left. Hence, if $R(n)/\Gamma$ is compact, E^n/Γ is compact.

But $R(n)/\Gamma$ is compact if there exists $F^* \subset R(n)$ such that F^* is compact and every element of $R(n)$ differs from an element of F^* by a multiple of Γ. Clearly, we may choose $F^*=O(n) \times F$ where F is compact subset of R^n. Assuming that Γ acts with compact funda-

[2] We are using homogeneous coordinates and the multiplication is given by $(g_1, t_1)(g_2, t_2)=(g_1g_2, \operatorname{ad}(g_1)t_2+t_1)$.

mental domain on E^n gives an F in E^n which may be identified with F in R^n by the standard identification of R^n and E^n. The assertion is then trivially verified.

2. **Neighborhoods of the identity in $O(n)$.** In this section we will establish certain elementary properties of the orthogonal group $O(n)$. Let e always denote the identity element of $O(n)$. Some of this material has already appeared in print in [1] and is included for completeness of exposition.

LEMMA 1. *Let $O(n)$ denote the orthogonal group and let $\gamma, \eta \in O(n)$. Then there exists a neighborhood of the identity $U(e)$ such that if $\gamma, \eta \in U(e)$ and $\gamma\eta \neq \eta\gamma$, then γ will not commute with $\gamma\eta\gamma^{-1}\eta^{-1} = (\gamma, \eta)$.*

PROOF. Let us assume that γ and (γ, η) commute. Then γ and $\eta\gamma^{-1}\eta^{-1}$ commute. Hence η can be represented as a permutation of the invariant spaces of γ^{-1} or γ amongst themselves followed by a mapping of these spaces onto themselves. Hence, if η is sufficiently close to the identity, η can only map these invariant spaces onto themselves. Hence η and γ commute. This proves our lemma.

We will now state a general fact giving a general proof. This is the only fact from Lie-group theory we will use and if the reader is unfamiliar with it he can take it on faith or read it in [4, Chapter 2].

LEMMA 2. *Let G be a connected Lie group. Then there exists a neighborhood $W(S)$ of the identity in G such that for any $g_1, g_2 \in W(S)$, $(g_1, g_2) \in W(S)$ and the sequence*

$$(g_1(g_1, g_2)), (g_1(g_1(g_1, g_2))), \cdots$$

converges to the identity.

PROOF. Choose a canonical coordinate system about the identity in G. Then the coordinates of (g_1, g_2) can be expressed as a power series in the coordinates of g_1 and g_2 with quadratic leading term. This proves our assertion.

LEMMA 3. *There exist arbitrarily small neighborhoods U_α of the identity in $O(n)$ such that, for all $g \in O(n)$, $gU_\alpha g^{-1} = U_\alpha$.*

PROOF. Note merely that the set of elements of $O(n)$ whose eigenvalues ξ satisfy an inequality $|\xi - 1| < \epsilon$ is a neighborhood of the identity in $O(n)$.

DEFINITION. A neighborhood of the identity in $O(n)$ satisfying the conclusions of Lemmas 1, 2 and 3 will be called a stable neighborhood of the identity.

3. Lemmas on crystallographic groups.

LEMMA 4. *Let Γ be a crystallographic group and let $x \in E^n$. Then the set $\{\gamma(x)\}$ for $\gamma \in \Gamma$ cannot lie in a linear space of dimension less than n.*

PROOF. Assume the lemma is false and that $x_0 \in E^n$ exists such that $\{\gamma(x_0)\}$ lies in W, a proper linear subspace of E^n. By a new choice of origin in E^n we may assume $O(n)$ leaves x_0 fixed and then $\gamma \in \Gamma$, $\gamma = (g(\gamma), t(\gamma))$ must have $t(\gamma) \in W$.

Since Γ is a group, $g(W) = W$ for all $g = g(\gamma)$. Let $W^\perp$ be the orthogonal complement of W. Then, clearly, since points in $W^\perp$ at a distance d from the origin stay at a distance d, Γ cannot have a compact fundamental domain. This proves our assertion.

LEMMA 5. *Let Γ be an abelian crystallographic group; then Γ contains only pure translations.*

PROOF. Let $\gamma_0 \in \Gamma$ and let $\gamma_0 = (g(\gamma_0), t_0)$, where $g(\gamma_0) \neq e$. Then we can always choose an origin and a coordinate system in R^n such that, using homogeneous coordinates,

$$\gamma_0 = \begin{pmatrix} I & 0 & t_0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where I is the $(r \times r)$ identity matrix, $(\delta - I)$ is a nonsingular $s \times s$ matrix, t_0 is a $(1 \times r)$ matrix and 1 is a 1×1 matrix with 1 as an entry and $s + r = n$. Then, by Lemma 4, there exist $\gamma \in \Gamma$ such that

$$\gamma = \begin{pmatrix} A & 0 & t_1 \\ 0 & B & t_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where A is $(r \times r)$, B is $(s \times s)$, t_1 is $1 \times r$, and t_2 is a $(1 \times s)$ nontrivial matrix. Then, since Γ is abelian, $\gamma_1 \gamma_0 \gamma_1^{-1} = \gamma_0$ and this implies that $(\delta - I)t_2 = 0$. Since $(\delta - I)$ is nonsingular, this is impossible as $t_2 \neq 0$. This proves our assertion.

4. Main theorems.

PROPOSITION 1. *Let Γ be a discrete subgroup of $R(n)$ and let ψ denote the homomorphism of $R(n)$ onto $O(n)$ with kernel R^n. Then the identity component of the closure of $\psi(\Gamma)$ in $O(n)$ is abelian.*

PROOF. Let U be a stable neighborhood of the identity in $O(n)$ with the further property that for all $g\in U$ all the matrix values of g satisfy $|\xi_{ij}-1|<1/10$. In some orthonormal basis choose $\gamma_1, \gamma_2\in\Gamma$ with the property that $\psi(\gamma_i)\in U$, $i=1, 2$. Then $\gamma_i=\psi(\gamma_i)t(\gamma_i)$, which we will abbreviate x_iy_i.

Then

$$(\gamma_1, \gamma_2) = (x_1, x_2)\ \mathrm{ad}(x_1^{-1}x_2^{-1})[(\mathrm{ad}(x_2) - I)y_1 + (I - \mathrm{ad}(x_1))y_2],$$

where $\mathrm{ad}(x)$ is the automorphism of R^n induced by $x^{-1}R^nx$. We then form the sequence $\gamma_1, (\gamma_1, \gamma_2), (\gamma_1, (\gamma_1\gamma_2)), (\gamma_1, (\gamma_1, (\gamma_1, \gamma_2))), \cdots$. By our construction, the coefficients in $O(n)$ and R^n of this sequence are easily seen to be bounded. But by Lemma 1, this series can never be the identity, and, by Lemma 2, it can never become trivial at any point not the identity. Hence, since Γ is discrete, the identity component of the closure of $\psi(\Gamma)$ is abelian.

BIEBERBACH THEOREM 1. *Let Γ be a crystallographic group; then Γ satisfies the following three conditions:*

1. *$\Gamma\cap R^n$ is a vector space basis for R^n as a real vector space.*
2. *$\Gamma/\Gamma\cap R^n=F(\Gamma)$ is a finite group.*
3. *$F(\Gamma)$ has all integer entries with respect to any basis of R^n determined by the generators of $\Gamma\cap R^n$.*

PROOF. Assume first that $\Gamma\cap R^n$ is trivial. Then $\psi(\Gamma)$ is an isomorphism of Γ into $O(n)$ and we will let $I_0(\psi(\Gamma))$ denote the identity component of the closure of $\psi(\Gamma)$. Since $O(n)$ is compact, the closure of $\psi(\Gamma)$ can have only a finite number of components. Hence, since $I_0(\psi(\Gamma))$ is abelian, Γ contains a subgroup Γ_1 of finite index which is abelian. But then Γ_1, being of finite index in Γ, is also a crystallographic group. Hence, by Lemma 5, Γ_1 consists of pure translations. Thus we see that $\Gamma\cap R^n$ is nonempty.

Let $W\subset R^n$ be the subspace of R^n spanned by the pure translations of Γ, i.e., by $\Gamma\cap R^n$. Then, clearly, if $R(n)=O(n)\cdot R^n$ again and $\gamma\in\Gamma$ is given by $(g(\gamma), t(\gamma))$, $g(\gamma)\in O(n)$, $t(\gamma)\in R^n$, then $g(\gamma)$ leaves W invariant since $\Gamma\cap R^n$ is normal in γ. Note further that $\{g(\gamma)\}|W$ all $\gamma\in\Gamma$ is a finite group, for otherwise it would contain elements arbitrarily close to the identity which would, under inner automorphism with a basis of $\Gamma\cap R^n$, force Γ to be nondiscrete. From this we see that Γ induces an action on R^n/W which is that of a crystallographic group with no pure translations. By the first part, this implies the dimension of R^n/W is zero.

This discussion verifies part one and part two of the Bieberbach Theorem, part three follows trivially from parts one and two.

JORDAN THEOREM. *There exists a positive function of n, $n \geqq 0$, $f(n)$, such that, for every finite group $F \subset O(n)$, there exists an abelian normal subgroup $A(F)$ such that the order of $F/A(F)$ is less than $f(n)$.*

PROOF. Let U be a stable neighborhood of the identity as defined in §2. Let $A(F)$ be the subgroup of F generated by $F \cap U$. The definition of U insures that $A(F)$ is normal and abelian. Now assume $O(n)$ has Haar measure with total measure 1. Let the measure of $U > 1/m$, m an integer. Then it is easily seen that the order of $F/A(F)$ must be less than m.

THEOREM 2. *Let F_α, $\alpha = 1, \cdots, k$, be the set of subgroups of $O(n)$ which can be expressed as integer matrices with determinant ± 1 in* $\mathrm{GL}(n, R)$. *Then k is a finite cardinal.*

PROOF. A subgroup of a group satisfying our hypothesis again satisfies our hypothesis. Let A_α be the normal subgroup of F_α described in the Jordan Theorem. Since the order of F_α/A_α is bounded, there exist only a finite number of distinct groups of the form F_α/A_α, $\alpha = 1, \cdots, k$. If we can show there exist only a finite number of A_α, we will have proven our assertion as the group extensions must then also be finite. Now A_α is the abelian semisimple group. We will show that there are only a finite number of elements of $O(n)$ which can be in A_α for all α. Hence A_α must consist of a finite collection of groups. First note that n times the distinct characteristic polynomials is greater than the number of distinct elements of $O(n)$ in A_α. But since all roots have absolute value one and the coefficients of the characteristic polynomials are the elementary symmetric functions, they can take on at most a finite number of values. This completes the proof.

COROLLARY. *Let Γ be a crystallographic group and R^n the group of pure translations. Then, for each n, there exist only a finite number of groups $\Gamma/\Gamma \cap R^n$.*

BIEBERBACH THEOREM 2. *For each n, there exist only a finite number of crystallographic groups.*

PROOF. We have seen that Γ satisfies an exact diagram

$$1 \to Z^n \to \Gamma \to F \to 1,$$

where Z^n is n copies of the integers and F can range over a finite collection of groups. It is well known that for each finite group F there are only a finite number of nonisomorphic groups satisfying the above diagram. This completes our argument.

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EXTENSION OF NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS

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Let X be a Stein manifold, and let A be an analytic subset of X A well-known application of Cartan's Theorem B [2, Théorème 3 p. 52] states that each holomorphic function on A is the restriction of a holomorphic function on X. This paper presents a generalization of this application, namely that each normal family of holomorphic functions on A is the restriction of a normal family of holomorphic functions on X.

1. Let X be a topological space which is σ-compact, i.e., the union of a countable family of compact sets. Let $K(X)$ denote the set of all compact subsets of X. For $K \in K(X)$ and $f: X \to \mathbf{C}$ define $\|f\|_K = \sup\{|f(x)| \mid x \in K\}$. Define

$$B(X) = \{f \mid f: X \to \mathbf{C}, \|f\|_K < \infty \text{ for all } K \in K(X)\}.$$

Clearly $B(X)$ is a complex vector space, and $\{\| \ \|_K \mid K \in K(X)\}$ is a family of pseudonorms on $B(X)$ which then becomes a locally convex vector space. Since X is σ-compact, $B(X)$ is metrizable, and it is readily checked to be a Fréchet space.

Definition. Let V be a vector subspace of $B(X)$. We say that a set $F \subset V$ is normal with respect to V iff every sequence in F has a subsequence which converges in V.

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THE BIEBERBACH CASE IN GROMOV'S ALMOST FLAT MANIFOLD THEOREM

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1. Introduction (and abstract)

In 1976 M. Gromov has shown that every compact Riemannian manifold with normalized diameter whose sectional curvature is sufficiently close to zero is covered by a compact nilmanifold (= quotient of a nilpotent Lie group). [3] . This theorem, known as the almost flat manifold theorem has soon become famous not only because of its content but also because of the many unconventional methods Gromov has introduced to Riemannian geometry to get the proof.

The aim of the present notes is to explain how the ideas from Gromov's proof of the almost flat manifold theorem can be specialized to give a proof of the Bieberbach theorem. Since this specialization is much more accessible than the almost flat manifold theorem, one can very nicely explain some of Gromov's ideas in this context. It is also interesting to compare this new proof with older proofs of Bieberbach's theorem.

2. The Bieberbach theorem

We fix some notation. A euclidean motion $\alpha\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by $\alpha x = Ax + a$, $A \in O(n)$, $a \in \mathbb{R}^n$. We call $A = r(\alpha)$ the rotational part and $a = t(\alpha)$ the translational part of the motion. To each rotation A corresponds an orthogonal decomposition

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \dots \oplus E_k$$

such that A restricted to E_i is a rotation through the angle θ_i ; in the orientation reversing case E_k is eigenspace of A for the eigenvalue -1, we include this in the case $\theta_k = \pi$. Then

$$\sphericalangle\, (x, Ax) = \theta_i \quad \text{for all} \quad x \in E_i \quad .$$

These so called main rotational angles are arranged in increasing order:

$$0 = \theta_0 < \theta_1 < \dots < \theta_k \quad .$$

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The dimension of E_0 may be zero. The main rotational angles give rise to the following biinvariant distance function (Finsler metric) in the orthogonal group:

$$\|A\| := \theta_k = \max_{\|x\|=1} \measuredangle (x,Ax) \ , \quad d(A,B) = \|AB^{-1}\| \ .$$

From this metric we derive a distance function for the entire group of motions by

2.1 $\quad \|\alpha\| := \max\{\|r(\alpha)\|, \text{const}\cdot|t(\alpha)|\} \ , \quad d(\alpha,\beta) = \|\alpha\beta^{-1}\| \ .$

There is a degree of freedom in the choice of the constant. It will be fixed later according to the momentary needs.

A crystallographic group is a discrete group of euclidean motions with compact fundamental domain.

2.2 <u>Theorem</u> (Bieberbach) [1]. Let G be a crystallographic group in $\mathbb{R}^n$.

(i) Each $\alpha \in G$ has either $A = \text{id}$ or $d(A,\text{id}) \geq \frac{1}{2}$.

(ii) The group Γ of pure translations in G is a normal subgroup of finite index. G/Γ has order $\leq 2\cdot(4\pi)^{\dim SO(n)}$.

(iii) In addition to (i): If $\alpha \in G$, $r(\alpha) \in SO(n)$ and $0 < \theta_1 < \ldots < \theta_k$

are the main rotational angles of $A = r(\alpha)$ then

$$\theta_\varkappa \geq \frac{1}{2}(4\pi)^{\varkappa-k} \ , \quad \varkappa = 1,\ldots,k \ .$$

The original version of Bieberbach's theorem consists only of the statement that G/Γ has finite index. It was used by Bieberbach to solve the 18^{th} Hilbert problem:

2.3 <u>Corollary</u> (Bieberbach) [1]. For each n there exist only finitely many isomorphism classes of crystallographic groups in $\mathbb{R}^n$.

In the formulation 2.2 of the Bieberbach theorem the most important part is 2.2 (i): The translations in G are those motions which have a rotational part smaller than $\frac{1}{2}$. This characterization is Gromov's discovery; the proof depends as all other proofs of the Bieberbach theorem on commutator estimates, but Gromov combines these with the a priori bound 2.5 on the length of nontrivial commutators. The further statements 2.2 (ii) and 2.2 (iii) follow rather easily in 2.9 and 2.10. In particular the bound 2.2 (ii) on the order of G/Γ implies that there are only finitely many possibilities for the group of rotational parts; this is the main part of the finiteness theorem 2.3. The remaining part is a group cohomology argument dealing with nonisomorphic extensions of $\mathbb{Z}^n$ by finite groups.

Proof of the Bieberbach theorem (Following Gromov).
We introduce the finite subset

$$G_\rho^\varepsilon = \{\alpha \in G \mid \|r(\alpha)\| < \varepsilon \ , \ |t(\alpha)| < \rho\}$$

of G, where $0 < \varepsilon \leq \frac{1}{2}$ and $\rho > 0$ (large), and denote by $\langle G_\rho^\varepsilon \rangle$ the smallest subgroup of G which contains G_ρ^ε. The working tools will be lemmas 3,4,5 in section 1
The proof is divided into two parts:
2.4 For any $R > 0$ we can find some $\rho \geq R$ such that for all $x \in \mathbb{R}^n$ with $|x| \leq \frac{3}{4}\rho$ there is $\alpha \in G_\rho^\varepsilon$ with $|t(\alpha) - x| \leq \rho/4$.

2.5 $\langle G_\rho^\varepsilon \rangle$ is d-nilpotent with $d \leq 3^{n^2}$.

By d-nilpotent we mean that all d-fold commutators $[\ldots[\beta_1,\beta_2],\ldots,\beta_d]$ are trivial ($[\alpha,\beta] = \alpha\beta\overline{\alpha}^1\overline{\beta}^1$).

Hence instead of showing that the pure translations provide a vector space basis of $\mathbb{R}^n$, it is first shown that (the translational parts of) the almost translations ($=G_\rho^\varepsilon$) do, and instead of showing commutativity one starts with nilpotency. The reason why this procedure carries over to more general situations is that both, 2.4 and 2.5 are proved by means of estimates rather than by equations.

2.4 and 2.5 together suffice to show 2.2 (i) and in particular that G_ρ^ε is in fact a set of pure translations.

Assume there is $\gamma \in G$ with $r(\gamma) = C$, $t(\gamma) = c$ such that $\|C\| = \theta \in (0,\frac{1}{2})$. Then decompose $\mathbb{R}^n$ into $E \oplus E^\perp$ where E is an invariant plane of maximal rotation and let $x = x^E + x^\perp$ be the corresponding decomposition of vectors in $\mathbb{R}^n$. Put $\varepsilon = \frac{1}{10}(\sin\frac{\theta}{2})^d$ and choose $\rho \geq 2|c|$ in 2.4 so that one can find $\alpha \in G$ with $\|A\| \leq \varepsilon$ and $|a - x| \leq \frac{\rho}{4}$ for $x \in E$, $|x| = \frac{3}{4}\rho$; consequently $|c| \leq |a| \leq 2|a^E|$. Consider the iterated commutators

$$\alpha_k = [\ldots[\alpha,\gamma],\ldots,\gamma] \quad \text{(k-fold)} \ , \quad k = 1,\ldots,d \ .$$

From 4.3 we have the estimate

$$\|A_{k+1}\| = \|[A_k,C]\| \leq 2\|A_k\|\cdot\|C\| \leq \|A_k\| \leq \cdots < \|A\| \leq \varepsilon$$

which we use in the decomposition

$$a_{k+1} = (\mathrm{id}-C)a_k^E + (\mathrm{id}-C)a_k^\perp + (\mathrm{id}-[A_k,C])Ca_k + A_kC(\mathrm{id}-\overline{A}_k^1)\overline{C}^1c$$

to obtain first inductively

$$|a_{k+1}| \le (\|C\| + \|A_{k+1}\|)\cdot|a_k| + \|A_k\|\cdot|c| \le |a| \quad .$$

Then, since $E, E^\perp$ are invariant under C we can use the last two estimates to obtain

$$\begin{aligned} |a_d^E| &\ge |(\mathrm{id}-C)a_{d-1}^E| - \|[A_{d-1},C]\|\cdot 1a_{d-1}\| - \|A_{d-1}\|\cdot|c| \\ &\ge 2|a_{d-1}^E|\sin\tfrac{\theta}{2} - 2\epsilon|a| \\ &\ge (2\sin\tfrac{\theta}{2})^d\cdot|a^E| - 2\epsilon|a|\cdot\sum_{k=0}^{d-1}(2\sin\tfrac{\theta}{2})^k \\ &\ge |a^E|(\sin\tfrac{\theta}{2})^d \quad . \end{aligned}$$

Now $|a_d| > 0$, which contradicts 2.5 and proves 2.2 (i) .

2.6 A pigeon hole argument (Proof of 2.4).

Put $\rho_i = (R+r)\cdot 10^{i+1}$, $i = 0,\dots,2\cdot\mathrm{int}(2\pi/\epsilon)^{\dim SO(n)} = N(\epsilon)$, where r is the diameter of the fundamental domain of G .(This is the only point in the proof where compactness of R^n/G is used). Define $\mathcal{U}_i = \{\alpha\in G|\,|t(\alpha)| < \rho_i\}$. For each $x\in R^n$, $|x| \le \frac{3}{4}\rho_i$ choose $\alpha_i\in G$ with $a_i = t(\alpha_i)$ next to x ; then $|a_i - x| \le r$ and $r+\rho_{i-1} < \frac{1}{4}\rho_i$ imply for all $\beta\in\mathcal{U}_{i-1}$ that $|t(\alpha_i\circ\overline{\beta}^1) - x| < \frac{1}{4}\rho_i$ and $|t(\alpha_i\circ\overline{\beta}^1)| < \rho_i$. Therefore if 2.4 were false for all the ρ_i we would have for each i some $\alpha_i\in G$ with $\|r(\alpha_i\circ\overline{\beta}^1)\| > \epsilon$ for all $\beta\in\mathcal{U}_{i-1}$. In particular we get $N(\epsilon)+1$ elements $r(\alpha_i)\in O(n)$ with pairwise distance $>\epsilon$, contradicting lemma 4.4.

2.7 The short basis (Proof of 2.5)

We fix the constant in 2.1 to be ϵ/ρ . Then $\|\alpha\| < \epsilon$ for all $\alpha\in G_\rho^\epsilon$, and 4.3 implies

$$\|[\alpha,\beta]\| < \min\{\|\alpha\|,\|\beta\|\} \qquad (\alpha,\beta\in G_\rho^\epsilon) \quad .$$

A <u>short basis</u> $\{\alpha_1,\dots,\alpha_d\}$ is defined inductively by choosing a nontrivial

$$\alpha_1\in G_\rho^\epsilon \quad \text{with minimal } \|\alpha_1\| \quad ,$$

$$\alpha_{i+1}\in G_\rho^\epsilon - \langle\{\alpha_1,\dots,\alpha_i\}\rangle \quad \text{with minimal } \|\alpha_{i+1}\| \quad .$$

($\langle\{\alpha_1,\dots,\alpha_i\}\rangle$ is the smallest subgroup of G containing $\{\alpha_1,\dots,\alpha_i\}$.)

The basis is finite since G_ρ^ϵ is finite. The important point is that d has an upper bound which is independent of ρ and ϵ: If we could find $i<j\le d$ such that $d(\alpha_i,\alpha_j)<\|\alpha_j\|$, then $\|\alpha_j\bar\alpha_i^{-1}\|<\|\alpha_j\|<\epsilon$ hence $\alpha_j\bar\alpha_i^{-1}\in G_\rho^\epsilon$ and also $\alpha_j\bar\alpha_i^{-1}\in\langle\{\alpha_1,\dots,\alpha_{j-1}\}\rangle$ since $\|\alpha_j\|$ is minimal in the complement. Now $\alpha_j=(\alpha_j\bar\alpha_i^{-1})\circ\alpha_i\in\langle\{\alpha_1,\dots,\alpha_{j-1}\}\rangle$ is impossible. Hence the elements of a short basis satisfy the pairwise distance condition of 4.5 so that $d\le 3^{n+\dim SO(n)}$.

This d is also a bound on the length of nonvanishing commutators since $\|[\alpha_i,\alpha_j]\|<\min(\|\alpha_i\|,\|\alpha_j\|)$ implies first

$$(2.8)\qquad [\alpha_i,\alpha_j]\in\langle\{\alpha_1,\dots,\alpha_{i-1}\}\rangle\qquad (i<j)\ .$$

Then use induction on the wordlength based on the formulas $[\alpha\,\beta,\gamma]=[\beta,\gamma]\cdot[[\gamma,\beta],\alpha]\cdot[\alpha,\gamma]$ and $[\bar\alpha^{-1},\gamma]=[\bar\alpha^{-1},[\gamma,\alpha]]\cdot[\gamma,\alpha]$ and an induction on i to show that $\langle\{\alpha_1,\dots,\alpha_i\}\rangle$ is i-nilpotent.

2.9 Proof of 2.2 (ii)

The translations in G - clearly a normal subgroup, have been described as the set of all α with $\|A\|<\frac{1}{2}$. From 2.4 $\mathbb{R}^n/\Gamma$ is compact hence G/Γ is finite. The homomorphism $r\colon G\to O(n)$ induces an isomorphism between G/Γ and a discrete subgroup of $O(n)$ whose elements satisfy the pairwise distance condition of 4.4 with $\epsilon=\frac{1}{2}$. Therefore G/Γ has order $\le N(\frac{1}{2})$ (2.6.)

2.10 Proof of 2.2 (iii)

Consider pairwise orthogonal 2-planes $R_1\subseteq E_1,\dots,R_k\subseteq E_k$ through the origin such that A as restricted to R_i is a rotation by θ_i. Let S_i^1 be the unit circle in R_i. Fix $\varkappa\le k-1$. Then A acts isometrically on the flat torus $T_\varkappa=S^1_{\varkappa+1}\times\dots\times S^1_k$ not only with respect to the Riemannian but also with respect to the Finsler distance $d(x,y)=\max\{\measuredangle(x_i,y_i)\mid i=\varkappa+1,\dots,k\}$, ($x_i$ = orthogonal projection of x to R_i). The function $d(x,Ax)$ is constant on $T_\varkappa$. Since each torus has the same volume as the Finsler ball of radius π in its tangent space and since points of pairwise distance $\frac{1}{2}$ give disjoint balls of radius $\frac{1}{4}$ we have the volume ratio $m_\varkappa=\mathrm{int}(4\pi)^{k-\varkappa}$ as a bound on the number of such points. It follows that for some power A^m, $0<m\le m_\varkappa$, we have $d(x,A^mx)<\frac{1}{2}$ for all $x\in T_\varkappa$, which implies $|\measuredangle(x,A^mx)|<\frac{1}{2}$ for all $x\in E_{\varkappa+1}\oplus\dots\oplus E_k$. Therefore, if we had

$0<\theta_\varkappa<\frac{1}{2}(4\pi)^{\varkappa-k}$ it would follow from $0<\theta_1<\ldots<\theta_\varkappa<\frac{1}{2}m_\varkappa$ that $\sphericalangle(x,A^m x)<\frac{1}{2}$ for all $x\in R^n$, i.e. $\|A^m\|<\frac{1}{2}$ but $A^m \neq \mathrm{id}$, a contradiction to 2.2 (i).

3. Earlier proofs

In this section we sketch Bieberbach's original proof [1] and the one given in Wolf's book [4]. Both use commutator estimates though in different form. To simplify the description we use 4.3.

3.1 The structure of Bieberbach's approach consists of the following observations (X p. 317 and XII p. 328, Math. Ann. 70 (1911)).

(i) All main rotation angles occurring in G are rational $(\in \pi\mathbb{Q})$.

(ii) An infinite discrete group of motions has elements without fixed points.

The two propositions are proved independently. From (i) it follows that each infinite subgroup of G contains translations, and by a not too complicated induction argument Bieberbach then concludes:

(iii) If all translations of G are contained in a subspace E of R^n, then also all translational parts are contained in E.

At this point the proof is complete: Since G has compact fundamental domain E must be R^n. While the proof of (ii), based on the commutator estimate (Hilfssatz on p. 328) makes no trouble we like to comment on (i), which is the heart of Bieberbach's arguments. The way of proving (i) is by showing that irrational angles would imply the existence of infinitesimal sequences, i.e. sequences in G which do not contain the identity but which converge to it. First $\alpha\in G$ is chosen with the maximal possible number of irrational angles $\theta_1,\ldots,\theta_\lambda$. By taking powers it is achieved that all other angles are zero. By a change of origin there is a 2λ-dimensional invariant subspace $E\subseteq R^n$ such that $t(\alpha)\in E^\perp$ and $r(\alpha)|E^\perp=\mathrm{id}$. Since G has compact fundamental domain there is $\gamma\in G$ with $t(\gamma)\notin E^\perp$. This γ does not commute with any power $\alpha^m (m\neq 0)$. Certainly one can construct a set of powers of α such that the rotational parts form an infinitesimal sequence. The problem is to have the translational parts converge also. This is achieved together with γ in the following way.

By Minkowski's theorem on simultaneous rational approximation there exist for all $j=1,2,\ldots$ integers $x_1(j),\ldots,x_\lambda(j)$ and $n(j)$ such that simultaneously

$$\left|2\pi\frac{x_\iota(j)}{n(j)}-\theta_\iota\right|\leq\frac{1}{j\cdot n(j)}\qquad \iota=1,\ldots,\lambda\ .$$

Now for each fixed m (which serves as parameter) Bieberbach considers the sequence of m-fold commutators

$$\gamma_m^{(i)} = [\ldots[\gamma,\alpha^{n(j)}],\ldots,\alpha^{n(j)}] , \qquad j = 1,2,\ldots$$

Due to Minkowski's inequality the powers $\alpha^{n(j)}$ have small rotational angles, and from this by an involved calculation the following orders of magnitude are shown

$$\|r(\gamma_m^{(j)})\| = O(j^{1-m}) , \quad |t(\gamma_m^{(j)})| = O(j^{\lambda+2-m}), \quad m \geq 2 .$$

Therefore the proof of (i) is complete if for $m = \lambda + 3$ the sequence $\{\gamma_m^{(j)}\}_{j=1}^{\infty}$ does not contain the identity. Now by the particular choice of α and γ one finds these sequences free from the identity for $m = 2$ and 3. Yet there may be a minimal $m \geq 4$ such that $\{\gamma_m^{(j)}\}_{j=1}^{\infty}$ is not infinitesimal. If this happens, various cases must be considered. If $m = 4$ and

$$[\gamma_2^{(j)},\gamma_3^{(j)}] \neq 1 \ (j \geq j_0) \ , \text{ then } \ \{[\gamma_3^{(j)},\gamma_1^{(j)}\gamma_2^{(j)}]\}_{j=1}^{\infty}$$

is infinitesimal instead. If $m = 4$ and $[\gamma_2^{(j)},\gamma_3^{(j)}] = 1$ then $\{\gamma_3^{(j)}\}$ is infinitesimal. For $m = 5$ one can take $\{[\gamma_3^{(j)},\gamma_4^{(j)}\gamma_1^{(j)}]\}$. And finally for $m \geq 6$ the sequence looked for is $\{\gamma_{m-1}^{(j)} \cdot (\gamma_{m-2}^{(j)})^{-1}\}$.

It is interesting, how a little more information about γ simplifies the proof of (i). From the pigeon hole argument 2.6 one can choose γ such that in addition $\|r(\gamma)\| < \frac{1}{2}$. Then $\gamma_m^{(j)} \neq 1$ for all m , $(j \geq 3)$; for otherwise by the lemma below, $\gamma_2^{(j)}$ and a fortiori each further $\gamma_m^{(j)}$ is a translation which due to the choice of α and γ has always a nonzero component in E , a contradiction. Hence $\{\gamma_m^{(j)}\}_{j=1}^{\infty}$ is always infinitesimal, in particular for $m = \lambda + 3$. However there is a still simpler argument: Look at the series $\{\gamma_m^{(j)}\}_{m=1}^{\infty}$ instead of $\{\gamma_m^{(j)}\}_{j=1}^{\infty}$. As mentioned, it does not contain the identity. By the commutator estimate (c.f. 2.7) it converges to the identity. Thus it is infinitesimal.

3.2 Bieberbach's proof succeeded by extracting translations from G by means of powers (based on the non-existence of irrational angles). The logical structure od Gromov's proof is different. He first <u>defines</u> a subgroup $(\langle G_\rho^{\frac{1}{2}} \rangle)$ of finite index (by the pigeon hole argument) in G and then proves that the subgroup is already a group of translations (by the short basis trick). Wolf's proof also starts by defining a suitable normal subgroup $G^* = r^{-1}(T) \subseteq G$ where $T \subseteq SO(n)$ is the identity component of the closure of $r(G)$. G/G^* is almost immediately finite: Since T is closed and $SO(n)$ is compact, only finitely many different sets $r(\gamma)\cdot T$ occur as γ runs through G . The task is again to show that G^* is purely translational: First one observes ([4] p. 100)

Lemma If $A,B \in SO(n)$, $\|A\|, \|B\| < \pi/2$ then

$$[[A,B],B] = 1 \text{ implies } [A,B] = 1 .$$

(This lemma is not used by Gromov since due to occurring homotopy errors there is no analogue for non flat situations). Together with the commutator estimate (c.f. 2.7) one finds T toral in $SO(n)$. Hence the subspace $W = \{x \in \mathbb{R}^n | T(x) = x\}$ is characterized as the fixed point set of a single rotation $r(\gamma_0)$, $\gamma_0 \in G^*$, and by a change of origin one may assume $t(\gamma_0) \in W$. Since T is abelian, one checks that $t(\gamma) \in W$ for all further $\gamma \in G^*$ also. Since $\mathbb{R}^n/G^*$ is compact this is possible only if $W = \mathbb{R}^n$. Hence $T = r(G^*) = \{id\}$, i.e. G^* is a set of translations.

4. The group of motions.

The lemmas of this section will be proved with differential geometric techniques. We recall the following facts:

4.1 The orthogonal group $O(n)$ is a Lie group with identity component $SO(n)$. Its Lie algebra $so(n)$ is the space of skewsymmetric matrices $X,Y,\ldots$ and is canonically identified with the space of left invariant vector fields, using that the brackets of left invariant vector fields are left invariant.

(1) $\quad \mathrm{ad}X\,(Y) := [X,Y] = XY - YX$.

The exponential map $\exp\colon\ so(n) \to SO(n)$, $\exp X = id + \sum_{k=1}^{\infty} \frac{X^k}{k!}$ relates to ad and conjungation $K_A\colon\ B \to ABA^{-1}$ as follows:

(2) $\quad \exp Y \cdot \exp X \cdot \exp(-Y) = \exp(dK_{\exp Y} X)$

(3) $\quad \mathrm{Exp}\,(t\,\mathrm{ad}Y) := id + \sum_{k-1}^{\infty} \frac{1}{k!}\,(t\,\mathrm{ad}Y)^k = (d\,K_{\exp tY})_{id}$.

Denote by D^L the left invariant connection for which left invariant vector fields are parallel, then

$$D_X Y := D^L_X Y + \frac{1}{2}\ [X,Y]$$

defines a torsion free biinvariant connection with parallel curvature tensorfield R:

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]] \quad .$$

Obviously

$$R(J) := D^L_{\dot c}\, D^L_{\dot c}\, J + D^L_{\dot c}[\dot c,J] = D_{\dot c}\, D_{\dot c}\, J + R(J,\dot c)\dot c$$

for vector fields J along geodesics $t \to c(t) = \exp tX$. The solutions of $R(J) = 0$ are the Jacobifields and are either obtained as

(4) $$J(t) = dL_{c(t)} \cdot k^L(t) \ , \quad k^L\colon \mathbb{R} \to so(n) \ , \quad \ddot k^L + [X,\dot k^L] = 0 \ ,$$

$(L_A(B) := A \cdot B)$ where $dL_{c(t)}$ is parallel translation along c with respect to the connection D^L , or as

(5) $$J(t) = P_t \cdot k(t), \ k\colon \mathbb{R} \to so(n), \quad \ddot k - \frac{1}{4}\,(\mathrm{ad}X)^2 k = 0$$

where

(6) $$P_t := dL_{c(t)} \circ \mathrm{Exp}(-\frac{t}{2}\mathrm{ad}X)$$

is parallel translation along c with respect to the connection D . The differential $d\exp$ can be described with Jacobi fields as follows

(7) $$(d\exp)_{tX}\, Y = \frac{1}{t}\, J(t) \quad \text{if} \quad J(0) = 0, \ \frac{D}{dt}\, J(0) \quad (= \frac{D^L}{dt}\, J(0)) = Y \quad .$$

4.2 If for $S \in so(n)$ we put

$$\|S\| = \max\{|Sv|;\ v \in \mathbb{R}^n \ , \ |v| = 1\} \quad ,$$

then from (1)

(8) $$\|[S,T]\| \le 2\|S\| \cdot \|T\| \ .$$

By left translating this norm to all other tangent spaces we obtain a Finsler metric for $O(n)$ whose distance function

$$d(A,B) = \max\{ \measuredangle\,(v,Av) \mid \ v \in \mathbb{R}^n \ , \ |v| = 1\} \, .$$

has already been introduced in section 2. The diameter of $SO(n)$ and the injectivity radius of exp with respect to the Finsler metric are π . Since the distance function is biinvariant, $(dK_A)_{id}$: $so(n) \to so(n)$ is a norm isometry and it follows from (3)

(9) $\qquad \|\mathrm{Exp}\,(\mathrm{ad}Y)\cdot X\| = \|X\|$, $\quad X,Y \in so(n)$.

Hence both parallel translations $dL_{c(t)}$ (by definition) and P_t (by (6) and (9)) are norm preserving.

If $J(t) = dL_{c(t)}\, k^L(t)$ is a Jacobifield (4) , then k^L satisfies $\dot{k}^L(t) = \mathrm{Exp}\,(t\mathrm{ad}X)\cdot\dot{k}(0)$, $\|\dot{k}^L(t)\| = \|\dot{k}(0)\|$ (9) and therefore

$$\|J(t)\| = \|k^L(t)\| \le t\|\frac{D^L}{dt}J(0)\| = t\|Y\| \quad .$$

(10) $\quad \|(d\exp)_{tX}Y\| \le \|Y\|$,

i.e. exp does not increase lengths in the Finsler metric.

4.3 Lemma

$$d([A,B],\mathrm{id}) \le 2d(A,\mathrm{id})\cdot d(B,\mathrm{id}), \quad A,B \in SO(n) \quad .$$

Proof Let $A = \exp X$, $B = \exp Y$ and connect A with BAB^{-1} by the curve (c.f. (2) and (3))

$$t \to \gamma\,(t) = \exp\,(\mathrm{Exp}\,(t\,\mathrm{ad}\,Y)\cdot X), \qquad t \in [0,1] \quad .$$

From the biinvariance of the Finsler metric

$$d([A,B],\mathrm{id}) = d(A,BAB^{-1}) \le \int_0^1 \|\dot\gamma(t)\|dt \quad .$$

Since exp does not increase lengths (10)

$$\|\dot\gamma(t)\| \le \|\frac{d}{dt}\,(\mathrm{Exp}(t\,\mathrm{ad}\,Y)\cdot X\,\| \overset{(3)}{=} \|\mathrm{Exp}(t\,\mathrm{ad}\,Y)\cdot[Y,X]\|$$

$$\overset{(9)}{=} \|[X,Y]\| \overset{(8)}{\le} 2\|X\|\cdot\|Y\| = 2\,d(A,\mathrm{id})\cdot d(B,\mathrm{id}) \quad .$$

4.4 Lemma

For $\varepsilon > 0$ there exist at most $N(\varepsilon) = 2\,\mathrm{int}\,(2\pi/\varepsilon)^{\dim SO(n)}$ rotations in $O(n)$ with pairwise distances $\ge \varepsilon$.

Proof. It suffices to prove $N(\varepsilon)/2$ as upper bound on $SO(n)$. Since metric balls $B_{\varepsilon/2}$ of radius $\varepsilon/2$ around the considered elements have pairwise disjoint interior and equal volumes, it follows from $B_\pi = SO(n)$ that

$$\operatorname{vol} B_\pi / \operatorname{vol} B_{\varepsilon/2}$$

is an upper bound. To get it explicitly we estimate $\det(d\exp)_{tX}$ in the standard Riemannian metric (which provides the volume function on $SO(n)$); the Levi-Civita connection is D. Norms with respect to the Riemannian metric are denoted by $|\cdot|$. We use an orthonormal Basis $\{Y_1,\dots,Y_m\} \subseteq so(n)$ of eigenvectors with eigenvalues $\lambda_1^2,\dots,\lambda_m^2$ of the nonnegative symmetric operator $-(\operatorname{ad}X)^2$, $m = \dim so(n)$. If J is the Jacobifield (7) for $Y = Y_i$, then in (5) obviously $k(t) = \frac{2}{\lambda_i}\sin\frac{t}{2}\lambda_i$ is a solution. Therefore since the Levi-Civita parallel translation P_t perserves $|.|$ we conclude from (6) and (7) that the Jacobifields corresponding to $Y_1,\dots,Y_m$ are pairwise orthogonal along $c(t) = \exp tX$ and satisfy

$$|(d\exp)_{tX}\cdot Y| = t^{-1}|J(t)| = t^{-1}|k(t)| = (\frac{t\lambda_i}{2})^{-1}\sin t\lambda_i/2\ , \qquad t \leq \pi\ .$$

Since $\|\operatorname{ad}X\| \leq 2\,\|X\| \leq 2\pi$ (1), the eigenvalues of $-(\operatorname{ad}X)^2$ are $\leq 4\pi^2$. Hence

$$\det(d\exp)_{tX} = \prod_{i=1}^{m} \frac{\sin}{\mathrm{id}}(\frac{t}{2}\lambda_i)$$

is not increasing and $\operatorname{vol} B\pi/\operatorname{vol} B_t \leq (\pi/t)^m$, q.e.d.

4.5 Lemma

There exist at most $3^{n+\dim SO(n)}$ euclidean motions $\alpha,\beta,\dots$ which pairwise satisfy the condition (c.f.2.1)

$$d(\alpha,\beta) \geq \max\{\|\alpha\|,\|\beta\|\}\ .$$

Proof Consider m such motions α_i and corresponding pairs $w_i = (S_i,a_i) \in so(n)\times\mathbb{R}^n$ where $\exp S_i = A_i = r(\alpha_i)$, $a_i = t(\alpha_i)$. Introducing the norm $\|(S,a)\| = \max\{\|S\|, c\cdot|\ \ |\}$ (the constant is irrelevant) in the vectorspace $so(n)\times\mathbb{R}^n$ we find m points $\tilde{w}_i = \|w_i\|^{-1}w_i$ on the unit sphere satisfying

$$\|\tilde{w}_i - w_j\| \geq \|w_j\|^{-1}\|w_i - w_j\| - \|\,\|w_j\|^{-1}w_i - \tilde{w}_i\| \geq 1$$

(if w.l.o.g. $\|w_j\| \leq \|w_i\|$), since by (10)

$$\|w_i - w_j\| \geq d(\alpha_i,\alpha_j) \geq \max\{\|\alpha_i\|,\ \|\alpha_j\|\} = \max\{\|w_i\|,\|w_j\|\} \quad .$$

It follows that the open balls of radius $1/2$ around the $\tilde{w}_i$ are pairwise disjoint and contained in a ball of radius $3/2$. Now m cannot exceed the volume ratio $3^{\dim(so(n)\times R^n)}$. q.e.d.

Remark. There is no finite bound if the condition is replaced by $d(\alpha,\beta) \geq \epsilon \max\{\|\alpha\|,\|\beta\|\}$ $\epsilon < 1$. In many cases as e.g. in the proof of Gromov's theorem, it is desirable to have an open condition. One such condition is

$$d(\alpha,\beta) \geq \max\{\|\alpha\| - \epsilon\|\beta\|,\|\beta\| - \epsilon\|\alpha\|\} \quad .$$

The number of motions is then bounded above by $(\frac{3-\epsilon}{1-\epsilon})^{n+\dim SO(n)}$, the proof is the same.

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ALMOST FLAT MANIFOLDS

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1. Introduction

1.1. We denote by V a connected n-dimensional complete Riemannian manifold, by $d = d(V)$ the diameter of V, and by $c^+ = c^+(V)$ and $c^- = c^-(V)$, respectively, the upper and lower bounds of the sectional curvature of V. We set $c = c(V) = \max(|c^+|, |c^-|)$.

We say that V is ε-flat, $\varepsilon \geq 0$, if $cd^2 \leq \varepsilon$.

1.2. Examples.

a. Every compact flat manifold is ε-flat for any $\varepsilon \geq 0$.

b. Every compact nil-manifold possesses an ε-flat metric for any $\varepsilon \geq 0$. (*A* manifold is called a nil-manifold if it admits a transitive action of a nilpotent Lie group; see 4.5.)

The second example shows that for $n \geq 3$, $\varepsilon > 0$ there are infinitely many ε-flat n-dimensional manifolds with different fundamental groups.

1.3. Define inductively $ex_i(x) = \exp(ex_{i-1}(x))$, $ex_0(x) = x$, and set $\hat{\varepsilon}(n) = \exp(-ex_j(n))$, where $j = 200$. (We are generous everywhere in this paper because the true value of the constants is unknown.)

1.4. Main Theorem. *Let V be a compact $\hat{\varepsilon}(n)$-flat manifold, and π its fundamental group. Then*:

(a) *There exists a maximal nilpotent normal divisor $N \subset \pi$*;

(b) $\operatorname{ord}(\pi/N) \leq ex_3(n)$;

(c) *the finite covering of V corresponding to N is diffeomorphic to a nil-manifold.*

Corollary. *If V is $\hat{\varepsilon}(n)$-flat, then its universal covering is diffeomorphic to R^n. If V is $\hat{\varepsilon}(n)$-flat and π is commutative, then V is diffeomorphic to a torus.*

1.5. Manifolds of positive and almost positive curvature. For such manifolds one expects the properties (a) and (b) from Main theorem 1.4, but we are able to prove only the following:

(i) If V is a manifold of nonnegative sectional curvature ($c^- \geq 0$), then its fundamental group π and every subgroup of π can be generated by 3^n elements.

(ii) If $d(V) \leq \mathscr{D}$, $c^-(V) \geq -K$, $K \geq 0$, then π can be generated by $N \leq 3^n\, ex_2(nK\mathscr{D}^2)$ elements; if π is a free group and $K\mathscr{D}^2 \leq \hat{\varepsilon}(n)$, then π is generated by one element.

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1.6. Manifolds of almost negative curvature. The universal coverings of such manifolds are expected to be contractable. If $n = 2$, it is so for V with $c^+(V) \leq 1$, $d(V) < \frac{1}{4}\pi$ (S. Mayers, see [4]), but for $n = 3$ we have

Counterexample. For given $\varepsilon > 0$ there exists a manifold V diffeomorphic to the sphere S^3 such that $d(V) \leq \varepsilon$, $c^+(V) \leq \varepsilon$. (See [5].)

1.7. The volume and the injectivity radius. A slight modification of Cheeger's arguments from [1], [2] shows that the lower bound on the volume vol (V) or on the injectivity radius reduces drastically the number of almost flat manifolds (compare with Examples 1.2):

(a) The number of distinct up to diffeomorphism manifolds with $d(V) \leq 1$, vol $(V) \geq K^{-1}$, $c(V) \leq K$, $K \geq 0$, is less than $ex_6(n + K)$, Cheeger [1].

(b) If $d(V) \leq 1$, vol $(V) \geq K^{-1}$, $K \geq 0$ and $c(V) \leq \hat{\varepsilon}(n + K)$, then V is diffeomorphic to a flat manifold.

1.8. The second statement is a weak pinching theorem. For positive curvature there is much better result:

If $c^+(V) \leq 1$, $c(V) \geq 0.97$, then V is diffeomorphic to a manifold of a constant positive curvature (Grove, Karcher, Ruh [7]).

The following is known for the negative case:

If $c^+(V) \leq -1$, $c^-(V) \geq -1 - \kappa$, $\kappa \geq 0$, then in the following three cases V is diffeomorphic to a manifold of constant negative curvature:

(a) $\kappa \leq (ex_7(n + d(V)))^{-1}$; (E. Heintze, see [8]).

(b) $\kappa \leq (ex_7(n + \text{vol}(V)))^{-1}$ and $n \neq 3$ (for $n = 3$ it is unknown).

(c) n is even and $\kappa \leq (ex_9(n + |\chi(V)|))^{-1}$, where $\chi(V)$ is the Euler characteristic.

Proof. In view of the Margulis-Heintze theorem (see the next section) one can apply to (a) Cheeger's arguments as in the previous section. About (b) see [6]. The case (c) follows from "b" and the Gauss-Bonnet theorem.

1.9. About the proof of the main theorem. Our arguments imitate the proof of the Bieberbach theorem (see [9]). The first application of the discrete group technique to geometry is due to Margulis who proved (but has never published) the following analog of the Kazdan-Margulis theorem (see [9]):

If V is compact, $c^+(V) < 0$, $c^-(V) > -1$, then $\text{vol}(V) \geq C_n^{-1}, C_n \leq ex_4(n)$. (Margulis is not responsible for that particular C_n.)

This fact was independently discovered by Ernst Heintze (see [8]).

To prove that theorem Margulis established the following:

The Margulis Lemma. *Let V be as above, and suppose $\alpha, \beta \in \pi = \pi_1(V, v_0)$ can be represented by loops of the length $\leq C^{-1}$. If $C \geq ex_2(n)$, there is a natural number m such that $\alpha^m, \beta^m \subset \pi$ generate a nilpotent group.*

The ideas of Margulis lying behind his lemma are crucial for our proof of the Main Theorem. I am also very much indebted to Yu. Burago, J. Cheeger, D. Gromoll, V. Eidlin, W. Meyer and J. Milnor for discussions having led to a simplification of the proof. I am essentially thankful to Professor H. Karcher for his constructive criticism and suggestions. In particular, the present versions of statements 2.3, 2.5, 2.6, 2.8.and 7.2 are due to him.

2. Almost positive curvature

2.1. For a group Γ with a function $\gamma \to \|\gamma\| \in \boldsymbol{R}_+$ we denote the "ball" $(\|\ \|)^{-1}[0, \rho]$ by $\Gamma_\rho \subset \Gamma$. We say that Γ is discrete with respect to $\|\ \|$ if all balls are finite.

We call $\gamma_1, \gamma_2, \cdots, \gamma_s \in \Gamma$ a short basis (or short generators) and the sequence of subgroups $e = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_s = \Gamma$ a short filtration with respect to $\|\ \|$, if Γ_i is generated by $\gamma_1, \cdots, \gamma_i$ and $\|\gamma_{i+1}\|$ is minimal for all γ from the complement $\Gamma \backslash \Gamma_i$.

2.2. From now on we fix a point $v_0 \in V$, denote the tangent space at v_0 by T, and set $\pi = \pi_1(V, v_0)$. For a geodesic $\lambda: [0, l] \to V$ with $\lambda(0) = v_0$ we denote by $t(\lambda) \in T$ the corresponding tangent vector with length $(t(\lambda)) =$ length (λ). For $\alpha \in \pi$ we denote by $\|\alpha\|$ the length of the shortest loop representing α.

2.3. Let $\alpha, \beta \in \pi$, and λ, μ be the corresponding shortest loops with ϕ the angle between $t(\lambda)$ and $t(\mu)$. Put $\rho = \max(\|\alpha\|, \|\beta\|)$ and $\kappa^2 = \max(0, -c^-(V))$. If $\cos\phi \geq \cosh \kappa\rho \cdot (1 + \cosh \kappa\rho)^{-1}$ (i.e., for $\kappa = 0$ if $\phi \leq \frac{1}{3}\pi$), then $\|\alpha^{-1}\beta\| \leq \max(\|\alpha\|, \|\beta\|)$.

Proof. Apply the Toponogov comparison theorem to the universal covering $\tilde{V}$.

2.4. Proof of 1.5 (i). Take the short basis $\gamma_1, \cdots, \gamma_s \in \pi$ and the corresponding shortest loops $\lambda_1, \cdots, \lambda_s$. From 2.3 it follows that all angles between $t(\lambda_i)$ and $t(\lambda_j)$, $1 \leq i < j \leq s$, are at least $\pi/3$ and so $s \leq \mathrm{vol}(S^n)/\mathrm{vol}(B^n_{\pi/6}) \leq 3^n$.

2.5. If $\rho \geq 2d(V)$, then the ball $\pi_\rho \subset \pi$ generates π, since every loop strictly longer than ρ can be decomposed into two shorter ones.

2.6. Therefore we can estimate the number of generators in 1.5 (ii) by using ϕ from $\cos\phi = \cosh(2\kappa\mathscr{D}) \cdot (1 + \cosh 2\kappa\mathscr{D})^{-1}$ by

$$s \leq \mathrm{vol}(S)/\mathrm{vol}(B^n_{\phi/2}) \leq 3^n \cdot \cosh^n(\kappa\mathscr{D}) .$$

For the last statement we need an algebraic fact.

2.7. For a group Γ with generators $\gamma_1, \cdots, \gamma_s$ we denote by $N^k(\gamma_1, \cdots, \gamma_s)$ the smallest number N such that every subgroup in Γ generated by words of length $\leq k$ admits a system of N generators. Denote by $N^k(\Gamma)$ the minimum of all $N^k(\gamma_1, \cdots, \gamma_s)$ with respect to all systems of generators of Γ.

If Γ is free and noncommutative, then $N^k(\Gamma) \geq k$. This is obvious and in fact $N^k(\Gamma)$ grows exponentially.

2.8. End of the proof of 1.5. For a short basis $\gamma_1, \cdots, \gamma_s \subset \pi$ we conclude as before $N^k(\gamma_1, \cdots, \gamma_s) \leq 3^n \cdot \cosh^n(\kappa \cdot k\mathscr{D})$. Now, if $\kappa \cdot \mathscr{D} < 3^{-2n}$, then this upper bound for $N^k(\gamma_1, \cdots, \gamma_s)$ is, for noncommutative π, incompatible with $k \leq N^k(\gamma_1, \cdots, \gamma_s)$, (e.g., at $k = 3^n$).

3. q-isometries

3.1. A set in a metric space X is said to be δ-dense if it intersects every ball of radius δ. A discrete set $\Delta \subset X$ is said to be σ-uniformly δ-dense if for any two balls $A, B \subset X$ of radius δ the numbers i, j of points in $A \cap \Delta$, $B \cap \Delta$ satisfy

$$\sigma^{-1} \leq i/j \leq \sigma .$$

A map f from one metric space to another is called a R-restricted q-isometry if for any two points x, y with dist $(x, y) \leq R$ we have

$$q^{-1} \leq \frac{\operatorname{dist}(f(x), f(y))}{\operatorname{dist}(x, y)} \leq q .$$

3.2. For a complete Riemannian C^∞-manifold X, a discrete set $\Delta \subset X$ and a finite C^∞-function $\psi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ we construct a map $\varphi: X \to H = l^2(\Delta)$ (= the space of l^2-functions on Δ): $(\varphi(x))\,(y) = \psi\,(\operatorname{dist}(x, y))$, $x \in X$, $y \in \Delta$. Further we fix ψ with properties: ψ is supported in the interval $[0.1, 1]$ if $x \in [\frac{1}{3}, \frac{2}{3}]$, then $\psi(x) = x$ and $\psi(x) + |\psi'(x)| + |\psi''(x)| \leq 100$, $x \in [0, 1]$.

3.3. Let X_1 and X_2 be manifolds as above of dimension n, and $\Delta_1 \subset X$, $\Delta_2 \subset X$ be σ-uniformly δ-dense sets. Denote by R_0 the minimum of the injectivity radii Rad (X_1), Rad (X_2), and by K the maximum of the curvatures $c(X_1)$ and $c(X_2)$. Let $f: \Delta_1 \to \Delta_2$ be a bijective R-restricted q-isometry. If $\sigma \leq 2$, $\delta \leq \exp(-10n)$, R, $R_0 \geq 10$, $q \leq 1 + \exp(-10n)$, $K \leq \exp(-10n)$, then there exists a diffeomorphism $F: X_1 \to X_2$.

Proof. Using $f: \Delta_1 \to \Delta_2$ we identify Δ_1 with Δ_2, and set $H = l^2(\Delta_1) = l^2(\Delta_2)$. It is easy to see that the maps $\varphi_1: X_1 \to H$ and $\varphi_2: X_2 \to H$ are smooth imbeddings, the image X_1' of the first map is contained in a normal tubular neighborhood of the image X_2' of the second map, and the normal projection $X_1' \to X_2'$ is a diffeomorphism.

3.4. Remark. Our construction for F is metrically invariant. So if f commutes with an isometrical action of a group in Δ_1 and Δ_2, then so does F. (We suppose here that a group acts isometrically on X_1 and X_2, and Δ_1, Δ_2 are invariant sets.)

3.5. Notice that 1.7 (a) immediately follows from 3.3 and the Cheeger inequality: If $d(V) \leq 1$, then Rad $(V) \geq$ vol $(V)\,(ex_2(n + K)^{-1}$; see [1].

4. Lie groups

4.1. The group of motions. We normalize the biinvariant metric in $O(n)$ by the condition $d(O(n))$ (= diam $(O(n)) = 1$, and denote by $M(n)$ the group of rigid motions of $\boldsymbol{R}^n$ with the metric induced by the decomposition $M(n) = O(n) \times \boldsymbol{R}^n$. We denote the projections $M(n) \to O(n)$ and $M(n) \to \boldsymbol{R}^n$ by "rot" and "trans" respectively. In all three groups we denote by $\|\alpha\|$ the distance from α to the identity element, and by B_a, $a \geq 0$, the ball of radius a centered at the identity element.

By $[\alpha, \beta]$ we denote the commutator of α and β. For $A \in O(n)$ by $E_{\max}(A) \subset \boldsymbol{R}^n$ we denote the eigenspace corresponding to the (complex) eigenvalue λ maximizing the distance: dist $(\lambda, 1)$.

4.2. The following properties of the commutators are obvious and well known (see [9]):

(a) $\|[\alpha, \beta]\| \leq C_n\|\alpha\|\cdot\|\beta\|$, where α, β from $O(n)$ or $M(n)$, $\|\alpha\|, \|\beta\| \leq 1$ and $C_n \leq ex_2(n)$;

(b) Let $A \in O(n)$, $b \in E_{\max}(A)$, and $\alpha: x \mapsto Ax$, $\beta: x \mapsto x + b$, $x \in R^n$ be the motions from $M(n)$. Set $\alpha_1 = [\alpha, \beta]$, $\alpha_i = [\alpha, \alpha_{i-1}]$. Then $\|\alpha_i\| \geq n^{-i}\|A\|^i\|b\|$.

Nilpotent groups

4.3. Let L be an n-dimensional simply connected nilpotent Lie group, and l its Lie algebra. Equip l with an Euclidean structure, and L with the corresponding left invariant metric. Expressing curvature of L in terms of l we have

4.4. If $\|[x, y]\| \leq c\|x\|\|y\|$, $x, y \in l$, $c \geq 0$, then the curvature $c(L)$ satisfies $c(L) \leq 100c^2$.

4.5. Take a triangular basis $x_1, \cdots, x_n \in l$ (i.e., $[x, x_i] \in l_{i-1}$, $x \in l$, and l_{i-1} is spanned by $x_1, \cdots, x_{i-1}$), and for $x = \sum_{i=1}^n a_i x_i$ set $\|x\|^2 = \sum_{i=1}^n \mu_i a_i^2$, $\mu_i \geq 0$.

If $\mu_{i-1} \leq \mu_i^n$ and μ_n is small, then the curvature $c(L)$ is small because of 4.4, and for given uniform discrete subgroup $\Gamma \subset L$ the diameter $d(L/\Gamma)$ is also small. This provides the second example in 1.2.

4.6. For vectors $x_1, \cdots, x_k \in \boldsymbol{R}^n$, $k \leq n$ we denote by $\mathscr{D}(x_1, \cdots, x_k)$ the volume of the k-dimensional parallelepiped spanned by $x_1, \cdots, x_k$. We say that a system of independent vectors $x_1, \cdots, x_k$ is regular if $\|x_i\| \leq 3^{i-1}\|x_j\|$, $1 \leq i < j \leq k$, and $\mathscr{D}(x_1, \cdots, x_k) \geq A_n \prod_{i=1}^k \|x_i\|$, $A_n^{-1} = ex_2(n)$.

4.7. Consider an n-dimensional lattice $\Lambda \subset \boldsymbol{R}^n$ equipped additionally with the structure of a nilpotent group without torsion. Let $\lambda_1, \cdots, \lambda_n \subset \Lambda$ be a basis in Λ such that the sublattices $\Lambda_i = \{\sum_{j=1}^i m_j\lambda_j\}$ are also invariant subgroups with respect to the nilpotent group structure, $[\Lambda, \Lambda_i] \subset \Lambda_{i-1}$, $i = 1, \cdots, n$, and $\lambda_1^{m_1}\lambda_2^{m_2} \cdots \lambda_n^{m_n} = \sum_{i=1}^n m_i\lambda_i$, $m_i = \cdots, -1, 0, 1, \cdots, i = 1, \cdots, n$.

Realize Λ now (see [9]) as a uniform discrete subgroup in a nilpotent group L and associate with the basis $\lambda_1, \cdots, \lambda_n \in \Lambda \subset R^n$ a left invariant metric in L as follows: take $x_1, \cdots, x_n \in l$ with $\exp(x_i) = \lambda_i \in \Lambda \subset L$, equip l with the Euclidean structure induced by the isomorphism $\boldsymbol{R}^n \to l$ extending $\lambda_i \to x_i$, and take the corresponndig metric in L. For $\lambda, \mu \in \Lambda \subset L$ we denote the distance with respect to this metric by $d_L(\lambda, \mu)$.

4.8. Suppose that for $\lambda, \mu \in \Lambda \subset R^n$ with $\|\lambda\|, \|\mu\| \leq \rho > 0$ we have $\|[\lambda, \mu]\| \leq c\|\lambda\|\|\mu\|$. If the basis $\lambda_1, \cdots, \lambda_n \in \Lambda \subset R^n$ is regular and $\rho/\|\lambda_n\| \geq ex_3(n)$, then $c(L) \leq (c')^2$, $c' = c\cdot ex_5(n)$, and for $\lambda \in \Lambda \subset R^n$ with $\|\lambda\| \leq (c\, ex_6(n))^{-1}$ we have $q^{-1} \leq d_L(e, \lambda)/\|\lambda\| \leq q$, where $e \in \Lambda \subset L$ is the identity element and $q \leq \exp(c \cdot \|\lambda\| \cdot ex_6(n))$.

Proof. The product in the nilpotent group $\Lambda \subset \boldsymbol{R}^n$ is given by a polyinomial $P: \Lambda \times \Lambda \to \Lambda$ of degree $\leq n$. Extending this polynomial to $\boldsymbol{R}^n \times \boldsymbol{R}^n$ provides on $\boldsymbol{R}^n$ the structure of a nilpotent Lie group isomorphic to L. The bracket in the Lie algebra may be expressed in terms of the coefficients of P and so by an obvious interpolation argument inequalities $\|[\lambda, \mu]\| \leq c\|\lambda\|\|\mu\|$ in the ball in Λ yield the analogous inequality for l:

$$\|[x, y]\| \leq 10^{-2}c'\|x\|\|y\|, \qquad x, y \in l.$$

This, together with **4.4**, proves the first statement of the lemma and the same interpolation arguments prove the second.

5. Pseudogroups

5.1. A pseudogroup is by definition a set Γ with a product $\alpha \cdot \beta \in \Gamma$ defined for some pairs $\alpha, \beta \in \Gamma$ and having the following properties:

There is the unique identity element $e \in \Gamma$, and every $\gamma \in \Gamma$ has a unique inverse.

If the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are defined, they are equal and are written as $\alpha\beta\gamma$. Generally, the notation $\gamma_1\gamma_2 \cdots \gamma_k$ means that the product is defined for any setting of brackets.

5.2. Example. A symmetric subset of a group, containing the identity element, is a pseudogroup.

5.3. Any pseudogroup Γ can be viewed as a presentation (by generators and relations) of a group $\pi = \pi(\Gamma)$. If the natural map $\Gamma \to \pi$ is injective, we say that Γ is injective. The pseudogroups from the above example are injective.

5.4. A symmetric subset of a pseudogroup containing the identity is again a pseudogroup, but we use the term "subpseudogroup" only for sets closed with respect to the multiplication.

5.5. A function $\gamma \mapsto \|\gamma\| \in \boldsymbol{R}_+, \gamma \in \Gamma$, is called a norm if it is symmetric ($\|\gamma^{-1}\| = \|\gamma\|$), positive outside the identity element, and $\|\alpha\beta\| \leq \|\alpha\| + \|\beta\|$.

We introduce the radius $\operatorname{rad}(\Gamma) = \max_{\gamma \in \Gamma} \|\gamma\|$, and say that Γ is radial if for $\alpha, \beta \in \Gamma$ with $\|\alpha\| + \|\beta\| \leq \operatorname{rad}(\Gamma)$ the product $\alpha \cdot \beta$ is defined.

5.6. Example: the local fundamental pseudogroup. Denote by Ω the H-space of all piecewise smooth loops in V based at $v_0 \in V$ with the composition denoted by $\varphi \circ \psi$ for $\varphi, \psi \in \Omega$. Denote by Ω_ρ, $\rho > 0$ the set of loops of length less than or equal to ρ and by $\Gamma = \pi_\rho$ the set of all geodesic loops in Ω_ρ. We denote by $\|\gamma\|, \gamma \in \Gamma$, the length of γ. If $\rho^2 c^+(V) \leq 0.1$ we define for $\alpha, \beta \in \Gamma$ with $\alpha \circ \beta \in \Omega_\rho$ the product $\alpha\beta \in \Gamma$: $\alpha\beta$ is the shortest loop homotopic in Ω_ρ to $\alpha \circ \beta$. The pseudogroup Γ so defined is discrete (see 2.1) and radial, and if $\rho > 4d(V)$ then $\pi(\Gamma)$ is canonically isomorphic to $\pi_1(V, v_0)$; but it may be not injective (see 1.6).

Our major concern is the injectivity for the almost flat case. To prove that we shall later need the following two facts. For their proof note that a pseudogroup is trivially injective if it can be described as a pseudogroup of transformations of some set.

5.7. Let Γ be discrete and radial (we use the notation from 3.1).

(a) If subpseudogroup $\Delta \subset \Gamma$ is injective and δ-dense in Γ, then the ball $\Gamma_\rho \subset \Gamma$ (with the induced pseudogroup structure) is injective for $\rho \leq 0.1 \operatorname{rad}(\Gamma) - 10\delta$.

(b) Suppose N, $A \subset \Gamma$ are injective subpseudogroups, N is invariant ($\Gamma N \Gamma^{-1} \subset N$ when the product is defined), the map $(\nu, \alpha) \mapsto \nu \cdot \alpha \in \Gamma$, $\nu \in N$, $\alpha \in A$, is injective (where it is defined) and every $\gamma \in \Gamma_\rho \subset \Gamma$, $\rho \leq \operatorname{rad} \Gamma$, admits the decomposition $\gamma = \nu\alpha$, $\nu \in N$, $\alpha \in A$. Then the ball $\Gamma_{\rho_0} \subset \Gamma$ is injective for $\rho_0 \leq 0.1\, \rho$.

5.8. Nilpotency. We say that a set $A \subset \Gamma$ is nilpotent if in the sequence $A_0 = A$, $A_i = [A, A_{i-1}]$ all commutators are defined and there exists a number d such that $A_d = \{e\}$. A minimal such d is denoted by nil (A).

A system of generators $\gamma_1 \cdots \gamma_s \in \Gamma$ is called a nilpotent basis if all commutators $[\gamma_i, \gamma_j]$, $1 \leq i, j \leq s$, are defined and $[\gamma_i, \gamma_j] \in \Gamma_{i-1}$, where by Γ_i we denote the subseudogroup generated by $\gamma_1 \cdots \gamma_i$.

Let Γ be a discrete pseudogroup of radius R, and $A \subset \Gamma_\rho \subset \Gamma$ a symmetric set containing the identity element. If A has a nilpotent basis $\alpha_1, \cdots, \alpha_s \in A$, and $R \geq \rho\, ex_2(s)$, then nil $(A) \leq s$.

This is obvious.

6. Pseudogroups of motions

6.1. A map $h\colon \Gamma \to G$ from a discrete radial pseudogroup to a Lie group G (both with the norms $\|\ \|$) is called an ε-homomorphism if

$$h(e) = e\,, \qquad h(\gamma^{-1}) = (h(\gamma))^{-1};$$

if $\alpha\beta\gamma = e$, $\alpha, \beta, \gamma \in \Gamma$, then $\|h(\alpha)h(\beta)h(\gamma)\| \leq \varepsilon \|\alpha\| \|\beta\|$.

6.2. Let $r\colon \Gamma \to O(n)$ be an ε-homomorphism (about $O(n)$ see 4.1), and let $\rho_0, \rho_1, \theta, \mu$ be given numbers with $0 \leq \rho_0 < \rho_1 \leq \operatorname{rad} \Gamma$, $0 < \theta, \mu < 1$.

If $\rho_0\rho_1^{-1} \leq \mu^N$, $N \geq (10 + \theta^{-1})^{3k}$, $k = \dim O(n) = \frac{1}{2}n(n-1)$, and $\rho_1^2\varepsilon \leq 0.1\theta$, then there exists a ρ, $\rho_0 \leq \rho \leq \rho_1$, such that the inverse image $r^{-1}(B_\theta) \subset \Gamma$ of the ball $B_\theta \subset O(n)$ is δ-dense in $\Gamma_\rho \subset \Gamma$ with $\delta \leq \mu\rho$.

Proof. This follows from the possibility of covering $0(n)$ by N balls of the radius $\frac{1}{3}\theta$.

6.3. Let $r\colon \Gamma \to 0(n)$ be an ε-homomorphism with image in the ball $B_\theta \subset O(n)$, $\theta \leq \exp(-n)$. If $\rho \leq \operatorname{rad}(\Gamma)$ and $\varepsilon \leq 0.1\,(\theta\rho^{-2})$, then $\|r(\gamma)\| \leq 10\theta\rho^{-1} \|\gamma\|$, $\gamma \in \Gamma$.

Proof. If $\alpha, \alpha^2 \cdots \alpha^i \in B_\theta$, then $\|\alpha^i\| = i\|\alpha\|$. Given this, the inequality

$\|r(\gamma^i)\| \leq \theta$, with $i = \text{ent}\,(\rho/\|\gamma\|)$, yields the proof.

6.4. For an ε-homomorphism $m: \Gamma \to M(n)$ we set $t(\gamma) = \text{trans}\,(m(\gamma)) \in \boldsymbol{R}^n$ and $r(\gamma) = \text{rot}\,(m(\gamma)) \in 0(n)$, $\gamma \in \Gamma$. We suppose that $\|t(\gamma)\| = \|\gamma\|$.

6.5. Let m be as above, and let θ, ρ be positive numbers. Denote by $N \subset \Gamma_\rho \subset \Gamma$ the pseudogroup generated in Γ_ρ by $\Gamma_\rho \cap r^{-1}(B_\theta)$, $B_\theta \subset O(n)$. If $\theta + \rho \leq \exp(-ex_2 n)$, $\text{rad}\,\Gamma \geq \rho\; ex_3(d)$, $d = 10^k$, $k = \dim M(n) = \frac{1}{2}n(n+1)$, and $\varepsilon \leq 0.01$, then nil $(N) \leq d$.

Proof. In N take a short basis $\gamma_1, \cdots, \gamma_p \in N$ with respect to the function $\gamma \to \|m(\gamma)\|$. As in 2.4 we conclude that $p \leq d$; from 4.2 (a) it follows that this basis is nilpotent, and applying 5.8 we finish the proof.

6.6. Let m be an ε-homomorphism as in 6.4, let $\Gamma_\rho \subset \Gamma$, $\rho \leq 1$ be the ball with nil $(\Gamma_\rho) \leq d$, and let θ', δ', δ, $\theta \geq 0$ be real numbers with $ex_3(n + d + \theta^{-1}) \leq (\varepsilon + \theta' + (\delta + (\delta + \delta')/\rho))^{-1}$. If the set $r^{-1}(B_{\theta'}) \subset \Gamma$ is δ'-dense in Γ_ρ, and the image of $t: \Gamma \to \boldsymbol{R}^n$ is δ-dense in the ball $B_\rho \subset \boldsymbol{R}^n$, then $\|r(\gamma)\| \leq \theta$, $\gamma \in \Gamma_\rho$.

Proof. Take $x \in E_{\max}(r(\gamma))$ (see 4.1), $\gamma \in \Gamma_\rho$, with $\|x\| = \frac{1}{2}\rho$ and $\alpha \in r^{-1}(B_{\theta'})$ with $\|t(\alpha) - x\| \leq \delta + \delta' + 2\varepsilon$. Consider $\alpha_1 = [\alpha, \gamma], \cdots, \alpha_i = [\alpha_{i-1}, \gamma], \cdots$. If $\|r(\gamma)\| > \theta$, then using 4.2 (*b*) we conclude: $\|\alpha_i\| \geq h^{-i}(\theta/2)^i\,\|\alpha\|$, $i = 1, \cdots, d$, but the condition nil $(\Gamma_\rho) \leq d$ yields $\|\alpha_d\| = \|e\| = 0$, and the contradiction proves the lemma.

6.7. A discrete set $\Gamma \subset \boldsymbol{R}^n$ equipped with a pseudogroup structure is called an ε-lattice of radius $R = R(\Gamma) = \max_{\gamma \in \Gamma} \|\gamma\|$ if the origin in $\boldsymbol{R}^n$ serves as the identity element in Γ, the product $\alpha\beta$ is defined for $\alpha, \beta \in \Gamma$ with $\|\alpha\| + \|\beta\| \leq \frac{1}{2}R$, and $\|\alpha\beta - \alpha - \beta\| \leq \varepsilon\|\alpha\|\,\|\beta\|$. Here $\|\ \|$ means the norm in $\boldsymbol{R}^n$ but as a function on Γ it may not satisfy the conditions in 5.5, and we do not suppose that Γ (as a pseudogroup) has any norm at all. Notice also that $\Gamma \subset \boldsymbol{R}^n$ is not necessarily symmetric: $\gamma^{-1} \neq -\gamma$.

Example. Let $m: \Gamma \to M(n)$ be an ε-homomorphism as in 6.4 with $\|r(\gamma)\| \leq \nu\,\|\gamma\|$, $\gamma \in \Gamma$, and let the map $t: \Gamma \to \boldsymbol{R}^n$ be injective. Then its image is an ε-lattice with $\varepsilon' \leq (\varepsilon + \nu)\exp(n + 10)$.

6.8. For an ε-lattice $\Gamma \subset \boldsymbol{R}^n$ we call the system of generators $\gamma_1, \cdots, \gamma_k \in \Gamma$ a normal basis if the following conditions are satisfied:

1. If the commutator $[\gamma, \gamma_i]$, $\gamma \in \Gamma$, $i = 1, \cdots, k$, is defined, then $[\gamma, \gamma_i] \in \Gamma_{i-1}$, where Γ_i is the subpseudogroup generated by $\gamma_1, \cdots, \gamma_i$.
2. If $\|\gamma\| \leq \exp(-ex_2(n))\,R(\Gamma)$, then there exists a unique representation $\gamma = \gamma_1^{m_1}\gamma_2^{m_2} \cdots \gamma_k^{m_k}$.
3. The system of vectors $\gamma_1, \cdots, \gamma_k$ is regular (see 4.6).

6.9. Consider an ε-lattice $\Gamma \subset \boldsymbol{R}^n$ with a normal basis $\gamma_1, \cdots, \gamma_n \in \Gamma$. For $\gamma \in \Gamma$ represented as $\gamma = \gamma_1^{m_1} \cdots \gamma_n^{m_n}$ denote the sum $\gamma = \sum_{i=1}^n m_i\gamma_i$ by $\lambda = \lambda(\gamma)$. A simple calculation shows

$$q^{-1}\|\lambda\| \leq \|\gamma\| \leq q\,\|\lambda\|\,, \text{ with } 1 \leq q \leq 1 + \tau, \tau \geq 0\,,$$
$$ex_5(n + \tau^{-1}) \geq (\varepsilon\,\|\gamma\|)^{-1}\,.$$

If the commutator $[\alpha, \beta] \in \Gamma$ and $\lambda(\alpha)$, $\lambda(\beta)$, $\lambda([\alpha, \beta])$, $\alpha, \beta \in \Gamma$, are defined, then $\|\lambda([\alpha, \beta])\| \leq \varepsilon' \|\lambda(\alpha)\| \|\lambda(\beta)\|$, where $ex_7(n + (\varepsilon')^{-1}) \geq \varepsilon^{-1}$.

6.10. Let $\Gamma \subset \boldsymbol{R}^n$ be an ε-lattice of radius R. If Γ is δ-dense in the ball $B_R \subset \boldsymbol{R}^n$ and $(\varepsilon R + \delta R^{-1})^{-1} \geq ex_6(n)$, then there exists a normal basis $\gamma_1, \cdots, \gamma_n$ in Γ.

Proof. Take a nontrivial $\gamma_1 \in \Gamma \subset \boldsymbol{R}^n$ with minimal norm, and consider $\boldsymbol{R}^{n-1} \subset \boldsymbol{R}^n$ orthogonal to γ_1. Obviously (compare with 6.5) γ_1 belongs to the "center" of Γ. For $\gamma \in \Gamma$ with $\|\gamma\| \leq \frac{1}{3}R$ consider the trajectory $\{\gamma_1^i\gamma\}$, $i = \cdots, -1, 0, 1, \cdots$, as far as it is defined, and take $\tilde{\gamma} \in \{\gamma_1^i\gamma\}$ with the properties: $\langle\tilde{\gamma}, \gamma_1\rangle \geq 0$, $\langle\gamma_1^{-1}\tilde{\gamma}, \gamma_1\rangle < 0$. Such a $\tilde{\gamma}$ exists and it is unique. Denote by $\gamma' \subset \boldsymbol{R}^{n-1}$ the orthogonal projection of $\tilde{\gamma}$ to $\boldsymbol{R}^{n-1}$, and by $\Gamma' \subset \boldsymbol{R}^{n-1}$ the set of all such $\gamma' \in \boldsymbol{R}^{n-1}$. Setting $\gamma_1'\gamma_2' = (\tilde{\gamma}_1\beta_2)'$ we equip Γ' with a pseudogroup structure. It is easy to see that Γ' is ε'-pseudogroup of radius R' where $\varepsilon' \leq 20\varepsilon$, $R' \geq \frac{1}{4}R$.

Now, by induction having constructed the normal basis $\gamma_2', \cdots, \gamma_n' \in \Gamma'$, we take $\gamma_1, \tilde{\gamma}_2, \cdots, \tilde{\gamma}_n$ for the normal basis in Γ, and verfy the properties 1–3 in 6.8 again by an obvious induction.

6.11. Consider an ε-homomorphism $m: \Gamma \to M(n)$ as in 6.4. If $\varepsilon^{-1} \geq ex_2(n + 1)$, rad $\Gamma \geq 10$, then the restriction of $t: \Gamma \to \boldsymbol{R}^n$ to the unit ball $\Gamma_{\rho=1} = \Gamma_1 \subset \Gamma$ is injective, and we identify Γ_1 with the image of that restriction $t: \Gamma_1 \to \boldsymbol{R}^n$.

Let $\Gamma_1 \subset \boldsymbol{R}^n$ be δ-dense in the unit ball $B_1 \subset \boldsymbol{R}^n$ where $(\delta + \varepsilon)^{-1} \geq ex_{80}(n)$. Then there exists a subpseudogroup $N_1 \subset \Gamma_1$ with the following properties:

1. N_1 is δ'-dense in B, with $\delta' \leq ex_4(n)\delta$.
2. If $\gamma \in N$, then $\|r(\gamma)\| \leq \nu \|\gamma\|$ where $\exp_7(n + \nu^{-1}) = (\varepsilon + \delta)^{-1}$.
3. If $\|r(\gamma)\| \leq \exp(-ex_4(n))$, $\gamma \in \Gamma_1$, then $\gamma \in N$. (Notice that $\theta > \nu$.)
4. Both pseudogroups Γ_1 and N_1 are injective; the group $\pi(N_1) \subset \pi(\Gamma_1)$ is a maximal nilpotent subgroup and the maximal invariant nilpotent subgroup at the same time; $\pi(N_1)$ has no torsion, rank $(\pi(N_1)) = n$ and ord $(\pi(\Gamma_1)/\pi(N_1)) \leq ex_3(n)$.

Proof. Take the ball $\Gamma_\rho \subset \Gamma$ with $\rho = \exp(-ex_{40}(n))$, and generate N_1 by the intersection $\Gamma_\rho \cap r^{-1}(B_\theta)$, $B_\theta \in O(n)$, $\theta = \exp(-ex_4(n))$.

From 6.2 it follows that N_1 is δ''-dense in Γ_1 with $\delta'' = \exp(-ex_{20}(n))$, and properties 2 and 3 for $\gamma \in \Gamma_\rho$ follow from 6.3, 6.5, 6.6. Property 2 shows that $\Gamma_\rho \subset \boldsymbol{R}^n$ is an ε'-lattice, and ε' is small enough to apply 6.10 (see the example in 6.7) and to construct a normal basis in N_ρ. The existence of the normal basis, together with 6.6, 5.7 and properties 2, 3, yields property 4 with the exception of the last inequality, but that inequality is reduced now to the following obvious fact:

If a maximal nilpotent subgroup $N \subset \pi$ is invariant and has no torsion, rank $(N) = n$ and the group $G = \pi/N$ is finite, then ord $(G) \leq ex_3(n)$.

Noticing that $\pi(\Gamma_\rho) = \pi(\Gamma_1)$ and $\pi(N_\rho) = \pi(N_1)$ we extend all properties of Γ_ρ to Γ_1, again using 6.6. Notice in the end that the inequality ord $(\pi(\Gamma_1)/\pi(N))$ $\leq ex_3(n)$ yields property 1 with $\delta' < \delta''$.

7. The proof of the main theorem

7.1. We return now to the manifold V with a fixed point $v_0 \in V$ (see 5.6). We identify the tangent space of V at v_0 with $\boldsymbol{R}^n$, and denote the linear and the affine holonomy maps by $r\colon \Omega \to O(n)$ and $m\colon \Omega \to M(n)$ respectively

Consider a contractable loop $w \in \Omega$, $w\colon [0, 1] \to V$ and a deformation $w_t\colon [0, 1] \to V$, with $w_t \in \Omega$, $t \in [0, 1]$, $w_{t=0} = w$ and $w_{t=1}$ the constant map. The family w_t can be viewed as a map of a 2-dimensional disk to V. Denote by S the area of that map and denote by L the maximum of the lengths of w_t, $t \in [0, 1]$.

7.2. From $|R(x, y)z| \leq 2 \cdot c(V) \cdot |x \wedge y| \cdot |z|$ for the curvature tensor and assuming $c(V) \leq \varepsilon$ we have

$$\|r(w)\| \leq 2 \cdot \varepsilon \cdot S ,$$
$$\|m(w)\| \leq L \cdot (e^{2\varepsilon S} - 1) + 2\varepsilon S .$$

Together with simple comparison arguments (see [3]) it yields:

7.3. If $c(V) \leq 10^{-10}\varepsilon$, $0 \leq \varepsilon \leq 1$, then the restrictions of the maps r and m to the local fundamental pseudogroup $\Gamma = \pi_\rho$, $\rho \leq 10$ (see 5.6) are ε-homomorphisms, m enjoys the properties from 6.4, and the image of $t\colon \Gamma \to \boldsymbol{R}^n$ is δ-dense in $B_\rho \subset \boldsymbol{R}^n$ with $\delta \leq 2d(V)$.

7.4. Now everything is ready for the proof of 1.4. We can suppose that $(d(V) + c(V))^{-1} \geq ex_{199}(n)$, and can apply 6.11 to $\Gamma = \pi_\rho$ because of 7.3. This gives (a) and (b) of 1.4.

Take N_1 as in 6.11, and realize $\pi(N_1)$ as a uniform discrete subgroup in a nilpotent Lie group L. Take in $N_1 \subset \Gamma_1 \subset \boldsymbol{R}^n$ (see 6.11) (viewed as an ε-lattice) a normal basis $\gamma_1, \cdots, \gamma_n$, and identify $\pi(N_1)$ with the lattice $\Lambda \subset \boldsymbol{R}^n$ spanned by $\gamma_1, \cdots, \gamma_n$, matching $\gamma = \gamma_1^{m_1} \cdots \gamma_n^{m_n}$ to $\lambda = \sum_{i=1}^n m_i\gamma_i$.

Now equip L with the metric associated with that basis (see 4.7), and consider the map f from $N = \pi(N_1) \subset L$ to the universal covering $(\tilde{V}, \tilde{v}_0)$ of (V, v_0), given by $f(\gamma) = \gamma(\tilde{v}_0)$. ($N$ lies in $\pi_1(V, v_0)$ and so acts in $\tilde{V}$.) Applying 4.8 and 6.9 we conclude that f is an R-restricted q-isometry satisfying all properties of 3.3 (L corresponds to X_1 in 3.3, $\tilde{V}$ to X_2, N to Δ_1, and $\operatorname{Im}(f)$ to Δ_2), and applying 3.3, 3.4 we construct the diffeomorphism $F\colon L \to \tilde{V}$ commuting with the action of N and so inducing the diffeomorphism of L/N to $\tilde{V}/N$.

8. Appendix: The proof of the Margulis theorem

8.1. The Margulis lemma follows (up to ex_i-nonsense) from 7.3, 6.2, and 6.5. To prove the theorem we need two obvious facts about $\pi = \pi(V, v_0)$ for $c^+(V) < 0$.

8.2. A. Every nilpotent subgroup of π is cyclic.

B. For every cyclic subgroup $N \subset \pi$ there exists an $\alpha \in \pi$ such that $\|\alpha\nu\alpha^{-1}\| \geq 1$, $\nu \in N$, (about $\|\ \|$ see 2.2).

8.3. Now take the shortest $\gamma \in \pi_1(V, v_0)$. If $\|\gamma\|^{-1} \leq ex_2(n)$, then the injectivity radius at $v_0 \in V$ satisfies $\mathrm{Rad}\,(V, v_0) \geq \frac{1}{2}(ex_2(n))^{-1}$. This yields the Margulis theorem. Otherwise we take the maximal cyclic subgroup $N \subset \pi$ with $\gamma \in N$ and $\alpha \in \pi$ as in 8.2B. Realize α by a loop: w: $[0, 1] \to V$, and for $\nu \in N$ denote by ν_t, $t \in [0, 1]$, the shortest loop at the point $w(t) \in V$ homotopic to the loop $w^{-1}_{|[0,t]} \circ \tilde{\nu} \circ w_{|[0,t]}$, where $w_{[0,t]}$: $[0, t] \to V$ is the restriction of w and $\tilde{\nu}$ is the geodesic loop at v_0 realizing ν. By continuity there is a $t_0 \in [0, 1]$ such that $\min_{\nu \in N} (\|\nu_{t_0}\|) = (ex_2(n))^{-1}$. Using the Margulis lemma and 8.2A we conclude that at the point $w(t_0) \in V$ the length of any geodesic loop is at least $(ex_2(n))^{-1}$, and the proof is finished.

Those arguments (up to minor details) are due to Margulis, and for the homogenous case to Kazdan and Margulis (see [9]).

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REPORT ON M. GROMOV'S ALMOST FLAT MANIFOLDS (*)

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1. Introduction

A basic theme in Riemannian geometry is the following question : To what extend do assumptions on local invariants determine global properties ? Very important such assumptions are bounds for the curvature of the metric - recall that in Riemann's normal coordinates the curvature tensor is obtained as the second derivative of the metric. Examples of known results are :

(i) The only surfaces which carry positive curvature metrics are S^2 and $P^2(\mathbb{R})$, because $2\pi . \chi(M) = \int_M K d\theta$.

(ii) A complete simply connected Riemannian manifold M^n of nonpositive curvature is diffeomorphic to $\mathbb{R}^n$, because the Riemannian exponential map $\exp_p$ has maximal rank on the tangent space T_pM and is in fact a covering map.

(iii) More specifically, if M^n has zero curvature ("flat") then $\exp_p$ is an isometric covering map, i.e. the fundamental group $\pi_1(M,p)$ operates as a discrete - and for compact M : uniform - group of isometries on $\mathbb{R}^n$. From Bieberbach's classification of such groups it follows that compact flat manifolds are covered by flat tori.

(iv) If M^n is complete, noncompact and has positive curvature then convexity arguments show that M^n is diffeomorphic to $\mathbb{R}^n$.

(v) If M^n is simply connected, complete and has curvature bounds $\frac{1}{4} < K \leq 1$ then M^n is homeomorphic to S^n . For even dimensions ≥ 4 the result is sharp since $P^n(\mathbb{C})$ carries a metric with $\frac{1}{4} \leq K \leq 1$.

(vi) If M^n is complete and has curvature bounds $0.7 \leq K \leq 1$ then the following holds : The universal covering $\widetilde{M}^n$ is diffeomorphic to S^n in such a way that the action of $\pi_1(M,p)$ on $\widetilde{M}$ is conjugate to an orthogonal action on S^n , i.e. M^n is diffeomorphic to a space of constant curvature.

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(vii) In principle similar results hold if the model space S^n is replaced by any of the other symmetric spaces of compact type, but the precise formulation is more elaborate.

The purpose of this lecture is to explain the proof of the following theorem of M. Gromov [6] which differs from all the previous results by the fact that the model space is not known a priori but has to be constructed in the proof. ([6] is a general reference throughout the paper.)

1.1. THEOREM.- Let M be a compact n-dimensional Riemannian manifold, assume that the sectional curvatures K of M are bounded in terms of the diameter $d(M)$:

$$|K| \leq \varepsilon \cdot d(M)^{-2} \quad \text{with} \quad \varepsilon \leq \exp(-\exp(\exp(2n^2))) \text{ (present estimate).}$$

Then there is a finite - at most $2 \cdot (6\pi)^{\frac{1}{2}n(n-1)}$ fold - covering of M which is diffeomorphic to a compact quotient of a nilpotent Lie group.

There are many more manifolds than the compact flat ones which allow for every $\varepsilon > 0$ an ε-flat metric, i.e. one which satisfies $|K| \leq \varepsilon \cdot d(M)^{-2}$.

1.2. Example.- On the nilpotent Lie algebra $\mathfrak{g} = \left\{ A = \begin{pmatrix} 0 & \cdot & & a_{ij} \\ & & \cdot & \\ & & & \cdot \\ 0 & & & 0 \end{pmatrix} \right.$; $a_{ij} \in \mathbb{R}$, $\left. 1 \leq i < j \leq n \right\}$ define the following family of scalar products :

$$\|A\|_q^2 = \sum_{i<j} a_{ij}^2 \cdot q^{2(j-i)}$$

and extend them by left translation to the corresponding nilpotent Lie group G of upper triagular matrices. From the estimate $\|[A,B]\|_q \leq 2 \cdot (n-2) \cdot \|A\|_q \cdot \|B\|_q$ one derives the following q-independant bound for the curvature tensors R_q of these left invariant metrics

$$\|R_q(A,B)\, C\|_q \leq 24(n-2)^2 \cdot \|A\|_q \cdot \|B\|_q \cdot \|C\|_q ,$$

or $\|R_q\|_q \leq 24 \cdot (n-2)^2$.

Each compact quotient $\Gamma\backslash G$ can be given an arbitrarily small diameter by appropriate choice of q ; therefore $\Gamma\backslash G$ is ε-flat for each $\varepsilon > 0$. If one takes for Γ the integer subgroup of G , then Γ is not a Bieberbach group since the rank of its free Abelian subgroups is too small and therefore $\Gamma\backslash G$ does not carry any flat metric.

1.3. The first steps of Gromov's proof. Because of the strong curvature assumptions the maximal rank radius r_m of the Riemannian exponential map is much larger than the diameter of M . Therefore many short geodesic loops exist and Gromov defines a product between short loops at p which satisfies the relations of a group where it is defined. From this torso one can generate the fundamental group $\pi_1(M,p)$ abstractly : by generators and relations. Each short loop at p is mapped onto its holonomy motion and this map is almost compatible with the Gromov product since small curvature implies that parallel translation varies only slightly with the change of the path. Therefore commutators of loops almost behave as commutators of motions, i.e. iterated commutators converge to the identity if the rotational part of the corresponding holonomy motion is small ($\le \frac{1}{3}$). Every set consisting of loops with rotational parts $\le \frac{1}{3}$ will therefore generate a nilpotent subgroup of $\pi_1(M,p)$ if the homotopy errors are not too large. Moreover the degree of nilpotency of all such subgroups has the a priori bound $d = (\frac{40}{13})^{\frac{1}{2}n(n+1)}$ which is derived by a counting argument in the group of motions. - We continue this summary in 2.15 after the more detailled explanations of chapter 2 have been given.

2. Products of short loops

From Riemannian geometry we have

2.1. Rauch's THEOREM [5].- Curvature bounds $-\lambda^2 \le K \le \Lambda^2$ imply for the Riemannian exponential map exp at p (for $v, w \in T_pM$)

$$|w| \cdot \frac{\sin \Lambda|tv|}{\Lambda|tv|} \le |(d\exp)_{tv} \cdot w| \le |w| \cdot \frac{\sinh \lambda|tv|}{\lambda|tv|} ,$$

$(d\exp)_{tv}$ has maximal rank if $|tv| < \pi \cdot \Lambda^{-1}$ $(\le \pi \cdot \varepsilon^{-1/2} \cdot d(M)$ in 1.1).

2.2 Klingenberg's Long-Homotopy-lemma [5].- Let r_m be the maximal rank radius of $\exp_p$; assume $\exp_p v = \exp_p w$. Then any homotopy which joins the geodesic arcs $\exp tv$ and $\exp tw$ $(0 \le t \le 1)$ contains a curve of length $\ge r_m$.

2.3. DEFINITION.- A homotopy which contains only curves shorter than the maximal rank radius r_m of the exponential map is called a <u>short homotopy</u>. The corresponding equivalence classes are called <u>short homotopy classes</u>.

From 2.2 and the standard shortening process by geodesic segments we have

2.4. Every short homotopy class of closed curves at p contains <u>exactly one</u> geodesic loop at p.

2.5. DEFINITION.- Let α and β be geodesic loops at p ; assume that the sum of their lengths is less than the maximal rank radius r_m , e.g. $|\alpha| + |\beta| < \pi \cdot \varepsilon^{-1/2} \cdot d(M)$. Let $\beta \cdot \alpha$ be the closed curve "first α then β ", as usual. <u>Gromov's product</u> $\beta * \alpha$ is the unique (!) geodesic loop in the <u>short</u> homotopy class of $\beta \cdot \alpha$.

If one lifts the curve $\beta \cdot \alpha$ to T_pM by $\exp_p$, then the ray to the endpoint of this curve is mapped by $\exp_p$ onto the loop $\beta * \alpha$. Clearly α^{-1} is the loop α parametrized backwards and associativity holds as long as the sum of the lengths of the factors is $< r_m (\geq \pi \ \varepsilon^{-1/2} \ d(M))$. Every closed curve can be decomposed (in $\pi_1(M,p)$) into a product of curves shorter than $2d(M) + \eta$ ($\eta > 0$ chosen) ; therefore $\pi_1(M,p)$ is generated by geodesic loops $\leq 2d(M) + \eta$. Under the mild additional condition $5 \leq \pi \cdot \varepsilon^{-1/2}$ it can already be proved that all relations in $\pi_1(M,p)$ are products of relations which are given by <u>short</u> homotopies between loops of length $< 5 \cdot d(M)$. Therefore the short loops $(< 5 \cdot d(M))$ with Gromov product generate a group isomorphic to $\pi_1(M,p)$.

2.6. DEFINITION.- Let c be a curve and let a vectorfield X along c satisfy the differential equation $\frac{D}{dt} X(t) = \dot{c}(t)$. The map $m(c) : T_{c(0)} \longrightarrow T_{c(1)}M$ given by $X(0) \longrightarrow X(1)$ is called <u>affine translation</u> [10] along c . $m(c)$ is a motion, since its linear part is Levi-Civita translation along c .

2.7. <u>Path dependence of translations</u> [2]. Let c_1 , c_2 be two curves from $c_i(0) = p$ to $c_i(1) = q$; assume the existence of a smooth homotopy from c_1 to c_2 with area $\leq F$ and longest curve $\leq L$. Let $X_i(t)$ be Levi-Civita parallel along c_i and $X_1(0) = X_2(0)$; let $Y_i(t)$ be affine parallel along c_i with $Y_i(0) = 0$. Let $\|R\|$ be a bound for the curvature tensor along the homotopy. Then

$$|\sphericalangle (X_1(1), X_2(1))| \leq \|R\| \cdot F ,$$

$$|Y_i(1)| \leq \text{length } (c_i) , \quad |Y_1(1) - Y_2(1)| \leq L \cdot \|R\| \cdot F .$$

Our most important application of 2.7 is to homotopies which are given by geodesic segments spanned in geodesic triangles. L is the sum of two edgelengths and F is obtained from

2.8. Aleksandrow's area comparison [1]. Consider a geodesic triangle and span any surface with geodesic segments. Assume a curvature bound $K \leq \Lambda^2$ along the surface. Consider a triangle with the same edgelengths as the given one in the plane of constant curvature Λ^2 (if $\Lambda^2 > 0$ this requires a circumference $< 2\pi\Lambda^{-1}$). Then the area of the spanned surface is not larger than the area of this constant curvature triangle. In particular, if two edgelengths, a , b are $\leq \pi \cdot (3\Lambda)^{-1}$ then $F \leq 0.7ab$ ($\leq 0.5ab$ if $\Lambda = 0$).

To conveniently express how closely the Gromov product $\beta * \alpha$ and the composition of the holonomy motions $m(\beta) \circ m(\alpha)$ are related we use the following Finsler metrics :

2.9. DEFINITION.- For A , $B \in SO(n)$ define $d(A,B) = \max\{ |\sphericalangle (B^{-1}AX,X)| ; 0 \neq X \in R^n \}$; the corresponding norm in the tangent space $T_{id}SO(n)$ of skew symmetric matrices is $|S| = \max\{ |SX| ; X \in R^n , |X| = 1 \}$. For motions $\widetilde{A}_i(X) = A_i \cdot X + a_i$ define $\widetilde{d}(\widetilde{A}_1 , \widetilde{A}_2) = \max(d(A_1 , A_2) , 3\Lambda \cdot |a_1 - a_2|)$. (Λ^2 should be thought of as a curvature bound ; the factor 3Λ makes the definition independent of renormalizations of the metric of M ; it is also convenient in 2.12.) Abbreviate $d(A , id) = \|A\|$; $\widetilde{d}(\widetilde{A} , id) = \|\widetilde{A}\|$.

We summarize 2.5 - 2.9 (note $|K| \leq \Lambda^2 \Longrightarrow \|R\| \leq \frac{4}{3} \Lambda^2$) :

2.10. Homotopy errors. Let α , β be geodesic loops with $\beta * \alpha$ defined. Let $r(\alpha)$ and $t(\alpha)$ be rotational and translational part of the holonomy motion 2.6. Assume curvature bounds $|K| \leq \Lambda^2$. Then

$$d(r(\beta * \alpha) , r(\beta) \circ r(\alpha)) \leq \Lambda^2 |t(\alpha)| \cdot |t(\beta)| ,$$

$$|r(\beta) \cdot t(\alpha) + t(\beta) - t(\beta * \alpha)| \leq (|t(\alpha)| + |t(\beta)|) \cdot \Lambda^2 |t(\alpha)| \cdot |t(\beta)| .$$

For commutators better estimates are true than follow from 2.10. One needs

2.11. Comparison of Riemannian and Euclidean translation [9]. Let $w(t)$ be a parallel vector field along the geodesic $c(t) = \exp tv$. Assume $|K| \leq \Lambda^2$. Then

$$d(\exp(v+w(0)),\exp_{c(1)} w(1)) \le \frac{1}{3}\Lambda(|v||w| \sinh \Lambda(|v| + |w|) .$$

First the translational part of the commutator $[\beta,\alpha] = \beta^{-1} * \alpha^{-1} * \beta * \alpha$ is estimated directly with 2.1 and 2.11 ; then this information is used to get a good bound on the homotopy error of the rotational part from 2.7 and 2.8. Gromov does not seem to use 2.11.

2.12. Commutator estimates [2]. Let α , β be short geodesic loops (2.5) at p and assume $|K| \le \Lambda^2$. Then

$$|t([\beta,\alpha])| \le \frac{2}{3}|t(\alpha)||t(\beta)|\cdot \Lambda \sinh \Lambda(|t(\alpha)| + |t(\beta)|) + 2\sin(\tfrac{1}{2}\|r(\beta)\|)\cdot|t(\alpha)| +$$
$$+ 2\sin(\tfrac{1}{2}\|r(\alpha)\|)\cdot|t(\beta)| ,$$

$$d(r([\beta,\alpha]), r(\beta)^{-1}\cdot r(\alpha)^{-1}\cdot r(\beta)\cdot r(\alpha)) \le \Lambda^2(2|t(\alpha)||t(\beta)| + (|t(\alpha)| + |t(\beta)|)\cdot|t[\beta,\alpha]|).$$

Assume in addition $|m(\alpha)|$, $|m(\beta)| \le \frac{1}{3}$ (hence $|t(\alpha)|$, $|t(\beta)| \le (9\Lambda)^{-1}$ (2.9)), then $\|m([\beta,\alpha])\| \le 2.4\|m(\alpha)\|\cdot\|m(\beta)\| \le 0.8 \min(\|m(\alpha)\| , \|m(\beta)\|)$.

This result is very powerful. It shows that - after handling the homotopy errors - one can work with commutators of loops almost in the same way as with commutators of motions (we recall $\|[\widetilde{A}, \widetilde{B}]\| \le 2\|\widetilde{A}\|\cdot\|\widetilde{B}\|$). This use of commutators seems to go back to Margulis who derived from 2.12-type estimates a lower bound for the volume of a compact negatively curved Riemannian manifold. Gromov uses 2.12 to generate nilpotent subgroups of the fundamental group. Very surprisingly the following holds :

2.13. A priori estimate [2]. The degree of nilpotency of all subgroups of $\pi_1(M,p)$ which are generated from sets of loops which satisfy $\|m(\alpha)\| \le \frac{1}{3}$ has a bound

$$d \le \left(\frac{40}{13}\right)^{\frac{1}{2}n(n+1)} \le 1.76^{n(n+1)} .$$

Proof. Choose economic generators as follows : α_1 is such that $\|m(\alpha_1)\|$ is minimal (in the generating set U). If $\alpha_1,\dots,\alpha_j$ are already chosen, then consider the set U_j of Gromov-products of these and choose α_{j+1} in $U \setminus U_j$ such that $\|m(\alpha_{j+1})\|$ is minimal. After finitely many steps one has a so called short basis $\alpha_1,\dots,\alpha_k$ for U . Because of 2.12 one can show by induction that the degree of nilpotency of the generated group $\langle\alpha_1,\dots,\alpha_k\rangle$ cannot be larger than k . From the construction follows

$$\|m(\alpha_i^{-1} * \alpha_j)\| \ge \max(\|m(\alpha_i)\| , \|m(\alpha_j)\|) ,$$

and with 2.10

$$\|m(\alpha_i)^{-1} \circ m(\alpha_j)\| \geq \max(\|m(\alpha_i)\|, \|m(\alpha_j)\|) - \frac{1}{9}\|m(\alpha_i)\| \cdot \|m(\alpha_j)\| \geq$$
$$\geq \max(\|m(\alpha_i)\| - \frac{1}{27}\|m(\alpha_j)\|, \|m(\alpha_j)\| - \frac{1}{27}\|m(\alpha_i)\|).$$

There are at most as many motions which pairwise satisfy these inequalities as there are unit vectors (Finsler length) in the tangent space of this group which satisfy $|w_i - w_j| \geq \frac{26}{27}$. The balls of radius $\frac{13}{27}$ around such w_i are disjoint and contained in a ball of radius $\frac{40}{27}$. The volume ratio $\left(\frac{40}{13}\right)^{\frac{1}{2}n(n+1)}$ of the balls gives an upper bound for the number of vectors w_i.

2.14. We have formulated 2.13 for the generated group. It is important to observe, that the inductive proof in fact shows : if d is the length of a short basis, and if a d-fold commutator of loops is defined in the sense of 2.5, then this d-fold commutator is already 0 as a loop (while 2.13 only says that this loop is 0 in $\pi_1(M)$).

2.15. The next steps of Gromov's proof. We have constructed nilpotent subgroups of $\pi_1(M)$; next, one has to find one such subgroup which can be embedded as a uniform discrete subgroup Γ into an n-dimensional nilpotent Lie group G. Observe that such a Lie group can be identified with $\mathbb{R}^n$ such that the product is given by Malcev's polynomials [11] of degree $\leq n$. These polynomials are uniquely determined if one knows their values on sufficiently many points of an uniform discrete subgroup of G. Gromov shows that a selected set of short loops, called Γ_{ρ_1}, can be found and (in 3.4) be identified with so large a ball of an integer lattice in $\mathbb{R}^n$ that the products of these loops determine Malcev polynomials [11] which define a product on $\mathbb{R}^n$ turning it into a nilpotent Lie group G. The mentioned set Γ_{ρ_1} of loops is such that the Gromov product behaves almost as the translational parts of the loops do (3.2.5). Therefore one can choose a basis in the same way as in a translational group and express the short loops in Γ_{ρ_1} as words in the basis elements ; these words allow the identification of the short loops with the lattice points of a large ball, even in such a way that loop length and lattice length almost coincide (3.4.2). - The set Γ_{ρ_1} of loops is constructed in 3.2 ; this construction requires curvature assumptions (see 3.2.3) which are so strong that homotopy errors at all other parts of the proof turn out to be almost neglegible.

3. Small rotational parts

3.1. A Dirichlet choice. We have to find a radius ρ_o with the following properties : for every $v \in T_pM$, $|v| = 3\rho_o$, one has a loop α with $|t(\alpha) - v| \le \rho_o + d(M)$ and $\|r(\alpha)\| \le \eta_1 = (2.6\pi)^{-d}$ (recall $d = 1.76^{n(n+1)}$ from 2.13).

The smallness of η_1 is explained in 3.2. To estimate the index of the constructed subgroup in $\pi_1(M)$ one needs $\rho_o \ge 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ (see 3.3).
One can find $\rho_o \le 4^N \cdot 2(6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ with $N \le \exp(\exp(n^2))$.

Proof. First, a lifting argument shows that the translational parts of loops at p are $d(M)$-dense within the ball of radius r_m $(\ge \pi \varepsilon^{-1/2} d(M))$ in which $\exp_p$ has maximal rank. However the nearest loop to a given $v \in T_pM$ need not have small rotational part, but it suffices if its rotational part occurs η_1-almost among loops of length $\le \rho_o$. (Homotopy errors are neglected since they cause a neglegible contribution.) Let $B_{\frac{1}{2}\eta_1}$ be a (Finsler-) ball of radius $\frac{1}{2}\eta_1$ in $SO(n)$; there are at most $N = \text{vol } O(n) \big/ \text{vol } B_{\frac{1}{2}\eta_1} \le 2\left(\frac{2\pi}{\eta_1}\right)^{\dim SO(n)} < \exp(\exp n^2)$ rotations in $O(n)$ with pairwise distance $\ge \eta_1$. Therefore, if $\rho_{o,1} = 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ does not have the desired property, one tries $\rho_{o,2} = 4 \cdot \rho_{o,1}$; after at most N such 4-fold increases one must have found a suitable ρ_o , since it cannot be true at each step that one finds a rotational part for a loop of length between $2\rho_o$ and $4\rho_o$ which does not η_1-almost occur among the loops $\le \rho_o$.

3.2. The almost translational set of loops. Consider the set Γ_{ρ_1} of loops with lengths $\le \rho_1 = e^{3n^2} \rho_o$ and rotational parts $\le \frac{1}{3}$. (The large ratio $\frac{\rho_1}{\rho_2}$ is needed in 4.1 to have sufficiently many products available to determine the Malcev polynomials.) Under the curvature assumption $3\Lambda\rho_1 \le \frac{1}{3}$ we have 2.14 for Γ_{ρ_1} , i.e. a short basis of length $\le d = 1.76^{n(n+1)}$ so that all d-fold commutators vanish. Let $\bar{\Gamma}_{\rho_1}$ be the set of all Gromov products of elements in Γ_{ρ_1} such that the products are inductively defined and have lengths $\leq \rho_1$. We claim :

3.2.1. All rotational parts in $\bar{\Gamma}_{\rho_1}$ are in fact $\leq 2^{-d}$; in particular $\bar{\Gamma}_{\rho_1} = \Gamma_{\rho_1}$.

<u>Proof</u>. Let $\delta \in \bar{\Gamma}_{\rho_1}$ be a loop with $\|r(\delta)\| = \theta > 2^{-d}$. Because of the inductive definition of $\bar{\Gamma}_{\rho_1}$ it is sufficient to assume $\theta \leq \frac{1}{3}$. We choose a vector $v \in T_pM$ with $|\sphericalangle(r(\delta)\cdot v, v)| = \theta$ and with 3.1 find a loop α such that $\|r(\alpha)\| \leq \eta_1$ and $|t(\alpha) - v| \leq \rho_o$. Consider the d-fold commutator $[\ldots[\alpha,\delta],\ldots,\delta]$; 2.14 shows that it is trivial ; on the other hand we can estimate its translational part directly and after some computation find it $\neq 0$ if the following is true :

3.2.2. $$\eta_1 \cdot \left(3.2d + (2\sin\frac{\theta}{2})^{-1} \cdot \frac{\rho_1}{\rho_o} \cdot (0.5 + 2\cdot 10^{-4}d) \cdot \left(\frac{2.4\,\theta}{2\sin\frac{\theta}{2}}\right)^d\right) < \frac{1}{2} .$$

From $\eta_1 \leq (2.6\pi)^{-d}$ follows that 3.2.2 is true for $\theta \in [2^{-d}, \frac{1}{3}]$, therefore these θ cannot occur in $\bar{\Gamma}_{\rho_1}$.

In the proof of 3.2.2 one has to use the estimates of homotopy errors from chapter 2 ; in particular one needs $3\Lambda|t(\alpha)| \leq \|r(\alpha)\|$ or $6\Lambda 4^N \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M) \leq \eta_1 = (2.6\pi)^{-d}$ (compare 3.1).

More explicitly,

3.2.3. $\Lambda^2 \cdot d(M)^2 \cdot \exp(\exp(\exp 2n^2)) \leq 1$ is a sufficient curvature assumption.

We repeat : this assumption is so strong that homotopy errors at all <u>other</u> parts of the proof do not significantly change the estimates.

An immediate consequence of 3.2.1 is (since at least $\rho_1 \cdot |t(\alpha)|^{-1}$ iterations of α are possible in Γ_{ρ_1}) :

3.2.4 If $\alpha \in \Gamma_{\rho_1}$ then $\|r(\alpha)\| \leq 2^{-d} \cdot \frac{|t(\alpha)|}{\rho_1}$.

Therefore we have the following <u>almost translational behaviour</u> ($\varepsilon \ll 1$ contains the homotopy error).

3.2.5. If $\alpha,\beta \in \Gamma_{\rho_1}$ then $|t(\alpha * \beta) - t(\alpha) - t(\beta)| \leq 2^{-d}\,\frac{|t(\alpha)| \cdot |t(\beta)|}{\rho_1} \cdot (1+\varepsilon)$.

Moreover we have from 2.12 (as a consequence of 2.11) already at this point a commutator estimate which Gromov derives only later.

3.2.6. If $\alpha, \beta \in \Gamma_{\rho_1}$ then $|t([\alpha,\beta])| \leq 2 \cdot 2^{-d} \cdot \frac{|t(\alpha)| \cdot |t(\beta)|}{\rho_1} \cdot (1+\varepsilon)$.

3.3. <u>The index estimate</u>. We estimate the index of the group Γ generated by Γ_{ρ_1} in $\pi_1(M,p)$ as follows :

(i) The finitely many loops at p of length $\leq 2 \cdot d(M)$ generate $\pi_1(M,p)$.

(ii) If all the words of wordlength $= \ell + 1$ in these generators occur already in equivalence classes mod Γ_{ρ_1} of the words of wordlength $\leq \ell$, then there are no further equivalence classes in longer words.

(iii) Two short loops ($\leq \rho_1$) are in the same equivalence class mod Γ_{ρ_1} if their rotational parts have a distance $\leq \frac{1}{3}$ in $O(n)$ (homotopy errors neglected). Therefore there are at most $W = \text{vol } O(n) \Big/ \text{vol } B_{\frac{1}{3}} \leq 2 \cdot (6\pi)^{\dim SO(n)}$ different equivalence classes among the short loops.

(iv) Words of wordlength $\leq W$ are still short as loops $(2W \cdot d(M) \leq 2\rho_0)$. Therefore (ii) must occur among the words of wordlength $\leq W$, so that there are not more than W equivalence classes mod Γ in $\pi_1(M)$.

3.4. <u>The lattice identification</u>. The almost translational behaviour 3.2.5 allows to pick generators in Γ_{ρ_1} in the same way as in a discrete translational subgroup of $\mathbb{R}^n$. Let δ_1 be the shortest loop in Γ_{ρ_1} ; δ_1 commutes with all other loops because of 3.2.6. For each $\delta \in \Gamma_{\rho_1}$ consider the orbit $\{\delta_1^i * \delta\} \subset \Gamma_{\rho_1}$. Scalar products $\langle t(\delta_1), t(\delta_1^i * \delta)\rangle$ and lengths $|t(\delta_1^i * \delta)|$ along the orbit can be controlled with 3.2.5 to find a unique representative $\tilde{\delta}$ in the orbit determined by $\langle t(\delta_1), t(\tilde{\delta})\rangle > 0$, $\langle t(\delta_1), t(\delta_1^{-1} * \tilde{\delta})\rangle \leq 0$. Starting from δ one needs at most $\left(1 + (1-2^{-d})^{-1} \circ \frac{|t(\delta)|}{|t(\delta_1)|}\right)$ multiplications by $\delta_1^{\pm 1}$ to reach $\tilde{\delta}$.

Let Γ' be the set of orthogonal projections of representatives $\tilde{\delta}$ onto the orthogonal complement of $t(\delta_1)$ in $\mathbb{R}^n = T_pM$ and define for $\alpha', \beta' \in \Gamma'$ the product $\alpha' * \beta'$ to be the projection of the representative of $\tilde{\alpha} * \tilde{\beta}$. Starting from $|\delta'| \leq \tilde{\delta} \leq 1{,}5 \cdot |\delta'|$ one proves that the inequalities 3.2.5 and 3.2.6 hold in Γ' with 2^{-d} replaced by $8 \cdot 2^{-d}$; note that even $8^n \cdot 2^{-d}$ is still much smaller than needed for the present arguments. To define a product $\alpha' * \beta'$ one needs the product $\tilde{\alpha} * \tilde{\beta}$ of somewhat longer elements in Γ , but for α' , $\beta' \in \Gamma'$ with $|\alpha'|$, $|\beta'| \leq \frac{1}{3} \cdot \rho_1$ the product is clearly defined. Therefore one is ready for an induction which for dimension reasons terminates after at most n steps : If inductively the basis $\delta_2', \ldots, \delta_n'$, for Γ' is already selected then choose $\delta_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_n$, as basis for Γ_{ρ_1} . Since the loops from Γ_{ρ_1} are ρ_o-dense in the $4\rho_o$-ball in $\mathbb{R}^n$ (see 3.1), and since we do not lose significantly from this relative denseness through n inductive steps (recall $d = 1.76^{n(n+1)}$), we will obtain exactly n generators $\delta_1, \ldots, \delta_n$ for Γ_{ρ_1}, which is Gromov's "normal basis". 3.2.6 shows at each inductive step that the shortest element is in the center ; therefore all loops $\delta \in \Gamma_{\rho_1}$ of length $\leq 3^{-n} \cdot \rho_1$ have a unique representation as a normal word $\delta_1^{k_1} * \ldots * \delta_n^{k_n}$. (The factor 3^{-n} stems from $|\tilde{\delta}| \leq 1.5|\delta'|$; it could be almost removed since for $|\tilde{\delta}| >> |\delta_1|$ a much sharper inequality is true.) Clearly we can identify the loop $\delta_1^{k_1} * \ldots * \delta_n^{k_n}$ with the n-tuple $(k_1, \ldots, k_n)$ or even with the lattice vector $\sum_{i=1}^{n} k_i . \delta_i$ in T_pM . This identification is much better than one might expect since the inductive choice of the normal basis gives

3.4.1. $\quad |\det(\delta_1, \ldots, \delta_n)| \geq 0.8^{n(n-1)} \cdot |\delta_1| \cdot \ldots \cdot |\delta_n|$.

From 3.2.5 and 3.4.1 we prove that the lattice-identification is very close to the translational part, namely (if $|t(\delta_1^{k_1} * \ldots * \delta_n^{k_n})| \leq 3^{-n} \cdot \rho_1$) :

3.4.2. $\quad \left|t(\delta_1^{k_1} * \ldots * \delta_n^{k_n}) - \sum_{i=1}^{n} k_i . \delta_i\right|_{T_pM} \leq 2^{-d} \cdot 2^{n^2} \cdot \frac{1}{\rho_1} \cdot \left|\sum_{i=1}^{n} k_i . \delta_i\right|^2$.

We interpret now Gromov's product of loops as a product between the lattice points $\sum k_i . \delta_i$ of T_pM and since lattice length and loop length almost coincide by 3.4.2 we have :

3.4.3. Inequalities 3.2.5 and 3.2.6 hold for lattice vectors of length $\leq 3^{-n} \cdot \rho_1$ if loop length $|t(\delta_1^{k_1} * \ldots * \delta_n^{k_n})|$ is replaced by lattice length $|\sum k_i . \delta_i|$ and ε is increased slightly.

Finally we note that at each inductive step the shortest vector is $\leq 2\rho_o$, therefore we have for the normal basis

3.4.4. $|\delta_i| < 2\rho_o \bullet (1.5)^{i-1}$ $(i = 1,\dots,n)$.

4. The nilpotent Lie group

4.1. The Malcev polynomials. 3.4.3 shows that commutators $[\delta_i, \delta_j]$ are generated by $\delta_1,\dots,\delta_{\min(i,j)-1}$. Therefore the product of two words $\delta_1^{k_1} * \dots * \delta_n^{k_n} * \delta_1^{\ell_1} * \dots * \delta_n^{\ell_n}$ is a new word $\delta_1^{p_1} * \dots * \delta_n^{p_n}$ where the p_i are polynomials of degree $\le n+1-i$ in the exponents $k_1,\dots,k_n$, $\ell_1,\dots,\ell_n$ [11]. (Commutators are so much shorter than their factors that the rearranging of the product into its normal form does not change its length very much ; therefore the rearranging can be considered an algebraic procedure as in [11].) We want to use these so called "Malcev polynomials" to extend the product from a ball in the lattice $\sum k_i.\delta_i$ to all of $\mathbb{R}^n$ and thus obtain the desired n-dimensional nilpotent Lie group G . If one knows associativity, inverses and the nilpotency relations on sufficiently many lattice points then the polynomials expressing these relations are satisfied on all of $\mathbb{R}^n$ and therefore define the nilpotent Lie group structure on $\mathbb{R}^n$.

The inverse is given by a polynomial of degree $\le n$, associativity is expressed by a polynomial of degree n^3 and the vanishing of the various n-fold commutators is expressed by polynomials of degree $\le n^{3n}$. Since commutators are shorter than their factors one stays in the domain where products are defined. Together with $\max\{ |\sum k_i.\delta_i| \,;\, |k_i| \le N\} \le n \bullet N \bullet 2\rho_o \bullet 1.5^{n-1}$ it follows that it is sufficient to have products defined for all loops of length $\le 2n \bullet n^{3n} \bullet 1.5^{n-1} \bullet \rho_o$. This leads to $\rho_1 = e^{3n^2} \bullet \rho_o$, the assumption made in 3.2. Therefore the Malcev polynomials are uniquely determined by the Gromov products of loops in Γ_{ρ_1} and they satisfy all relations to define a nilpotent Lie group structure on $\mathbb{R}^n$! The set Γ_{ρ_1} of loops $\le \rho_1$ with rotational part $\le \frac{1}{3}$ is identified in a product preserving way with a subset of this Lie group G , and the group Γ (which is abstractly generated from Γ_{ρ_1} with the short relations (2.5) between its elements) is identified as an uniform discrete subgroup of G via the integer lattice points $\sum k_i.\delta_i$ in $\mathbb{R}^n$.

4.2. Injectivity. Obviously Γ has a natural homomorphic image in $\pi_1(M)$; we need this to be an isomorphic one. Therefore one has to exclude the possibility that the other short loops, i.e. those with rotational parts $> \frac{1}{3}$, generate (in $\pi_1(M)$) additional relations between the elements of Γ . To achieve this we identify (in 4.2.1) all loops $\leq 3^{-n} \cdot \rho_1$ bijectively and product preserving with transformations of some set S . Clearly, the group generated from the loops is isomorphic to the group generated from the transformations ; therefore there are no further identifications in the generated group. Recall, that all relations in $\pi_1(M)$ are generated from the short relations between loops of length $\leq 5d(M)$ - which is $\leq 3^{-n} \cdot \rho_1$; this proves that the natural image of Γ in $\pi_1(M)$ is an isomorphic one. -

4.2.1. The definition of the set S . Consider two loops $\leq 3^{-n} \cdot \rho_1$ equivalent if they differ by a loop in Γ_{ρ_1} , then take A as a set of shortest representatives from these equivalence classes and put $S = A \times \Gamma$. To define the action of any loop b ($\leq 3^{-n} \cdot \rho_1$) on $(a,\delta) \in A \times \Gamma$ write $b * a = a' * \delta'$ $(a' \in A$, $\delta' \in \Gamma)$ and put $b \cdot (a,\delta) = (a' , \delta' * \delta)$. To check that this identifies the loops $\leq 3^{-n} \cdot \rho_1$ injectively and product preserving with transformations on S , one uses that Γ_{ρ_1} is fairly dense among all loops $\leq 3^{-n}\rho_1$ (see 3.4, in particular 3.4.2) and that Γ_{ρ_1} can be identified with its left actions on Γ (see end of 4.1).

4.3. The left invariant metric on G . We lift the "normal basis" $\delta_1,\dots,\delta_n \in G$ with the exponential map Exp of G to a basis of the tangent space T_eG and use this basis for an isometric identification of T_eG with T_pM ; then we left translate this metric to all of G . Next, the curvature tensor of this metric - or equivalently the norm of the Lie bracket - has to be estimated. We do not understand Gromov's "interpolation argument" , but we estimate the third order remainder term of the Campbell-Hausdorff power series inductively over the subgroups spanned by $\delta_1,\dots,\delta_i$:

4.3.1. If $H(X,Y)$ is defined by $\operatorname{Exp} X \cdot \operatorname{Exp} Y = \operatorname{Exp} H(X,Y)$, then we have

$$|H(X,Y) - X - Y - \tfrac{1}{2}[X,Y]| \leq |[X,Y]| \cdot \varepsilon \cdot (|X| + |Y|) ,$$

where $X \in T_eG$ is arbitrary, $Y \in T_e\mathrm{span}(\delta_1,\dots,\delta_i)$ and ε depends on the norm of the Lie bracket on $T_e\mathrm{span}(\delta_1,\dots,\delta_{i-1})$.

Consequently we have (side conditions as in 4.3.1) :

4.3.2. $$|H(e^{\mathrm{Ad}X} \cdot Y , -Y) - [X,Y]| \leq |[X,Y]| \cdot \varepsilon' \cdot (|X| + |Y|) .$$

Because of 4.3.2 and $\operatorname{Exp} X \cdot \operatorname{Exp} Y \cdot \operatorname{Exp}(-X) \cdot \operatorname{Exp}(-Y) = \operatorname{Exp} H(e^{\mathrm{Ad}X} \cdot Y , -Y)$ we can use the commutator estimates 3.4.3 to get, inductively over the subgroups $\mathrm{span}(\delta_1,\dots,\delta_i)$, estimates for the Lie bracket which are about as good as 3.2.6. (In other words : the elements $\delta_1,\dots,\delta_n$ are indeed so close to the identity in G that the higher than second order terms in the Campbell-Hausdorff series can be neglected for the computation of commutators.) In particular, the curvature of G is very small. (We do not give any more numbers, since the curvature assumption we were forced to make in 3.2.3 makes all estimates ridiculously small compared to what the present arguments would need.)

4.4. The Γ-equivariant diffeomorphism. Γ acts isometrically by left translations on G and - as the deck group of a finite covering of M - Γ also acts isometrically on the universal covering $\tilde{M}$. From the "normal basis" $\delta_1,\dots,\delta_n$ in Γ and the exponential maps of G and $\tilde{M}$ we obtain natural basis for T_eG and $T_p\tilde{M}$; therefore, after left translation by Γ, we have corresponding natural basis in the tangent spaces of all "lattice points" in G and $\tilde{M}$ which identify these tangent spaces almost isometrically. Then, with the exponential maps of G and $\tilde{M}$ we obtain maps from large balls around the lattice points in G onto corresponding balls in $\tilde{M}$. These local maps are compatible with the action of Γ and they are very close to isometries since the curvatures of G and $\tilde{M}$ are so small (see 2.1). Moreover, their differentials can be described by Jacobi fields, hence, again because of the small curvatures, these differentials are close to the identity (if we identify different tangent spaces by Levi-Civita parallel translation). Therefore a center-of-mass-average [9] of these local maps will produce a Γ-equivariant map of maximal rank from G to $\tilde{M}$, i.e. a Γ-equivariant diffeomorphism.

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GROMOV'S CONVERGENCE THEOREM AND ITS APPLICATION

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One of the basic questions of Riemannian geometry is that "If two Riemannian manifolds are similar with respect to the Riemannian invariants, for example, the curvature, the volume, the first eigenvalue of the Laplacian, then are they topologically similar?". Initiated by H. Rauch, many works are developed to the above question. Recently M. Gromov showed a remarkable theorem ([7] 8.25, 8.28), which may be useful not only for the above question but also beyond the above. But it seems to the author that his proof is heuristic and it contains some gaps (for these, see § 1), so we give a detailed proof of 8.25 in [7]. This is the first purpose of this paper. Second purpose is to prove a differentiable sphere theorem for manifolds of positive Ricci curvature, using the above theorem as a main tool.

For a d-dimensional Riemannian manifold M, we denote by K_M the sectional curvature, by $\mathrm{vol}(M)$ the volume, by $\mathrm{diam}(M)$ the diameter, by $d_M(m, n)$ the distance between m and n induced from Riemannian metric g and by i_M the injectivity radius.

A subset B is called δ-dense when for any point $m \in M$, there exists a point $n \in B$ with $d_M(m, n) \leqq \delta$. A subset B is called δ-discrete if $n_1, n_2 \in B$ $(n_1 \neq n_2)$ implies $d_M(n_1, n_2) \geqq \delta$. Let $M(d, \Delta, i_0)$ (resp. $M(d, \Delta, \rho, v)$) be the category of all complete Riemannian manifolds M with dimension $= d$, $|K_M| \leqq \Delta$ and $i_M \geqq i_0$ (resp. dimension $= d$, $|K_M| \leqq \Delta$, $\mathrm{diam}(M) \leqq \rho$, $\mathrm{vol}(M) \geqq v$).

The following theorem is seemingly different from 8.25 in [7] but the inwardness is essentially same.

THEOREM 1 (Gromov's convergence theorem). *Given* $d, \Delta, i_0 > 0$, $0 < R < \min(1/2\sqrt{\Delta}, i_0/2)$, *for any* $\delta > 0$, *there exist* $a = a(d, \Delta, i_0, R; \delta) > 0$ *and*

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$\varepsilon = \varepsilon(d, \Lambda, i_0, R; \delta) > 0$ *such that if* M, $M' \in M(d, \Lambda, i_0)$ *have an* ε*-dense,* $\varepsilon/10$*-discrete subset* $N[\varepsilon] = \{m_i\}_{i=1}^{N\varepsilon} \subset M$ *and* $N'[\varepsilon] = \{m'_i\}_{i=1}^{N\varepsilon} \subset M'$ *containing the same number of members with*

$$1 - a \leqq \frac{d_M(m'_i, m'_j)}{d_M(m_i, m_i)} \leqq 1 + a \qquad \text{for } 0 < d_M(m_i, m_i) \leqq R\,,$$

then there exists a diffeomorphism $F\colon M \to M'$ *with* $||dF_m(\xi)| - 1| < \delta$ *for* $\xi \in UM$, *where* UM *is the unit sphere bundle of* M.

We can estimate constants a, $\varepsilon > 0$ explicitly, but we omit it to avoid non-essential complexity. Here we call it Gromov's convergence theorem because he proved a convergence theorem (8.18 in [7]) with respect to the Hausdorff distance using this theorem as a main tool.

An easy application of Theorem 1 and Dirichlet drawer principle is,

THEOREM 2 (Cheeger's finiteness theorem). *The number* N *of the diffeomorphism classes of the manifolds in* $M(d, \Lambda, \rho, v)$ *is finite.*

This theorem was originally proved by J. Cheeger [2] except for $d = 4$. After this, in Cheeger-Ebin's book [3], it was stated in the above form without proof. It was also given by M. Gromov [6]. S. Peters [12] gave another (simple) proof.

The following is the differentiable sphere theorem mentioned above. Let Ric_M be the Ricci curvature of M.

THEOREM 3. *Given* $d, \Lambda > 0$, *there exists* $\delta_0 = \delta_0(d, \Lambda) > 0$ *such that if a compact d-dimensional Riemannian manifold* M *has the property that* $\mathrm{Ric}_M \geqq d - 1$, $|K_M| \leqq \Lambda$, $\mathrm{vol}\,(M) \geqq \omega_d - \delta_0$, *where* ω_d *is the volume of the d-dimensional unit sphere, then* M *is diffeomorphic to* S^d.

In [16], T. Yamaguchi obtained the same conclusion under a stronger assumption and in [9], Y. Itokawa showed that, under the essentially same assumption except for the estimate of the constant, M has the same homotopy type as S^d. (He only assumes the upper bound of K_M but under the condition of $\mathrm{Ric}_M \geqq d - 1$, the lower bound of K_M is automatically derived.) But it should be remarked that in [15], K. Shiohama proved that M is homeomorphic to S^d under a weaker assumption than ours.

Finally we remark that for the diameter or the first eigenvalue of the Laplacian $\lambda_1(M)$, the following pinching theorem is obtained by using the above one and the results of C. B. Croke [5] and A. Kasue [10].

COROLLARY. *Given $d, \Delta, v > 0$ there exist $\delta_1 = \delta_1(d, \Delta, v) > 0$ and $\delta_2 = \delta_2(d, \Delta, v) > 0$ such that if a d-dimensional Riemannian manifold M with* $\mathrm{Ric}_M \geqq d-1$, $|K_M| \leqq \Delta$, $\mathrm{vol}(M) \geqq v$ *has the property that* $\mathrm{diam}(M) \geqq \pi - \delta_1$ *or* $\lambda_1(M) \leqq d + \delta_2$. *then M is diffeomorphic to S^d.*

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Remark. After the preparation of this paper the author learned that D. L. Brittain also got the same result as Corollary independently.

[Donald L. Brittain, A diameter pinching theorem for positive Ricci curvature. (preprint.)]

§1. Outline of the proof of Theorem 1

Firstly we observe the case when $M, M' \in M(d, \Delta, i_0)$ is compact. For an ε-dense, $\varepsilon/10$-discrete subset $N[\varepsilon] = \{m_i\}_{i=1}^{N_\varepsilon}$, we define a map $f: M \to \boldsymbol{R}^{N_\varepsilon}$ using the distance from m_i. If ε is sufficiently small, then f is an embedding (§ 2). We can estimate $\delta > 0$ such that the normal exponential map Exp is a diffeomorphism on the δ-tubular neighborhood of $f(M)$; $B_\delta(f(M))$ (§ 4). For $M' \in M(d, \Delta, i_0)$ and for $f': M' \to \boldsymbol{R}^{N_\varepsilon}$ which is defined similarly to f, we see that $f(M) \subset B_\delta(f'(M'))$ and $f'(M') \subset B_\delta(f(M))$. From this, the normal projection $P: f(M) \to f'(M')$ can be defined (§ 5). Nextly, we see that the tangent spaces $T_p f(M)$ and $T_{p'} f'(M')$ are almost parallel, where $p' = P(p)$ (§ 6). Using this, it can be shown that $P: f(M) \to f'(M')$ is a diffeomorphism (§ 7). For $F = f'^{-1} \circ P \circ f$, we estimate $dF(\xi)|$ (§ 8). In the case when M is non compact, the diffeomorphism is given by the approximation arguments (§ 9).

Here the author would like to comment on Gromov's proof in [7] 8.25. Firstly he says that it suffices to estimate $\delta > 0$ so that Exp is locally diffeomorphic but it really needs to estimate $\delta > 0$ so that it is globally diffeomorphic. (We add Lemma 4.3.) Secondly P may cut the two points of $f(M)$, for this possibility, he says "good" one can be chosen without detailed arguments. (We add Section 6.) Thirdly for the argument of the estimate of $|dF(\xi)|$, it needs more arguments than that given there.

Though almost all arguments owe to Gromov [7], we give a full proof for the sake of completeness. It should be noted that the author also referred to T. Sakai [13].

§ 2. Definition of the embedding $f: M \to \boldsymbol{R}^{N_\varepsilon}$

We firstly prove the Theorem 1 in the case when M is compact.

Take constants $0 < r < R$ and $\kappa > 0$. Let $h: \boldsymbol{R} \to [0, 1]$ be a C^∞ function such that

$$\begin{aligned}
&h(t) = 1 \quad \text{if} \quad t \leqq 0,\ h(t) = 0 \quad \text{if} \quad t \geqq r \\
&\quad -\frac{4}{r} < h'(t) < -\frac{3}{r} \quad \text{if} \quad \frac{3r}{8} < t < \frac{5r}{8} \\
&\quad -\frac{4}{r} < h'(t) < 0 \quad \text{if} \quad \frac{2r}{8} < t \leqq \frac{3r}{8} \quad \text{or} \quad \frac{5r}{8} \leqq t < \frac{6r}{8} \\
&\quad -\kappa < h'(t) < 0 \quad \text{if} \quad 0 < t \leqq \frac{2r}{8} \quad \text{or} \quad \frac{6r}{8} \leqq t \leqq r .
\end{aligned}$$

Note that we may take $\kappa > 0$ arbitrarily small, which is needed in Section 8.

Put

$$k = \max\left(\left|h'(t)\left(\frac{1}{t} + \frac{\Delta t}{2}\right)\right|,\ |h''(t)|\right) \quad \text{and} \quad A = \left(1 - \frac{1}{3d^2}\right)^{1/2} .$$

In the following, we remark that the constants $c_i > 0$, $\beta > 0, \cdots$ which appear in the proof, are depending only on $d, \Delta, i_0, r, \delta > 0$ and $h(t)$.

Put

$$\varepsilon_1 = \min\left(\frac{r}{16}, \frac{s_\Delta(r)}{2}(1 - A^2)^{1/2}, \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right)\left(r\Delta + \frac{16}{r}\right)^{-1}\right),$$

where $s_\tau(t)$ is the function

$$\begin{aligned}
&\frac{1}{\tau^{1/2}} \sin(\tau^{1/2} t) , && \text{if } \tau > 0 , \\
&t , && \text{if } \tau = 0 , \\
&\frac{1}{(-\tau)^{1/2}} \sin h((-\tau)^{1/2} t) , && \text{if } \tau < 0 .
\end{aligned}$$

Using this $h(t)$ and an ε-dense, $\varepsilon/10$-discrete subset $N[\varepsilon] = \{m_i\}_{i=1}^{N_\varepsilon}$ with $\varepsilon < \varepsilon_1$, we define a C^∞ map $f = f_\varepsilon$: $M \to \boldsymbol{R}^{N_\varepsilon}$ by

$$f_\varepsilon(m) = (h(d_M(m_1, m)), \cdots, h(d_M(m_{N_\varepsilon}, m))) .$$

We show that f_ε is an embedding by the following two lemmas.

LEMMA 2.1. *f_ε has maximal rank at every point $m \in M$.*

Proof. Take an orthonormal basis $\{e_i\}_{i=1}^d$ of the tangent space T_mM to M at m and choose $\{m_{i_j}\}_{j=1}^d \subset N[\varepsilon]$ satisfying $d_M(\exp_m(r/2)e_j, m_{i_j}) < \varepsilon$. Put $t_j = |\exp_m^{-1} m_{i_j}|$ and $u_j = t_j^{-1} \exp_m^{-1} m_{i_j}$. Note that $3r/8 < d_M(m_{i_j}, m) < 5r/8$. Then, from the Rauch's comparison theorem (R. C. T.) (cf. [3] or [13] (1.2.20)), we see

$$\frac{s_\Delta(r)}{r} \cdot |(r/2)e_j - t_j u_j| \leqq d_M(m_{i_j}, \exp_m(r/2)e_j) < \varepsilon < \frac{s_\Delta(r)}{2}(1 - A^2)^{1/2}$$

and this implies $g(e_j, u_j) > A \geqq (1 - (1/3d^2))^{1/2}$. From this, we see $\{u_i\}_{i=1}^d$ are linearly independent. Since $\operatorname{grad} d_M|_{m_{i_j}} = u_j$, we can get the conclusion by

$$\begin{aligned} \text{the rank of } df \text{ at } m &= \operatorname{rank} df|_m \\ &= \operatorname{rank}(d \cdot h(d_M(m_{i_1}, \cdot))|_m, \cdots, d \cdot h(d_M(m_{i_d}, \cdot))|_m) \\ &\geqq \operatorname{rank}(h'(d_M(m_{i_1}, m))u_1, \cdots, h'(d_M(m_{i_d}, m))u_d) \\ &= d . \end{aligned}$$

q.e.d.

LEMMA 2.2. *f_ε is an embedding.*

Proof. If not, then there exist $m, n \in M$ with $m \neq n$ such that $f(m) = f(n)$. Since $d_M(m_i, m) = d_M(m_i, n)$ for all $m_i \in N[\varepsilon] \cap B_r(m) = N[\varepsilon] \cap B_r(n)$, we see $d_M(m, n) := \tilde{d} < 2\varepsilon < r/8$. Let γ be the minimal geodesic from m to n and put $z = \gamma((r/2) + \tilde{d})$. Then $z \in \bar{B}_{r/2}(n) - \bar{B}_{r/2}(m)$ and $B_{2\varepsilon}(z) \subset B_r(n) - \bar{B}_{r/4}(m)$, where $B_r(m)$ is the set of the point p with $d_M(p, m) < r$ and $\bar{B}$ is the closure of B. Take a point $p \in N[\varepsilon] \cap B_{2\varepsilon}(z)$ with $d' := d_M(p, n) \geqq r/2 - 2\varepsilon$, $d' < r/2$ and the vector $u \in T_nM$ that is the unit initial vector of the minimal geodesic λ from n to p. Now we estimate $g(u, \dot{\gamma}(\tilde{d}))$. From R.C.T., we get

$$\begin{aligned} |(r/2)\dot{\gamma}(\tilde{d}) - d'u| &= |\exp_n^{-1} z - \exp_n^{-1} p| \\ &\leqq \frac{r/2}{s_\Delta(r/2)} \cdot d_M(p, z) < \frac{r\varepsilon}{s_\Delta(r/2)} < \frac{r^2}{16 s_\Delta(r/2)} , \end{aligned}$$

from which follows

$$\begin{aligned}\frac{r}{2}\cdot g(\dot{\gamma}(\tilde{d}), u) &= g((r/2)\dot{\gamma}(\tilde{d}) - d'u, u) + d' \\ &\geqq d' - |(r/2)\dot{\gamma}(\tilde{d}) - d'u| > \frac{r}{2} - 2\varepsilon - \frac{r^2}{16s_\Delta(r/2)} \\ &\geqq \frac{r}{4}\cdot\left(1 - \frac{r}{4s_\Delta(r/2)}\right),\end{aligned}$$

namely

$$g(\dot{\gamma}(\tilde{d}), u) > \frac{1}{2}\cdot\left(1 - \frac{r}{4s_\Delta(r/2)}\right).$$

On the other hand, note that $d_M(p, \gamma(t)) < r$ for $0 \leqq t \leqq \tilde{d}$ and $d_M(p, \gamma(0)) = d_M(p, \gamma(\tilde{d}))$, then from the Rolle's theorem, there exists a point $m_1 = \gamma(t_1)$ $(0 < t_1 < \tilde{d})$ with $g(\dot{\gamma}(t_1), u_{t_1}) = 0$, where u_t is the unit initial vector of the minimal geodesic from $\gamma(t)$ to p. Then we have

$$\begin{aligned}g(\dot{\gamma}(\tilde{d}), u) &= \int_{t_1}^{\tilde{d}} \frac{d}{dt} g(\dot{\gamma}(t), u_t)dt \\ &\underset{(*)}{=} \int_{t_1}^{\tilde{d}} \operatorname{Hess} d_{M,p}(\dot{\gamma}(t), \dot{\gamma}(t))dt \\ &\underset{(*)}{\leqq} \int_{t_1}^{\tilde{d}} \left(\frac{1}{d_M(p, \gamma(t))} + \frac{\Delta}{2} d_M(p, \gamma(t))\right)dt \\ &< 2\varepsilon\left(\frac{8}{r} + \frac{r\Delta}{2}\right).\end{aligned}$$

After all we get

$$\varepsilon\left(r\Delta + \frac{16}{r}\right) > \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right).$$

It contradicts the fact

$$\varepsilon \leqq \left(\frac{1}{2} - \frac{r}{8s_\Delta(r/2)}\right)\left(r\Delta + \frac{16}{r}\right)^{-1}.$$

Except for $(*)$ we get the conclusion.

To show the inequality $(*)$, we need following sublemma. Put $d_{M,p}(\cdot) = d_M(p, \cdot)$.

SUBLEMMA ([7] 8.23 or [13] (1.4.4), iii). *If $|K_M| \leqq \Delta$, then the hessian of $d_{M,p}$ at $x = \operatorname{Hess} d_{M,p}(x, x) \leqq |x|^2(1/d_M(p, m) + (\Delta/2)d_M(p, m))$ for $x \perp \operatorname{grad} d_{M,p}|_m$ and $d_M(p, m) < r$.* q.e.d.

§ 3. Estimate of *df*

The contents of this section are detailed arguments developed by Gromov's hints.

(i) Estimate of the number of the elements in $N[\varepsilon]$, which are nearly orthonormal.

Firstly, we take $c_1 > 0$ with

$$c_1 \leqq \inf_{0<\varepsilon<\varepsilon_1/10} \frac{b_\Delta(\varepsilon/20)b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(4r)\cdot b_{-\Delta}(\varepsilon)} ,$$

where $b_\tau(t)$ is the volume of the ball with radius t in the space of the constant curvature τ. Note that c_1 can taken as positive because $\lim_{t\downarrow 0} b_\Delta(t/20)/b_{-\Delta}(t) = 20^{-d}$. Put $\tilde{N}_\varepsilon = \sup_m \sharp(B_{2r}(m) \cap N[\varepsilon])$, $\tilde{m}_i = \exp_m((r/2)e_i)$ and $D^i_m[\varepsilon] = B_{\varepsilon_1/2}(\tilde{m}_i) \cap N[\varepsilon]$.

LEMMA 3.1. *If* $\varepsilon \leqq \varepsilon_1/10$, *then* $c_1 \leqq \sharp(D^i_m[\varepsilon])/\tilde{N}_\varepsilon \leqq 1$.

Proof. From the fact

$$\bigcup_{q\in B_{\varepsilon_1/4}(\tilde{m}_i)\cap N[\varepsilon]} B_\varepsilon(q) \subset B_{\varepsilon_1/2}(\tilde{m}_i)$$

$$\bigcup_{q\in B_{2r}(m)\cap N[\varepsilon]} B_\varepsilon(q) \subset B_{4r}(m)$$

and the volume comparison theorem ([7] or [13]), we have

$$\sharp(D^i_m[\varepsilon]) \geqq \frac{b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(\varepsilon)}$$

$$\tilde{N}_\varepsilon \leqq \frac{b_{-\Delta}(4r)}{b_\Delta(\varepsilon/20)} .$$

Combining these, we get the conclusion.

(ii) Estimate of *df*.

LEMMA 3.2. *For* $\varepsilon < \varepsilon_1$, *there exist* c_2, $c_3 > 0$ *such that*

$$c_2\tilde{N}_\varepsilon^{1/2} \leqq |df_\varepsilon(\xi)| \leqq c_3\tilde{N}_\varepsilon^{1/2} \qquad \text{for any } \xi \in UM .$$

Proof. From the definition of f_ε, we see

$$df_{\varepsilon,m}(\xi) = (a_1 g(u_1, \xi), \cdots, a_{N_\varepsilon} g(u_{N_\varepsilon}, \xi)) ,$$

where $a_i = h'(d_M(m, m_i))$. We may put $c_3 = \sup_{0\leqq t\leqq r} |h'(t)|$. For the existence of c_2, we take the representatives $m_{k_i} \in D^i_m[\varepsilon]$ and put $u_{k_i} = \exp_m^{-1} m_{k_i}/|\exp_m^{-1} m_{k_i}|$. Let $\ell = \ell_{(k_1,\dots,k_d)}: T_m M \to \boldsymbol{R}^d$ be a linear map defined by

$$\ell(\xi) = (a_{k_1} g(u_{k_1}, \xi), \cdots, a_{k_d} g(u_{k_d}, \xi)) .$$

Then we see that it satisfies the following estimate

$$\min_{|\xi|=1} |\ell(\xi)| \geqq \frac{3}{2r} > 0 .$$

In fact, if we put $\alpha_{ij} = g(u_{k_i}, e_j)$ and $\xi = \sum_j \xi_j e_j$, then from the proof of Lemma 2.1 $\alpha_{ii} \geqq A$, $|\alpha_{ij}| \leqq (1 - A^2)^{1/2} (i \neq j)$ and $4/r \geqq |a_{k_i}| \geqq 3/r$. Thus, we get

$$\begin{aligned} |\ell(\xi)|^2 &= \sum_{i,j,\ell} a_{k_i}^2 \xi_j \xi_\ell \alpha_{ij} \alpha_{i\ell} \\ &= \sum_i a_{k_i}^2 \xi_i^2 \alpha_{ii}^2 + (\text{the other terms}) \\ &\geqq \left(\frac{3}{r}\right)^2 A^2 - d^2 \left(\frac{4}{r}\right)^2 (1 - A^2) \geqq \left(\frac{3}{2r}\right)^2 > 0 . \end{aligned}$$

On the other hand, from Lemma 4.1, we see

$$\sharp\{(k_1, \cdots, k_d) \,|\, m_{k_i} \in D_m^i[\varepsilon]\} \geqq \inf \sharp(D_m^i[\varepsilon]) \geqq c_1 \tilde{N}_\varepsilon .$$

Combining these, we get

$$|df(\xi)|^2 \geqq \sum_{(k_1, \cdots, k_d)} |\ell_{(k_1, \cdots, k_d)}(\xi)|^2 \geqq c_1 \left(\frac{3}{2r}\right)^2 \tilde{N}_\varepsilon .$$

Therefore we may put

$$c_2 = c_1^{1/2} \left(\frac{3}{2r}\right) . \qquad \text{q.e.d.}$$

Remark. We discuss here the dependence of r on c_1, c_2, c_3 when r is sufficiently small, which is essential in Section 8. Since the function $f(t) = b_\Delta(t/20)/b_{-\Delta}(t)$ is decreasing and we may assume $\varepsilon_1 \geqq r/50d$, we can take

$$\begin{aligned} c_1 &= (10^5 d)^{-d} \leqq \left(\frac{1}{40} \cdot \frac{1}{1600d}\right)^d \leqq \frac{b_\Delta(\varepsilon_1/200) b_\Delta(\varepsilon_1/4)}{b_{-\Delta}(\varepsilon_1/10) b_{-\Delta}(4r)} \\ &\leqq \inf_{0<\varepsilon<\varepsilon_1/10} \frac{b_\Delta(\varepsilon/20) b_\Delta(\varepsilon_1)}{b_{-\Delta}(4r) b_{-\Delta}(\varepsilon)} \\ c_2 &= c_1^{1/2} \left(\frac{3}{2r}\right) = \frac{3}{2r} (10^5 d)^{-d/2} , \\ c_3 &= \frac{4}{r} . \end{aligned}$$

§4. The tubular neighborhood of $f(M)$ and the normal exponential mapping

Let Exp: $Nf(M) \to \boldsymbol{R}^{N_\varepsilon}$ be the normal exponential map of the normal bundle $Nf(M)$. Put

$$B_\delta(Nf(M)) = \{(p, u) \in Nf(M) \,|\, |u| < \delta\} .$$

We estimate $\delta > 0$ such that $\mathrm{Exp}|_{B\delta(Nf(M))}$ is a diffeomorphism.

(i) Local estimate.

The following Lemma 4.1 owe to [7] and [13].

LEMMA 4.1. *There exists $c_4 > 0$ such that if $\varepsilon \leqq \varepsilon_1$ and $\delta \leqq c_4 \tilde{N}_\varepsilon^{1/2}$, then* $\mathrm{Exp}|_{B\delta(Nf(M))}$ *is an immersion.*

Proof. Suppose that $n \in \boldsymbol{R}^{N_\varepsilon}$ is a critical value of Exp. Namely there exists a curve $c(s) = f(m(s))$ in $f(M)$ and the normal vector field $n(s)$ along $c(s)$ such that $n = c(0) + n(0)$, $\dot{c}(0) + \dot{n}(0) = 0$. From $g(n(s), \dot{c}(s)) = 0$, we have

$$g(n(0), \ddot{c}(0)) = -g(\dot{n}(0), \dot{c}(0)) = |\dot{c}(0)|^2 .$$

Since $c(s)$ $(\cdots, h(d_M(m_i, m(s))), \cdots)$, we have

$$\ddot{c}(0) = \Big(\cdots, h''(d_M(m_i, m(0)))\Big(\frac{d}{ds}\Big|_{s=0} d_M(m_i, m(s))\Big)^2 + h'(d_M(m_i, m(0)))\Big(\frac{d^2}{ds^2}\Big|_{s=0} d_M(m_i, m(s))\Big) \cdots\Big) .$$

Recall that

$$\Big|\frac{d}{ds}\Big|_{s=0} d_M(m_i, m(s))\Big| = |g(\operatorname{grad} d_{M,m_i}, \dot{m}(0))| \leqq |\dot{m}(0)| ,$$

$$\Big|\frac{d^2}{ds^2}\Big|_{s=0} d_M(m_i, m(s))\Big| \leqq |\dot{m}(0)|^2\Big(\frac{1}{d_M(m_i, m(0))} + \frac{\Delta}{2} d_M(m_i, m(0))\Big) .$$

Note that $\max(|h'(t)(1/t + \Delta t/2)|, |h''(t)|) = k$. Then we see

$$|\dot{c}(0)|^2 \leqq |n(0)||\ddot{c}(0)| \leqq 2|n(0)||\dot{m}(0)|^2 k\tilde{N}_\varepsilon^{1/2} ,$$

and this implies,

$$\begin{aligned} d_M(n, f(M)) = |n(0)| &\geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} \cdot \frac{|\dot{c}(0)|^2}{|\dot{m}(0)|^2} \\ &\geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} |df_\varepsilon|^2 | \geqq \frac{1}{2k\tilde{N}_\varepsilon^{1/2}} c_2^2 \tilde{N}_\varepsilon = \frac{c_2^2}{2k} \cdot \tilde{N}_\varepsilon^{1/2} . \end{aligned}$$

Thus we get the conclusion by putting $c_4 = c_2^2/2k$.

Hereafter we denote by $\tilde{d}_M$, the distance on $f(M)$ defined by the induced Riemannian structure of $f(M)$ from $\boldsymbol{R}^{N_\varepsilon}$ and by d, the euclidean distance of $\boldsymbol{R}^{N_\varepsilon}$.

(ii) Relation between $\tilde{d}_M$ and d. (I)

LEMMA 4.2. *Fix $\alpha > 0$. If $\varepsilon \leqq \min(\varepsilon_1/100, \alpha/100c_3)$, then there exists $\tilde{\alpha} > 0$ such that if $\tilde{d}_M(p, q) \geqq \alpha \cdot \tilde{N}_\varepsilon^{1/2}$, then $d(p, q) \geqq \tilde{\alpha} \cdot \tilde{N}_\varepsilon^{1/2}$. For the case $\alpha = c_4/10$, we put $\tilde{\alpha} = 3c_5$.*

Proof. Since $\tilde{d}_M(p, q) \geqq \alpha \cdot \tilde{N}_\varepsilon^{1/2}$, we see $d_M(f^{-1}(p), f^{-1}(q)) \geqq \alpha/c_3$. Put $\varepsilon_2 = \min(r/10, \alpha/10c_3)$ and $\beta = |h(9\varepsilon_2) - h(\varepsilon_2)| > 0$.

Take the balls B_1, B_2 of radius ε_2 centered at $f^{-1}(p), f^{-1}(q)$ respectively. By the method similar to Section 3–(i), we find that there exists $\tilde{\beta} > 0$ such that

$$\sharp(B_i \cap N[\varepsilon])/\tilde{N}_\varepsilon \geqq \tilde{\beta} \qquad (i = 1, 2)$$

Therefore we get

$$(d(p, q))^2 = \sum_{i=1}^{N_\varepsilon} \{h(d_M(f^{-1}(p), m_i)) - h(d_M(f^{-1}(q), m_i))\}^2 \geqq \beta^2 \tilde{\beta} \tilde{N}_\varepsilon \, .$$

We have done if we take $\tilde{\alpha} \leqq \tilde{\beta}^{1/2}\beta$.

(iii) Global estimate.

LEMMA 4.3. *If $\varepsilon < \min(\varepsilon_1/100, c_4/1000c_3)$ and $\delta < c_5 \tilde{N}_\varepsilon^{1/2}$, then $\mathrm{Exp}|_{B_\delta(Nf(M))}$ is a diffeomorphism.*

Proof. Suppose that there exist (p, u), $(q, v) \in B_\delta(Nf(M))$ with $(p, u) \neq (q, v)$ and $\mathrm{Exp}(p, u) = \mathrm{Exp}(q, v) := x$. Then from Lemma 4.2, we see $\tilde{d}_M(p, q) \leqq c_4/10 \cdot \tilde{N}_\varepsilon^{1/2}$ because

$$\begin{aligned} d(p, q) &\leqq d(\mathrm{Exp}(p, u), \mathrm{Exp}(q, v)) + d(\mathrm{Exp}(p, u), p) + d(\mathrm{Exp}(q, v), q) \\ &\leqq |u| + |v| \leqq 2c_5 \tilde{N}_\varepsilon^{1/2} \, . \end{aligned}$$

Now we define a smooth map

$$F(s, t)\colon [0, 1] \times [0, 1] \longrightarrow \boldsymbol{R}^{N_\varepsilon}$$

by $F(s, t) = (1 - t)\gamma(s) + tx$, where $\gamma(s)$ is the minimal geodesic from p to q in $f(M)$.

Since

$$\begin{aligned} d(F(s, t), f(M)) &\leqq d(F(s, t), \gamma(s)) \leqq d(x, \gamma(s)) \\ &\leqq d(x, q) + d(q, \gamma(s)) \leqq d(x, q) + \tilde{d}_M(q, \gamma(s)) \end{aligned}$$

$$\leqq d(x, q) + \tilde{d}_M(p, q) \leqq c_5\tilde{N}_\varepsilon^{1/2} + \frac{c_4}{10}\cdot\tilde{N}_\varepsilon^{1/2} \leqq \frac{c_4}{2}\cdot\tilde{N}_\varepsilon^{1/2}\,.$$

we observe

$$F(s, t) \subset B_{(c_4/2)\cdot\tilde{N}_\varepsilon^{1/2}}(f(M)) = \operatorname{Exp}(B_{(c_4/2)\cdot\tilde{N}_\varepsilon^{1/2}}(Nf(M))\,.$$

The following sublemma is crucial in the proof. Put $B = B_{(c_4/2)\cdot\tilde{N}_\varepsilon^{1/2}}(Nf(M))$.

SUBLEMMA. *There exists a smooth map*

$$G(s, t)\colon [0, 1] \times [0, 1] \longrightarrow B$$

such that $\operatorname{Exp}(G(s, t)) = F(s, t)$.

Proof of the sublemma (cf. J. Schwartz [14] 1.23). Let I be the set of $t \in [0, 1]$ such that $G(s, t)$ can be defined for all $s \in [0, 1]$. Since $G(s, 0) = \gamma(s)$, $0 \in I \neq \phi$. It is sufficient to prove that I is open and closed.

We see that I is open by the following argument. Take $a \in I$. Since $\operatorname{Exp}|_B$ is an immersion and $\bigcup_s G(s, a)$ is compact, it can be covered by a family of finite open sets $\{U_i\}$, which are mapped by Exp diffeomorphically to open neighborhoods $\{V_i\}$ of $F(s, a_i)$ and $\bigcup_i V_i \supset \bigcup_s F(s, a)$. This implies $G(s, t)$ can be defined beyond a and I is open.

We show that I is closed. Since the closure of $B \subset B_{c_4\tilde{N}_\varepsilon^{1/2}}(Nf(M))$ is compact, there exists $A > 0$ such that $|d\operatorname{Exp}| \geqq A$. Then for all $(s, t) \in [0, 1] \times I$,

$$|G_t(s, t)| = |d\operatorname{Exp}^{-1}||F_t(s, t)| \leqq A^{-1}|F_t(s, t)| = A_s < \infty$$

where G_t, F_t mean the derivative with respect to t.

Integrating this we get

$$|G(s, t_1) - G(s, t_0)| \leqq A_s|t_1 - t_0|\,.$$

It implies $\lim_{t\to\sup I} G(s, t)$ exists and $G(s, \sup I)$ can be defined. It means I is closed whence the conclusion.

From this sublemma, we see $\operatorname{Exp}(G(s, 1)) = x$. But this contradicts the fact that $\operatorname{Exp}|_B$ is an immersion. Therefore $\operatorname{Exp}|_{B_\delta(Nf(M))}$ is a diffeomorphism. q.e.d.

§ 5. Definition of the projection P

Take another $M' \in M(d, \Delta, i_0)$, which has an ε-dense $\varepsilon/10$-discrete subset

$N'[\varepsilon] = \{m_i'\} \subset M'$ such that

$$1 - a \leqq \frac{d_{M'}(m_i', m_j')}{d_M(m_i, m_j)} \leqq 1 + a \qquad \text{for } 0 < d_M(m_i, m_j) < R\,.$$

We define f' for M' in the same way as f for M. From the definition of f and f' we get

$$\begin{aligned} d(f(m_k), f'(m_k')) &= \left(\sum_{i=1}^{N_\varepsilon} |h(d_M(m_i, m_k)) - h(d_{M'}(m_i', m_k'))|^2\right)^{1/2} \\ &\leqq \left(\sum_{i=1}^{N_\varepsilon} (a \cdot \sup|h'(t)|)^2\right)^{1/2} \leqq \frac{4a}{r} \tilde{N}_\varepsilon^{1/2}\,. \end{aligned}$$

The last inequality follows from the fact $|h'(t)| = 0$ if $t \geqq r$. Therefore we see

$$\begin{aligned} d(f(m), f'(M')) &\leqq d(f(m), f(m_k)) + d(f(m_k), f'(m_k')) \\ &\leqq \frac{4(a+\varepsilon)}{r} \tilde{N}_\varepsilon^{1/2}\,, \end{aligned}$$

where m_k is the point of $N[\varepsilon]$ with $d_M(m, m_k) \leq \varepsilon$. If a, $\varepsilon \leqq c_5 r/10$, then $f(M) \subset B_{c_5 \tilde{N}_\varepsilon^{1/2}}(f'(M'))$ and similarly $f'(M') \subset B_{c_5 \tilde{N}_\varepsilon^{1/2}}(f(M))$. From Lemma 4.3, the normal projection $P\colon B_{c_5 \tilde{N}_\varepsilon^{1/2}}(f'(M')) \to f'(M')$ is well defined. In the later section, we show that for sufficiently small a, $\varepsilon > 0$ $P|_{f(M)}\colon f(M) \to f'(M')$ is a diffeomorphism.

§6. $T_p f(M)$ and $T_{p'} f'(M')$ are almost parallel

(i) Relation between $\tilde{d}_M$ and d. (II)

Firstly we investigate the relation between $\tilde{d}_M$ and d. We have already done in Lemma 4.2, but here, we need the estimate of $\tilde{d}_M/d$ in the case when $d_M(x, y)$ is small, which is different from previous one.

LEMMA 6.1. *There exists $c_6 > 0$ such that if $\varepsilon < \varepsilon_1/10$ and $d_M(m, n) < \varepsilon_1/10$, then*

$$1 \leqq \frac{\tilde{d}_M(f(m), f(n))}{d(f(m), f(n))} \leqq c_6\,.$$

Proof. Let γ be the minimal geodesic from m to n. Put $d_1 = d_M(m, n)$ and $z = \gamma((r/2) + d_1)$. For $p \in B_{\varepsilon_1}(z) \cap N[\varepsilon]$ with $d_M(n, p) < r/2 - (\varepsilon_1/10)$, if $p' \in B_{\varepsilon_1/10}(p) \cap N[\varepsilon]$, then $p' \in B_{\varepsilon_1}(z) \cap N[\varepsilon]$ and $d_M(n, p') < r/2$. Thus, by the argument of the proof of Lemma 2.2, we see

$$g(\dot{\gamma}(d_1), u') > \frac{1}{4}\left(1 - \frac{r}{4s_\Delta(r/2)}\right),$$

$$g(\dot{\gamma}(d_1), u') - g(\dot{\gamma}(t), u_t) = \int_t^{d_1} \frac{d}{dt} g(\dot{\gamma}(t), u_t)dt < \frac{\varepsilon_1}{10}\left(\frac{16}{r} + r\Delta\right),$$

where u', u_t are the unit initial vector of the minimal geodesic from n, $\gamma(t)$ to p' respectively. This implies

$$\inf_{0 \leqq t \leqq d_1} g(\dot{\gamma}(t), u_t) \geqq \frac{1}{4}\left(1 - \frac{r}{4s_\Delta(r/2)}\right) - \frac{\varepsilon_1}{10}\left(\frac{16}{r} + r\Delta\right) := \beta_1 > 0.$$

Since $|h'(t)| > 3/r$ for $t \in [3r/8, 5r/8]$, and $3r/8 \leqq d_M(p', \gamma(t)) \leqq 5r/8$,

$$|h(d_M(p', m)) - h(d_M(p', n))| = \left| \int_0^{d_1} h'(d_M(p, \gamma(t)) g(\dot{\gamma}(t), u_t) dt \right|$$

$$\geqq \min\left(\frac{r}{10}, d_1\right) \cdot \beta_1 \cdot \frac{3}{r} \geqq \frac{3\beta_1 d_1}{10r}.$$

Combining this with the fact that there exists $c_7 > 0$ such that

$$\sharp(B_{\varepsilon_1/10}(p) \cap N[\varepsilon])/\tilde{N}_\varepsilon \geqq c_7,$$

which is obtained by the same method as Section 3-(i), we get, using the method similar to Section 4-(ii),

$$d(f(m), f(n)) \geqq c_7^{1/2} \cdot \frac{3\beta}{10r} \tilde{N}_\varepsilon^{1/2} d_1.$$

On the other hand, from Lemma 3.2, we get

$$\tilde{d}_M(f(m), f(n)) \leqq c_3 \tilde{N}_\varepsilon^{1/2} d_1.$$

These two estimates imply the conclusion.

For simplicity, we define some constants. For the later purpose, we introduce a new parameter $\sigma > 0$. For fixed $\sigma > 0$, we put

$$\mu = \max(8^{d-1} c_2^{-d} c_2^d c_6^d \sigma, 100\sigma(\Delta + 1)), \qquad \eta_1 = \frac{c_5}{100\mu} \tilde{N}_\varepsilon^{1/2}$$

$$\eta_2 = \frac{\eta_1}{1000\mu}, \qquad \eta_3 = \frac{\sigma c_6 \eta_3}{\mu c_2} \cdot \tilde{N}_\varepsilon^{-1/2},$$

$$\eta_4 = \frac{c_2 \eta_3}{c_6} \cdot \tilde{N}_\varepsilon^{1/2} = \frac{\eta_1}{\mu}, \qquad \eta_5 = \frac{\eta_1}{c_3} \cdot \tilde{N}_\varepsilon^{-1/2}.$$

In the later parts, we denote by $B_\tau(p)$ the ball with radius τ and centered p in $\boldsymbol{R}^{N_\varepsilon}$ and $B_\tau^Q(p)$ is the τ-neighborhood of p in Q with respect to the induced metric of a subset Q in $\boldsymbol{R}^{N_\varepsilon}$. Let $\tilde{P}\colon \boldsymbol{R}^{N_\varepsilon} \to T_p f(M)$ be the normal projection.

(ii) The position of $f(M)$ and $T_p f(M)$.

For $p_0 \in f(M)$, put $\tilde{p}_0 = \tilde{P}(p_0)$.

LEMMA 6.2. *If* $d(p, p_0) \leqq \eta \leqq 2\eta_1$, *then* $d(p_0, \tilde{p}_0) \leqq \eta/1000$.

Proof. Let $B(t, n)$ be the $(d+1)$-dimensional ball centered at $\operatorname{Exp}(p, tn)$ with the radius t in the $(d+1)$-dimensional subspace of $\boldsymbol{R}^{N_\varepsilon}$ spanned by a unit vector n normal to $T_p f(M)$ and the vectors in $T_p f(M)$. Then $B(t, n)$ is tangent to $T_p f(M)$ at p. Put $\tilde{B}(t) = \bigcup_n B(t, n)$.

CLAIM: *If* $t \leqq c_5 \tilde{N}_\varepsilon^{1/2}$, *then* $\tilde{B}(t) \cap f(M) = \{p\}$.

Proof. Suppose that $\tilde{B}(t) \cap f(M)$ contains another point q. Let n be the unit vector normal to $T_p M$ such that $\partial B(t, n) \cap f(M) - \{p\} \neq \phi$. Put $x = \operatorname{Exp}(p, tn)$. Then there exists $q' \in f(M)$ such that $p \neq q'$, $d(x, q') = d(x, f(M)) := t' \leqq t$. Note that the vector $v = \overrightarrow{q'x}$ is perpendicular to $T_{q'} f(M)$. Since $\operatorname{Exp}(q', t'v/|v|) = x$, it contradicts that $\operatorname{Exp}|_{\tilde{B}(t)}$ is a diffeomorphism.

Then this lemma follows from the following elementary fact. In general, let B be the ball in euclidean space with the radius a, tangent to an affine subspace H at p. If we take a point $q \in H$ with $d(p, q) \leqq a/b$ ($b \geqq 1000$), then $d(q, q') \leqq a/b^2$, where q' is a point of ∂B which projects normally on q. q.e.d.

(iii) $\tilde{P}(B_{\eta_1}^{f(M)}(p))$ occupies a "large portion" in $B_{\eta_1}^{T_p f(M)}(p)$.

Let $\langle \cdot, \cdot \rangle$ be the standard inner product of $\boldsymbol{R}^{N_\varepsilon}$.

LEMMA 6.3. *For any* $x \in U_p f(M)$, *there exists* $p_0 \in B_{\eta_1}^{f(M)}(p)$ *such that*

$$\langle \tilde{p}_0, x \rangle \geqq \eta_4 .$$

Proof. Put $A_{\eta_4}^x = \{v = tx + y \mid v \in B_{\eta_1}^{T_p f(M)}(p),\ |t| \leqq \eta_4,\ \langle x, y \rangle = 0\}$. It suffices to prove that $\tilde{P}(B_{\eta_1}^{f(M)}(p))$ is not contained in $A_{\eta_4}^x$. From Lemma 3.2, we see $B_{\eta_1}^{f(M)}(p) \supset f(B_{\eta_5}^M(f^{-1}(p)))$, where $B_\eta^M(\cdot)$ is the ball with radius η in M. Take a maximal η_3-discrete subset $\{n_i\}$ in $B_{\eta_5}^M(f(p))$. From the volume comparison theorem, we have

$$\#\{n_i\} \geqq \frac{b_\Delta(\eta_5)}{b_{-\Delta}(\eta_3/2)} \geqq \left(\frac{\eta_5}{\eta_3}\right)^d ,$$

because

$$s_\Delta(\eta_5) > \frac{\eta_5}{2\Delta^{1/2}} \quad \text{and} \quad s_{-\Delta} < \frac{\eta_3}{(2\Delta)^{1/2}} .$$

From Lemma 3.2, we observe that $\{f(n_i)\}$ is a $c_2\tilde{N}_\varepsilon^{1/2}\eta_3$-discrete subset with respect to $\tilde{d}_M$ in $f(B^M_{\eta_5}(f^{-1}(p)))$. From Lemma 6.1, it is an η_4-discrete subset with respect to d in $B_{\eta_2}(B^{T_pf(M)}_{\eta_1}(p))$. On the other hand, we consider η_4-discrete set $\{n'_i\}$ in $B_{\eta_2}(A^x_{\eta_4})$. Since $\eta_2 \leqq \eta_4/1000$, we easily see that $\{\tilde{P}(n'_i)\}$ is $\eta_4/2$-discrete in $A^x_{\eta_4+\eta_4} \subset A^x_{2\eta_4}$. Then,

$$\sharp\{n'_i\} = \sharp\{\tilde{P}(n'_i)\} \leqq \frac{\mathrm{vol}\,(A^x_{2\eta_4})}{b_0(\eta_4/2)} \leqq \left(\frac{4\eta_1}{\eta_4}\right)^{d-1},$$

From

$$\left(\frac{4\eta_1}{\eta_4}\right)^{d-1} = (4\mu)^{d-1} < \left(\frac{\mu c_2}{c_6 c_3}\right)^d = \left(\frac{\eta_5}{\eta_3\sigma}\right)^d,$$

there exists $(n_i) \notin B_{\eta_2}(A^x_{\eta_4})$, whence the conclusion.

(iv) Estimate of the "angle" between $T_pf(M)$ and $T_{p'}f(M')$.

Put $p' = P(p)$ and take $a \leqq \varepsilon < c_8 := \eta_2 r/10 \cdot \tilde{N}_\varepsilon^{-1/2}$. Hereafter we assume this. Then, for $\eta(\varepsilon) := (10\varepsilon/r)\tilde{N}_\varepsilon^{1/2} < \eta_2$,

$$f(M) \subset B_{\eta(\varepsilon)}(f'(M')) \quad \text{and} \quad f'(M') \subset B_{\eta(\varepsilon)}(f(M)) .$$

For $v \in U_pf(M)$ and $v' \in U_{p'}f'(M')$, let $\sphericalangle\,(v, v')$ be the angle between v and v', which is equal to $\cos^{-1}\langle v, v'\rangle$.

LEMMA 6.4. *For any $v \in U_pf(M)$, there exists $v' \in U_{p'}f'(M')$ such that*

$$\sphericalangle\,(v, v') \leqq \sin^{-1}\left(\frac{1}{25\sigma}\right) := \mu_\sigma .$$

Proof. If not, then there exists $v_0 \in U_pf(M)$ such that

$$\inf_{v' \in U_{p'}f'(M')} \sphericalangle\,(v_0, v') = \max_{v \in U_pf(M)} \Big(\inf_{v' \in U_{p'}f'(M')} \sphericalangle\,(v, v')\Big) > \mu_\sigma .$$

Let $H_{p'}$ be the plane through p' parallel to $T_pf(M)$ and $H = H_{p'} \cap T_{p'}f'(M')$. Then v_0 is perpendicular to H. In fact, let P': $T_pf(M) \to T_{p'}f'(M')$ be the normal projection and decompose v_0 as $v_0 = \lambda_1 v_1 + \lambda_2 v_2$, where $\lambda_1^2 + \lambda_2^2 = 1$, $v_1 \perp H$ and $v_2 \in H$. Since $|\tilde{P}'(\lambda_1 v_1 + \lambda_2 v_2)| = |\tilde{P}'(\lambda_1 v_1) + \lambda_2 v_2| \geqq |\tilde{P}'(v_1)|$ and $|\tilde{P}'(v_0)|$ is minimal, we see $\lambda_2 = 0$ and therefore v_0 is perpendicular to H. For $x = v_0$, we take $p_0 \in B^{f(M)}_{\eta_1}(p)$ satisfying $\langle \tilde{p}_0, v_0\rangle \geqq \eta_4$, by Lemma 6.3. Translate $\tilde{p}_0$ to $p'_1 \in H_{p'}$ and decompose $p'_0 = p'_1 + p'_2 + p'_3$, where p'_1 is v_0-component, $p'_2 \in H$ and p'_3 belongs to the orthogonal complement. Put $\tilde{P}'(p'_i) = q_i$. Then,

$$\begin{aligned} d(p_0, T_{p'}f'(M')) &> d(\tilde{p}_0, T_{p'}f'(M')) - d(\tilde{p}_0, p_0) \\ &= |p_0' - q_0| - \eta_2 - \eta_2 \geqq |p_1' - q_1| - 2\eta_2 \\ &\geqq \eta_4 \sin(\mu_\sigma) - 2\eta_2 \geqq 5\eta_2 - 2\eta_2 = 3\eta_2 . \end{aligned}$$

On the other hand, from $d(p, p_0) \leqq \tilde{d}_M(p, p_0) \leqq \eta_1$, we get

$$d(P(p_0), p') \leqq d(P(p_0), p_0) + d(p_0, p) + d(p, p') \leqq 2\eta_2 + \eta_1 \leqq 2\eta_1 .$$

Therefore, since Lemma 6.2 can be applied,

$$\begin{aligned} d(p_0, T_{p'}f'(M')) &\leqq d(p_0, P(p_0)) + d(P(p_0), T_{p'}f'(M')) \\ &\leqq \eta_2 + \eta_2 = 2\eta_2 . \end{aligned}$$

It is a contradiction. q.e.d.

§7. The diffeomorphism from M to M'

(i) $P|_{f(M)}$ is an injection.

LEMMA 7.1. *$P|_{f(M)}$ is injective.*

Proof. Suppose $P(p) = P(q) = p'$ with $p \neq q$. Note that the vector $\overrightarrow{pq}$ is perpendicular to $T_{p'}f'(M')$. From Lemma 6.4, there exists a unit normal vector n, which is parallel to the orthogonal complement of $T_{p'}f'(M')$ of $\overrightarrow{pq}$, such that

$$\measuredangle (n, \overrightarrow{pq}) \leqq \mu_\sigma .$$

Now, put $x = \mathrm{Exp}(p, c_5\tilde{N}_\varepsilon^{1/2}n)$. Since $\mathrm{Exp}|_{Bc_5\tilde{N}_\varepsilon^{1/2}}(Nf(M))$ is diffeomorphic, we see $d(x, p) < d(x, q)$. Let r be the point of the through x and q and $\overrightarrow{pr} \perp \overrightarrow{qx}$. Note that $d(p, r) \leqq d(p, q)$ and $\mu := \measuredangle (n, \overrightarrow{pq})$. Therefore,

$$d(p, q) \geqq d(p, r) \geqq c_5\tilde{N}_\varepsilon^{1/2} \cos(\mu) > 3\eta_3 .$$

On the other hand, since $f(M) \subset B_{\eta_2}(f'(M'))$ and $P(p) = P(q) = p'$,

$$d(p, q) \leqq d(p, p') = d(p', q) \leqq 2\eta_2 .$$

This is a contradiction. q.e.d.

(ii) $P|_{f(M)}$ is an immersion.

It sufficies to show the following.

LEMMA 7.2.

$$\frac{1 - \sin(\mu_\sigma)}{1 + \lambda} \leqq |dP(\xi)| \leqq \frac{1 + \sin(\mu_\sigma)}{1 - \lambda} \qquad \text{for } \xi \in UM ,$$

where $\lambda = 2\eta(\varepsilon)r/c_2^2\tilde{N}_\varepsilon^{1/2}$.

Proof. Firstly, we estimate the principal curvature of $f(M)$. For $x \in U_p f(M)$, let $c(s) = f(m(s))$ be the curve with $\dot{c}(0) = x$, $m(0) = m$. From the definition, the second fundamental form $H(x, x)$ is the normal component of $d^2/ds^2|_{s=0} c(s)$. Let $v^{\perp}$ be the normal component of the vector v.

$$\begin{aligned} H(x, x) &= \left(\frac{d^2}{ds^2}\Big|_{s=0} c(s)\right)^{\perp} = \left(\frac{d^2}{ds^2}\Big|_{s=0} f(m(s))\right)^{\perp} \\ &= \left(\cdots, h'(d_M(m_i, m)) \operatorname{Hess} d_{M,m_i}\left(\frac{\dot{m}(0)}{|\dot{c}(o)|}, \frac{\dot{m}(0)}{|\dot{c}(0)|}\right)\right. \\ &\quad \left. + h''(d_M(m, m))\left(g\left(\operatorname{grad} d_{M,m_i}, \frac{\dot{m}(0)}{|\dot{c}(0)|}\right)\right)^2, \cdots\right)^{\perp} . \end{aligned}$$

By the argument similar to Lemma 4.1,

$$|H(x, x)| \leqq 2k\tilde{N}_{\varepsilon}^{1/2} \cdot \frac{|\dot{m}(0)|^2}{|\dot{c}(0)|^2} \leqq \frac{2k}{c_2^2} \tilde{N}_{\varepsilon}^{-1/2} .$$

Nextly, let $x(s)$ be the curve on $f(M)$ with $\dot{x}(0) = \xi$ and put $y(s) = P(x(s))$. Then it can be written as $x(s) - y(s) = \ell(s)n(s)$, where $n(s)$ is the unit normal vector field along $y(s)$. Since $\xi - dP(\xi) = \dot{x}(0) - \dot{y}(0) = \dot{\ell}(0)n(0) + \ell(0)\dot{n}(0)$, we get

$$\begin{aligned} \tilde{P}'(\xi) &= \tilde{P}'(dP(\xi) + \dot{\ell}(0)n(0) + \ell(0)\dot{n}(0)) \\ &= dP(\xi) + \ell(0)\tilde{P}'(\dot{n}(0)) , \end{aligned}$$

where $\tilde{P}'$ is the normal projection to $T_{p'}f'(M')$.

Note that $\tilde{P}'(\dot{n}(0))$ is the tangential component of $\dot{n}(0)$. The above estimate implies,

$$\begin{aligned} |\tilde{P}'(\xi) - dP(\xi)| &= |\ell(0)\tilde{P}'(\dot{n}(0))| \leqq \frac{2k}{c_2^2}\eta(\varepsilon)\tilde{N}_{\varepsilon}^{-1/2}|dP(\xi)| \\ &= \lambda|dP(\xi)| . \end{aligned}$$

On the other hand, from Lemma 6.4, if we denote by $\tilde{\xi}$ the parallel translation from p to p' of ξ, then

$$|\tilde{\xi} - P'(\xi)| \leqq \sin(\mu_{\sigma}) .$$

Therefore

$$\begin{aligned} |dP(\xi) - \tilde{\xi}| &\leqq |dP(\xi) - \tilde{P}'(\xi)| + |\tilde{P}'(\xi) - \tilde{\xi}| \\ &\leqq \sin(\mu_{\sigma}) + \lambda|dP(\xi)| . \end{aligned}$$

From this, we get a conclusion.

Finally, we get the diffeomorphism $F\colon M \to M'$ by $F = f'^{-1} \circ P \circ f$.

§ 8. Estimate of *dF*

We show that $|dF|$ is close to 1, if we take sufficiently small $r > 0$, a, $\varepsilon > 0$.

(i) Triangle comparison theorem.

Following lemma is an easy consequence of triangle comparison theorem in [3] Chap. 2.

Let $\Delta(a, b, c) \subset M$ be the geodesic triangle whose segments are a, b, c and $\ell(a)$ be the length of a and $\sphericalangle(a, b)$ is the angle between a and b.

LEMMA 8.1. *For any $\delta' > 0$, there exist c_9, $c_{10} > 0$ such that if $\Delta(a, b, c) \subset M$ and $\Delta(a', b', c') \subset M'$ satisfy the following,*

i) $c_9 \geqq \ell(a),\ \ell(b),\ \ell(a'),\ \ell(b') \geqq c_9/10$,

ii) $|\ell(a) - \ell(a')|,\ |\ell(b) - \ell(b')|,\ |\ell(c) - \ell(c')| \leqq c_{10}$,

then $|\sphericalangle(a, b) - \sphericalangle(a', b')| \leqq \delta'$.

(ii) Estimate of $|d_M(m_i, m) - d_{M'}(m_i', F(m))|$.

LEMMA 8.2. *There exist c_{11}, $c_{12} > 0$ such that if $a \leqq \varepsilon < c_{12}$, then*

$$|d_M(m_i, m) - d_{M'}(m_i', F(m))| \leqq c_{11}\varepsilon\ .$$

Proof. Take $m_j \in N[\varepsilon]$ and $m_k' \in N'[\varepsilon]$ satisfying

$$d_M(m, m_j) \leqq \varepsilon \quad \text{and} \quad d_{M'}(F(m), m_k') \leqq \varepsilon\ .$$

From this,

$$\begin{aligned} d(f'(m_j'), f'(m_k')) &\leqq d(f'(m_j'), f(m_j)) + d(f(m_j), f(m)) \\ &\quad + d(f(m), P\circ f(m)) + d(P\circ f(m), f'(m_k')) \\ &\leqq \frac{4a}{r}\tilde{N}_\varepsilon^{1/2} + \varepsilon c_3\tilde{N}_\varepsilon^{1/2} + \eta(\varepsilon) + \varepsilon c_3\tilde{N}_\varepsilon^{1/2} \\ &\leqq \left(\frac{4}{r} + c_3 + \frac{10}{r} + c_3\right)\tilde{N}_\varepsilon^{1/2}\varepsilon := c_{13}\tilde{N}_\varepsilon^{1/2}\ . \end{aligned}$$

We recall Lemma 4.2 and take $\alpha = c_2\varepsilon_1/10$. For sufficiently small $a, \varepsilon > 0$, we see $c_{13}\varepsilon \leqq \tilde{\alpha}$. Thus we see $\tilde{d}_{M'}(f'(m_j'), f'(m_k')) \leqq (c_2\varepsilon_1/10)\tilde{N}_\varepsilon^{1/2}$ and from Lemma 3.2, $d_{M'}(m_j', m_k') \leqq \varepsilon_1/10$. So we can use Lemma 6.2, then,

$$\begin{aligned} d_{M'}(m_j', m_k') &\leqq \frac{c_6}{c_2}\tilde{N}_\varepsilon^{-1/2} d(f'(m_j'), f'(m_k')) \\ &\leqq \frac{c_6}{c_2}\tilde{N}_\varepsilon^{-1/2}\left(\frac{4a}{r}\tilde{N}_\varepsilon^{1/2} + \varepsilon c_3\tilde{N}_\varepsilon^{1/2} + \eta(\varepsilon) + \varepsilon c_3\tilde{N}_\varepsilon^{1/2}\right) \\ &\leqq \frac{c_6 c_{13}}{c_2}\varepsilon\ . \end{aligned}$$

From the above, we observe,

$$\begin{aligned}
&|d_M(m, m_i) - d_{M'}(F(m), m_i')| \\
&\quad \leqq |d_M(m_i, m_j) - d_{M'}(m_i', m_j')| + d_M(m, m_j) + d_{M'}(F(m), m_j') \\
&\quad \leqq 2ra + d_M(m, m_j) + d_{M'}(F(m), m_k') + d_{M'}(m_k', m_j') \\
&\quad \leqq 2r\varepsilon + \varepsilon + \varepsilon + \frac{c_6 c_{13}}{c_2}\varepsilon := c_{11}\varepsilon .
\end{aligned}$$

q.e.d.

(iii) Definition of the isometry I: $T_mM \to T_{F(m)}M'$.

$$\text{Put } u_i = \exp_m^{-1} m_i / |\exp_m^{-1} m_i|$$

and

$$u_i' = \exp_{F(m)}^{-1} m_i' / |\exp_{F(m)}^{-1} m_i'| .$$

Combining Lemma 8.1 and 8.2, we get for any $\delta'' > 0$, there exist c_{14}, $c_{15} > 0$ such that if $c_{14} \geqq d_M(m_i, m) \geqq c_{14}/10$ and $\varepsilon < c_{15}$, then

$$|\langle u_i, u_j\rangle - \langle u_i', u_j'\rangle| < \delta'' .$$

We choose $u_{i_1}, \cdots, u_{i_d}$ satisfying $\langle u_{i_j}, u_{i_j}\rangle \geqq 1 - (1/100d^2)$ and $|\langle u_{i_j}, u_{i_k}\rangle| \leqq 1/100d^2$, $(j \neq k)$. From these, we get the orthonormal basis $\{e_i\}_{i=1}^d$ of T_mM by Schmidt's orthogonalization. Namely $e_1 = u_{i_1}$,

$$e_{j+1} = \left(u_{i_{j+1}} - \sum_{k=1}^{d} \langle u_{i_{j+1}}, e_k\rangle e_k\right) \Big/ \left|u_{i_{j+1}} - \sum_{k=1}^{d} \langle u_{i_{j+1}}, e_k\rangle e_k\right|, \cdots .$$

We also get the orthonormal basis $\{e_i'\}_{i=1}^d$ of $T_{F(m)}M'$ from $\{u_{i_j}'\}_{i=1}^d$. Put $a_{jk} = \langle e_j, u_{i_k}\rangle$ and $a_{jk}' = \langle e_j', u_{i_k}'\rangle$. Then by inductive arguments, we see

$$|a_{jk} - a_{jk}'| \leqq (100d)^{j+k}\delta'' \leqq (100d)^{2d}\delta'' .$$

We define the isometry I: $T_mM \to T_{F(m)}M'$ by $I(e_i) = e_i'$.

(iv) Estimate of dF.

From the definition, we know

$$df_m(\xi) = (\cdots, h'(t_i) \textstyle\sum_j a_{ij}\xi_j, \cdots)$$

for $\xi = \sum \xi_j e_j \in U_mM$ and $t_i = d_M(m, m_i)$. Put $t_i' = d_{M'}(F(m), m_i'')$.

LEMMA 8.3. *For any $\delta > 0$, there exist c_{16}, c_{17}, $c_{18} > 0$ such that if $r < c_{16}$, $\kappa < c_{17}$ (see § 2), a, $\xi < c_{18}$, then,*

$$|dF(\xi) - I(\xi)| < \delta .$$

Proof. Firstly, we estimate $|df(\xi) - df'(I(\xi))|$. From the definition,

$$\begin{aligned}
|df(\xi) - df'(I(\xi))|^2 &= \sum_{i=1}^{N_\varepsilon} \Big(h'(t_i) \sum_j a_{ij}\xi_j - h'(t_i') \sum_j a_{ij}'\xi_j\Big)^2 \\
&\leqq \sum_{t_i, t_i' \in [r/8, 7r/8]} + \sum_{\text{otherwise}} .
\end{aligned}$$

From Lemma 8.2, there exists $c_{19} > 0$ such that if a, $\varepsilon < c_{19}$, then $|h'(t_i) - h'(t_i')| \leqq c_{17}/10d$. Thus, from $|h'(t)| \leqq 4/r$,

$$\begin{aligned}(\text{first term}) &\leqq \sum_{t_i, t_i' \in [2r/8, 6r/8]} \{h'(t_i)(\sum_j (a_{ij} - a_{ij}')\xi_j) \\ &\qquad\qquad + (h'(t_i) - h'(t_i')) \sum_j a_{ij}'\xi_j\}^2 \\ &\leqq \left(\frac{4}{r}(100d)^{2d}\delta'' d^2 + c_{17}\cdot \frac{d^2}{10d}\right)^2 \tilde{N}_\varepsilon .\end{aligned}$$

Note that if $t_i \in [0, r/8] \cup [7r/8, r]$, then $t_i' \in [0, 2r/8] \cup [6r/8, r] := J$. Since $c_{17} > \kappa > |h'(t)|$ on $t \in J$, we see

$$\begin{aligned}(\text{second term}) &\leqq (\sum_{t_i, t_i' \in J} |h'(t_i) + h'(t_i')|^2)(|\sum a_{ij}\xi_j| + |\sum a_{ij}'\xi_j|)^2 \\ &\leqq 4c_{17}^2 \cdot 4d^4 \tilde{N}_\varepsilon .\end{aligned}$$

Therefore,

$$\begin{aligned}|df(\xi) - df'(I(\xi))|^2 &\leqq \left(\left((100d)^{2d} + \frac{d}{10} + 4d^2\right)\cdot \frac{4}{r}\right)^2 (\delta'' + 2c_{17})^2 \tilde{N}_\varepsilon \\ &\leqq (100d)^{6d} \cdot r^{-2}(\delta'' + 2c_{17})^2 \tilde{N}_\varepsilon .\end{aligned}$$

Secondly, from Lemma 7.2, we find

$$|dP \circ df(\xi) - df(\xi)| \leqq 2\eta(\varepsilon) + \frac{\sin(\mu_\sigma) + \lambda}{1 + \lambda}|df(\xi)| .$$

For fixed $r > 0$, there exists $c_{20}, c_{21} > 0$ such that if a, $\varepsilon < c_{20}$, $\sigma > c_{21}$, then the righthand side of the above inequality is smaller than $(10^5 d)^{-d}(\delta/10c_3)|df(\xi)|$, by the definition of $\eta(\varepsilon)$ and μ_σ (§ 6, § 7).

Therefore since $c_2 = (10^5 d)^{-d/2}3/2r$, (§ 3 Remark),

$$\begin{aligned}&\frac{1}{\inf_\xi |df'(\xi)|}|dP \circ df(\xi) - df' \circ I(\xi)| \\ &\quad \leqq (c_2 \tilde{N}_\varepsilon^{1/2})^{-1}\left((100d)^{3d} r^{-1}(\delta'' + 2c_{17})\tilde{N}_\varepsilon^{1/2} + (10^5 d)^{-d}\frac{\delta}{10c_3}c_3 \tilde{N}_\varepsilon^{1/2}\right) \\ &\quad \leqq (10^5 d)^{5d}(\delta'' + 2c_{17}) + \frac{\delta}{10} .\end{aligned}$$

For $\delta'' > 0$ satisfying $(10^5 d)^{5d}\delta'' \leqq \delta/10$, take $c_{16} > 0$ as $c_{16} \leqq c_{14}$ and $c_{17} > 0$ as $(10^5 d)^{5d} 2c_{17} \leqq \delta/10$ and $c_{18} > 0$ as $c_{18} \leqq \min(c_{15}, c_{19}, c_{20})$.

Finally we get,

$$\begin{aligned}|dF(\xi) - I(\xi)| &= |df'^{-1} \circ dP \circ df(\xi) - I(\xi)| \\ &\leqq \frac{1}{\inf_\xi |df'(\xi)|}|dP \circ df(\xi) - df' \circ I(\xi)| < \delta . \qquad \text{q.e.d.}\end{aligned}$$

§9. In the case when M is noncompact

In the case when M is noncompact, let M_b be the set of all points m of M with $d_M(m, m_0) < b$ for fixed $m_0 \in M$. In the above, we get the map F_b: $M_{b-2r} \to M'_b$. Note that the estimate of constants do not depend on b, thus for fixed b_0, $F_b|_{M_{b_0}} = F_{b'}|_{M_{b_0}}$ for b, $b' \gg b_0$. Let F: $M \to M'$ be the inductive limit of F_b.

We see that F is a diffeomorphism. The injectivity and immersivity follows from those of F_b. Surjectivity follows from Lemma 8.3 and the implicit function theorem. q.e.d.

§10. Proof of Theorem 2

From the result of Heintze-Karcher [8] or Maeda [11], we get the estimate of the injectivity radius i_M in terms of d, Δ, ρ, v, namely,

$$i_M \geqq \min\left(\pi/\Delta^{1/2},\ \frac{\pi v}{\omega_d}\cdot\exp\left(-(d-1)\rho\Delta^{1/2}\right)\right).$$

Therefore we can use Theorem 1. Take a, $\varepsilon > 0$ which satisfy the assumption of Theorem 1. Let M_{N_1} be the set of elements in $M(d, \Delta, \rho, v)$, which have a minimal ε-dense subset $\{m_i\}_{i=1}^{N_1}$. From the volume comparison theorem, we see $N_1 \leqq b_{-\Delta}(\rho)/b_\Delta(\varepsilon/2) := N_0$. Therefore it suffices to estimate the number of the diffeomorphism classes in M_{N_1} for $N_1 \leqq N_0$.

Now, take a function

$$\Phi\colon M_{N_1} \longrightarrow Q = \prod_{k=1}^{N_1(N_1-1)} [\log(\varepsilon/2), \log(\rho)]$$

defined by

$$\Phi(M) = \{\log(d_M(m_i^k, m_j^k))\}_{k=1}^{N_1(N_1-1)},$$

where Q is the direct product of the intervals $[\log(\varepsilon/2), \log(\rho)]$ and k is a loxicographic order of (i, j). We define the distance d_Q on Q by,

$$d_Q(x, y) = \max_{1\leqq k\leqq N_1(N_1-1)} |x_k - y_k|,$$

where $x = \{x_i\}$, $y = \{y_i\}$.

Then, Theorem 1 says that if $d_Q(\Phi(M), \Phi(M')) \leqq -\log(1-a) := b_1$, then M and M' are diffeomorphic. Therefore it is sufficient to estimate the cardinality of maximal set P_{N_1} in Q, of which elements α, β $(\alpha \neq \beta)$ satisfy $d_Q(\alpha, \beta) > b_1$,

$$\# P_{N_1} \leqq \left(\frac{2b_2}{b_1}\right)^{N_1(N_1+1)} \leqq \left(\frac{2b_2}{b_1}\right)^{N_0(N_0+1)},$$

where $b_2 = \log(\rho) - \log(\varepsilon/2)$. After all we can estimate the number of the diffeomorphism classes of $M(d, \Delta, \rho, v)$, which is smaller than $N_0(2b_2/b_1)^{N_0(N_0+1)}$.

q.e.d.

§11. Outline of the proof of Theorem 3

Let M be a compact d-dimensional Riemannian manifold with $|K_M| \leqq \Delta$ and $\mathrm{Ric}_M \geqq d-1$. Let $m, n, m_1, m_2, \cdots$, be the points of M and $p, q, p_1, p_2, \cdots$, be the points of S^d. We denote by $TD(m)$ the interior of the tangential cut locus i.e., $TD(m) =$ the interior of $\{v \in T_m M \mid d_M(m, \exp_m v) = |v|\}$. For the linear isometry I: $T_p S^d \to T_m M$, we define the map $F = \exp_m \circ I \circ \exp_p^{-1}$: $B_\pi(p) \to M$. Put $D' = \exp_p(I^{-1}(TD(m)))$. From the theorem of Myers, we see $D' \subset B_\pi(p)$. Moreover if the closure of D' is not contained in $B_\pi(p)$, then $\mathrm{diam}_M = \pi$, so M is isometric to S^d by Cheng's Theorem [2]. We may argue the case when the closure of D' is contained in $B_\pi(p)$.

We give an outline of the proof of Theorem 3. From $|K_M| \leqq \Delta$, $|dF|$ can be estimated in D'. We see that $\mathrm{vol}(S^d - D')$ is small and $|dF|$ is close to 1 on much part in D' —this is "good" part—, using the fact $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta$. Since the volume of the "bad" part is small, we can choose $\varepsilon/2$-dense, $\varepsilon/4$-discrete subset $\{p_i\}$ of S^d in D' such that the geodesic connecting the points of $\{p_i\}$ intersects small "bad" part. So we see that $d_{S^d}(p_i, q_j)$ is not much smaller than $d_M(m_i, m_j)$, where $m_i = F(p_i)$. Therefore, if we see that

(1) $\{m_i\}$ is ε-dense, $\varepsilon/10$-discrete in M.

(2) $d_{S^d}(p_i, p_j)$ is not much larger than $d_M(m_i, m_j)$,

then, from Theorem 1, we find that M is diffeomorphic to S^d. We show (1) by the following arguments. If not, then there exists a point $n \in M$ such that $\min d_M(n, m_i)$ is larger than $3\varepsilon/2$. Since F does not much expand on "good" part and so $B_{\varepsilon/4}(n)$ is intersect only "bad" part. But since "bad" part is very small, it cannot cover $B_{\varepsilon/4}(n)$. This contradicts the fact F is surjection. Assume that (2) does not hold, namely there exist p_i, p_j such that $d_{S^d}(p_i, p_j)$ is much larger than $d_M(m_i, m_j)$. Let B_1, B_2 be the ball with the center p_i, p_j, of which radius is a half of $d_{S^d}(p_i, p_j)$. From the assumption, we see that $\mathrm{vol}(B_1 \cup B_2)$ is much larger than $\mathrm{vol}(F(B_1 \cup B_2))$. It contradicts the fact $\mathrm{vol}(M) > \mathrm{vol}(S^d) - \delta$.

§ 12. Estimate of dF

LEMMA 12.1. i) $|\det F| \leqq 1$ *on* D'.

ii) *For any* $\delta_3 > 0$, *there exists* $L = L(d, \Delta\colon \delta_3) > 0$ *such that*

$$|dF| \leqq L \qquad \text{on } B_{\pi-\delta}(p)\,.$$

Proof. From $\mathrm{Ric}_M \geqq d-1$, i) follows from the volume comparison theorem (cf. [7] or [13]). For ii), we quote from [1] 6.4.1, that is $|(d\exp_m)_{rv}w| \leqq |w|(s_{-\Delta}(t)/r)$ on M, where $|v| = 1$, $v \perp w$ and this inequality holds as long as $s_{(1/2)(-\Delta+\Delta)}(r) = r$ is positive. Since $|(d\exp_p)_{rv}w| = |w|(\sin(r)/r)$ on S^d, we may put $L = s_{-\Delta}(\pi - \delta_3)/\sin(\pi - \delta_3)$. q.e.d.

Put $\bar{A}[\delta_4] = \{q \in D' \,\|\, dF_q| > 1 + \delta_4\}$ and $\bar{B}[\delta_4] = \{q \in D' \,\|\det dF_q| < 1 - \delta_4\}$. Notice that $\bar{A}$ does not mean the closure of A here.

LEMMA 12.2. *For any* δ_4, $\delta_5 > 0$, *there exists* $\delta_6 = \delta_6(d, \Delta; \delta_5) > 0$ *such that if* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_6$, *then* $\mathrm{vol}(\bar{A}[\delta_4]) < \delta_5$, $\mathrm{vol}(\bar{B}[\delta_4]) < \delta_5$ *and* $\mathrm{vol}(S^d - D') < \delta_5$.

Since the proof of this lemma is elementary but complicated, so we only give here an outline and the detailed proof is left over to Section 14. It seems to be able to prove more easily.

From Lemma 12.1, F is volume decreasing. With $F(D') = M$ and $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta$, we see that the $\mathrm{vol}(\bar{B}[\delta_4]) < \delta_5$ and $\mathrm{vol}(S^d - D') < \delta_5$. To show the first inequality, we observe that the arguments of the equality case of the volume comparison theorem in [8] can be modified to the near-equality case. So we find K_M is close to 1 on much part. From this, using Rauch's comparison theorem, we see $|dF|$ is close to 1 on much part.

§ 13. Proof of Theorem 3

(i) Construction of ε-dense set $\{p_i\}$ on S^d.

LEMMA 13.1. *For any* δ_7, $\delta_8 > 0$, *there exists* $\delta_9 = \delta_9(d, \Delta; \delta_7, \delta_8) > 0$ *and a* δ_7*-dense subset* $\{p_i\}$ *of* S^d *in* $B_{\pi-\delta_7/10}(p)$ *such that if* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_9$, *then*

$$\frac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \leqq 1 + \delta_8 \qquad \textit{for } d_{S^d}(p_i, p_j) < \frac{\pi}{20}\,.$$

Proof. We may assume $0 < \delta_8 < \delta_7 < 1$. Take *a* $\delta_7/2$-dense, $\delta_7/2$-discrete subset $\{q_i\}_{i=1}^N$ of S^d in $B_{\pi-\delta_7/20}(p)$. Put $N=\sharp\{q_i\}$ and $B_i=B_{\delta_7/100}(q_i)$. Note that $B_i\subset B_{\pi-\delta_7/20}(p)$. Take

$$\delta_{10}\leqq\left(\frac{d}{20N}\cdot\frac{10}{\pi}\cdot b_1\left(\frac{\delta_8}{100}\right)\cdot\left(\frac{\delta_8}{1000}\right)^{d-1}\right)^{1/(d-1)}.$$

We define

$$\Lambda[\delta_8]=\{q\in B_{\pi-\delta_{10}}(p)\,|\,|dF_q|\leqq 1+\delta_8/2\}=B_{\pi-\delta_{10}}(p)-\overline{A}[\delta_8/2]\,.$$

From Lemma 12.2, there exists $\delta_9>0$ such that if $\operatorname{vol}(M)\geqq\operatorname{vol}(S^d)-\delta_9$, then

$$\operatorname{vol}(B_{\pi-\delta_{10}}(p)-\Lambda[\delta_8])<\frac{1}{20N}\cdot b_1\left(\frac{\delta_8}{100}\right)\cdot\frac{2}{d}\cdot\left(\frac{\alpha}{4}\right)^d\sin^{1-d}\left(\frac{\pi}{10}\right),$$

where $\alpha=\delta_7\delta_8/200L$ and $L=L(\delta_{10})=s_{-\Delta}(\pi-\delta_{10})/\sin(\pi-\delta_{10})$ in Lemma 12.1.

Hereafter we denote by $\gamma_{p,q}$ the minimal geodesic from p to q. Then, we observe that for $q_i'\in B_i$, $q_j'\in B_j$, if $\gamma_{q_i',q_j'}\subset B_{\pi-\delta_{10}}(p)$, then

$$\begin{aligned}d_M(F(q_i'),F(q_j'))&\leqq\int_{\gamma_{q_i',q_j'}}|dF|\,dt\\&=\int_{\Lambda[\delta_8]\cap\gamma_{q_i',q_j'}}|dF|\,dt+\int_{\gamma_{q_i',q_j'}-\Lambda[\delta_8]}|dF|\,dt\\&\leqq(1+\delta_8/2)d_{S^d}(q_i',q_j')+L\cdot m(\gamma_{q_i',q_j'}-\Lambda[\delta_8]):=A_1\end{aligned}$$

where $m(\cdot)$ is the canonical measure on $\gamma_{q_i',q_j'}$.

If $m(\gamma_{q_i',q_j'}-\Lambda[\delta_8])\leqq\alpha$, then

$$\begin{aligned}\frac{d_M(F(q_i'),F(q_j'))}{d_{S^d}(q_i',q_j')}&\leqq\frac{A_1}{d_{S^d}(q_i',q_j')}\\(*)\qquad&\leqq 1+\frac{\delta_8}{2}+\frac{L\alpha}{d_{S^d}(q_i',q_q')}\\&\leqq 1+\frac{\delta_8}{2}+\frac{\delta_8}{2}=1+\delta_8\,.\end{aligned}$$

In the following, we prove that p_i can be taken in $\Lambda[\delta_8]\cap B_i$. For the existence of $p_1\in B_1\cap\Lambda[\delta_8]$, we only note the inequality $\operatorname{vol}(B_{\pi-\delta_{10}}-\Lambda[\delta_8])<\operatorname{vol}(B_1)$.

Nextly, suppose that there exist points $p_1,p_2,\cdots,p_k$ ($p_i\in B_i$) such that

$$\frac{d_M(F(p_i),F(p_j))}{d_{S^d}(p_i,p_j)}\leqq 1+\delta_8\qquad\text{for } d_{S^d}(p_i,p_j)\leqq\frac{\pi}{20}\,.\quad(1\leqq i,j\leqq k)$$

Then, we show that there exists $p_{k+1} \in B_{k+1}$ which satisfies

$$\frac{d_M(F(p_{k+1}), F(p_i))}{d_{S^d}(p_{k+1}, p_i)} \leqq 1 + \delta_8 \qquad \text{for } i \leqq k\,.$$

In fact, if not, then for any $q \in B_{k+1}$, there exists $p_i \in B_i$ such that

$$\frac{d_M(F(q), F(p_i))}{d_{S^d}(q, p_i)} > 1 + \delta_8\,.$$

Then from $(*)$, $m(\gamma_{q,p_i} - \Lambda[\delta_8]) > \alpha$ or $\gamma_{q,p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi$, where $\tilde{p}$ is the antipodal point of p. Let S_i^1 be the set of $q \in B_{k+1}$ such that $m(\gamma_{q,p_1} - \Lambda[\delta_8]) > \alpha$ and S_i^2 be the set of $q \in B_{k+1}$ such that $\gamma_{p_1,q} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi$ and $S_i = S_i^1 \cup S_i^2$. Since, by the assumption, $B_{k+1} \subset \bigcup_i S_i$, we may assume that

$$(**) \qquad \operatorname{vol}(S_1) = \max_i \operatorname{vol}(S_i) \geqq \frac{1}{2N} \cdot \operatorname{vol}(B_{k+1})\,.$$

Let C^i be the cone consisting of the points of $\gamma_{p_1,q}(q \in S_1^i)$ and $\tilde{C}^i = \exp_{p_1}^{-1}(C^1)$. Put $E_t^i = C^i \cap B_t(p_1)$. Since $m(\gamma_{q,p_1} - \Lambda[\delta_8]) > \alpha$, for $q \in S_1^1$, from the Fubini's theorem, we observe

$$\begin{aligned}
&\operatorname{vol}(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) \\
&\quad \geqq \int_{U_{p_1}S^d \cap \tilde{C}_1 \ni v} \left(\int_0^{\pi-\delta_{10}} \chi_{\gamma_v - \Lambda[\delta_8]}(t) \cdot \sin^{d-1}(t)\, dt \right) dv_{U_{p_1}S^d}\,.
\end{aligned}$$

where γ_v is the geodesic emanating from p_1 with initial vector v, $\chi_A(t)$ is the characteristic function of the set A and $dv_{U_pS^d}$ is the canonical measure on U_pS^d induced from Lebesgue measure on T_pS^d.

$$\begin{aligned}
&\geqq \int_{U_{p_1}S^d \cap \tilde{C}_1} \left(\left(\int_0^{\alpha/2} + \int_{\pi-\delta_{10}-\alpha/2}^{\pi-\delta_{10}} \right) \sin^{d-1}(t)\, dt \right) dv_{U_{p_1}S^d} \\
&\geqq \int_{U_{p_1}S^d \cap \tilde{C}_1} \left(\int_0^{\alpha} \left(\frac{t}{2} \right)^{d-1} dt \right) dv_{U_{p_1}S^d} \\
&= \int_{U_{p_1}S^d \cap \tilde{C}_1} \frac{2}{d} \left(\frac{\alpha}{2} \right)^d dv_{U_{p_1}S^d}\,.
\end{aligned}$$

Namely,

$$\int_{U_{p_1}S^d \cap \tilde{C}_1} dv_{U_{p_1}S^d} \leqq \operatorname{vol}(B_{\pi-\delta_{10}} - \Lambda[\delta_8]) \cdot \frac{d}{2} \cdot \left(\frac{2}{\alpha} \right)^{\alpha}.$$

On the other hand, since $d_{S^d}(p_1, B_{\delta_{10}}(\tilde{p})) > \delta_8/100$, we see

$$\operatorname{vol}(E_{\pi/10}^2) \leqq \operatorname{vol}(E)\,,$$

where E is the cone in S^d, which contains $B_{\delta_{10}}(\tilde{p})$ far from its summit with distance $\delta_8/100$ and the length of generating line is smaller than $\pi/10$. From the spherical trigonometry, we calculate

$$\mathrm{vol}\,(E) \leqq \frac{1}{d}\cdot\frac{\pi}{10}\cdot\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1}.$$

Thus we estimate, from $m(\gamma_v \cap B_1) < \delta_8/50$.

$$\begin{aligned}
\mathrm{vol}\,(S_1) &\leqq \mathrm{vol}\,(E^1_{\pi/10}\cap B_1) + \mathrm{vol}\,(E^2_{\pi/10}) \\
&= \int_{U_{p_1}S^d\cap\tilde{C}_1\ni v}\left(\int_\gamma \chi_{(\gamma_v\cap B_1)}(t)\sin^{d-1}(t)\,dt\right)dv_{U_{p_1}S^d} + \mathrm{vol}\,(E^2_{\pi/10}) \\
&\leqq \int_{U_{p_1}S^d\cap\tilde{C}_1}\sin^{d-1}\left(\frac{\pi}{10}\right)\cdot\frac{\delta_8}{50}\,dv_{U_pS^d} + \mathrm{vol}\,(E^2_{\pi/10}) \\
&\leqq \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8])\cdot\frac{d}{2}\cdot\left(\frac{2}{\alpha}\right)^d\cdot\sin^{d-1}\left(\frac{\pi}{10}\right)\cdot\frac{\delta_8}{50} \\
&\quad + \frac{1}{d}\cdot\frac{\pi}{10}\cdot\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1} \\
&\leqq \frac{1}{10N}\cdot b_1\left(\frac{\delta_8}{100}\right) < \frac{1}{2N}\mathrm{vol}\,(B_{k+1}),
\end{aligned}$$

namely,

$$\mathrm{vol}\,(S_1) < \frac{1}{2N}\cdot\mathrm{vol}\,(B_{k+1}),$$

It contradicts $(**)$. q.e.d.

(ii) Proof of Theorem 3.

We take a, $\varepsilon > 0$, which satisfy the assumption of Theorem 1. For $\delta_7 = \varepsilon/2$, take $\delta_8 > 0$ satisfying $\delta_8 \leqq \min\,((1/2)b_\Delta(\delta_7/10)\omega_d^{-1}, a/10)$. Let $\{p_i\}$ and $\alpha > 0$ be the same as in Lemma 13.1. From Theorem 1, it suffices to prove that there exists $\delta > 0$ such that, if $\mathrm{vol}\,(M) \geqq \mathrm{vol}\,(S^d) - \delta$, then $\{F(p_i)\}$ is an ε-dense, $\varepsilon/10$-discrete in M and it satisfies

$$\frac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \geqq 1 - a\,, \qquad \text{for } 0 < d_{S^d}(p_i, p_j) < \frac{\pi}{20}.$$

CLAIM 1: *$\{F(p_i)\}$ is $2\delta_7$ $(=\varepsilon)$-dense in M.*

Proof of Claim 1. If not, then there exists $n \in M$ such that

$$B_{\delta_7/10}(n) \cap (\bigcup_i B_{3\delta_7/2}(F(p_i))) = \phi\,.$$

Put

$$B'_i = \{q \in B_{\delta_7}(p_i) \mid q \in \gamma_{q',p_i},\ q' \in \partial B_{\delta_7}(p_i),$$
$$m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \alpha \text{ or } \gamma_{q',p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi\}$$

and $\tilde{B}_i = B_{\delta_7}(p_i) - B'_i$. From $(*)$ in the proof of Lemma 13.1, we see $F(\bigcup_i \tilde{B}_i) \subset (\bigcup B_{3\delta_7/2}(F(p_i)))$.

From the similar argument to Lemma 13.1, we see

$$\mathrm{vol}\,(B'_i) \leqq \frac{d}{2}\cdot\left(\frac{4}{\alpha}\right)^d \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) + \frac{1}{d}\left(\frac{\pi}{10}\right)\left(\frac{1000\delta_{10}}{\pi\delta_8}\right)^{d-1} := A_1$$
$$\mathrm{vol}\,(\tilde{B}_i) \geqq \mathrm{vol}\,(B_{\delta_7}(p_i)) - A_1\,.$$

Note that

$$\mathrm{vol}\,(F(\tilde{B}_i \cap D' - \bar{B}[\delta_8]) \geqq (1-\delta_8)\,\mathrm{vol}\,(\tilde{B}_i \cap D' - \bar{B}[\delta_8]),$$

where $\bar{B}[\delta_8]$ appears in Lemma 12.2.

From this, we have

$$\begin{aligned}
\mathrm{vol}\,(M) &\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + \mathrm{vol}\,(\bigcup_i F(\tilde{B}_i)) \\
&\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\,\mathrm{vol}\,(\bigcup (B_i \cap D' - \bar{B}[\delta_8])) \\
&\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\mathrm{vol}\,(\bigcup_i \tilde{B}_i)) - \mathrm{vol}\,(S^d - D') - \mathrm{vol}\,(\bar{B}[\delta_8]) \\
&\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + (1-\delta_8)(\mathrm{vol}\,(\bigcup_i B_{\delta_7}(p_i))) - NA_1 - \mathrm{vol}\,(S^d - D') \\
&\quad - \mathrm{vol}\,(\bar{B}[\delta_8]) \\
&\geqq \mathrm{vol}\,(B_{\delta_7/10}(n)) + \mathrm{vol}\,(S^d) - \delta_8\,\mathrm{vol}\,(S^d) - NA_1 - \mathrm{vol}\,(S^d - D') \\
&\quad - \mathrm{vol}\,(\bar{B}[\delta_8])
\end{aligned}$$

where $N = \#\{p_i\}$.

From Lemma 12.2, there exists $\delta_{11} > 0$ such that if $\mathrm{vol}\,(M) \geqq \mathrm{vol}\,(S^d) - \delta_{11}$, then

$$\begin{aligned}
\delta_8\,\mathrm{vol}\,(S^d) + NA_1 + \mathrm{vol}\,(S^d - D') + \mathrm{vol}\,(\bar{B}[\delta_8]) &< b_\Delta(\delta_7/10) \\
&\leqq \mathrm{vol}\,(B_{\delta_7/10}(n)).
\end{aligned}$$

(The constants are determined in following order, $\delta_7 \to \delta_8 \to \delta_{10} \to L \to \alpha \to \delta_{11}$.)

Therefore, we see,

$$\mathrm{vol}\,(M) > \mathrm{vol}\,(S^d) + \mathrm{vol}\,(B_{\delta_7/10}(n)) - b_\Delta(\delta_7/10) \geqq \mathrm{vol}\,(S^d) \geqq \mathrm{vol}\,(M)\,.$$

It is a contradiction.

CLAIM 2: $\dfrac{d_M(F(p_i), F(p_j))}{d_{S^d}(p_i, p_j)} \geqq 1 - \delta_8 > 1 - a\,.$

Proof of Claim 2. If not, then we may assume $d_M(F(p_1), F(p_2)) < (1-\delta_8)d_{S^d}(p_1, p_2)$, Put $d' = d_{S^d}(p_1, p_2)$ and $d'' = d_M(F(p_1), F(p_2))$. There exists $\delta_{12} > 0$ such that

$$b_1\left(\frac{d'}{2}\right) - b_1\left(\frac{d'}{2} - \delta_{12}\right) < \frac{1}{10} \cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right).$$

For this δ_{12}, similarly as Lemma 13.1, there exists $\eta > 0$ such that if $d_M(F(q'), F(p_i)) > d'/2$ for $q' \in \partial B_{d'/2-\delta_{12}}(p_i)$, then

$$m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \eta \quad \text{or} \quad \gamma_{q',p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi .$$

Put

$$B = \bigcup_{i=1}^{2} (B_{d'/2-\delta_{12}}(p_i) - \{q \in B_{d'/2-\delta_{12}}(p_i) \mid q \in \gamma_{q',p_i},$$
$$q' \in \partial B_{d'/2}(p_i),\ m(\gamma_{q',p_i} - \Lambda[\delta_8]) > \eta \quad \text{or} \quad \gamma_{q',p_i} \cap B_{\delta_{10}}(\tilde{p}) \neq \phi\}) .$$

and

$$A_2 = \frac{d}{2}\left(\frac{4}{\eta}\right)^d \mathrm{vol}\,(B_{\pi-\delta_{10}}(p) - \Lambda[\delta_8]) + \frac{1}{d}\left(\frac{\pi}{10}\right)\left(\frac{1000\delta_{10}}{\pi\delta_8}\right).$$

Then we observe $F(B) \subset (B_{d'/2}(F(p_1)) \cup B_{d'/2}(F(p_2)))$ and

$$\begin{aligned} &\mathrm{vol}\,(F(B_{d'/2-\delta_{12}}(p_1) \cup B_{d'/2-\delta_{12}}(p_2))) - A_2 \\ &\quad \leqq \mathrm{vol}\,(F(B)) \leqq \mathrm{vol}\,(B_{d'/2}(F(p_1)) \cup B_{d'/2}(F(p_2))) \\ &\quad \leqq \mathrm{vol}\,(B_{d'/2}(F(p_1))) + \mathrm{vol}\,(B_{d'/2}(F(p_2))) \\ &\qquad - \mathrm{vol}\,(B_{d'/2-d''/2}(z)) , \end{aligned}$$

where z is the mid point of the minimal geodesic from $F(p_1)$ to $F(p_2)$.

These inequalities imply that

$$\begin{aligned} \mathrm{vol}\,(M) &\leqq \mathrm{vol}\,(F(D' - (B_{d'/2}(p_1) \cup B_{d'/2}(p_2)))) \\ &\quad + \mathrm{vol}\,(F(D' \cap (B_{d'/2}(p_1) \cup B_{d'/2}(p)))) \\ &\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cap B_{d'/2}(p_2))) \\ &\quad + \mathrm{vol}\,(B_{d'/2}(p_1) - B_{d'/2-\delta_{12}}(p_1)) \\ &\quad + \mathrm{vol}\,(B_{d'/2}(p_2) - B_{d'/2-\delta_{12}}(p_2)) \\ &\quad + \mathrm{vol}\,(F(B_{d'/2-\delta_{12}}(p_1) \cup B_{d'/2-\delta_{12}}(p_2)) \\ &\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cup B_{d'/2}(p_2))) \\ &\quad + 2\left(b_1\left(\frac{d'}{2}\right) - b_1\left(\frac{d'}{2} - \delta_{12}\right)\right) + A_2 \\ &\quad + \mathrm{vol}\,(B_{d'/2}(F(p_1)) + \mathrm{vol}\,(B_{d'/2}(F(p_2)) \\ &\quad - \mathrm{vol}\,(B_{d'/2-d''/2}(z)) \\ &\leqq \mathrm{vol}\,(S^d - (B_{d'/2}(p_1) \cup B_{d'/2}(p))) \\ &\quad + \mathrm{vol}\,(B_{d'/2}(p_1)) + \mathrm{vol}\,(B_{d'/2}(p_2)) + A_2 \\ &\quad - \frac{1}{2} b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right) \\ &\leqq \mathrm{vol}\,(S^d) + A_2 - \frac{1}{2} b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right). \end{aligned}$$

Note that the second term of A_2 can be small if we take sufficiently small δ_{10}. From Lemma 12.2, there exists $\delta_{13} > 0$ such that if $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_{13}$, then

$$A_2 < \frac{1}{4} \cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right).$$

We take

$$\delta_0 = \min\left(\delta_{13}, \frac{1}{5} \cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right)\right)$$

in Theorem 3. Then if $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_0$, then

$$\mathrm{vol}(M) < \mathrm{vol}(S^d) - \frac{1}{4} \cdot b_\Delta\left(\frac{d'}{2} - \frac{d''}{2}\right) < \mathrm{vol}(S_d) - \delta_0 .$$

It is a contradiction. $\varepsilon/10$-discreteness of $\{F(p_i)\}$ follows immediately from Claim 2 with $a \leqq \varepsilon^2/10$. q.e.d.

Corollary follows from the above and the following two theorems.

THEOREM A. (C. B. Croke [5], Theorem B.) *Let M be a compact d-dimensional Riemannian manifold with* $\mathrm{diam}(M) \leqq D < \pi$ *and* $\mathrm{Ric}_M \geqq d-1$. *Then there exists $C(d, D) > 1$ such that $\lambda_1(M) \geqq C(d, D) \cdot d$.*

THEOREM B. (A. Kasue [10], Theorem 4.1.) *Given $d, \Delta, v_0 > 0$ with $\Delta > 1$, $v_0 < \omega_d$, for any $V \in (v_0, \omega_d)$, there exists a constant $\rho = \rho(d, \Delta, v_0; V) > 0$ with $\rho < \pi$ such that if d-dimensional Riemannian manifold M has the property that* $\mathrm{Ric}_M \geqq d-1$, $|K_M| \leqq \Delta$, $\mathrm{vol}(M) \geqq v_0$ and $\mathrm{diam}(M) \geqq \rho$, *then* $\mathrm{vol}(M) \geqq V$.

§14. Proof of Lemma 12.2.

We firstly take constants which satisfy the following.

$$\delta_{22} < \log(1+\delta_4), \quad \delta_3 < \frac{\delta_5}{4\,\mathrm{vol}(S^{d-1})} \quad \text{and} \quad \log\left(\frac{s_{-\Delta}(\delta_3)}{\sin(\delta_3)}\right) < \frac{\delta_{22}}{3},$$

$$\delta_{21} < \frac{\delta_{22}}{2}, \quad \delta_{20} < \min\left(\delta_3, \frac{\delta_3^2\delta_{21}}{36\pi(\Delta+1)}\right), \quad \delta_{19} < \frac{\sin(\delta_3)\delta_{21}}{3(16d^2+1)},$$

$$\delta_{18} < \frac{\delta_{19}}{2}, \quad \delta_{17} < \left(\frac{\delta_3\delta_{18}}{6}\right)^2,$$

$$\delta_{16} < \min\left(\frac{4\Delta}{\delta_3^2}, \frac{\delta_{17}\delta_{20}}{10d^3}, \frac{\delta_{18}}{3}, \frac{\delta_{18}^2}{12K_3}, \frac{\delta_{19}^2}{12K_1^2K_4}, \frac{\delta_{21}}{3}\right), \quad \delta_{15} < \frac{\delta_{20}}{10},$$

$$\delta_{14} < \min\left(\delta_4, 1 - \exp\left(-\frac{\delta_{20}\delta_{16}}{10}\right)\right),$$

$$K_4 > \frac{d^3 K_3}{\delta_{21}}, \quad K_3 > \frac{3}{2}\pi d\Delta\left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\right)^2 + \frac{K_2 s_{-\Delta}(\pi)}{s_\Delta(\delta_3)},$$
$$K_2 > \left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)} + s_{-\Delta}(\pi/2\Delta^{1/2})\right)(s_\Delta(\pi/2\Delta^{1/2}))^{-1},$$
$$K_1 > \frac{s_{-\Delta}(\pi)^{d-1}}{\sin^{d-1}(\delta_3)s_\Delta(\delta_3)^{d-1}(1-\delta_7)}.$$

Then we can conclude by putting

$$\delta_6 = \min\left(\frac{\delta_5\delta_{14}\delta_{15}}{6\pi}\sin^{d-1}\left(\frac{\delta_5}{3}\right), \ \delta_5\delta_{14}\right).$$

Put

$$\begin{aligned}\bar{C}[\delta_3, \delta_{14}, \delta_{15}] &= \{v \in U_pS^d \mid \gamma(t) = \exp_p tv, \\ &\qquad m(\gamma([0, \pi-\delta_3]) \cap (\bar{B}[\delta_{14}] \cup S^d - D')) > \delta_{15}\}, \\ \bar{D}[\delta_3, \delta_{14}, \delta_{15}] &= U_pS^d - \bar{C}[\delta_3, \delta_{14}, \delta_{15}],\end{aligned}$$

and

$$D[\delta_3, \delta_{14}, \delta_{15}] = I(\bar{D}[\delta_3, \delta_{14}, \delta_{15}]).$$

CLAIM 1: *If* $\mathrm{vol}(M) \geqq \mathrm{vol}(S^d) - \delta_6$, *then*

$$\mathrm{vol}(\bar{B}[\delta_4]) \leqq \mathrm{vol}(\bar{B}[\delta_{14}]) \leqq \frac{\delta_6}{\delta_{14}} < \delta_5, \qquad \mathrm{vol}(S^d - D') < \delta_6$$
$$\mathrm{vol}_{(U_pS^d)}(\bar{C}[\delta_3, \delta_{14}, \delta_{15}]) \leqq \frac{3\delta_6}{\delta_{14}\delta_{15}\sin^{d-1}(\delta_{15}/3)} < \frac{\delta_5}{2\pi}.$$

where $\mathrm{vol}_{(U_pS^d)}$ *means the canonical measure on* U_pS^d.

Proof of Claim 1. Since

$$\begin{aligned}\mathrm{vol}(S^d) - \delta_6 \leqq \mathrm{vol}(M) &= \int_M dv_M = \int_{D'} |\det dF|\, dv_{S^d} \\ &\leqq \int_{\bar{B}[\delta_{14}]} (1-\delta_{14}) dv_{S^d} + \int_{D'-\bar{B}[\delta_{14}]} dv_{S^d} \\ &= \mathrm{vol}(S^d) - \mathrm{vol}(S^d - D') - \delta_{14}\,\mathrm{vol}(\bar{B}[\delta_{14}]),\end{aligned}$$

we see

$$\mathrm{vol}(\bar{B}[\delta_{14}]) < \frac{\delta_6}{\delta_{14}} \quad \text{and} \quad \mathrm{vol}(S^d - D') < \delta_6.$$

From the Fubini's theorem,

$$\begin{aligned}
&\operatorname{vol}(\bar{B}[\delta_{14}] \cup S^d - D') \\
&= \int_{U_pS^d \ni v} \left(\int \chi_{(\gamma_v(t) \cap \bar{B}[\delta_{14}] \cup (S^d - D'))}(t) \sin^{d-1} t \, dt \right) dv_{U_pS^d} \\
&\leqq \int_{\bar{C}[\delta_3, \delta_{14}, \delta_{15}]} \frac{\delta_{15}}{3} \cdot \sin^{d-1}\left(\frac{\delta_{15}}{3}\right) dv_{U_pS^d} ,
\end{aligned}$$

namely

$$\begin{aligned}
\operatorname{vol}_{(U_pS^d)}(\bar{C}[\delta_3, \delta_{14}, \delta_{15}]) &\leqq \frac{3 \operatorname{vol}(\bar{B}[\delta_{14}] \cup S^d - D')}{\delta_{15} \sin^{d-1}(\delta_{15}/3)} \\
&\leqq \frac{3\delta_6}{\delta_{14}\delta_{15} \sin^{d-1}(\delta_{15}/3)} < \frac{\delta_5}{2\pi} .
\end{aligned}$$

q.e.d.: Claim 1.

For $v \in D[\delta_3, \delta_{14}, \delta_{15}]$, put $\gamma(t) = \exp_m tv$ and $\bar{\gamma}(t) = \exp_p tI^{-1}(v)$. Let $U_i(t)$ (resp. $\bar{U}_i(t)$) $(1 \leqq i \leqq d-1)$ be the linearly independent parallel vector fields along tv (resp. $tI^{-1}(v)$) which is perpendicular to v (resp. $I^{-1}(v)$). Put $Y_i(t) = d\exp_m(tU_i(t))$, $\bar{Y}_i(t) = d\exp_p(t\bar{U}_i(t))$ and $W_i(t) = P_t \circ I \circ P_{-t}\bar{Y}_i(t)$, where P_t and P_{-t} are the parallel translations along $\gamma(t)$ and $\bar{\gamma}(t)$ respectively. For $\bar{\gamma}(s_0) \in D' - \bar{B}[\delta_7]$, we put

$$\begin{aligned}
E^{\gamma}_{s_0}[\delta_{16}] &= E^{\gamma}_{s_0}[\delta_3, \delta_{14}, \delta_{15}, \delta_{16}] \\
&= \{\gamma(s) \mid s \in [0, s_0],\ (\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' \\
&\qquad \leqq (\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)' + \delta_{16}\} .
\end{aligned}$$

CLAIM 2: $m(\gamma([0, s_0]) - E^{\gamma}_{s_0}[\delta_{16}]) \leqq \dfrac{-\log(1-\delta_{14})}{\gamma_{16}} < \dfrac{\delta_{20}}{10}$.

Proof of Claim 2. It is an easy consequence of the following two inequalities,

$$(\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)' \leqq (\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' ,$$

$$\log|\bar{Y}_1(s_0) \wedge \cdots \wedge \bar{Y}_{d-1}(s)| \leqq (\log|Y_1(s_0) \wedge \cdots \wedge Y_{d-1}(s_0) - \log(1-\delta_{14}) ,$$

q.e.d.: Claim 2

In the following, we fix $s_1 \in E^{\gamma}_{s_0}[\delta_{16}]$. We may assume $s_1 \geqq \pi - \delta_3 - \delta_{20}/10 - \delta_{15} > \pi/2$. Since the value

$$(\log|\bar{Y}_1(s) \wedge \cdots \wedge \bar{Y}_{d-1}(s)|)' - (\log|Y_1(s) \wedge \cdots \wedge Y_{d-1}(s)|)'$$

does not change when we replace Y_i and $\bar{Y}_i$ by linear combination, so we may assume that $\{Y_i(s_i)\}$ and $\{\bar{Y}_i(s_1)\}$ are orthonormal.

We denote by $I_{s_1}(Y_i, Y_i)$ the index form of Y_i along $\gamma|_{[0,s_1]}$.

CLAIM 3: *If* $\gamma(s_1) \in \gamma[0, s_0]) - E^{\gamma}_{s_0}[\delta_{16}]$, *then*

$$I_{s_1}(W_i, W_i) \leqq I_{s_1}(Y_i, Y_i) + \delta_{16} .$$

Proof of Claim 3. From the argument of Heintze-Karcher [8], we see

$$\begin{aligned}
&(\log |Y_1(s_1) \wedge \cdots \wedge Y_{d-1}(s_1)|)' \\
&= \sum_{i=1}^{d-1} I_{s_1}(Y_i, Y_i) \qquad (\{Y_i(s_1)\} \text{ are orthonormal.}) \\
&\leqq \sum_{i=1}^{d-1} I_{s_1}(W_i, W_i) \qquad (\text{the index lemma.}) \\
&\leqq \sum_{i=1}^{d-1} I_{s_1}(\overline{Y}_i, \overline{Y}_i) = (\log |\overline{Y}_1(s_1) \wedge \cdots \wedge \overline{Y}_{d-1}(s_1)|)' \\
&= (\log |Y_1(s_1) \wedge \cdots \wedge Y_{d-1}(s_1)|)' + \delta_{16} \\
&= \sum_{i=1}^{d-1} I_{s_1}(Y_i, Y_i) + \delta_{16} .
\end{aligned}$$

Thus with the index lemma, $I_{s_1}(Y_i, Y_i) \leqq I_{s_1}(W_i, W_i)$, we get

$$I_{s_1}(W_i, W_i) \leqq I_{s_1}(Y_i, Y_i) + \delta_{16} \qquad \text{for each } i .$$

q.e.d.: Claim 3

Since $\{Y_j(s)\}$ is a basis of $T_{\gamma(s)}M$, we may put $W_i(s) = \sum_{j=1}^{d} f_{ij}(s) Y_j(s)$. For fixed i, we define

$$\begin{aligned}
F^i_{s_1}[\delta_{17}] &= F^i_{s_1}[\delta_3, \varepsilon_{14}, \delta_{15}, \delta_{16}, \delta_{17}] \\
&= \left\{\gamma(s) \,|\, s \in [0, s_1],\ \left|\sum_{j=1}^{d} f'_{ij}(s) Y_j(s)\right|^2 < \delta_{17}\right\} .
\end{aligned}$$

CLAIM 4: (i) $m(\gamma([\delta_3, s_1]) - F^i_{s_1}[\delta_{17}]) < \dfrac{\delta_{16}}{\delta_{17}}$.

(ii) *If* $\gamma(s) \in F^i_{s_1}[\delta_{17}]$, *then,*

$$\left|\int_0^s \sum_{j=1}^{d} f''_{ij} f_{ik} g(Y_j, Y_k)' dt\right| < \delta_{18} .$$

Proof of Claim 4. From the arguments of Cheeger-Ebin [3] (Chap 1, §8, 1.21), we have

$$I_{s_1}(W_i, W_i) = I_{s_1}(Y_i, Y_i) + \int_0^{s_1} \left|\sum_{j=1}^{d} f'_{ij} Y_j\right|^2 dt ,$$

therefore,

$$\int_0^s \left|\sum_{j=1}^{d} f'_{ij} Y_j\right|^2 dt \leqq \delta_{16} \qquad \text{for } s \leqq s_1 .$$

This implies (i).

By the integration by parts, we observe,

$$\begin{aligned}\int_0^s \left|\sum_{j=1}^d f'_{ij} Y_j\right|^2 dt &= \left[\sum_{j,k=1}^d f'_{ij} f_{ik} g(Y_j, Y_k)\right]_0^s \\ &\quad - \int_0^s \sum_{j,k=1}^d f''_{ij} f_{ik} g(Y_j, Y_k) dt \\ &\quad - \int_0^s \sum_{j,k=1}^d f'_{ij} f_{ik} (g(Y'_j, Y_k) + g(Y_j, Y'_k)) dt .\end{aligned}$$

For the estimate of

$$\left|\int_0^s \sum_{j,k=1}^d f''_{ij} f_{ik} g(Y_j, Y_k) dt\right| ,$$

firstly we see $g(Y'_j, Y_k) = g(Y_j, Y'_k)$ by taking the derivation of the both sides. (cf. [3] p. 25 (**))

Nextly, from R.C.T., we have

$$\begin{aligned}|\bar{Y}_i(s)| &= |\bar{Y}'_i(0)| \sin(s) = |\bar{Y}_i(s_i)| \cdot \frac{\sin(s)}{\sin(s_1)} = \frac{\sin(s)}{\sin(s_1)} < \frac{2}{\delta_3} , \\ |Y_k(s)| &\leqq |Y'_k(0)| s_{-\Delta}(s) \leqq |Y_k(s_1)| \cdot \frac{s_{-\Delta}(s)}{s_\Delta(s_1)} = \frac{s_{-\Delta}(s)}{s_\Delta(s_1)} \leqq \frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\end{aligned}$$

Thirdly, we estimate $|f_{ik}|$. Put $Y_i(s) = \sum a_{ik} e_k$, $W_i(s) = \sum b_{ik} e_k$, where $\{e_i\}_{i=1}^d$ is the orthonormal basis of $T_{\gamma(s)}M$. From $W_i = \sum f_{ij} Y_j$, we get $b_{ik} = \sum f_{ij} a_{jk}$. Let B^j_ℓ be the matrix such that the ℓ-th column of $A = (a_{jk})$ is replaced by $b_{j\ell}$. By Cramer's formula, $f_{ik} = \det B^i_k / \det A$. Note that $\det A = |Y_1 \wedge \cdots \wedge Y_d|$ and

$$\max_{i,k} |\det B^i_k| \leqq \max_{i,k} (|W_i| \prod_{j \neq k} |Y_j|) \leqq \frac{\sin(s)}{\sin(s_1)} \left(\frac{s_{-\Delta}(s)}{s_\Delta(\delta_3)}\right)^{d-1} .$$

Since $\bar{\gamma}(s) \in D' - \bar{B}[\delta_7]$,

$$|Y_1 \wedge \cdots \wedge Y_d| \leqq |Y_1 \wedge \cdots \wedge Y_d| (1 - \delta_7) \leqq \sin^d(s)(1 - \delta_7) .$$

It implies

$$|f_{ik}| \leqq \max |\det B^i_k / \det A| \leqq \frac{1}{\sin^d(s)(1 - \delta_7)} \cdot \frac{\sin(s)}{\sin(s_1)} \left(\frac{s_{-\Delta}(s)}{s_\Delta(s_1)}\right)^{d-1} \leqq K .$$

Fourthly we have

$$|Y'_k(s_1)| \leqq K_2$$

by the following arguments.

We may assume $\Delta > 1$. Decompose $Y_k(s)$ as $Y_k(s) = Z_1(s) + Z_2(s)$, where

$Z_i(s)$ are Jacobi fields with $Z_1(s_1) = Z_2'(s_1) = 0$, $Z_2(s) = Y_k(s_1) = 1$ and $Z_1'(s_1) = Y_k'(s_1)$. From the Berger's comparison theorem ([2] 1.29),

$$|Z_2(s_1 - \pi/2\Delta^{1/2})| \leqq c_{-\Delta}(\pi/2\Delta^{1/2}) .$$

Then,

$$\begin{aligned}|Z_1(s_1 - \pi/2\Delta^{1/2})| &\leqq |Y_k(s_1 - \pi/2\Delta^{1/2})| + |Z_2(s_1 - \pi/2\Delta^{1/2})| \\ &\leqq \frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)} + c_{-\Delta}(\pi/2\Delta^{1/2}) .\end{aligned}$$

Thus, we get

$$|Y_k'(s_1)| = |Z_1'(s_1)| \leqq \left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)} + c_{-\Delta}(\pi/2\Delta^{1/2})\right)(s_\Delta(\pi/\Delta^{1/2}))^{-1} \leqq K_2$$

Fifthly we have

$$(*)\quad \begin{aligned}\int_0^s \sum_{k=1}^d |Y_k'|^2 dt &\leqq \int_0^{s_1} \sum_{k=1}^d |Y_k'|^2 dt \\ &= \sum_{k=1}^d \int_0^{s_1} g(R(Y_k, \dot\gamma)\dot\gamma, Y_k)dt + g(Y_k'(s_1), Y_k(s_1))dt \\ &\leqq d\cdot \int_0^{s_1} (3/2)\Delta |Y_k|^2 dt + |Y_k'(s_1)||Y_k(s_1)|dt \\ &\leqq \frac{3}{2}\pi d\Delta\left(\frac{s_{-\Delta}(\pi)}{s_\Delta(\delta_3)}\right)^2 + \frac{K_2 s_{-\Delta}(\pi)}{s_\Delta(\delta_3)} \leqq K_3 .\end{aligned}$$

Therefore, we get, from $W_i = \sum f_{ij} Y_j$,

$$\begin{aligned}&\left|\int_0^s \sum_{j,k=1}^d f_{ij}'' f_{ik} g(Y_j, Y_k)\, dt\right| \\ &\leqq \int_0^s \left|\sum_{j=0}^d f_{ij}' Y_j\right|^2 dt + \left|\sum_{i=1}^d f_{ij}'(s) g(Y_j(s), W_i(s))\right| \\ &\quad + 2\left|\int_0^s \sum_{j=1}^d f_{ij}' f_{ik} g(Y_j, Y_k')\, dt\right| \\ &\leqq \delta_{16} + \left|\sum_{j=1}^d f_{ij}'(s) Y_j(s)\right| |W_i(s)| \\ &\quad + 2\left(\int_0^s \left|\sum_{j=1}^d f_{ij}' Y_j\right|^2 dt\right)^{1/2} \left(\int_0^s \left|\sum_{j=1}^d f_{ik} Y_k'\right|^2 dt\right)^{1/2} \\ &\leqq \delta_{16} + \frac{2\delta_{17}^{1/2}}{\delta_3} + 2(\delta_{16} K_3)^{1/2} < \delta_{18} .\end{aligned}$$

q.e.d.: Claim 4

We put

$$G_{s_1}^i[K_4] = \left\{\gamma(s) \in F_{s_1}^i[\delta_{17}] \,\middle|\, \sum_{j=1}^d |Y_j'(s)|^2 \leqq K_4\right\} .$$

Then, from Claim 4 and $(*)$, we see,

$$m(G^i_{s_1}[K_4]) \geqq s_1 - \frac{\delta_{16}}{\delta_{17}} - \frac{K_3}{K_4} .$$

CLAIM 5: *If $\gamma(s) \in G^i_{s_1}[K_4]$, then*

$$\left|\int_0^s g(\bar{R}(\bar{Y}_i, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{Y}_i) - g(R(Y_i, \dot{\gamma})\dot{\gamma}, Y_i)dt\right| \leqq \delta_{19} .$$

Proof of Claim 5. From $g(\bar{Y}_i'', \bar{Y}_i) = g(W_i'', W_i)$ and

$$W_i'' = \left(\sum_{j=1}^d f_{ij} Y_j\right)'' = \sum_{j=1}^d (f_{ij}'' Y_j + 2f_{ij}' Y_j' + f_{ij} Y_j'') ,$$

we find if $\gamma(s) \in G^i_{s_1}[K_4]$, then,

$$\begin{aligned} &\left|\int_0^s g(\bar{R}(\bar{Y}_i, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{Y}_i) - g(R(W_i, \dot{\gamma})\dot{\gamma}, W_i)dt\right| \\ &= \left|\int_0^s g(\bar{Y}_i'', \bar{Y}_i) - \sum_{j=1}^d f_{ij} g(Y_j'', W_i)dt\right| \\ &= \left|\int_0^s \sum_{j=1}^d f_{ij}'' g(Y_j, W_i) + 2 \sum_{j,k=1}^d f_{ij}' f_{ik} g(Y_j, Y_k')dt\right| \\ &\leqq \left|\int_0^s \sum_{j=1}^d f_{ij}'' g(Y_j, W_i)dt\right| + 2\left|\int_0^s \sum_{j=1}^d f_{ij}' Y_j dt\right| \max(|f_{ik}||Y_k'|) \\ &= \delta_{18} + 2\delta_{17}K_1K_4^{1/2} < \delta_{19} . \end{aligned}$$

q.e.d.: Claim 5

We put $G^{\{Y\}} = \bigcap_{i=1}^d G^i_{s_1}[K_4]$. We take another orthonormal basis $\{X_k^{ij}\}$ at $\gamma(s_1)$ with $X_1^{ij} = (X_i + Y_j)/(|Y_i + Y_j|)$ and repeat the above arguments for each (i, j). Put $G^\gamma = \bigcap_k G^{\{X_k^{ij}\}}$ and

$$G = \bigcap_{\dot{\gamma}(0) \in D[\delta_3, \delta_{14}, \delta_{15}]} G^\gamma .$$

Then, since $s_1 \geqq \pi - \delta_3 - \delta_{20}/10 - \delta_{15}$, we see

$$(**) \qquad m(G^\gamma) \geqq \pi - \delta_3 - \frac{\delta_{20}}{10} - \delta_{15} - d^3\left(\frac{\delta_{16}}{\delta_{17}} + \frac{K_3}{K_2}\right) > \pi - \delta_3 - \delta_{20} .$$

On the other hand, we find if $\gamma(s) \in G^\gamma$, then for any $\bar{X} \in T_{\bar{\gamma}(s)} S^d$,

$$\left|\int_0^s g(R(W_X, \dot{\gamma})\dot{\gamma}, W_X) - g(\bar{R}(\bar{X}, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{X})\, dt\right| \leqq \pi(16d^2 + 1)|X|^2\delta_{19} ,$$

where $W_X = P_s \circ I \circ P_{-s}(\bar{X})$. It is easily derived from the following inequality,

$$\left|\int_0^s K\left(\sum_{i=1}^d \lambda_i Y_i, \sum_{i=1}^d \lambda_i Y_i\right)dt\right| \leqq \sum_{i=1}^d \lambda_i^2 \left|\int_0^s K(Y, Y)\, dt\right|$$

$$+ 2\sum_{i=1}^{d}|\lambda_i\lambda_j|\left(\left|\int_0^s K(Y_i+Y_j,\,Y_i+Y_j)\,dt\right| + \left|\int_0^s K(Y_i,\,Y_i)\,dt\right| + \left|\int_0^s K(Y_j,\,Y_j)\,dt\right|\right),$$

where $K(Z_1, Z_2) := g(\bar{R}(Z_1, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, Z_2) - g(R(W_{Z_1}, \dot{\gamma})\dot{\gamma}, W_{Z_2})$ and $\sum_{i=1}^{d}\lambda_i^2 = 1$.

CLAIM 6: *For $\gamma(t) \in G_\gamma$, $|dF_{\bar{\gamma}(t)}| < 1 + \delta_4$.*

Proof of Claim 6. Similarly as above, put $Y(t) = d\exp_m(tU(t))$ and $\bar{Y}(t) = d\exp_p(tI^{-1}(U(t)))$. Take $\gamma(t_1) \in G^\gamma$ with $t_1 \geqq \delta_3$ and put

$$V(t) = \frac{Y(t)}{|Y(t_1)|}\bar{V}(t) - \frac{\bar{Y}(t)}{|\bar{Y}(t_1)|} \quad \text{and} \quad W(t) = P_t \circ I \circ P_{-t}\bar{V}(t) = \sum_{i=1}^{d} f_i V_1 ,$$

where $\{V_i\}$ are the linearly independent Jacobi fields such that $\{V_i(t_1)\}$ are orthonormal. For fixed

$$s_1 \geqq \pi - \delta_3 + \frac{\log(1-\delta_{14})}{\varepsilon_{16}} - \delta_{15},$$

put

$$\bar{V}_1(t) = \frac{\bar{Y}(t)}{|Y(s_1)|} = \bar{V}(t)\cdot\frac{|\bar{Y}(t_1)|}{|\bar{Y}(s_1)|} \quad \text{and} \quad W_1(t) = P_t \circ I \circ P_{-t}\bar{V}_1(t).$$

Then, similarly as above, we see

$$|I_{t_1}(V, V) - I_{t_1}(W, W)| \leqq \delta_{16}$$

and therefore

$$\begin{aligned}
&|I_{t_1}(V, V) - I_{t_1}(\bar{V}, \bar{V})| \\
&\quad \leqq \delta_{16} + \left|\int_0^{t_1}(g(R(W, \dot{\gamma})\dot{\gamma}, W) - g(R(\bar{V}, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{V}))\,dt\right| \\
&\quad \leqq \delta_{16} + \left|\int_0^{t_1}(g(R(W_1, \dot{\gamma})\dot{\gamma}, W_1) - g(R(\bar{V}_1, \dot{\bar{\gamma}})\dot{\bar{\gamma}}, \bar{V}_1))\,dt\right|\cdot\frac{|\bar{Y}(s_1)|}{|\bar{Y}(t_1)|} \\
&\quad \leqq \delta_{16} + (16d^2+1)\delta_{19}\left(\frac{\sin(s_1)}{\sin(\delta_3)}\right) < \delta_{21}\,.
\end{aligned}$$

Namely,

$$|(\log|Y(t_1)|)' - (\log|\bar{Y}(t_1)|)'| \leqq \delta_{21}\,.$$

For $\gamma(t) \notin G^\gamma$, since the value $(\log|Y(t)|)'$ does not change when $Y(t)$ replace by constant multiple of $Y(t)$, for $t \leqq t_1$, we see

$$\begin{aligned}(\log|Y(t)|)' &= \frac{Y(t)}{Y(t_1)} I_t(V, V) \\ &\leqq \int_0^{t_1} \left|\sum_{i=1}^{d} f_i' V_i\right|^2 dt + \int_0^{t_1} g(R(W, \dot\gamma)\dot\gamma, W)\, dt \\ &\leqq \delta_{16} + 2\varDelta\left(\frac{2}{\delta_3}\right)^2 < 3\varDelta\pi\left(\frac{2}{\delta_3}\right)^2,\end{aligned}$$

and similarly,

$$(\log|\overline{Y}(t)|)' \leqq 3\pi\left(\frac{2}{\delta_3}\right)^2.$$

Integrating these, we get

$$\log|Y(t)| - \log|\overline{Y}(t)| \leqq \log\left(\frac{s_{-\varDelta}(\delta_3)}{\sin(\delta_3)}\right) + \delta_{21}\pi + \left(\frac{2}{\delta_3}\right)^2 3(\varDelta + 1)\pi\delta_{20} \leqq \delta_{22}.$$

Therefore we see

$$|dF_{\bar\gamma(t)}| = \frac{|Y(t)|}{|\overline{Y}(t)|} \leqq \exp(\delta_{22}) < 1 + \delta_4. \qquad \text{q.e.d.: Claim 6}$$

Note that $\overline{A}[\delta_4] \subset D' - F^{-1}(G) := \tilde{A}[\delta_4]$.

CLAIM 7: $\operatorname{vol}(\overline{A}[\delta_4]) \leqq \operatorname{vol}(\tilde{A}[\delta_4]) < \delta_5$.

Proof of Claim 7. Since $m(F^{-1}(G^\gamma)) = m(G^\gamma)$, we have, from Claim 1 and $(**)$,

$$\begin{aligned}\operatorname{vol}(\overline{A}[\delta_4]) &\leqq \int_{\overline{C}[\delta_3, \delta_{14}, \delta_{15}]} \left(\int_0^\pi \sin^{d-1} t\, dt\right) dv_{U_pS^d} \\ &\quad + \int_{\overline{D}[\delta_3, \delta_{14}, \delta_{15}]} \left(\int_{(\gamma([0,\pi]) - G^\gamma)} \sin^{d-1} t\, dt\right) dv_{U_pS^d} \\ &\leqq \operatorname{vol}_{(U_pS^d)}(\overline{C}[\delta_3, \delta_{14}, \delta_{15}])\pi \\ &\quad + \max m(\gamma([0,\pi]) - G^\gamma)\operatorname{vol}(S^{d-1}) \\ &\leqq \frac{\delta_5}{2\pi}\pi + (\delta_{25} + \delta_3)\operatorname{vol}(S^{d-1}) \leqq \delta_5. \qquad \text{q.e.d.: Claim 7}\end{aligned}$$

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COLLAPSING RIEMANNIAN MANIFOLDS TO ONES OF LOWER DIMENSIONS

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0. Introduction

In [7], Gromov introduced a notion, Hausdorff distance, between two metric spaces. Several authors found that interesting phenomena occur when a sequence of Riemannian manifolds M_i collapses to a lower dimensional space X. (Examples of such phenomena will be given later.) But, in general, it seems very difficult to describe the relation between topological structures of M_i and X. In this paper, we shall study the case when the limit space X is a Riemannian manifold and the sectional curvatures of M_i are bounded, and shall prove that, in that case, M_i is a fiber bundle over X and the fiber is an infranilmanifold. Here a manifold F is said to be an infranilmanifold if a finite covering of F is diffeomorphic to a quotient of a nilpotent Lie group by its lattice.

A complete Riemannian manifold M is contained in class $\mathscr{M}(n)$ if **dim** $M \leqslant n$ and if the sectional curvature of M is smaller than 1 and greater than -1. An element N of $\mathscr{M}(n)$ is contained in $\mathscr{M}(n, \mu)$ if the injectivity radius of N is everywhere greater than μ.

Main Theorem. *There exists a positive number $\varepsilon(n, \mu)$ depending only on n and μ such that the following holds.*

If $M \in \mathscr{M}(n)$, $N \in \mathscr{M}(n, \mu)$, and if the Hausdorff distance ε between them is smaller than $\varepsilon(n, \mu)$, then there exists a map $f\colon M \to N$ satisfying the conditions below.

(0-1-1) *(M, N, f) is a fiber bundle.*

(0-1-2) *The fiber of f is diffeomorphic to an infranilmanifold.*

(0-1-3) *If $\xi \in T(M)$ is perpendicular to a fiber of f, then we have*

$$e^{-\tau(\varepsilon)} < |df(\xi)|/|\xi| < e^{\tau(\varepsilon)}.$$

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Here $\tau(\varepsilon)$ *is a positive number depending only on* ε, n, μ *and satisfying* $\mathbf{lim}_{\varepsilon\to 0}\tau(\varepsilon) = 0$.

Remarks. (1) In the case when N is equal to a point, our main theorem coincides with [6, 1.4].

(2) In the case when the dimension of M is equal to that of N, the conclusion of our main theorem implies that f is a diffeomorphism and that the Lipschitz constants of f and f^{-1} are close to 1. Hence, in that case, our main theorem gives a slightly stronger version of [7, 8.25] or [8, Theorem 1]. (In [7] or [8], it was assumed that the injectivity radii of both M and N were greater than μ, but here we assume that one of them is greater than μ.)

Next we shall give some examples illustrating the phenomena treated in our main theorem.

Examples. (1) Let $T_i^2 = \mathbb{R}^2/\mathbb{Z} \oplus (1/i)\mathbb{Z}$ be flat tori. Then $\mathbf{lim}_{i\to\infty} T_i^2 = S^1$ $(= \mathbb{R}/\mathbb{Z})$ and T^2 is a fiber bundle over S^1.

(2) (See [9].) Let (M, g) be a Riemannian manifold. Suppose S^1 acts isometrically and freely on M. Let g_ε denote the Riemannian metric such that $g_\varepsilon(v, v) = \varepsilon \cdot g(v, v)$ if v is tangent to an orbit of S^1 and $g_\varepsilon(v, v) = g(v, v)$ if v is perpendicular to an orbit of S^1. Then $\mathbf{lim}_{\varepsilon\to 0}(M, g_\varepsilon) = (M/S^1, g')$ for some metric g'. In this example, the fiber bundle in our main theorem is $S^1 \to M \to M/S^1$.

(3) Let G be a solvable Lie group and Γ its lattice. Put $G_0 = G$, $G_1 = [G, G]$, $G_2 = [G_1, G_1], \cdots, G_{i+1} = [G_1, G_i]$. Take a left invariant Riemannian metric g on G. Let g_ε denote the left invariant Riemannian metric on G such that $g_\varepsilon(v, v) = \varepsilon^{i\cdot 2^i} \cdot g(v, v)$ if $v \in T_e(G)$ is tangent to G_i and perpendicular to G_{i+1}. (Here e denotes the unit element.) Then $\mathbf{lim}_{\varepsilon\to 0}(\Gamma \setminus G, g_\varepsilon)$ is equal to the flat torus $\Gamma \setminus G/G_1$, and the sectional curvatures of g_ε are uniformly bounded. In this example, the fiber bundle in our main theorem is $(G_1 \cap \Gamma)\setminus G_1 \to \Gamma \setminus G \to \Gamma \setminus G/G_1$.

Finally, we shall give an example of collapsing to a space which is not a Riemannian manifold.

(4) (This example is an amplification of [7, 8.31].) Let (G_i, Γ_i) be a sequence of pairs consisting of nilpotent Lie groups G_i and their lattices Γ_i. Let (M, g) be a compact Riemannian manifold and φ_i a homomorphism from Γ_i to the group of isometries of (M, g). Put $T = \bigcap_i(\overline{\bigcup_{j\geqslant i}\varphi_j(\Gamma_j)})$. Here the closure, $\overline{\bigcup_{j\geqslant i}\varphi_j(\Gamma_j)}$, is taken in the sense of compact open topology. It is proved in [1, 7.7.2] that there exists a sequence of left invariant metrics g_i on G_i such that the sectional curvatures of g_i $(i = 1, 2, \cdots)$ are uniformly bounded and that $\mathbf{lim}_{i\to\infty}(\Gamma_i \setminus G_i, \bar{g}_i) =$ point. On $M \times G_i$, we define an equivalence relation $\sim$ by $(\varphi_i(\gamma^{-1})(x), g) \sim (x, \gamma g)$. Let $M \times_{\Gamma_i} G_i$ denote the set of equivalence

classes. Then it is easy to see

$$\lim_{i\to\infty}\left(M\times_{\Gamma_i}G_i,\ \overline{g\times g_i}\right)=(M/T,\bar{g}).$$

In this example, there also exists a map from $M\times_{\Gamma_i}G_i$ to M/T.

This example gives all possible phenomena which can occur at a neighborhood of each point of the limit. In fact, using the result of this paper, we shall prove the following in [5]:

Let M_i be a sequence of compact m-dimensional Riemannian manifolds such that the sectional curvatures of M_i are greater than -1 and smaller than 1. Suppose $\lim_{i\to\infty}M_i$ is equal to a compact metric space X. Then, for each sufficiently large i, there exists a map $f\colon M_i\to X$ satisfying the following.

(1) For each point p of X, there exists a neighborhood U which is homeomorphic to the quotient of $\mathbb{R}^n$ by a linear action of a group T. Here T denotes an extension of a torus by a finite group.

(2) Let Y denote the subset of X consisting of all points which have neighborhoods homeomorphic to $\mathbb{R}^k$. Then $(f_i|_{f_i^{-1}(Y)}, f_i^{-1}(Y), Y)$ is a fiber bundle with an infranilmanifold fiber F.

(3) Suppose p has a neighborhood homeomorphic to $\mathbb{R}^n/T$. Then $f_i^{-1}(p)$ is diffeomorphic to F/T.

The global problem on collapsing is still open even in the case of fiber bundles.

Problem. Let F be an infranilmanifold and (M,N,f) a fiber bundle with fiber F. Give a necessary and sufficient condition for the existence of a sequence of metrics g_i on M such that the sectional curvatures are greater than -1 and smaller than 1 and that $\lim_{i\to\infty}(M,g_i)$ is homeomorphic to N.

The organization of this paper is as follows. In §1, we shall construct the map f. In §2, we shall prove that (M,N,f) is a fiber bundle. In §3, we shall prove a lemma on triangles on M. This lemma will be used in the argument of §§2, 4, and 5. In §4, we shall verify (0-1-3). In §5, we shall prove (0-1-2). Our argument there is an extension of one in [1] or [6].

In [7, Chapter 8] and [9] (especially in [7, 8.52]), several results which are closely related to this paper are proved or announced, and the author is much inspired from them. Several related results are obtained independently in [3] and [4]. The result of this paper is also closely related to Thurston's proof of his theorem on the existence of geometric structures on 3-dimensional orbifolds. The lecture by T. Soma on it was also very helpful to the author.

Notation. Put $R=\min(\mu,\pi)/2$. The symbol ε denotes the Hausdorff distance between M and N. Let σ be a small positive number which does not depend on ε. We shall replace the numbers ε and σ by smaller ones, several times in the proof. The symbol $\tau(a\,|\,b,\cdots,c)$ denotes a positive number depending only on $a, b,\cdots, c, R, \mu$ and satisfying $\lim_{a\to 0}\tau(a\,|\,b,\cdots,c) = 0$ for each fixed $b,\cdots,c$. For a Riemannian manifold X, a point $p \in X$, and a positive number r, we put

$$B_r(p, X) = \{x \in X \,|\, d(x, p) < r\},$$
$$BT_r(p, X) = \{\xi \in T_p(X) \,|\, |\xi| < r\}.$$

Here $T_p(X)$ denotes the tangent space. For a curve $l\colon [0, T] \to X$, we let $(Dl/dt)(t)$ denote the tangent vector of l at $l(t)$. For two vectors $\xi, \xi' \in T_p(X)$, we let $\operatorname{ang}(\xi, \xi')$ denote the angle between them. All geodesics are assumed to have unit speed.

1. Construction of the map

First remark that Rauch's comparison theorem (see [2, Chapter 1, §1]) immediately implies the following.

(1-1-1) For each $p \in M$ and $p' \in N$ the maps $\mathbf{exp}|_{BT_{2R}(p,M)}$ and $\mathbf{exp}|_{BT_{2R}(p',N)}$ have maximal rank. Here $\mathbf{exp}$ denotes the exponential map.

(1-1-2) On $BT_{2R}(p, M)$ [resp. $BT_{2R}(p', N)$], we define a Riemannian metric induced from M [resp. N] by the exponential map. Then, the injectivity radii are greater than R on $BT_R(p, M)$ and $BT_R(p', N)$.

Secondly we see that, by the definition of the Hausdorff distance, there exists a metric d on the disjoint union of M and N such that the following holds: The restrictions of d to M and N coincide with the original metrics on M and N respectively, and for each $x \in N$, $y \in M$ there exist $x' \in M$, $y' \in N$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$. It follows that we can take subsets Z_N of N and Z_M of M, a set Z, and bijections $j_M\colon Z \to Z_M$, $j_N\colon Z \to Z_N$, such that the following holds.

(1-2-1) The 3ε-neighborhood of Z_N [resp. Z_M] contains N [resp. M].

(1-2-2) If z and z' are two elements of Z, then we have

$$d\big(j_N(z), j_N(z')\big) > \varepsilon \quad \text{and} \quad d\big(j_M(z), j_M(z')\big) > \varepsilon.$$

(1-2-3) For each $z \in Z$, we have

$$d\big(j_N(z), j_M(z)\big) < \varepsilon.$$

Now, following [8], we shall construct an embedding $f_N\colon N \to \mathbb{R}^Z$. Put $r = \sigma R/2$. Let κ be a positive number determined later, and $h\colon \mathbb{R} \to [0, 1]$ a

C^∞-function such that

(1-3) $h(0) = 1$ and $h(t) = 0$ if $t \geqslant r$,

$$\tfrac{4}{r} < h'(t) < -\tfrac{3}{r} \quad \text{if } \tfrac{3r}{8} < t < \tfrac{5r}{8},$$
$$-\tfrac{4}{r} < h'(t) < 0 \quad \text{if } \tfrac{2r}{8} < t \leqslant \tfrac{3r}{8} \text{ or } \tfrac{5r}{8} \leqslant t < \tfrac{6r}{8},$$
$$\kappa < h'(t) < 0 \quad \text{if } 0 < t < \tfrac{2r}{8} \text{ or } \tfrac{6r}{8} \leqslant t \leqslant r.$$

We define a C^∞-map $f_N : N \to \mathbb{R}^Z$ by $f_N(x) = (h(d(x, j_N(z))))_{z \in Z_N}$. In [8], it is proved that, if ε and σ are smaller than a constant depending only on R, μ, and n, then f_N satisfies the following facts (1-4-1), (1-4-2), (1-4-3), and (1-4-4). The numbers C_1, C_2, C_3, C_4 below are positive constants depending only on R, μ, and n.

(1-4-1) f_N is an embedding [8, Lemma 2.2].

(1-4-2) Put

$$B_C(Nf_N(N)) = \{(p, u) \in \text{the normal bundle of } f_N(N) \mid |u| < C\},$$
$$K = \sup_{x \in N} \#(B_r(p, N) \cap j_N(Z_N)).$$

Then the restriction of the exponential map to $B_{C_1 K^{1/2}}(Nf_N(N))$ is a diffeomorphism [8, Lemma 4.3].

(1-4-3) For each $\xi' \in T_{p'}(N)$ satisfying $|\xi'| = 1$, we have

$$C_2 K^{1/2} < |df_N(\xi')| < C_3 K^{1/2} \quad [8, \text{Lemma } 3.2].$$

(1-4-4) Let $x, y \in N$. If $d(x, y)$ is smaller than a constant depending only on σ, μ, and n, then we have

$$K^{1/2} \cdot d(x, y) \leqslant C_4 \cdot d_{\mathbf{R}^n}(f_N(x), f_N(y)) \quad [8, \text{Lemma } 6.1].$$

The next step is to construct a map from M to the $C_1 K^{1/2}$-neighborhood of $f_N(N)$. The map $x \to (h(d(x, j_M(z))))_{z \in Z}$ has this property. But unfortunately this map is not differentiable when the injectivity radius of M is smaller than r, and is inconvenient for our purpose. Hence we shall modify this map. For $z \in Z$ and $x \in M$, put

$$d_z(x) = \int_{y \in B_\varepsilon(j_M(z), M)} d(y, x)\, dy / \mathrm{Vol}(B_\varepsilon(j_M(z), M)),$$
$$f_M(x) = (h(d_z(x)))_{(z \in Z)}.$$

Assertion 1-5. *d_z is a C^1-function and for each $\xi \in T_x(M)$ we have*

$$\xi(d_z) = \frac{\int_A \xi(d(y, \cdot))\, dy}{\mathrm{Vol}(A)}.$$

Here $A = \{y \in B_\varepsilon(j_M(z), N) \mid y$ is not a cut point of $x\}$.

Assertion 1-5 is a direct consequence of the following two facts: *d_z is a Lipschitz function; the cut locus is contained in a set of smaller dimension.* (*Remark that d_z is not necessarily of C^2-class.*)

Lemma 1-6. *$f_M(M)$ is contained in the $3\varepsilon K^{1/2}$-neighborhood of $f_N(N)$.*

Proof. Let x be an arbitrary point of M. The definition of d_z implies $|d(j_M(z), x) - d_z(x)| < \varepsilon$. Take a point x' of N such that $d(x, x') < \varepsilon$. Then condition (1-2-3) implies that $|d(j_M(z), x) - d(j_N(z), x')| < 2\varepsilon$. It follows that $|d(j_N(z), x') - d_z(x)| < 3\varepsilon$. The lemma follows immediately.

Lemma 1-6, combined with facts (1-4-1) and (1-4-2), implies that $f_N^{-1} \circ \pi \circ \mathrm{Exp}^{-1} \circ f_M = f$ is well defined, where $\pi : N(f_N(N)) \to f_N(N)$ denotes the projection. This is the map f in our main theorem.

2. (M, N, f) is a fiber bundle

The proof of the following lemma will be given in the next section. Let δ, δ', and ν be positive numbers satisfying $\delta \leqslant \delta'$.

Lemma 2-1. *Let $l_i : [0, t_i] \to M (i = 1, 2)$ be geodesics on M such that $l_1(0) = l_2(0)$, and $l_i' : [0, t_i'] \ (i = 1, 2)$ be minimal geodesics on N such that $l_1'(0) = l_2'(0)$. Suppose*

$$\text{(2-2-1)} \qquad d(l_i(0), l_i(t_i)) - t_i < \nu,$$

$$\text{(2-2-2)} \qquad d\big(l_i(0), l_i'(0)\big) < \nu,$$

$$\text{(2-2-3)} \qquad d\big(l_i(t_i), l_i'(t_i')\big) < \nu,$$

$$\text{(2-2-4)} \qquad \delta R/10 < t_1 < \delta R \quad \text{and} \quad \delta' R/10 < t_2 < \delta' R.$$

Then we have

$$\left| \operatorname{ang}\left(\frac{Dl_1}{dt}(0), \frac{Dl_2}{dt}(0) \right) - \operatorname{ang}\left(\frac{Dl_1'}{dt}(0), \frac{Dl_2'}{dt}(0) \right) \right|$$
$$< \tau(\delta) + \tau(\nu \mid \delta, \delta') + \tau(\varepsilon \mid \delta, \delta').$$

Now we shall show that (M, N, f) is a fiber bundle. It suffices to see that f_M is transversal to the fibers of the normal bundle of $f_N(N)$. (Here we identified the tubular neighborhood to the normal bundle.) For this purpose, we have only to show the following lemma.

Lemma 2-3. *For each $p \in M$ and $\xi' \in T_{f(p)}(N)$, there exists $\xi \in T_p(M)$ satisfying*

$$|df_M(\xi) - df_N(\xi')| / |df_N(\xi')| < \tau(\sigma) + \tau(\varepsilon \mid \sigma).$$

To prove Lemma 2-3, we need Lemmas 2-4 and 2-9.

Lemma 2-4. *Suppose* $\sigma \leqslant \delta$, $\nu < \sigma/100$. *Let* $l_3: [0, t_3] \to M$, $l_3': [0, t_3'] \to N$ *be minimal geodesics satisfying the following*

(2-5-1) $$d\big(l_3(0), l_3'(0)\big) < \nu,$$

(2-5-2) $$d\big(l_3(t_3), l_3'(t_3')\big) < \nu,$$

(2-5-3) $$\delta R/10 < t_3, t_3' < \delta R.$$

Then we have

$$\frac{\left| df_M\left(\frac{Dl_3}{dt}(0)\right) - df_N\left(\frac{Dl_3'}{dt}(0)\right)\right|}{\left| df_N\left(\frac{Dl_3'}{dt}(0)\right)\right|} < \tau(\sigma) + \tau(\nu \mid \sigma, \delta) + \tau(\varepsilon \mid \sigma, \delta).$$

Proof. Put $p = l_3(0)$, $\xi = (Dl_3/dt)(0)$, $\xi' = (Dl_3'/dt)(0)$. For an arbitrary element z of Z satisfying

(2-6) $$d(p, j_M(z)) > r + 2\nu \quad \text{or} \quad d(p, j_M(z)) < r/8 - 2\nu,$$

we have, by (1.3), that

(2-7) $$|\xi(h(d(j_N(z), \cdot)))| < \kappa, \quad |\xi(h(\tilde{d}_x(\cdot)))| < \kappa,$$

in some neighborhoods of $l_3'(0)$ and $l_3(0)$, respectively. Next we shall study the case when $z \in Z$ does not satisfy (2-6). Assume that an element y of $B_\varepsilon(j_M(z), M)$ is not contained in the cut locus of p. Let $l_4: [0, t_4] \to M$ and $l_4': [0, t_4'] \to N$ denote minimal geodesics joining $l_3(0)$ to y and $l_3'(0)$ to $j_N(z)$ respectively. Since $\sigma R/10 < r/8 - 2\varepsilon - 2\nu < r + 2\varepsilon + 2\nu < \sigma R$, we have $\sigma R/10 < t_4 < \sigma R$, $\delta R/10 < t_3 < \delta R$. Hence, Lemma 2-1 implies

$$\big|\xi'(d(j_N(z), \cdot)) - \xi(d(y, \cdot))\big| < \tau(\sigma) + \tau(\nu \mid \sigma, \delta) + \tau(\varepsilon \mid \sigma, \delta).$$

Therefore, by using Assertion 1-5, we have

(2-8) $$\big|\xi'(d(j_N(z), \cdot)) - \xi(d_z(\cdot))\big| < \tau(\sigma) + \tau(\nu \mid \sigma, \delta) + \tau(\varepsilon \mid \sigma, \delta).$$

Then, Lemma 2-4 follows from (2-7), (2-8), and (1-4-3) if we take κ sufficiently small.

Lemma 2-9. *For each* $p \in M$, *we have* $d(p, f(p)) < \tau(\varepsilon)$.

Proof. By the definition of f and Lemma 1-6, we have

(2-10) $$d_{\mathbf{R}^n}(f_M(p), f_N(f(p))) < 3\varepsilon K^{1/2}.$$

Let $q \in N$ be an element satisfying $d(p, q) < \varepsilon$. Then, by the proof of Lemma 1-6, we have

(2-11) $$d_{\mathbf{R}^n}(f_M(p), f_N(q)) < 3\varepsilon K^{1/2}.$$

Inequalities (2-10) and (2-11) imply

$$d_{\mathbf{R}^n}(f_N(q), f_N(f(p))) < 6\varepsilon K^{1/2}.$$

Therefore (1-4-4) implies

$$d(q, f(p)) < 6C_4\varepsilon.$$

The above inequality, combined with $d(p,q) < \varepsilon$, implies the lemma.

Proof of Lemma 2-3. By assumption, there exist geodesics $l_3: [0, t_3] \to M$, $l_3': [0, t_3'] \to N$ such that $l_3(0) = p$, $l_3'(0) = f(p)$, $d(l_3(t_3), l_3'(t_3')) < \varepsilon$, $(Dl_3'/dt)(0) = \xi'$, and $\sigma R/10 < t_3, t_3' < \sigma R$. Lemma 2-9 implies $d(l_3(0), l_3'(0)) < \tau(\varepsilon)$. Therefore, Lemma 2-4 implies

$$\left| df_N(\xi') - df_M\left(\frac{Dl_3}{dt}(0)\right)\right| \Big/ |df_N(\xi')| < \tau(\sigma) + \tau(\varepsilon|\sigma),$$

as required.

3. A triangle comparison lemma

To prove Lemma 2-1, we need the following:

Lemma 3-1. *Let $l_i: [0, t_i] \to M$ $(i = 5, 6)$ be geodesics on M such that $l_5(0) = l_6(0)$. Suppose*

(3-2-1) $$l_5(0) = l_5(t_5),$$

(3-2-2) $$|d(l_6(0), l_6(t_6)) - t_6| < \nu,$$

(3-2-3) $$\delta^2 R < t_5 < 2\delta R \quad \textit{and} \quad \delta R/10 < t_6 < \delta R.$$

Then we have

$$\left|\operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_6}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Proof. Let $l_6': [-t_6/\delta, t_6/\delta] \to N$ be a minimal geodesic satisfying $d(l_6(0), l_6'(0)) < \varepsilon$ and $d(l_6(t_6), l_6'(t_6)) < 3\varepsilon + \nu$. (The existence of such a geodesic follows from (3-2-2).) Take a minimal geodesic $l_7: [0, t_7] \to M$ satisfying $l_7(0) = l_5(0)$ and $d(l_7(t_7), l_6'(t_6/\delta)) < \varepsilon$. Let $l_8: [0, t_8] \to M$ be a minimal geodesic joining $l_6(t_6)$ to $l_7(t_7)$. Then, since $|t_6 + t_8 - t_7| < \tau(\nu) + \tau(\varepsilon)$, and since l_7 is minimal, it follows that

(3-3) $$\operatorname{ang}\left(\frac{Dl_6}{dt}(t_6), \frac{Dl_8}{dt}(0)\right) < \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Let $l_9: [0, t_6/\delta] \to M$ denote the geodesic such that $l_9|_{[0, t_6]} = l_6$. Put $t_9 = t_6/\delta$ $(< R)$. Inequality (3-3) and the fact $|t_7 - t_9| < \tau(\nu) + \tau(\varepsilon)$ imply

$$d(l_7(t_7), l_9(t_9)) < \tau(\nu|\delta) + \tau(\varepsilon|\delta).$$

Hence, by the minimality of l_7, we obtain

$$(3\text{-}4) \qquad |d(0, l_9(t_9)) - t_9| < \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta).$$

Now let $\tilde{l}_i : [0, t_i] \to BT_R(l_1(0), M)$ ($i = 5, 9$) denote the lifts of l_i such that $\tilde{l}_i(0) = 0$. Then, (3-4) implies

$$(3\text{-}5) \qquad d(\tilde{l}_5(t_5), \tilde{l}_9(t_9)) > d(\tilde{l}_5(0), \tilde{l}_9(t_9)) - \tau(\nu\,|\,\delta) - \tau(\varepsilon\,|\,\delta).$$

On the other hand, by (3-2-3), we have

$$(3\text{-}6) \qquad t_5/t_9 < 20\delta \quad \text{and} \quad \delta^2 R < t_5.$$

Inequalities (3-5), (3-6), and Toponogov's comparison theorem (see [2, Chapter 2]) imply

$$(3\text{-}7) \qquad \operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_6}{dt}(0)\right) > \pi/2 - \tau(\delta) - \tau(\nu\,|\,\delta) - \tau(\varepsilon\,|\,\delta).$$

Next, let $l_{10} : [0, t_{10}] \to M$ be a minimal geodesic satisfying $l_5(0) = l_{10}(0)$ and $d(l'_6(-t_6/\delta), l_{10}(t_{10})) < \varepsilon$. Then, since

$$|d(l_6(t_6), l_{10}(t_{10})) - (t_6 + t_{10})| < \tau(\nu) + \tau(\varepsilon),$$

it follows that

$$(3\text{-}8) \qquad \left|\operatorname{ang}\left(\frac{Dl_6}{dt}(0), \frac{Dl_{10}}{dt}(0)\right) - \pi\right| < \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta).$$

On the other hand, by the method used to show (3-7), we can prove

$$(3\text{-}9) \qquad \operatorname{ang}\left(\frac{Dl_5}{dt}(0), \frac{Dl_{10}}{dt}(0)\right) > \pi/2 - \tau(\delta) - \tau(\nu\,|\,\delta) - \tau(\varepsilon\,|\,\delta).$$

The lemma follows immediately from inequalities (3-7), (3-8), (3-9).

Remark that to prove Lemma 2-1 we may assume $\delta = \delta'$. When $t_2, t'_2 < \delta R$, clearly we can take $\delta = \delta'$, and when $t_2, t'_2 \geqslant \delta R$, Assertion 3-10 implies that we can replace l_2, l'_2 by $l_2|_{[0,\delta R]}, l'_2|_{[0,\delta R]}$.

Assertion 3-10. $d(l_2(\delta R), l'_2(\delta R)) < \tau(\nu\,|\,\delta, \delta') + \tau(\varepsilon\,|\,\delta, \delta')$.

Proof. Take minimal geodesics $l'_{11} : [0, R] \to N$ and $l_{11} : [0, t_{11}] \to M$ satisfying $l'_2(0) = l'_{11}(0)$, $d(l_2(\delta R), l'_{11}(\delta R)) < 2\nu + 2\varepsilon$, $l_2(0) = l_{11}(0)$, and $d(l_{11}(t_{11}), l'_{11}(t'_2)) < \varepsilon$. Let $l_{12} : [0, t_{12}] \to M$ denote the minimal geodesics joining $l_2(\delta R)$ to $l_{11}(t_{11})$. Then, since $|t_{12} + \delta R - t_{11}| < \tau(\nu) + \tau(\varepsilon)$ and since l_{11} is minimal, it follows that

$$\operatorname{ang}\left(\frac{Dl_2}{dt}(\delta R), \frac{Dl_{12}}{dt}(0)\right) < \tau(\nu\,|\,\delta, \delta') + \tau(\varepsilon\,|\,\delta, \delta').$$

Hence we have

$$d(l_2(t_2), l_{11}(t_2)) < \tau(\nu\,|\,\delta, \delta') + \tau(\varepsilon\,|\,\delta, \delta').$$

On the other hand, by assumption, we have

$$d\big(l_2(t_2), l_2'(t_2')\big) < \nu, \qquad d\big(l_{11}(t_{11}), l_{11}'(t_2')\big) < \varepsilon.$$

Then, we conclude

$$d\big(l_2'(t_2'), l_{11}'(t_2')\big) < \tau(\nu\,|\,\delta, \delta') + \tau(\varepsilon\,|\,\delta, \delta').$$

Therefore, applying Toponogov's comparison theorem to N, we obtain

$$d\big(l_2'(\delta R), l_{11}'(\delta R)\big) < \tau(\nu\,|\,\delta, \delta') + \tau(\varepsilon\,|\,\delta, \delta').$$

The assertion follows from the above inequality and the fact $d(l_2(\delta R), l_{11}'(\delta R)) < \varepsilon$.

Therefore, in the rest of this section, we shall assume $\delta = \delta'$. Take a minimal geodesic $l_{13}: [0, t_{13}] \to M$ joining $l_1(t_1)$ to $l_2(t_2)$. Let $\tilde{l}_i: [0, t_i] \to BT_R(l_1(0), M)$ ($i = 1, 2, 13$) denote the lifts to l_i such that $\tilde{l}_i(0) = 0$ ($i = 1, 2$) and $\tilde{l}_{13}(0) = \tilde{l}_1(t_1)$.

Assertion 3-11. *We have* $d(\tilde{l}_{13}(t_{13}), \tilde{l}_2(t_2)) < (\tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta)) \cdot \delta$.

Proof. Put $\iota = d(\tilde{l}_{13}(t_{13}), \tilde{l}_2(t_2))$. We may assume $\delta^2 R < \iota$. Take another lift $\hat{l}_2$ of l_2 satisfying $\hat{l}_2(t_2) = \tilde{l}_{13}(t_{13})$. Let $\tilde{l}_i: [0, t_i] \to BT_R(l_1(0), M)$ ($i = 14, 15$) denote the minimal geodesics joining $\tilde{l}_2(t_2)$ to $\tilde{l}_{13}(t_{13})$ and $\tilde{l}_1(0)$ to $\hat{l}_2(0)$ respectively. Then Lemma 3-1 implies

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_2}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

$$\left|\operatorname{ang}\left(\frac{D\hat{l}_2}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(t_{15})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_2}{dt}(t_2), \frac{D\tilde{l}_{14}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

$$\left|\operatorname{ang}\left(\frac{D\hat{l}_2}{dt}(t_2), \frac{D\tilde{l}_{14}}{dt}(t_{14})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_1}{dt}(0), \frac{D\tilde{l}_{15}}{dt}(0)\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_{13}}{dt}(t_{13}), \frac{D\tilde{l}_{14}}{dt}(t_{14})\right) - \pi/2\right| < \tau(\delta) + \tau(\nu\,|\,\delta) + \tau(\varepsilon\,|\,\delta),$$

Hence, a standard argument using Toponogov's comparison theorem implies

$$\begin{aligned} &d(\tilde{l}_{13}(0), \tilde{l}_1(t_1)) \\ &\quad > \iota\{1 - \tau(\delta) - \tau(\nu\,|\,\delta) - \tau(\varepsilon\,|\,\delta)\} - \delta\{\tau(\delta) - \tau(\nu\,|\,\delta) - \tau(\varepsilon\,|\,\delta)\}. \end{aligned}$$

But $\tilde{l}_{13}(0) = \tilde{l}_1(t_1)$. The assertion follows immediately.

Now we are in the position to complete the proof of Lemma 2-1. Assertion 3-11 implies

$$|d(\tilde{l}_1(t_1), \tilde{l}_2(t_2)) - d(l'_1(t_1), l'_2(t_2))| < 2\varepsilon + \delta\{\tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta)\}.$$

On the other hand, we have

$$|t_i - t'_i| < 2\nu \quad \text{and} \quad \delta R/10 < t_i < \delta R \qquad (i = 1, 2).$$

Hence, Toponogov's comparison theorem implies

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_1}{dt}(0), \frac{D\tilde{l}_2}{dt}(0)\right) - \operatorname{ang}\left(\frac{Dl'_1}{dt}(0), \frac{Dl'_2}{dt}(0)\right)\right| < \tau(\delta) + \tau(\nu|\delta) + \tau(\varepsilon|\delta),$$

as required.

4. f is an "almost Riemannian submersion"

In this section we shall verify (0-1-13). First we shall prove the following:

Lemma 4-1. $|df| < 1 + \tau(\sigma) + \tau(\varepsilon|\sigma)$.

Proof. Since the second fundamental form of $f_N(N)$ is smaller than $\tau(\sigma)$, the norm of the restriction of the exponential map to $B_{4\varepsilon K^{1/2}}(Nf_N(N))$ is greater than $1 - \tau(\sigma) - \tau(\varepsilon|\sigma)$ (for details, see the proof of [8, Lemma 7.2]). Therefore Lemma 4-1 follows from Lemma 2-3 and the definition of f.

Let $p \in M$, $q = f(p)$. Put $k =$ (the dimension of N). We introduce a new small positive constant θ and assume $\sigma < \theta$. Take points $z'_1, z'_2, \cdots, z'_k$ of N such that $d(q, z'_i) = \theta R$ and that the set of vectors $\operatorname{grad}_q(d(z'_1, \cdot)), \cdots, \operatorname{grad}_q(d(z'_k, \cdot))$ is an orthonormal base of $T_q(N)$. Let z_i be a point of M such that $d(z_i, z'_i) < \varepsilon$. For $x \in B_{\theta^2 R}(p, M)$, put

$$g_i(x) = \int_{y \in B_\varepsilon(z_i, M)} d(x, y)\, dy / \mathbf{Vol}(B_\varepsilon(z_i, M)),$$

and let $\Pi_1(x)$ denote the linear subspace of $T_x(M)$ spanned by $\operatorname{grad}_x(g_1), \cdots, \operatorname{grad}_x(g_k)$, and $\Pi_2(x)$ the orthonormal complement of $\Pi_1(x)$. $P_i \colon T_x(M) \to \Pi_i(x)$ denotes the orthonormal projections.

Lemma 4-2. *For each* $\xi \in \Pi_1(x)$ *satisfying* $|\xi| = 1$, *we have*

$$||df(\xi)| - |\xi|| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Proof. By Lemmas 2-4, 2-9, and the definitions of f_M, f_N and g_i, we can prove

$$\left|df_M(\operatorname{grad}_x(g_i)) - df_N\left(\operatorname{grad}_{f(x)}\left(d(z'_i, \cdot)\right)\right)\right| < (\tau(\sigma) + \tau(\varepsilon|\sigma)) \cdot K^{1/2}.$$

Therefore, by the definition of f, we have

$$|df(\mathrm{grad}_x(g_i)) - \mathrm{grad}_{f(x)}(d(z_i', \cdot))| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

It follows that

$$||df(\mathrm{grad}_x(g_i))| - 1| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

This inequality, combined with Lemma 4-1, implies Lemma 4-2.

The following lemma is a direct consequence of Lemmas 4-1 and 4-2 and the fact $\mathbf{dim}\,\Pi_2(p) = \mathbf{dim}\,N$.

Lemma 4-3. *Let $x \in B_{\theta^2 R}(p, M)$. Then for each $\xi \in T_x(M)$ tangent to the fiber, we have*

$$|P_1(\xi)|/|\xi| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Now, (0-1-3) follows immediately from Lemmas 4-1, 4-2, and 4-3.

In the rest of this section, we shall prove several lemmas required in the argument of the next section.

Lemma 4-4. *Let $x \in B_{\theta^x R}(p, M)$ and let $\xi \in \Pi_1(x)$ be a vector with $|\xi| = 1$. Then we have*

$$|d(x, \mathbf{exp}_x(s\xi)) - s| < \tau(\sigma) - \tau(\varepsilon|\sigma)$$

and

$$|d(f(x), f(\mathbf{exp}_x(s\xi))) - s| < \tau(\sigma) - \tau(\varepsilon|\sigma)$$

for each s smaller than R.

Proof. Put $\xi' = df(\xi)$, and $l'(t) = \mathbf{exp}(t\xi'/|\xi'|)$. Lemma 4-2 implies $||\xi'| - 1| < \tau(\sigma) + \tau(\varepsilon|\sigma)$. Let $l: [0, R] \to M$ be a minimal geodesic satisfying $d(l(R), l'(R)) < 4\varepsilon + R(|\xi'| - 1)$. Put $\eta = (Dl/dt)(0)$. By Lemma 2-3 and the definition of f, we have

$$|df(\eta) - \xi'| < \tau(\sigma) + \tau(\varepsilon|\sigma). \tag{4-5}$$

Hence we have $||df(\eta)| - |\eta||/|\eta| < \tau(\sigma) + \tau(\varepsilon|\sigma)$, Therefore, Lemmas 4-1, 4-2 imply

$$|P_1(\eta) - \eta| < \tau(\sigma) + \tau(\varepsilon|\sigma). \tag{4-6}$$

Inequalities (4-5), (4-6), combined with the facts $\xi \in \Pi_1(x)$, $df(\xi) = \xi'$, and Lemmas 4-1, 4-2, imply $|\eta - \xi| < \tau(\sigma) + \tau(\varepsilon|\sigma)$. Furthermore, by the definition of η, we have

$$|d(f(x), f(\mathbf{exp}_x(s\eta))) - s| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

The lemma follows immediately from these facts.

Lemma 4-7. *Let $x \in B_{\theta^2 R}(p, M)$, and $\xi_1, \xi_2 \in \Pi_1(x)$ be vectors such that $|\xi_1| = |\xi_2| < \sigma R$. Then we have*

$$|d(\mathbf{exp}(\xi_1), \mathbf{exp}(\xi_2)) - 2 \cdot |\xi_1| \cdot \mathbf{sin}(\mathrm{ang}(\xi_1, \xi_2)/2)| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Proof. By Lemma 4-4, we have

$$|d(q, f(\mathbf{exp}(\xi_i))) - |\xi_i|| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

On the other hand, Lemmas 4-1 and 4-2 imply

$$|\mathrm{ang}(df(\xi_1), df(\xi_2)) - \mathrm{ang}(\xi_1, \xi_2)| < \tau(\sigma) + \tau(\varepsilon|\sigma).$$

Hence, applying Toponogov's comparison theorem to N, we obtain the lemma.

Lemma 4-8. *Let $x \in B_{\theta^2 R}(p, M)$ and $\xi \in \Pi_2(x)$ be a vector with $|\xi| = 1$. Then we have*

$$d(f(\mathbf{exp}(s\xi)), f(x)) < (\tau(\sigma) + \tau(\theta) + \tau(\varepsilon|\sigma, \theta)) \cdot s$$

for each positive number s smaller than $\theta^2 R$.

Proof. Put $l_{16}(t) = \mathbf{exp}(t\xi)$. Since $\xi \in \Pi_2(x)$, we have

$$\mathrm{ang}(\xi, \mathrm{grad}_x(g_i)) = \pi/2. \tag{4-9}$$

Lemma 4-8 follows immediately from Lemmas 4-1, 4-2, 4-3, and the following:

Assertion 4-10. *For each $t < s$, we have*

$$\left|\mathrm{ang}\left(\frac{Dl_{16}}{dt}(t), \mathrm{grad}_{l_{16}(t)}(g_i)\right) - \pi/2\right| < \tau(\varepsilon|\theta) + \tau(\theta).$$

Proof. Let $l_k : [0, t_k] \to M$ $(k = 17, 18)$ be minimal geodesics joining x and $l_{16}(t)$ to z_i respectively. By the definition of g_i, we can take l_{17} and l_{18} so that they satisfy

$$\mathrm{ang}\left(\frac{Dl_{17}}{dt}(0), -\mathrm{grad}_x(g_i)\right) < \tau(\varepsilon|\theta), \tag{4-11}$$

$$\mathrm{ang}\left(\frac{Dl_{18}}{dt}(0), -\mathrm{grad}_{l_{16}(t)}(g_i)\right) < \tau(\varepsilon|\theta). \tag{4-12}$$

Let $\tilde{l}_j$ $(j = 16, 17, 18)$ denote the lifts of l_j $(j = 16, 17, 18)$ to $B_R(x, M)$ satisfying $\tilde{l}_{16}(0) = \tilde{l}_{17}(0) = 0$ and $\tilde{l}_{18}(0) = \tilde{l}_{16}(t)$, and let $\tilde{l}_{19} : [0, t_{19}] \to B_R(x, M)$ denote the minimal geodesic joining $\tilde{l}_{17}(t_{17})$ to $\tilde{l}_{18}(t_{18})$. Put $l_{19} = \mathbf{exp}_x \tilde{l}_{19}$. Then Lemma 3-1 implies that one of the following holds:

$$t_{19} < \theta^2 R, \tag{4-13-1}$$

$$\begin{aligned} &\left|\mathrm{ang}\left(\frac{Dl_{17}}{dt}(t_{17}), \frac{Dl_{19}}{dt}(0)\right) - \pi/2\right| < \tau(\theta) + \tau(\varepsilon|\theta), \\ &\left|\mathrm{ang}\left(\frac{Dl_{18}}{dt}(t_{18}), \frac{Dl_{19}}{dt}(t_{19})\right) - \pi/2\right| < \tau(\theta) + \tau(\varepsilon|\theta). \end{aligned} \tag{4-13-2}$$

If (4-13-2) holds, then applying Toponogov's comparison theorem to $B_R(x, M)$, we obtain

$$t > (1 - \tau(\varepsilon|\theta) - \tau(\theta)) \cdot t_{19}.$$

Then, in each case, we have $d(\tilde{l}_{17}(t_{17}), \tilde{l}_{18}(t_{18})) = t_{19} < 2\theta^2 R$. Therefore, by a standard argument using Toponogov's comparison theorem, we can prove

(4-14)

$$\left|\operatorname{ang}\left(\frac{D\tilde{l}_{16}}{dt}(0), \frac{D\tilde{l}_{17}}{dt}(0)\right) - \operatorname{ang}\left(\frac{D\tilde{l}_{16}}{dt}(t), \frac{D\tilde{l}_{18}}{dt}(0)\right)\right| < \tau(\theta) + \tau(\varepsilon\,|\,\delta).$$

Assertion 4-10 follows immediately from (4-9), (4-11), (4-12), and (4-14).

5. The fiber in an infranilmanifold

In this section we shall verify (0-1-2). The following is a direct consequence of Lemma 2-9.

Lemma 5-1. *The diameter of the fiber, $f^{-1}(q)$, is smaller than $\tau(\varepsilon)$.*

If we can obtain an estimate of the second fundamental form of $f^{-1}(q)$, Lemma 5-1 combined with [6, 1.4] would imply (0-1-2). But as was remarked at §1, the map f is only of C^1-class and not necessarily of C^2-class. Hence, it is impossible to estimate the second fundamental form. Then, instead, we shall modify the proof of [6, 1.4] in order to verify (0-1-3). The detailed proof of [6, 1.4] is presented in [1]. Therefore, in the rest of this section, we shall follow [1], mentioning the required modifications.

We introduce a new positive constant ρ smaller than $\theta^2 R$. Let π_ρ denote the local fundamental pseudogroup introduced in [6, 5.6] or [1, 2.2.6] (in [1] the terminology, local fundamental pseudogroup, is not introduced, but the notation π_ρ is defined there). Here we take p as the base point. Following [1, 2.2.3], we let $*$ denote the Gromov's product on π_ρ. For a vector space V, the symbol $A(V)$ denotes the group of all affine transformations of V. Let $m: \pi_\rho \to A(T_p(M))$ denote the affine holonomy map introduced in [1, 2.3], r its rotation part, and t its translation part. The following lemma is proved in [1, 2.3.1].

Lemma 5-2. *For $\alpha, \beta \in \pi_\rho$, we have*

$$d(r(\beta)\circ r(\alpha), r(\beta * \alpha)) \leqslant |t(\alpha)|\cdot|t(\beta)|,$$

$$|t(m(\beta)\circ m(\alpha))| - |t(\beta * \alpha)| \leqslant |t(\alpha)|\,|t(\beta)|(|t(\alpha) + t(\beta)|).$$

Next we shall prove the following:

Lemma 5-3. *For each $\alpha \in \pi_\rho$, we have*

$$|P_1 \circ r(\alpha)\circ P_1 - P_1| < \tau(\theta) + \tau(\sigma\,|\,\theta) + \tau(\rho\,|\,\theta) + \tau(\varepsilon\,|\,\sigma, \theta).$$

Proof of Lemma 5-3. Put $s =$ (the length of α). Let ξ be an arbitrary element of $\Pi_1(p)$ satisfying $|\xi| = \theta R$. First we shall prove

(5-4) $$d(\mathbf{exp}(\xi), \mathbf{exp}(r(\alpha)(\xi))) < \tau(\rho\,|\,\theta).$$

In fact, let $\tilde{\xi} \in T_0(BT_R(p, M))$ be a vector satisfying $(d(\mathbf{exp}_p))(\tilde{\xi}) = \xi$, let a curve $\tilde{\alpha}: [0, s] \to BT_R(p, M)$ denote the lift of α satisfying $\tilde{\alpha}(0) = 0$, and let $\hat{\xi} \in T_{\tilde{\alpha}(s)}(BT_R(p, M))$ be a vector satisfying $d(\mathbf{exp}_p)(\hat{\xi}) = r(\xi)$. By the definition of r, the vector $\hat{\xi}$ is a parallel translation of $\tilde{\xi}$ along $\tilde{\alpha}$. Let $\tilde{\xi}(t) \in T_{\tilde{\alpha}(t)}(BT_R(p, M))$ denote the parallel translation of $\tilde{\xi}$ along $\tilde{\alpha}|_{[0,t]}$. Set $J_{t_0}(u) = D/dt|_{t=t_0} \mathbf{exp}_{\tilde{\alpha}(t)}(u \cdot \tilde{\xi}(t))$. Since $J_{t_0}(\cdot)$ is a Jacobi field along the geodesic $u \to \mathbf{exp}_{\tilde{\alpha}(t)}(u \cdot \tilde{\xi}(t_0))$, and since $|J_{t_0}(0)| = 1$, it follows that $|J_{t_0}(1)|$ has an upperbound depending only on n and $|\xi|$. Therefore, $\tilde{\xi}(s) = \hat{\xi}$ implies that

$$d(\mathbf{exp}(\tilde{\xi}), \mathbf{exp}(\hat{\xi})) < \int_0^s |J_t(1)|\, dt \leqslant \tau(\rho \,|\, \theta).$$

Inequality (5-4) follows immediately.

(5-4) and Lemma 4-4 imply

$$(5\text{-}5) \quad |d(p, \mathbf{exp}(r(\alpha)(\xi))) - |r(\alpha)(\xi)|| < \tau(\sigma) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma).$$

Next we shall prove the following:

Assertion 5-6. *We have*

$$|P_1(r(\alpha)(\xi)) - r(\alpha)(\xi)|/|r(\alpha)(\xi)| < \tau(\theta) + \tau(\sigma \,|\, \theta) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma, \theta).$$

Proof. Put $l_{20}(t) = \mathbf{exp}_p(t \cdot r(\alpha)(\xi)/|\xi|)$ and $t_{20} = |\xi|$. Let $l'_{20}: [0, t'_{20}] \to N$ denote the minimal geodesic satisfying $l'_{20}(0) = q$, $d(l_{20}(t_{20}), l'_{20}(t'_{20})) < \varepsilon$, and $l_{21}: [0, t_{21}] \to M$ be a minimal geodesic joining p to $\mathbf{exp}_p(r(\alpha)(\xi))$. Then, by inequality (5-5) and Lemma 2-9, we can apply Lemma 2-1, and obtain

$$(5\text{-}7) \quad \left|\mathrm{ang}\left(\frac{Dl_{21}}{dt}(0),\ r(\alpha)(\xi)\right)\right| < \tau(\theta) + \tau(\sigma \,|\, \theta) + \tau(\rho \,|\, \theta) + \tau(\varepsilon \,|\, \sigma, \theta).$$

On the other hand, by Lemma 2-4 and the definition of f, we have

$$\left|df\left(\frac{Dl_{21}}{dt}(0)\right) - \frac{Dl'_{21}}{dt}(0)\right| < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

It follows that

$$\left|\left|df\left(\frac{Dl_{21}}{dt}(0)\right)\right| - \left|\frac{Dl_{21}}{dt}(0)\right|\right| \Big/ \left|df\left(\frac{Dl_{21}}{dt}(0)\right)\right| < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

Therefore, Lemmas 4-1 and 4-2 imply

$$(5\text{-}8) \qquad \mathrm{ang}\left(\frac{Dl_{21}}{dt}(0),\ P_1\left(\frac{Dl_{21}}{dt}(0)\right)\right) < \tau(\sigma) + \tau(\varepsilon \,|\, \sigma).$$

Inequalities (5-7) and (5-8) immediately imply the assertion.

Now, Lemma 5-3 follows immediately from inequality (5-5) and Assertion 5-6.

We put $\tau = \tau(\theta) + \tau(\rho \mid \theta) + \tau(\sigma \mid \theta) + \tau(\varepsilon \mid \sigma, \rho, \theta)$. The following lemma corresponds to [1, Proposition 2.1.3].

Lemma 5-9. *For each $\xi \in \Pi_2(p)$ with $|\xi| < \rho$, there exists $\alpha \in \pi_\rho$ satisfying $|\xi - t(\alpha)| < \tau\rho$.*

Proof. By Lemma 4-8, we have

$$d(f(\mathbf{exp}(\xi)), q) < \tau \cdot |\xi|.$$

This formula and Lemma 5-1 imply that

$$d(\mathbf{exp}(\xi), p) < \tau(\varepsilon) + \tau \cdot |\xi|.$$

The lemma follows immediately.

Next we shall prove a lemma corresponding to [1, 2.2.7]. Following the notations there, we define a group $\hat{\pi}_\rho$ as follows. Let $W(\pi_\rho)$ be the free group of words in the elements of π_ρ; let $N_0(\pi_\rho)$ be the set of words $\alpha\beta\gamma^{-1}$ where $\gamma = \alpha * \beta$; let $N(\pi_\rho)$ be the smallest normal subgroup in $W(\pi_\rho)$ which contains $N_0(\pi_\rho)$. Put $\hat{\pi}_\rho = W(\pi_\rho)/N(\pi_\rho)$.

Lemma 5-10. *If ρ is smaller than a constant depending only on n and μ, and if σ and ε are smaller than a constant depending only on n and R, then there exists a natural isomorphism $\hat{\Phi}: \hat{\pi}_\rho \to \pi_1(f^{-1}(q))$.*

Proof. Since f is a fiber bundle and since any μ balls in N are contractible, we see that $\pi_1(f^{-1}(q))$ is isomorphic to the image of $\pi_1(B_C(p, M))$ in $\pi_1(B_{C'}(p, M))$, where $\sigma, \varepsilon < \tau(C) < C < C'/2 < C' < \mu$. Using this remark, we can prove Lemma 5-10 by the same method as [1, Proposition 2.2.7].

Using Lemmas 5-2, 5-9, and 5-10, the arguments of [1, Chapters 3 and 4] stand with little change. Then, we obtain the following result which corresponds to [1, 4.6.5].

Lemma 5-11. *We can choose ρ such that the following holds.*

(i) *The natural map $\pi_\rho \to \hat{\pi}_\rho$ is injective and $\hat{\pi}_\rho = \pi_1(f^{-1}(q), p)$.*

(ii) *$\hat{\pi}_\rho$ has a nilpotent, torsion free normal subgroup $\hat{\Gamma}_\rho$ of finite index. We put $\Gamma_\rho = \hat{\Gamma}_\rho \cap \pi_\rho$.*

(iii) *Γ_ρ is generated by m loops $\gamma_1, \cdots, \gamma_m$ such that each element $\gamma \in \Gamma_\rho$ can uniquely be written as a normal word $\gamma = \gamma_1^{l_1} \cdots \gamma_m^{l_m}$; these generators are adapted to the nilpotent structure, i.e.*

$$\gamma_j \cdot \langle \gamma_1, \cdots, \gamma_i \rangle \cdot \gamma_j^{-1} = \langle \gamma_1, \cdots, \gamma_i \rangle \qquad (1 \leqslant i \leqslant j \leqslant m).$$

Here m denotes the dimension of $f^{-1}(q)$.

Furthermore, Corollary 3.4.2 in [1] implies the following.

Lemma 5-12. *If $\alpha \in \Gamma_\rho$, then $|r(\alpha)| < \tau$.*

Next we shall follow the argument of [1, Chapter 5]. By Corollary 5.1.3 of [1], we have the following:

Lemma 5-13. *The structure of nilpotent groups on* $\hat{\Gamma}_\rho = (\mathbb{Z}^n, \cdot)$ *can be extended to* $\mathbb{R}^n$. *Namely there exists a nilpotent Lie group* $G = (\mathbb{R}^n, \cdot)$ *such that* $\hat{\Gamma}_\rho$ *is a lattice of* G.

Following [1, 5.1.4], we shall introduce a left invariant metric on G.

Definition 5-14. Put $X_i = P_2(t(\gamma_i))$, $Y_i = \mathbf{exp}^{-1}(\gamma_i) \in L$. Here L denotes the Lie algebra of G. We introduce a scalar product on L such that the linear map given by $X_i \to Y_i$ is an isometry between $\Pi_2(p)$ and L, and extend this product by left translation to a Riemannian metric on G.

Let $\overline{B}$ be a subset of M containing $B_{2\rho}(p, M)$ and satisfying $\pi_1(\overline{B}) = \pi_1(f^{-1}(q))$. Let B denote the universal covering space of $\overline{B}$, and $\pi : B \to \overline{B}$ the projection. Take a point $\tilde{p}$ in $\pi^{-1}(p)$. By the method of [1, 5.4], we can prove the following two lemmas.

Lemma 5-15. *For each* $\alpha \in \Gamma_\rho$, *we have*

$$|d(\tilde{p}, \alpha(\tilde{p})) - d_G(e, \alpha)| < \tau.$$

Here d_G *is the distance induced from the metric defined in* 5-14, *and* e *denotes the unit element.*

Lemma 5-16. *The absolute value of the sectional curvature of* G *has an upperbound depending only on the dimension.*

Let $f_G : G \to L^2(\Gamma_\rho)$ be the map defined by $x \to (h(d_G(x, \gamma(\tilde{p}))))_{\gamma \in \Gamma_\rho}$, where h is a function satisfying condition (1-3), and as the number r in (1-3) we take a constant depending only on ρ, R, and n. The restriction of f_G to $B_\rho(e, G)$ is an embedding. Let $d_B : B \to L^2(\Gamma_\rho)$ denote the map defined by $x \to (h(d(x, \gamma(\tilde{p}))))_{\gamma \in \Gamma_\rho}$. Now using Lemmas 5-15 and 5-16 we can repeat the argument of §§1, 2 and obtain the following. The symbol C_5 below denotes a constant depending only on ρ, R and, n.

Lemma 5-17. *Let* B' *be the* C_5*-neighborhood of* $\{\gamma(\tilde{p}) \mid \gamma \in \Gamma_{\rho - C_5}\}$. *Then there exists a map* $\Phi : B' \to B_\rho(e, G)$ *such that the following hold*:

(5-18-1) Φ *has maximal rank.*

(5-18-2) *If* $x \in B'$, $\gamma \in \hat{\Gamma}_\rho$, $\gamma(x) \in B'$, *then* $\gamma(\Phi(x)) = \Phi(\gamma(x))$.

(5-18-3) *If* $x \in B'$, $\xi \in T_x(B')$ *satisfy* $d\Phi(gx) = 0$, *then we have*

$$\operatorname{ang}(d\pi(\xi), \Pi_2(x)) < \tau$$

(*see Lemma* 4.3).

Now we are in the position to complete the proof of (0-1-2). Put $\tilde{F} = \pi^{-1}(f^{-1}(q))$. By Lemma 5-1, we may assume $\tilde{F} \subset B'$ replacing ε by a smaller one if necessary. Hence, by Lemma 5-17, we obtain a map $\tilde{F}/\hat{\Gamma}_\rho \to G/\hat{\Gamma}_\rho$. Fact (5-18-3) and Lemma 4-3 imply that this map is a covering map. Hence $\tilde{F}/\hat{\Gamma}_\rho$ is a nilmanifold. On the other hand, $\tilde{F}/\hat{\Gamma}_\rho$ is a finite covering of $f^{-1}(q)$. Therefore $f^{-1}(q)$ is an infranilmanifold. Thus the verification of (0-1-2) is completed.

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Collapsing Riemannian manifolds to ones with lower dimension II

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§ 0. Introduction.

The purpose of this paper is to investigate the phenomena that a sequence of Riemannian manifolds M_i converges to ones with lower dimension, N, with respect to the Hausdorff distance, which is introduced in [**11**]. We have studied this phenomena in [**7**] and proved there that M_i is a fibre bundle over N with infranilmanifold fibre. In this paper, we study which fibre bundle it is, and give a necessary and sufficient condition. We will describe it in Theorem 0-1 and 0-7.

THEOREM 0-1. *Let M_i be a sequence of $n+m$-dimensional compact Riemannian manifolds and N be an n-dimensional compact Riemannian manifold. Assume*

(0-2-1) *M_i converges to N with respect to the Hausdorff distance,*

(0-2-2) *$|$sectional curvature of $M_i| \leqq 1$.*

Then, for sufficiently large i, there exists a map $\pi_i: M_i \to N$ such that the following hold.

(0-3-1) *π_i is a fibre bundle.*

(0-3-2) *$\pi_i^{-1}(p)=G/\Gamma$, where G is a nilpotent Lie group and Γ is a discrete group of affine transformations of G satisfying $[\Gamma: G\cap\Gamma]<\infty$. Here we put the (unique) connection on G which makes all right invariant vector field parallel, and G is regarded to be a group of affine transformations on G by right multiplication.*

(0-3-3) *The structure group of π_i is contained in the skew product of $C(G)/(C(G)\cap\Gamma)$ and $\mathrm{Aut}\,\Gamma$, where $C(G)$ denotes the center of G.*

REMARK 0-4. Statements (0-3-1) and (0-3-2) were proved in [**7**].

REMARK 0-5. [**7**, 0-1-3] also holds. Namely π_i is an almost Riemannian submersion in the sense stated there.

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Remark 0-6. It is well known that the group $\pi_k(\mathrm{Diff}(G/\Gamma))$ is not finitely generated in general, but $\pi_k(C(G)/(C(G)\cap\Gamma)\tilde{\times}\mathrm{Aut}\,\Gamma)$ is always finitely generated. Therefore, there exist a lot of fibre bundles which satisfy (0-3-1) and (0-3-2) but do not satisfy (0-3-3).

Theorem 0-7. *Let M be an $n+m$-dimensional manifold, N an n-dimensional complete Riemannian manifold with bounded sectional curvature, and $\pi: M\to N$ be a smooth map. Suppose that π satisfies* (0-3-1), (0-3-2) *and* (0-3-3). *Then, there exists a family of Riemannian metrics g_ε on M such that the following hold.*

(0-8-1) *The sequence of Riemannian manifolds (M, g_ε) converges to the Riemannian manifold N, with respect to the Hausdorff distance.*

(0-8-2) *There exists a constant C independent of ε such that*

$$|\text{sectional curvature of } (M, g_\varepsilon)| \leqq C\,.$$

Theorems 0-1 and 0-7, combined with [**9**, Theorem 0-6], imply the following:

Theorem 0-9. *For each m and D, there exists a positive constant $\varepsilon(n, D)$ such that the following holds. Suppose an m-dimensional Riemannian manifold M satisfies*

(0-10-1) *volume of $M \leqq \varepsilon(m, D)$,*

(0-10-2) *diameter of $M \leqq D$,*

(0-10-3) *$|$sectional curvature of $M| \leqq 1$,*

(0-10-4) $\pi_k(M) = 1$, *for* $k \geqq 2$.

Then, $\mathrm{Minvol}\,M=0$, *where* $\mathrm{Minvol}\,M$ *is defined in* [**10**].

Theorem 0-9 is a partial answer to the following

Problem 0-11. *Does there exist ε_m such that* $\mathrm{Minvol}\,M\leqq\varepsilon_m$ *implies* $\mathrm{Minvol}\,M=0$ *?*

If we can remove the conditions (0-10-2) and (0-10-4), we will have the affirmative answer.

The organization of this paper is as follows. Sections 1 to 5 are devoted to the proof of Theorem 0-1. The outline of these sections is in § 1. In the course of the proof, we shall prove some results on eigenfunctions of Laplace operator, which improve one of [**6**]. These results may have an independent interest. In § 6, we shall prove Theorem 0-7. In § 7, we shall give an orbifold version of Theorem 0-1. The proof of Theorem 0-9 is in § 7. In § 8, we add some remarks concerning the case when the limit space is not a manifold.

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NOTATION. For a Riemannian manifold M, $\mathrm{Vol}\, M$ denotes the volume of M, $\mathrm{Diam}\, M$ denotes the diameter of M. For a metric space X and $x \in X$ we put

$$B_D(x, X) = \{y \in X \mid d(x, y) < D\}.$$

$B(C)$ stands for $B_C(0, \boldsymbol{R}^n)$. For two metric spaces X, Y, $d_H(X, Y)$ denotes the Hausdorff distance between them which is defined in [**11**], $\lim_{i\to\infty} {}_H X_i = X$ means $\lim_{i\to\infty} d_H(X, X_i) = 0$.

§ 1. Outline of the proof.

Our main Theorem 0-1 is a consequence of the following:

THEOREM 1-1. *Let M_i and N be as in Theorem 0-1. Then, for each sufficiently large i, there exists a fibration $\pi_i: M_i \to N$ such that the following hold.*

(1-2-1) *For each $p \in N$, there exists a flat connection on $\pi_i^{-1}(p)$, which depends smoothly on p.*

(1-2-2) *There exists a nilpotent Lie group G and a group of affine transformations Γ of G such that $\pi_i^{-1}(p)$ is affinely diffeomorphic to G/Γ and that $[\Gamma : \Gamma \cap G] < \infty$.*

Theorem 1-1 is a generalization of Ruh's result [**14**], which corresponds to the case when N is a point.

Theorem 0-1 is a corollary of Theorem 1-1. In fact, let $\pi_i: M_i \to N$ be as in Theorem 1-1. Then, by (1-2-1) and (1-2-2), we can find $(U_j, \phi_{i,j})$ such that

(1-3-1) U_j, $j = 1, 2, \cdots$ is an open covering of N,

(1-3-2) $\phi_{i,j}$ is a diffeomorphism between $\pi_i^{-1}(U_j)$ and $U_i \times G/\Gamma$,

(1-3-3) the restriction of $\phi_{i,j}$ to each fibre gives an affine diffeomorphism between $\pi_i^{-1}(p)$ and $\{p\} \times G/\Gamma$.

By (1-3-3), the transition function of π_i with respect to the chart $(U_j, \phi_{i,j})$ is contained $\mathrm{Aff}(G/\Gamma)$, the group of affine diffeomorphism of G/Γ. We may assume that G is simply connected. Then, we have the following:

LEMMA 1-4. *There exists a split exact sequence*

$$1 \longrightarrow G/\Gamma \cap C(G) \longrightarrow \mathrm{Aff}(G/\Gamma) \longrightarrow \mathrm{Aut}\, \Gamma \longrightarrow 1\,.$$

Here $C(G)$ denotes the center of G.

We omit the proof, which is straightforward. Let $\mathrm{Aff}'(G/\Gamma)$ be the subgroup of $\mathrm{Aff}(G/\Gamma)$ generated by $C(G)/\Gamma\cap C(G)$ and $\mathrm{Aut}\,\Gamma$. Then we have $\mathrm{Aff}(G/\Gamma)/\mathrm{Aff}'(G/\Gamma)\cong \boldsymbol{R}^k$. Therefore the structure group of the $\mathrm{Aff}(G/\Gamma)$ bundle $\pi_i: M_i\to N$ can be reduced to $\mathrm{Aff}'(G/\Gamma)$. And $\mathrm{Aff}'(G/\Gamma)$ is a skew product of $C(G)/\Gamma\cap C(G)$ and $\mathrm{Aut}(\Gamma)$. This implies Theorem 0-1.

The proof of Theorem 1-1 occupies Sections 2 to 5. Since it is long, we shall give an outline first. The proof uses a parametrized version of Ruh's argument in [**14**]. To apply it, we have to improve the result of [7] and to prove that the fibres of the fibre bundles $f_i: M_i\to N$ obtained there are almost flat. ([7, 0-1-2] implies that fibres are diffeomorphic to almost flat manifolds. But, in [7], we did not obtain the estimate of the curvatures of the fibres.) Namely we shall prove Lemma 1-6 below. As will be remarked at the beginning of §5, we can assume, without loss of generality, that

$$|\nabla^k R(M_i)| < C_k . \tag{1-5}$$

Here $R(M_i)$ is the curvature tensor, $|\ |$ the C^0-norm, and C_k a constant independent of i. For $x\in M_i$, we let $\exp_{x,r}: B(r)\to M_i$ denote the exponential map at x. We fix a coordinate system $(U_j, \phi_j): U_j\subseteq \boldsymbol{R}^n$, $\phi_j: U_j\to N$.

LEMMA 1-6. *Let M_i and N be as in Theorem 0-1. Assume that M_i satisfies (1-5). Then, for sufficiently large i, there exists a fibration $\pi_i: M_i\to N$ such that π_i is an almost Riemannian submersion in the sense of* [7, 0-1-3], *and that*

$$\left|\frac{\partial^{|\alpha|}(\phi_j\circ\pi_i\circ\exp_{x,r})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right| \leqq C_\alpha \tag{1-7}$$

holds for each multiindex α. Here C_α denotes a constant independent of i.

(1-7) and the fact that π_i is a Riemannian submersion imply that the sectional curvatures of the fibres of π_i are uniformly bounded. Hence, the fibres are almost flat for sufficiently large i. Therefore, [**14**] shows that there exists a flat connection on each fibre satisfying (1-2-2). A little more argument is required to obtain a connection on $\pi_i^{-1}(p)$ depending smoothly on p. This is done in §5.

The proof of Lemma 1-6 is performed in Sections 2 to 4. Recall that in [7] we used embeddings M_i, $N\hookrightarrow \boldsymbol{R}^Z$ in order to construct the fibration $M_i\to N$. The embeddings there were constructed by making use of the distance function from a point. To obtain an embedding satisfying (1-7), we have to approximate this embedding by one with bounded higher derivatives. The approximation we used in [7] is not sufficient for this purpose, because it is not of C^2-class. In this paper, we use another embedding constructed by making use of eigenfunctions of Laplace operator. This embedding is appropriate for our purpose since eigenfunctions enjoy uniform estimate of higher derivatives. In order to apply the argument of [7, §§1, 2] to our embedding, we need to study the convergence of eigenfunctions. In [6], we introduce a notion, measured Hausdorff topology and proved that the k-th eigenvalue of the Laplace operator on M_i converges to that of the operator $P_{(N,\mu)}$ defined in [6, §0], if M_i converges to (N, μ) with respect to the measured Hausdorff topology. We also proved an "L^2-convergence" of eigenfunctions there. But, for our purpose, L^2-convergence is not sufficient. We have to prove a "C^1-convergence". (Precise statement will be given as Theorem 3-1.) For this purpose, we shall begin with proving that eigenfunctions of $P_{(N,\mu)}$ are smooth. [6, Theorem 0.6] implies that the measure μ is a multiple of the volume element Ω_N by a continuous function χ_N. If χ_N is of C^1-class, our operator $P_{(N,\mu)}$ is written as

$$P_{(N,\mu)}\varphi = \Delta_N\varphi - \langle d\varphi, d\chi_N\rangle/\chi_N\,. \tag{1-8}$$

Therefore, to prove that the eigenfunctions of $P_{(N,\mu)}$ are smooth, it suffices to show that χ_N is smooth. This is done in §2. In §3, we shall prove the "C^1-convergence". The proof of Lemma 1-6 is completed in §4.

REMARK. In 1984, S. Gallot proposed to embed Riemannian manifolds using heat kernels, in order to study Hausdorff convergence. The embedding we use in this paper is essentially the same as Gallot's.

§2. Smoothing density functions.

LEMMA 2-1. *Let M_i be a sequence of $n+m$-dimensional compact Riemannian manifolds satisfying* (0-2-2) *and* (1-5), *and X be a metric space, μ a probability measure on it. Suppose M_i converges to (X, μ) with respect to the measured Hausdorff topology defined in* [6, 0.2 B]. *Then there exists a function χ_X on X such that*

(2-2-1) $\mu=\chi_X\times$(*the volume element of X*),

(2-2-2) *χ_X is of C^∞-class,*

(2-2-3) *χ_X satisfies* [6, 0.7.1 *and* 0.7.3].

PROOF. In [6, 0.6], we have already proved (2-2-1) and (2-2-3). By the argument in [6, §3], it suffices to show (2-2-2) in the case when X is a compact Riemannian manifold N. Put $V_i=\mathrm{Vol}\,M_i$, $\mu_{M_i}=\Omega_{M_i}/V_i$, where Ω_{M_i} denotes the volume element of M_i. By the definition of measured Hausdorff topology, we can take ε_i-Hausdorff approximation $f_i: M_i\to N$ such that $(f_i)_*(\mu_{M_i})$ converges to μ with respect to the weak* topology. (Here $\varepsilon_i\to 0$. The definition of the Hausdorff approximation is in [8, 1.6].) In view of [7], we may assume that f_i is a fibration. Then, by [**6**, § 3], the functions $p \mapsto \mathrm{Vol}(f_i^{-1}(p))/V_i$, $i = 1, 2, \cdots$ on N converge, with respect to the C^0-norm, to a continuous function χ_N satisfying (2-2-1) and (2-2-3). We shall prove that χ_N is of C^∞-class. Choose (not necessary continuous) section $\phi_i: N \to M_i$ to f_i. Take an arbitrary point p_0 of N and put $p_i = \phi_i(p_0)$. We shall prove that χ_N is of C^∞-class at p_0. Put $B = B(1)$. Let $\mathrm{Exp}_i: B \to M_i$ be the composition of a linear isometry $B \to T_{p_i}(M_i)$ and the exponential map $T_{p_i}(M_i) \to M_i$. Let g_i denote the Riemannian metric on B induced by Exp_i from the metric on M_i. In view of (1-5), we may assume, by taking a subsequence if necessary, that g_i converges to a metric g_0 with respect to the C^∞-topology. Now, recall the argument in [8, § 3], where we constructed a sequence of local groups G_i converging to a Lie group germ G, such that

(2-3-1) G_i acts by isometry on the pointed metric space $((B, g_0), 0)$,

(2-3-2) $((B, g_i), 0)/G_i$ is isometric to a neighborhood of p_i in M_i,

(2-3-3) G acts by isometry on $((B, g_0), 0)$,

(2-3-4) $((B, g_0), 0)/G$ is isometric to a neighborhood of p_0 in N.

Let $P_i: (B, g_i) \to M_i$, $P: (B, g_0) \to N$ denote natural projections. (In fact, $P_i = \mathrm{Exp}_i$.) In our case, since N is a manifold, the action of G on B is free. Let $\mathfrak{g}$ denote the Lie algebra of G. Choose a basis $X_1, \cdots, X_m$ of $\mathfrak{g}$. We can regard X_i as a Killing vector field on (B, g_0). For $x \in B$, we put

$$\tilde{\chi}(x) = |X_1(x) \wedge \cdots \wedge X_m(x)|. \tag{2-4}$$

Since the nilpotent Lie algebra $\mathfrak{g}$ is unimodular, it follows that $\tilde{\chi}$ is G-invariant. Therefore, there exists a function χ on a neighborhood of p_0 such that $\chi \circ p = \tilde{\chi}$. Clearly χ is of C^∞-class. Hence, to prove Lemma 2-1, it suffices to show the following:

LEMMA 2-5. *χ_N/χ is a constant function on a neighborhood of p_0.*

PROOF. Put

$$G_i' = \left\{\gamma \in G_i \mid d_{(B, g_i)}(\gamma(0), 0) < \frac{1}{2}\right\} \tag{2-6-1}$$

$$G' = \left\{\gamma \in G \mid d_{(B, g_0)}(\gamma(0), 0) < \frac{1}{2}\right\}. \tag{2-6-2}$$

There exist a neighborhood U of p_0 in N and a C^∞-map $s: U \to B$ such that

(2-7-1) $s(p_0) = 0$,

(2-7-2) $P \circ s = \text{identity}$,

(2-7-3) $d_{(B, g_0)}(s(q), 0) = d_N(q, p_0)$ holds for $q \in N$.

Put

(2-8-1) $E_i(q, \delta) = \{x \in B \mid$ there exists $\gamma \in G'_i$ such that $d_{(B, g_i)}(x, \gamma s(q)) < \delta\}$,

(2-8-2) $E_0(q, \delta) = \{x \in B \mid$ there exists $\gamma \in G'$ such that $d_{(G, g_0)}(x, \gamma s(q)) < \delta\}$.

SUBLEMMA 2-9. *There exists a positive number C independent of q such that*

$$\lim_{\delta\to 0}\lim_{i\to\infty}\left|\frac{\mathrm{Vol}(E_i(q, \delta))}{\# G'_i\cdot\delta^n\cdot\mathrm{Vol}(f_i^{-1}(q))}-C\right|=0.$$

The proof of the sublemma will be given at the end of this section. Next we see that

$$\lim_{i\to\infty}\sup_{q\in U}\left|\frac{\mathrm{Vol}(E_i(q, \delta))}{\mathrm{Vol}(E_0(q, \delta))}-1\right|=0 \tag{2-10}$$

holds for each $\delta>0$. Thirdly, we put

$$G'(q)=\{\gamma s(q) \mid \gamma\in G'\}.$$

Then, clearly we have

$$\lim_{\delta\to 0}\mathrm{Vol}((E_0(q, \delta))/\delta^n)=W_n\mathrm{Vol}(G'(q)), \tag{2-11}$$

$$\frac{\mathrm{Vol}(G'(q))}{\chi(q)}=\frac{\mathrm{Vol}(G'(q'))}{\chi(q')}, \tag{2-12}$$

for $q, q' \in U$. Here $n = \dim N$, $W_n = \mathrm{Vol}\, B^n(1)$. (2-11) and (2-12) imply

$$\lim_{\delta\to 0}\frac{\mathrm{Vol}(E_0(q, \delta))\cdot\chi(q')}{\mathrm{Vol}(E_0(q', \delta))\cdot\chi(q)}=1. \tag{2-13}$$

From Sublemma 2-9, Formulas (2-10) and (2-13), we conclude

$$\lim_{i\to\infty}\frac{\mathrm{Vol}(f_i^{-1}(q))\chi(q')}{\mathrm{Vol}(f_i^{-1}(q'))\chi(q)}=1.$$

On the other hand, we have

$$\lim_{i\to\infty}\sup_{q, q'\in N}\left|\frac{\mathrm{Vol}(f_i^{-1}(q)\cdot\chi_N(q'))}{\mathrm{Vol}(f_i^{-1}(q'))\chi_N(q)}-1\right|=0.$$

Therefore,

$$\frac{\chi_N(q)\chi(q')}{\chi_N(q')\chi(q)}=1.$$

This implies Lemma 2-5.

PROOF OF SUBLEMMA 2-9. Put $s_i = P_i \circ s : U \to M_i$. Choose an open subset $V_i(\delta)$ of B such that the following hold.

(2-14-1) If $\gamma \in G'_i$, $\gamma \neq 1$, then $\gamma V_i(\delta) \cap V_i(\delta) = \emptyset$.

(2-14-2) $P_i(V_i(\delta))$ is a dense subset of $B_\delta(s_i(q), M_i)$.

(2-14-3) $V_i(\delta)\subset B_\delta(s(q), (B, g_i))$ and if $x\in V_i(\delta)$, $\gamma\in G'_i$, then

$$d(\gamma(x),\ s(q))\geqq d(x,\ s(q)).$$

Put $E'_i(q, \delta)=\{\gamma(x) \mid \gamma\in G'_i,\ x\in V_i(\delta)\}$. Then, by the definition of $V_i(\delta)$ and $E_i(q, \delta)$, we have $\overline{E'_i(q, \delta)}=\overline{E_i(q, \delta)}$. Hence, by (2-14-1), we have

$$\text{(2-16)} \qquad \mathrm{Vol}(V_i(\delta))=\frac{\mathrm{Vol}(E_i(q, \delta))}{\# G'_i}.$$

On the other hand, put

$$c_i=\sup_{p\in U} d(s_i(p), p_i), \qquad d_i=\sup_{p\in U} \mathrm{Diam}\, f_i^{-1}(p).$$

Then, $\lim_{i\to\infty}c_i=\lim_{i\to\infty}d_i=0$. It is easy to see

$$\text{(2-17)} \qquad f_i^{-1}(B_{\delta-d_i-c_i}(q, N))\subset B_\delta(s_i(q), M_i)\subset f_i^{-1}(B_{\delta+d_i+c_i}(q, N)),$$

(2-15), (2-16), and (2-17) imply

$$\text{(2-18)} \qquad \lim_{i\to\infty}\frac{\# G'_i\cdot\int_{p\in B_\delta(q, N)}\mathrm{Vol}(f_i^{-1}(p))\cdot\Omega_N}{\mathrm{Vol}(E_i(q, \delta))}=1$$

where Ω_N is the volume element of N. Since the family of functions $p\mapsto \log(\mathrm{Vol}(f_i^{-1}(p)))$, $i=1, 2, \cdots$, is equicontinuous ([6, Lemma 3.2]), it follows that

$$\text{(2-19)} \qquad \lim_{\delta\to 0}\sup_{i=1,2,\cdots}\left|\frac{\int_{p\in B_\delta(q, N)}\mathrm{Vol}(f_i^{-1}(p))\cdot\Omega_N}{\delta^n W_n\mathrm{Vol}(f_i^{-1}(q))}-1\right|=0.$$

The sublemma follows immediately from (2-18) and (2-19). Q. E. D.

§ 3. C^1-convergence of eigenfunctions.

THEOREM 3-1. *Let M_i and (X, μ) be as in Lemma 2-1. Then, there exist smooth maps $f_i: M_i\to X$ such that the following hold.*

(3-2-1) *f_i satisfies [7, (0-1-1), (0-1-2), (0-1-3)], if X is a Riemannian manifold.*

(3-2-2) *$(f_i)_*(\mu_{M_i})$ converges to μ with respect to the weak* topology, where $\mu_{M_i}=\Omega_{M_i}/\mathrm{Vol}(M_i)$.*

(3-2-3) *Let $\varphi_{i,k}$ be a k-th eigenfunction of the Laplace operator on M_i satisfying $\sup_{x\in M_i}|\varphi_{i,k}(x)|=1$. Then there exist functions $\varphi'_{i,k}$ on X such that*

(a) *$\varphi'_{i,k}$ is a k-th eigenfunction of $P_{(X,\mu)}$,*

(b) *for each $p_i\in M_i$, we have*

$$|\varphi_{i,k}(p_i)-\varphi'_{i,k}(f_i(p_i))|<\varepsilon_i(k),$$

(c) *for each vector $V_i\in T(M_i)$, we have*

$$|V_i(\varphi_{i,k})-(f_i)_*(V_i)(\varphi'_{i,k})|<\varepsilon_i(k)\cdot|V_i|,$$

where $\varepsilon_i(k)$ *denotes positive numbers depending only on* i *and* k *and satisfying* $\lim_{i\to\infty}\varepsilon_i(k)=0$.

REMARK. In the case when X is a manifold, (3-2-1) means that f_i is a fibration with infranilmanifold fibre.

First, we shall prove C^0-convergence, (b). We begin with the following Ascoli-Arzelà type lemma.

LEMMA 3-3. *Let* X_i *and* X *be compact metric spaces,* $\psi_i: X\to X_i$ ε_i*-Hausdorff approximation,* $\lim\varepsilon_i=0$, *and* φ_i *be continuous functions on* X_i. *Assume*

(3-4-1) φ_i, $i=1, 2, 3, \cdots$, *are uniformly bounded,*

(3-4-2) φ_i, $i=1, 2, 3, \cdots$, *are equi-uniformly continuous. Namely for each* $\varepsilon>0$, *there exists* $\delta>0$ *independent of* i, x *and* y *such that* $d(x, y)<\delta$, $x, y\in X_i$ *implies* $|\varphi_i(x)-\varphi_i(y)|<\varepsilon$.

Then, there exist a subsequence i_j *and a continuous function* φ *on* X *such that*

$$\lim_{j\to\infty}\sup_{x\in X}|\varphi(x)-\varphi_{i_j}\circ\psi_{i_j}(x)|=0.$$

The proof is an obvious analogue of that of Ascoli-Arzelà's theorem, and hence is omitted. Next we need the following:

LEMMA 3-5. $\varphi_{i,k}$, $i=1, 2, 3\cdots$, *are equi-uniformly continuous for each* k.

PROOF. By [**6**, 4.3], we have

$$|V(\varphi_{i,k})|<k\cdot|V|\,\|\varphi_{i,k}\|_{L^2}/\mathrm{Vol}(M_i)^{1/2}$$

for each $V\in T(M_i)$. The lemma follows immediately. Q.E.D.

Now we shall prove (3-2-1), (3-2-2) and (3-2-3) (a) and (b). We constructed, in [**7**, Theorem 0-1], the map f_i satisfying (3-2-1) and (3-2-2). Suppose that we can not find f_i satisfying (3-2-3) (a) and (b). Then, there exist $\theta>0$ and a subsequence i_j such that

$$\sup_{x\in M_{i_j}}|\varphi_{i_j,k}(x)-\varphi\circ f_{i_j}(x)|>\theta \tag{3-6}$$

holds for each j and each k-th eigenfunction φ of $P_{(X,\mu)}$. On the other hand, Lemmas 3-3 and 3-5 imply that we may assume, by taking a subsequence if necessary, the existence of a continuous function φ_∞ on X such that

$$\lim_{j\to\infty}\sup_{x\in M_{i_j}}|\varphi_{i_j,k}(x)-\varphi_\infty\circ f_{i_j}(x)|=0. \tag{3-7}$$

Moreover, [**6**, Theorem 0.4] implies that the L^2-distance between $\varphi_{i_j}\circ\psi_j$ and the k-th eigenspace of $P_{(X,\mu)}$ converges to 0, where $\psi_j: X\to M_{i_j}$ is a measurable map satisfying $f_{i_j}\circ\psi_j=$identity. Therefore, (3-7) implies that φ_∞ is a k-th eigenfunction of $P_{(X,\mu)}$. This contradicts (3-6).

REMARK. We have not yet used Assumption 1-5.

To prove (3-2-3) (c), we first remark the following elementary inequality

LEMMA 3-8. *Let $\varphi:(a-\varepsilon, b+\varepsilon)\to \boldsymbol{R}$ be a C^2-function satisfying*

$$\sup_{t\in[a,b]}\left|\frac{d^2\varphi}{dt^2}\right|\leqq C.$$

Then we have

$$\left|\frac{d\varphi}{dt}(a)-\frac{\varphi(b)-\varphi(a)}{b-a}\right|\leqq C\cdot(b-a).$$

Secondly, [**6**, 4.3.2] implies the following.

LEMMA 3-9. *There exists a constant C_k independent i such that the following holds. Let $l:[0,1]\to M_i$ be a geodesic with unit speed. Then*

$$\sup_{t\in[0,1]}\left|\frac{d^2(\varphi_{i,k}\circ l)}{dt^2}\right|< C_k.$$

By a method similar to [**6**, §7], we may assume that X is a manifold, N. Then, since the k-th eigenspace of $P_{(N,\mu)}$ is finite dimensional and consists of smooth functions, it follows that

$$\sup_{t\in[0,1]}\left|\frac{d^2(\varphi'_{i,k}\circ l)}{dt^2}\right|< C'_k \tag{3-10}$$

holds for each geodesic $l:[0,1]\to N$ with unit speed.

Now let $V_i\in T(M_i)$ be a unit vector. We put $l_i(t)=\exp(t\cdot V_i)$, $l'_i(t)=\exp(t\cdot(f_i)_*(V_i)/|(f_i)_*(V_i)|)$. Then, by [**7**, §4], we have

$$\lim_{i\to\infty}\sup_{t\in[0,1]} d(f_i l_i(t), l'_i(t))=0, \tag{3-11}$$

$$\limsup_{i\to\infty}|(f_i)_*(V_i)|\leqq 1. \tag{3-12}$$

Let δ be an arbitrary small positive number. Lemmas 3-8 and 3-9 imply

$$\left|V_i(\varphi_{i,k})-\frac{\varphi_{i,k}\circ l_i(\delta)-\varphi_{i,k}\circ l_i(0)}{\delta}\right|\leqq C_k\cdot\delta. \tag{3-13}$$

On the other hand, by Lemma 3-8, Formulae (3-10), (3-12), we have

$$\limsup_{i\to\infty}\left|(f_i)_*(V_i)(\varphi'_{i,k})-\frac{\varphi'_{i,k}\circ l'_i(\delta)-\varphi'_{i,k}\circ l'_i(0)}{\delta}\right|\leqq C'_k\cdot\delta. \tag{3-14}$$

Furthermore (3-2-3) (b) and (3-11) imply

$$\lim_{i\to\infty}\sup_{t\in[0,1]}|\varphi_{i,k}\circ l_i(t)-\varphi'_{i,k}\circ l'_i(t)|=0. \tag{3-15}$$

From Formulae (3-13), (3-14), (3-15), we conclude

$$\lim_{i\to\infty} | V_i(\varphi_{i,k})-(f_i)_*(V_i)(\varphi'_{i,k})| \leqq (C_k+C'_k)\delta .$$

Q. E. D.

§ 4. Estimating derivatives of the fibration.

In this section we shall prove Lemma 1-6. Let M_i and N be as in Theorem 0-1. By [**1**], we obtain, for each $\delta>0$, metrics $g_{i,\delta}$ on M_i such that

(4-1-1) $$|g_{i,\delta}-g_i| < \tau(\delta) ,$$

(4-1-2) $$|\nabla^k R(M_i, g_{i,\delta})| < C(k, \delta) .$$

Here g_i denotes the original Riemannian metric on M_i, and $\tau(\delta)$, $C(k, \delta)$ are positive numbers independent of i and satisfying $\lim_{\delta\to 0}\tau(\delta)=0$. By taking a subsequence if necessary, we may assume $(M_i, g_{i,\delta})$, $i=1, 2, \cdots$, converge to a metric space N_δ with respect to the Hausdorff distance. Then, [**8**, Lemma 2-3] implies that N_δ is diffeomorphic to N and

(4-2) $$\lim_{\delta\to 0} d_L(N, N_\delta) = 0 ,$$

where d_L denotes the Lipschitz distance defined in [**11**]. Therefore, it suffices to show Lemma 1-6 for $M_{i,\delta}$ and N_δ. Hereafter we shall write M_i and N in place of $M_{i,\delta}$ and N_δ. Thus, we verified that we can assume (1-5) while proving Lemma 1-6.

By [**6**, Corollary 2-11], we may assume, by taking a subsequence if necessary, that M_i converges to $(N, \chi_N\Omega_N)$ with respect to the measured Hausdorff topology. Then, Lemma 2-1 implies that χ_N is smooth. Hence the operator $P_{(N,\chi_N\Omega_N)}$ is elliptic with smooth coefficients. It follows the following:

LEMMA 4-3. *There exists J such that the map $I_0: N\to \boldsymbol{R}^J$ defined by $I_0(P)=(\varphi_1(P), \cdots, \varphi_J(P))$ is a smooth embedding. Here φ_k denotes a k-th eigenfunction of $P_{(N,\chi_N\Omega_N)}$.*

Next, we apply Theorem 3-1 to obtain eigenfunctions $\varphi_{i,k}$ and $\varphi'_{i,k}$ satisfying (3-2-3). Put

$$I'_i(x) = (\varphi_{i,1}(x), \cdots, \varphi_{i,J}(x)) .$$

Then, there exists a sequence of isometries L_i of $\boldsymbol{R}^J$ such that $L_i\circ I'_i$ converges to I_0 with respect to the C^1-topology. We have the following:

LEMMA 4-4. *There exist smooth maps $I_i: M_i\to \boldsymbol{R}^J$, $I_0: N\to \boldsymbol{R}^J$ such that*

(4-5-1) *I_0 is an embedding,*

(4-5-2) $$\lim_{i\to\infty} \sup_{x\in M_i} |I_i(x)-I_0\circ f_i(x)| = 0 ,$$

(4-5-3) $$\lim_{i\to\infty}\sup_{V\in T(M_i)}|(I_i)_*(V)-(I_0\circ f_i)_*(V)|=0\,,$$

(4-5-4) $$|\Delta^k I_i|\leqq C^k|I_i|\,.$$

Here $f_i: M_i\to N$ *is a fibration of* §3, *and* C *is a constant independent of* i *and* k.

PROOF. Put $I_i=L_i\circ I_i'$. We have already proved (4-5-1), $\cdots$, (4-5-3). Formula (4-5-4) follows from the definition of I_i and the estimate of the eigenfunctions of Laplace operators (see [**6**]). Q. E. D.

Now, put

$$B_\delta N(N)=\{(p,u)\in\boldsymbol{R}^J \mid |u|<\delta,\ u \text{ is perpendicular to } (I_0)_*(T_p(N))\}.$$

Let $E: B_\delta N(N)\to\boldsymbol{R}^J$ denote the map $E(p,u)=I_0(p)+u$. Then, by (4-5-1), we can choose δ such that $E: B_\delta N(N)\to\boldsymbol{R}^J$ is a diffeomorphism to its image. Then, by (4-5-2), we see that, for sufficiently large i, we have $I_i(M_i)\subset E(B_\delta N(N))$. Thus, the map $\pi_i=P\circ E^{-1}\circ I_i$ is well defined, ($P: E(B_\delta N(N))\to N$ is defined by $P(p,u)=p$). As in [**7**, §2], the fact (4-5-3) implies that π_i is a fibration. Facts (4-5-4) and (4-1-2) imply that π_i satisfies (1-7). The proof of Lemma 1-6 is now complete.

§5. The construction of a smooth family of connections.

In this section, we shall complete the proof of Theorem 1-1. Then, Lemma 1-6 implies the following:

LEMMA 5-1. *Let* $\pi_i: M_i\to N$ *be as in Lemma* 1-6. *Then, there exists a constant* C *independent of* i, *such that*

$$|\textit{the second fundamental form of } \pi_i^{-1}(p)|<C\,.$$

On the other hand, we have

(5-2) $$\lim_{i\to\infty}\sup_{p\in N}\operatorname{Diam}(\pi_i^{-1}(p))=0\,.$$

Hence, by [**14**], we can construct, for each i and $p\in N$, a flat connection on $\pi_i^{-1}(p)$ such that $\pi_i^{-1}(p)$ is affinely diffeomorphic to G/Γ, where G and Γ are as in Theorem 1-1. Hence it suffices to modify these connections so that they depend smoothly on p. If the flat connection constructed in [**14**] were canonical, then there would be nothing to show. But, unfortunately, the connection there depends on the choice of the base point on an almost flat manifold. Therefore, we should check carefully the construction there. In [**14**], the construction of the connection is divided into three steps. In the first step, a flat connection ∇' with small torsion tensor is constructed. The connection ∇' is used, in the second step, to construct a flat connection with parallel torsion tensor. In the third step, it is shown that almost flat manifolds equipped with a flat connection with parallel torsion tensor are affinely diffeomorphic to G/Γ. Roughly speaking, we do not have to modify the arguments in the second and the third steps, because connections constructed there depend smoothly on the data given in the first step.

Now, we shall present the parametrized version of the first step. First we change the normalization of the metric of the fibres. (Our normalization so far was $|\text{curvature}|\leqq 1$, Diameter$\to 0$. The normalization in [**14**] was Diameter $=1$, $|\text{curvature}|\to 0$.)

LEMMA 5-3. *Let $\pi_i: M_i\to N$ be as in Lemma 1-6. Then, there exists a smooth family of Riemannian metrics $g_i(p)$ on $\pi_i^{-1}(p)$ such that*

$$\text{(5-4-1)} \qquad \operatorname{Diam}(\pi_i^{-1}(p), g_i(p)) = 1,$$

$$\text{(5-4-2)} \qquad |\nabla^k R(g_i(p))| \leqq \varepsilon_{i,k},$$

where $\lim_{i\to\infty}\varepsilon_{i,k}=0$.

Secondly, we introduce the C^k-norm on $\pi_i^{-1}(p)$ as follows. Take $x\in\pi_i^{-1}(p)$ and let $\operatorname{Exp}_x: B(100)\to\pi_i^{-1}(p)$ be the exponential map. Let A be a tensor on $f_i^{-1}(p)$. We define $|A|_{C^k}$ to be the C^k-norm of the coefficients of $E^*(A)$. This definition is independent of x modulo constant multiple. Then (5-4-2) implies

$$\text{(5-4-3)} \qquad |R(g_i(p))|_{C^k} \leqq \varepsilon_{i,k}\,.$$

Thirdly we put $p_j\in N$, $V_j=B_\mu(p_j, N)$, $U_j=B_{2\mu}(p_j, N)$, where μ is the one third of the injectivity radius of N. Assume $\bigcup V_j=N$. Let $s_{i,j}: U_j\to M_i$ be smooth sections to π_i. Then, using $s_{i,j}(p)$ as a base point of $\pi_i^{-1}(p)$, we can follow the argument of [**14**, p. 5, p. 6] and obtain the following:

LEMMA 5-5. *For each i and j, there exists a smooth family of connections $\nabla^{(i,j)}(p)$ on $\pi_i^{-1}(p)\,(p\in U_j)$ such that*

(5-6-1) $\nabla^{(i,j)}(p)$ *is flat,*

(5-6-2) $|T^{(i,j)}(p)|_{C^k}<\varepsilon_{i,k}$, *where $T^{(i,j)}(p)$ is the torsion tensor of $\nabla^{(i,j)}(p)$,*

(5-6-3) $\nabla^{(i,j)}(p)$ *is a metric connection with respect to the metric $g_i(p)$.*

Fourthly, we shall estimate the tensor $\nabla^{(i,j)}(p)-\nabla^{(i,j')}(p)$, and prove

$$\text{(5-6-4)} \qquad |\nabla^{(i,j)}(p)-\nabla^{(i,j')}(p)|_{C^k} < \varepsilon_{i,k}\,.$$

By the construction of $\nabla^{(i,j)}(p)$ (which is presented in [**14**, p. 5, p. 6]), it suffices to estimate the parallel transform (Sublemma 5-7). Let $\tilde{g}_{i,j}(p)$ be the metric on $B(100)$ induced by the exponential map $\operatorname{Exp}_{s_{i,j}(p)}: T_{s_{i,j}(p)}(\pi_i^{-1}(p))\to\pi_i^{-1}(p)$. For $x\in B(100)$, we identify $\boldsymbol{R}^n$ and $T_x(B(100))$ in an obvious way. Then, for

$x,\ y\in B(100)$, the parallel translation along the shortest geodesic $p_{x,y}^{i,j,p}: T_x(B(100)) \to T_y(B(100))$ with respect to the metric $\tilde{g}_{i,j}(p)$, can be regarded as an element of $GL(n, \boldsymbol{R})$. Put

$$Q_{x,y}^{i,j,p}(Z) = P_{x,z}^{i,j,p} - P_{y,z}^{i,j,p}\,.$$

$Q_{x,y}^{i,j,p}$ is a matrix valued function. Now, (5-6-4) follows from the following:

SUBLEMMA 5-7. *There exists* $\varepsilon_k(\delta)$ *independent of* i, j, p *such that if* $|x-y| < \delta$ *then* $|Q_{x,y}^{i,j,p}(Z)|_{C^k} < \varepsilon_k(\delta)$. *Here* $\lim_{\delta\to 0}\varepsilon_k(\delta)=0$.

PROOF. If Sublemma does not hold, there exist $x_l,\ y_l,\ z_{(l)}^{(0)}\in B(100),\ i_l,\ j_l,\ \theta>0$ and a multiindex α such that

$$\text{(5-8-1)}\qquad \left|\frac{\partial^{|\alpha|}(P_{x_l,z}^{i_l,j_l})}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}} - \frac{\partial^{|\alpha|}(P_{y_l,z}^{i_l,j_l})}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\right|_{z=z_{(l)}^{(0)}} > \theta\,,$$

$$\text{(5-8-2)}\qquad \lim_{l\to\infty} d(x_l,\ y_l) = 0\,.$$

By taking a subsequence, we may assume that $\lim x_l = \lim y_l = W$, $\lim z_{(l)}^{(0)} = z^{(0)}$ and $\tilde{g}_{i_l,j_l}(p)$ converges to g_∞ with respect to the C^∞-topology. Then we have

$$\text{(5-9)}\qquad \lim_{l\to\infty}\left(\frac{\partial^{|\alpha|}P_{x_l,z}^{i_l j_l}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\bigg|_{z=z_{(l)}^{(0)}}\right) = \frac{\partial^{|\alpha|}P_{w,z}^{\infty}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\bigg|_{z=z^{(0)}} = \lim_{l\to\infty}\left(\frac{\partial^{|\alpha|}P_{y_l,z}^{i_l,j_l}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\bigg|_{z=z_{(l)}^{(0)}}\right),$$

where P^∞ denotes the parallel translation with respect to g_∞. (5-9) contradicts (5-8-1). Q.E.D.

Thus, we have verified (5-6-4). Finally we shall prove the following:

LEMMA 5.10. *There exists a smooth family of connections* $\nabla_i'(p)$ *on* $\pi_i^{-1}(p)$ $(p\in N)$ *such that*

(5-11-1) $\nabla_i'(p)$ *is flat,*

(5-11-2) $|T_i'(p)|_{C^k}\leq\varepsilon_{i,k}$, *where* $T_i'(p)$ *is the torsion tensor of* $\nabla_i'(p)$,

(5-11-3) $\nabla_i'(p)$ *is a metric connection with respect to the metric* $g_i(p)$.

PROOF. For simplicity, we assume $V_1\cup V_2=N$. First we shall find a gauge transformation $O_{p,i}$ such that $\nabla^{(i,1)}(p)=O_{p,i}^{-1}\circ\nabla^{(i,2)}(p)\circ O_{p,i}$ holds for $p\in U_1\cap U_2$. Here $O_{p,i}$ is a section of the fibre bundle $\mathrm{Aut}(F(\pi_i^{-1}(p)))=F(\pi_i^{-1}(p))\times_{\mathrm{Ad}}O(m)$, where $F(\pi_i^{-1}(p))$ is the frame and $m=\dim\pi_i^{-1}(p)$. We have two monodromy representations $\tilde{\rho}_1^{(p,i)},\ \tilde{\rho}_2^{(p,i)}: \Gamma\to O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ with respect to the flat connections $\nabla^{(i,1)}(p)$ and $\nabla^{(i,2)}(p)$, respectively. (Here we recall $\pi_i^{-1}(p)=G/\Gamma$. And $O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ denotes the set of linear isometries of $T_{s_{i,1}(p)}(\pi_i^{-1}(p))$.) By the construction of $\nabla^{(i,j)}(p)$ presented in [**14**, p. 5, p. 6] we see $\tilde{\rho}_1^{(p,i)}(\Gamma\cap G)=\tilde{\rho}_2^{(p,i)}(\Gamma\cap G)=1$. Hence there exist a projection $P: \Gamma\to\Lambda$ to a finite group Λ and representations $\rho_1^{(p,i)},\ \rho_2^{(p,i)}: \Lambda\to O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ such that $\rho_1^{(p,i)}\circ P=$

$\tilde{\rho}_1^{(p,i)}$, $\rho_2^{(p,i)} \circ P = \tilde{\rho}_2^{(p,i)}$. Then, since $\# \Lambda < \infty$ and $\rho_1^{(p,i)}$ and $\rho_2^{(p,i)}$ are close to each other, there exists $\alpha_i(p) \in O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ depending smoothly on p such that $\rho_2^{(p,i)}(\gamma) = \alpha_i(p)^{-1} \rho_1^{(p,i)}(\gamma) \cdot \alpha_i(p)$, and $\alpha_i(p)$ converges to identity with respect to the C^∞-topology when i tends to ∞. Now we define $O_{p,i}(x) : T_x(\pi_i^{-1}(p)) \to T_x(\pi_i^{-1}(p))$, for $x \in \pi_i^{-1}(p)$, as follows. Let $l : [0, 1] \to \pi_i^{-1}(p)$ be an arbitrary curve connecting x to $s_{i,1}(p)$, and P_1, $P_2 : T_x(\pi_i^{-1}(p)) \to T_{s_{i,1}(p)}(\pi_i^{-1}(p))$ denote the parallel translations along l with respect to the connections $\nabla^{(i,1)}(p)$ and $\nabla^{(i,2)}(p)$, respectively. We put

$$O_{p,i}(x)(V) = P_2^{-1}(\alpha_i(p)^{-1} \cdot P_1(V)) . \tag{5-12}$$

Using $\alpha_i(p)^{-1} \cdot \tilde{\rho}_1^{(p,i)} \cdot \alpha_i(p) = \tilde{\rho}_2^{(p,i)}$, it is easy to verify that $O_{p,i}(x)$ does not depend on the choice of l. The equality $\nabla^{(i,1)}(p) = O_{p,i}^{-1} \circ \nabla^{(i,2)}(p) \circ O_{p,i}$ is also obvious from the definition. By construction, $O_{p,i}$ converges to the identity with respect to the C^∞-topology. Therefore, the section $\log O_{p,i}$ to $F(\pi_i^{-1}(p)) \times_{\mathrm{ad}} \mathfrak{o}(m)$ is well defined, (where $\mathfrak{o}(m)$ is the Lie algebra of $O(m)$ and $m = \dim \pi_i^{-1}(p)$), and $\log O_{p,i}$ satisfies

$$|\log O_{p,i}|_{C^k} \leqq \varepsilon_i(k) . \tag{5-13}$$

Take a smooth function $\phi : N \to [0, 1]$ such that $\phi \equiv 1$ on a neighborhood of $\overline{V_1 \setminus U_2}$ and that $\phi \equiv 0$ on a neighborhood of $\overline{V_2 \setminus U_1}$. Put $O'_{p,i} = \exp(\phi(p) \log O_{p,i})$, for $p \in U_1 \cap U_2$. We define $\nabla'_i(p)$ by

$$\nabla'_{(i)}(p) \begin{cases} = O'^{-1}_{p,i} \circ \nabla^{(i,2)}(p) \circ O'_{p,i} & p \in U_1 \cap U_2 \\ = \nabla^{(i,2)}(p) & p \in V_2 - U_1 \\ = \nabla^{(i,1)}(p) & p \in V_1 - U_2 . \end{cases}$$

(5-12) implies that $\nabla'_i(p)$ depends smoothly on p. (5-13) implies (5-11-2). Facts (5-11-1) and (5-11-3) are obvious from the construction. Q. E. D.

Thus we have proved the parametrized version of the first step in [**14**]. The rest of the argument is completely parallel to [**14**]. We use Newton's method to obtain a sequence of flat connections $\nabla'_{i,k}(p)$ and a connection $\nabla_i(p)$ such that

(5-14-1) $\nabla'_{i,0}(p) = \nabla'_i(p)$,

(5-14-2) $\lim_{k \to \infty} |\nabla'_{i,k}(p) - \nabla_i(p)|_{C^2} = 0$,

(5-14-3) $\nabla_i(p)(T_i(p)) = 0$, where $T_i(p)$ is the torsion tensor of $\nabla_i(p)$.

(In [**14**] the convergence of $\nabla'_{i,k}(p)$ to ∇_i is the C^0-convergence. But, in our case, we can prove the C^k-convergence for an arbitrary k, thanks to (5-11-2).) By (5-14-2) $\nabla_i(p)$ is a C^2-family of connections. It is easy to modify it to a C^∞-family. Then (5-14-3) implies, as in [**14**, p. 13], that $\nabla_i(p)$ is the connection we have been looking for. The proof of Theorem 1-1 is now completed.

§ 6. The construction of a collapsing family of metrics.

In this section, we shall prove Theorem 0-7. Let $\pi: M \to N$ be a fibre bundle satisfying (0-3-1), (0-3-2), (0-3-3). T denotes the structure group of the fibration π. Then T is an extension of a torus T_0 by a discrete group Λ contained in $\mathrm{Aut}\,\Gamma$, where Γ and G are as in (0-3-2). Choose a T connection of π. It gives a decomposition of $T_x(M)$ to its horizontal subspace $H_x(M)$ and vertical subspace $V_x(M)=T_x(\pi^{-1}\pi(x))$. We put

$$g_\varepsilon(V, W) = g_N(\pi_*(V), \pi_*(W)), \quad \text{if } V, W \in H_x(M), \tag{6-1-1}$$

$$g_\varepsilon(V, W) = 0, \quad \text{if } V \in H_x(M),\ W \in V_x(M). \tag{6-1-2}$$

Here g_N denotes the Riemannian metric of N. We shall define $g_\varepsilon(V, W)$ for $V, W \in V_x(M)$.

Let $\pi_1: P_1 \to N$ be the principal T-bundle associated to π, and $\pi_2: P_2 \to N$ be the principal Λ-bundle induced from π_1. (Namely $P_2=P_1/T_0$.) Let $\mathfrak{g}$ be the Lie algebra of G. Put $\mathfrak{g}_0'=\mathfrak{g}$, $\mathfrak{g}_{k+1}'=[\mathfrak{g}_k', \mathfrak{g}]$, and $\mathfrak{g}_k=\mathfrak{g}_k'+(\text{center of } \mathfrak{g})$ if $\mathfrak{g}_k'\neq 0$, $\mathfrak{g}_k=0$ if $\mathfrak{g}_k'=0$. We have $[\mathfrak{g}, \mathfrak{g}_k]\subset \mathfrak{g}_{k+1}$. If $\mathfrak{g}_K=0$, $\mathfrak{g}_{K-1}\neq 0$, then $\mathfrak{g}_{K-1}=$center of $\mathfrak{g}$. Since $\Lambda\subset \mathrm{Aut}\,\Gamma$, Malcev's rigidity theorem (see [**13**, p. 34]) implies $\Lambda \subset \mathrm{Aut}\, G$. Hence Λ acts on $\mathfrak{g}$ by isomorphism. It follows that Λ preserves the filtration $\mathfrak{g}=\mathfrak{g}_0\supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_K=0$. Put $E=P_2\times_\Lambda \mathfrak{g}, \cdots, E_K=P_2\times_\Lambda \mathfrak{g}_K$. Then $\pi_0: E\to N$, $\pi_k: E_k \to N$ are vector bundles. Fix a metric h_1 on E and let F_k be the intersection of E_{k-1} and the orthogonal complement of E_k. Then, F_k, $k=1, 2, \cdots$ are orthogonal to each other and $\bigoplus F_k=E$. We define h_ε by

$$h_\varepsilon(V, W) = \delta_{k,k'}(\varepsilon^{2^k})^2 h_1(V, W) \tag{6-2}$$

for $V\in F_k$, $W \in F_{k'}$. Let $U_i \subset N$, $\phi_i: \pi^{-1}(U_1)\to U_i\times G/\Gamma$ be a coordinate chart and $s_{i,j}(p)\in T\ (p\in U_i \cap U_j)$ be the transition function. Namely, if $\phi_i(p)=(p, g)$ then $\phi_j(p)=(p, s_{j,i}(p)\cdot g)$. Let $\phi_i': \pi_0^{-1}(U_i)\to U_i\times \mathfrak{g}$ be a coordinate chart. By definition we can take ϕ_i' so that the transition function of this chart is $P(s_{i,j})$, where $P: T\to \Lambda=T/T_0$ is the natural projection. Namely

$$\phi_i'(u) = (p, P(s_{i,j}(p))\cdot a) \quad \textit{if} \quad \phi_j'(u)=(p, a). \tag{6-3}$$

For $V, W \in \mathfrak{g}$, $p\in U_i$, we put

$$h_{\varepsilon,i}(p)(V, W) = h_\varepsilon(\phi_i'^{-1}(p, V), \phi_i'^{-1}(p, W)).$$

The quadratic form $h_{\varepsilon,i}(p)$ gives a right invariant metric $\tilde{g}_{\varepsilon,i}(p)$ on G. Hence it induces a Riemannian metric on $G/(G\cap\Gamma)$. By Lemma 1-4, $\Gamma/(G\cap\Gamma)$ is a finite subgroup of $\mathrm{Aut}(G)$. Therefore, we can choose h_1 so that $h_{\varepsilon,i}(p)$ is preserved by $\Gamma/(G\cap\Gamma)\subset \mathrm{Aut}(\mathfrak{g})$. Then, $\tilde{g}_{\varepsilon,i}(p)$ induces a Riemannian metric on $\{p\}\times G/\Gamma$. This metric, together with (6-1-1) and (6-1-2), determines a Riemannian metric $g_{\varepsilon,i}$ on $U_i\times G/\Gamma$. Then, using (6-3) and the fact that T_0 is contained in the center of G, we can easily verify that $g_{\varepsilon,i}$ can be patched together and gives a Riemannian metric g_ε on M. The equality $\lim_{\varepsilon\to 0}{}_H(M, g_\varepsilon)=N$ is obvious. Thus, we are only to show that the sectional curvatures of g_ε have an upper and a lower bound independent of ε. Since the problem is local, we have only to study $U_i\times G/\Gamma$. Hence it suffices to obtain an estimate of sectional curvatures of $(U_i\times G, \tilde{g}_{\varepsilon,i})$. (Hereafter we omit the index i.) Now, let $e_1', \cdots, e_n'$ be an orthonormal frame of vector fields on U, and $e_1, \cdots, e_n$ denote their horizontal lifts to $U\times G$. Choose an orthonormal basis $X_1(p), \cdots, X_m(p)$ of $(\mathfrak{g}, h_1(p))$, such that there exists a nondecreasing map $O:\{1, \cdots, m\}\to \boldsymbol{Z}^+$ satisfying $X_i(p)\in F_{O(i)}(p)$, where $F_k(p)$ denotes the orthogonal complement of $\mathfrak{g}_k$ in $(\mathfrak{g}_{k-1}, h_1(p))$. We may assume that $X_i(p)$ depends smoothly on p. These elements $X_i(p)$ determine, through the right action of G, a vector field on $\{p\}\times G$. Thus, we obtain a vector field f_i on $U\times G$. Then, $(e_1, \cdots, e_n, f_1, \cdots, f_m)$ is an orthonormal frame of vector fields on $(U\times G, \tilde{g}_1)$ and $(e_1, \cdots, e_n, \varepsilon^{-2^{O(i)}}f_1, \cdots, \varepsilon^{-2^{O(m)}}f_m)$ is one on $(U\times G, \tilde{g}_\varepsilon)$. We shall calculate commutators of those vector fields. First, since our connection of π is a T-connection, it follows that

$$\text{(6-4-1)} \qquad [e_i, e_j]=\sum_{k=1}^{n} a_{i,j}^k e_k+\sum_{O(k)=O(m)} b_{i,j}^k f_k\,,$$

where $a_{i,j}^k$ and $b_{i,j}^k$ are functions on U. Secondly, since $[\mathfrak{g}_k, \mathfrak{g}]\subset\mathfrak{g}_{k+1}$, we have

$$\text{(6-4-2)} \qquad [f_i, f_j]=\sum_{\substack{O(k)>O(i)\\ O(k)>O(j)}} C_{i,j}^k\cdot f_k\,,$$

where $C_{i,j}^k$ are functions on U. Next we shall calculate $[f_i, e_j]$. Let $Y_1, \cdots, Y_m$ be a basis of $\mathfrak{g}$. We may assume that Y_i is contained in $\mathfrak{g}_{O(i)-1}=\bigoplus_{k\in O(i)}F_k(p)$. The element Y_i of $\mathfrak{g}$, through the right action of G, induces a vector field f_i^* on $U\times G$. Since our connection of π is a T-connection and in particular is a G-connection, it follows that the horizontal lift is invariant by the right action of G. Therefore

$$\text{(6-5)} \qquad [e_i, f_j^*]=0\,.$$

On the other hand there exist functions $\alpha_{i,j}$ on U such that

$$\text{(6-6)} \qquad f_i(p, g)=\sum_{O(j)\geq O(i)}\alpha_{i,j}(p)\cdot f_j^*(p, g)\,.$$

We regard U as an open subset of $\boldsymbol{R}^n$, and put

$$\text{(6-7)} \qquad e_i'(p)=\sum_{j=1}^{n}\beta_{i,j}(p)\frac{\partial}{\partial p^j}\,.$$

Then, (6-5), (6-6) and (6-7) imply

$$[e_i, f_j](p, g) = \sum_{\substack{1 \leq k \leq n \\ O(l) \geq O(i)}} \beta_{j,k}(p) \frac{\partial \alpha_{i,l}}{\partial p^j} f_k^*(p, g).$$

Therefore, we have

$$[e_i, f_j] = \sum_{O(k) \geq O(i)} d_{i,j}^k f_k, \tag{6-4-3}$$

where $d_{i,j}^k$ are functions on U.

Now, let $e^1, \cdots, e^n, f_\varepsilon^1, \cdots, f_\varepsilon^m \in \Lambda^1(U \times G)$ be the dual base of $(e_1, \cdots, e_n, \varepsilon^{-2^{O(1)}} f_1, \cdots, \varepsilon^{-2^{O(m)}} f_m)$. Then, by (6-4-1), (6-4-2), (6-4-3), we have

$$de^i = \sum_{j,k} a_{jk}^i e^j \wedge e^k, \tag{6-8-1}$$

(6-8-2) if $O(i) \neq O(m)$, then

$$df_\varepsilon^i = \sum_{\substack{O(i) \geq O(j) \\ O(i) > O(k)}} C_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(j)} - 2^{O(k)}} \cdot f_\varepsilon^j \wedge f_\varepsilon^k + \sum_{O(i) \geq O(k)} d_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(k)}} e^j \wedge f_\varepsilon^k,$$

(6-8-3) if $O(i) = O(m)$, then

$$\begin{aligned} df_\varepsilon^i = & \sum_{\substack{O(i) \geq O(j) \\ O(i) > O(k)}} C_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(j)} - 2^{O(k)}} \cdot f_\varepsilon \wedge f_\varepsilon^k \\ & + \sum_{O(i) \geq O(k)} d_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(k)}} e^j \wedge f_\varepsilon^k + \sum b_{jk}^i \cdot \varepsilon^{2^{O(i)}} e^j \wedge e^k. \end{aligned}$$

We see that the coefficients a_{jk}^i, $c_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(j)} - 2^{O(k)}}$, $d_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(j)}}$, $b_{jk}^i \varepsilon^{2^{O(i)}}$ are bounded, with respect to the C^k-norm, while ε tends to 0. Therefore, we can prove that the sectional curvatures of g_ε are uniformly bounded thanks to the well known formula which expresses the curvature tensor in terms of these coefficients. The proof of Theorem 0-7 is now complete.

§ 7. The orbifold version of the main theorem.

For our application in § 8, we use a little more general result than Theorem 0-1. In other words we need to treat the case when M_i converges to a Riemannian orbifold.

DEFINITION 7-1. Let X be a metric space. We say that X is a *Riemannian orbifold* and $\{(U_i, \varphi_i, \Gamma_i)\}$ its *chart* if the following hold.

(7-2-1) U_i is an open subset of $\boldsymbol{R}^n$ equipped with a Riemannian metric.

(7-2-2) Γ_i is a finite group of isometries of U_i.

(7-2-3) φ_i is a map: $U_i \to X$ which induces an isometry: $U_i/\Gamma_i \to \varphi_i(U_i)$.

(7-2-4) $\{\varphi_i(U_i)\}$ is an open covering of X.

REMARK. The definition of the Riemannian orbifold here is not equivalent to one in [4]. The definition in [4] is a little more restrictive.

Next we shall define fibre bundles and their structure group in the category of orbifolds. We remark that if X is a Riemannian orbifold, we can modify its chart so that the following hold in addition.

(7-2-5) Suppose $\varphi_i(U_i)\cap\varphi_j(U_j)$, $i<j$. Then there exist a map $\varphi_{i,j}: \varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))\to\varphi_j^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))$, a homomorphism $\pi_{i,j}: \Gamma_i\to\Gamma_j$, and a subgroup $\Lambda_{i,j}\subset\Gamma_i$ such that:

(7-2-5-1) $\varphi_{i,j}(\gamma x)=\pi_{i,j}(\gamma)\varphi_{i,j}(x)$.

(7-2-5-2) $\varphi_{i,j}$ induces an isometry between $\varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))/\Lambda_{i,j}$ and $\varphi_j^{-1}(\omega_i(U_i)\cap\varphi_j(U_j))$.

(7-2-5-3) $\pi_{i,j}$ induces an isomorphism between $\Gamma_i/\Lambda_{i,j}$ and Γ_j.

(7-2-5-4) $\varphi_i(\varphi_{i,j}(x))=\varphi_i(x)$, for $x\in\varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))$.

DEFINITION 7-3. Let M, F be manifolds, X a Riemannian orbifold, and G a Lie group action on G. A map $f: M\to X$ is said to be a *fibre bundle*, F its *fibre*, G its *structure group*, if there exist a chart $\{(U_i, \varphi_i, \Gamma)\}$ of X satisfying (7-2-5), and $\{(g_{i,j}, \phi_i, \theta_i)\}$ such that:

(7-4-1) ϕ_i is a map: $U_i\times F\to f^{-1}\varphi_i(U_i)$.

(7-4-2) $g_{i,j}$ is a continuous map from $\varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))$ to G.

(7-4-3) θ_i is a homomorphism from Γ_i to G. We let Γ_i act on $U_i\times F$ by $\gamma(x, y)=(\gamma x, \theta_i(\gamma)y)$.

(7-4-4) $\phi_i(\gamma(x, y))=\gamma\phi_i(x, y)$ for $\gamma\in\Gamma_i$.

(7-4-5) ϕ_i induces a fibre preserving diffeomorphism between $(U_i\times F)/\Gamma_i$ and $f^{-1}\varphi_i(U_i)$.

(7-4-6) For $i<j<k$, $x\in\varphi_i^{-1}(\varphi_i(U_i)\cup\varphi_j(U_j)\cap\varphi_k(U_k))$, we have

$$g_{j,k}(\varphi_{i,j}(x))\cdot g_{i,j}(x)=g_{i,k}(x),$$

where $\varphi_{i,j}$ is an in (7-2-5).

(7-4-7) For $i<j$, $x\in\varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))$, $\gamma\in\Gamma_i$, $\pi_{i,j}: \Gamma_i\to\Gamma_j$, we have

$$\theta_j(\pi_{i,j}(\gamma))\cdot g_{i,j}(x)=g_{i,j}(\gamma x)\cdot\theta_i(\gamma).$$

(7-4-8) We define

$$\hat{\varphi}_{i,j}: \varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))\times F\longrightarrow\varphi_j^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j))\times F$$

by $\hat{\varphi}_{i,j}(x, y)=(\varphi_{i,j}(x), g_{i,j}(x)y)$. Then, we have

$$\phi_j\hat{\varphi}_{i,j}(x, y)=\phi_i(x, y), \qquad \text{for each } (x, y)\in\varphi_i^{-1}(\varphi_i(U_i)\cap\varphi_j(U_j)).$$

REMARK 7-5. In the case when $F=S^1$, $G=O(2)$. Definition 7-3 is equivalent to that of Seifert fibred space.

Now we have:

THEOREM 7-6. *Theorem* 0-1 *holds also in the case when the limit N there is replaced by a Riemannian orbifold X.*

SKETCH OF THE PROOF. Let FM_i be the frame bundles of M_i. FM_i converges to a Riemannian manifold Y on which $O(n)$ acts by isometry so that $Y/O(n)$ is isometric to X (see [8], § 10). By an argument in §§ 2, 3, 4, we obtain $O(n)$ equivariant fibrations: $FM_i \to Y$ with bounded higher derivatives. It induces a smooth map $f: M_i \to X$ with bounded higher derivatives. By an argument similar to one in § 1, we see that it suffices to construct a smooth family of flat connection on fibres such that their torsion tensors are parallel. There exists a natural stratification $\Sigma_i \subset X$ such that $\Sigma_i - \Sigma_{i-1}$ are Riemannian manifolds. By the argument of § 5, we can construct smooth family of connections with parallel torsion tensor over each $\Sigma_i - \Sigma_{i-1}$. We can extend this family to one over $B_{\varepsilon_i}(\Sigma_i) - B_{\varepsilon_{i-1}}(\Sigma_{i-1})$, where ε_i and $\varepsilon_{i-1}/\varepsilon_i$ are very small. By construction, those connections are close to Levi-Civita connection with respect to the C^∞ norm. Therefore, we can use the arguments of § 5 again to construct a desired family of connections over X. The conclusion holds.

THEOREM 7-7. *Theorem* 0-7 *holds also in the case when N is replaced by a Riemannian orbifold X.*

We omit the proof.

§ 8. A gap theorem for minimal volumes.

In this section we shall prove Theorem 0-9, by contradiction. We assume that there exists a sequence of n-dimensional Riemannian manifolds M_i such that

(8-1-1) $\operatorname{Diam} M_i \leqq D$,

(8-1-2) $\operatorname{Vol} M_i \leqq 1/i$,

(8-1-3) $|\text{sectional curvature of } M_i| \leqq 1$,

(8-1-4) $\operatorname{Minvol} M_i \geqq \varepsilon > 0$,

where ε is independent of i. Using [**9**, Theorem 0-6], we can find a subsequence M_{k_i}, and an aspherical Riemannian orbifold X/Γ such that

(8-1-5) $\lim_{i\to\infty} {}_H M_{k_i} = X/\Gamma$,

where an aspherical Riemannian orbifold stands for the quotient X/Γ of a contractible Riemannian manifold X by a properly discontinuous action of a group

Γ consisting of isometries of X. By a modification of the argument in §§ 1⋯5, we can generalize Theorem 0-1 to the case when the limit space is an orbifold. Hence we obtain a fibration $\pi_{p_i}: M_{k_i} \to X/\Gamma$ whose fibre is G/Γ and whose structure group is the extension of $C(G)/(C(G) \cap \Gamma)$ by $\mathrm{Aut}\,\Gamma$, where G and Γ are as in (0-3-2). Hence, Theorem 0-7 (more precisely its generalization to orbifold case) implies that there exist metrics g_ε on M_{k_i} such that

(8-2-1) $\lim_{\varepsilon \to 0} {}_H (M_{k_i}, g_\varepsilon) = X/\Gamma$,

(8-2-2) $|\text{sectional curvature of } g_\varepsilon| \leqq C$,

where C is a number independent of ε. On the other hand, (8-1-2) and [**11**, 8.30] imply $\dim X/\Gamma \lneqq \dim M_{k_i}$. Hence, by (8-2-1) we have

(8-2-3) $\lim_{\varepsilon \to 0} \mathrm{Vol}(M_{k_i}, g_\varepsilon) = 0$,

(8-2-2) and (8-2-3) contradict (8-1-4). Q. E. D.

§ 9. The case when the limit space is not a manifold.

So far, we have studied sequences of Riemannian manifolds converging to a manifold. In [8] we have studied more general situation. The method of this paper can be joined with one in [8] to prove the following:

THEOREM 9-1. *Let M_i be a sequence of $n+m$-dimensional Riemannian manifold satisfying* (0-2-2) *which converges to a metric space X with respect to the Hausdorff distance. Then, there exist a $C^{1,\alpha}$-manifold Y and $\tilde{\pi}_i: FM_i \to Y$, such that the following hold.* (*Here FM_i denotes the frame bundle.*)

(9-2-1) *$O(n+m)$ acts by isometry to Y. We have $X = Y/O(n+m)$.*

(9-2-2) *$\tilde{\pi}_i$ satisfies* (0-3-1), (1-2-1), (1-2-2).

(9-2-3) *$\tilde{\pi}_i$ is an $O(m+n)$-map, and the diagram*

$$\begin{tikzcd} FM_i \arrow[r, "\tilde{\pi}_i"] \arrow[d] & Y \arrow[d] \\ M_i \arrow[r, "\pi_i"] & X \end{tikzcd}$$

commutes.

(9-2-4) *Let $g \in O(n+m)$, $p \in Y$. Then the map $g: \tilde{\pi}_i^{-1}(p) \to \tilde{\pi}_i^{-1}(g(p))$ preserves affine structures.*

We omit the proof.

Unfortunately, our method in § 6 does not give the converse to Theorem

9-1. In other words, it seems that (9-2-1), $\cdots$, (9-2-4) is not a sufficient condition for the existence of a family of metrics g_ε on M_i and that $\lim_{\varepsilon\to 0} {}_H(M_i, g_\varepsilon)=X$ and that $|$sectional curvatures of $g_\varepsilon| \leqq C$.

In [**2**] and [**3**], Cheeger and Gromov developed another approach to study collapsing. They introduced the notion, F-structure there. Our Theorem 8-1 implies the following:

COROLLARY 9-3. *There exists a positive number $\varepsilon(n, D)$ such that the following holds. Suppose an n-dimensional Riemannian manifold M satisfies*

(9-4-1) $\mathrm{Vol}(M) \leqq \varepsilon(n, D)$,

(9-4-2) $\mathrm{Diam}(M) \leqq D$,

(9-4-3) $|$*sectional curvature of* $M| \leqq 1$.

Then M admits a pure F-structure of positive dimension.

REMARK 9-5. The assumption of Cheeger and Gromov in [**3**] is less restrictive than ours in the point that they do not assume the uniform bound of the diameter. Our conclusion is a little stronger. (In [**3**], the existence of F-structure is proved.)

REMARK 9-6. The converse to Corollary 9-3 is false. A counter example is given in [**2**, Example 1.9].

PROOF OF COROLLARY 9-3. We prove by contradiction. Assume M_i satisfies (9-4-2), (9-4-3) and $\lim_{i\to 0}\mathrm{Vol}(M_i)=0$, but M_i does not admit pure F-structure of positive dimension. By taking a subsequence if necessary, we may assume that M_i converges to a metric space X with respect to the Hausdorff distance. Therefore, by Theorem 9-1, we have Y, $\tilde{\pi}_i$, π_i satisfying (9-2-1), $\cdots$, (9-2-4). Let $G/\Gamma=\tilde{\pi}_i(P)$. Then $C(G)/(\Gamma\cap C(G))$ acts on each fibre. In view of (0-3-3), this action determines a pure (polarized) F-structure on FM_i. Then, (9-2-4) implies that this F-structure induces a pure F-structure on M_i. We shall prove that this F-structure is of positive dimension. Remark that we can assume (1-5). Let $x\in X$, $p_i\in\pi_i^{-1}(x)\subseteqq M_i$. We recall the argument in [**8**, § 3]. We have metrics g_i, g_∞ on $B=B(1)$, local groups H_i, and a Lie group germ H such that

(9-7-1) H_i acts by isometry on the pointed metric space $((B, g_i), 0)$,

(9-7-2) $(B, g_i)/H_i$ is isometric to a neighborhood of p_i on M_i,

(9-7-3) H acts by isometry on the pointed metric space $((B, g_\infty), 0)$,

(9-7-4) $(B, g_\infty)/H$ is isometric to a neighborhood of x in X,

(9-7-5) g_i converges to g_∞ with respect to the C^∞-topology.

Let $C(H_i)$ and $C(H)$ denote the centers of H_i and H, respectively. By construction, the dimension of the orbit through p_i of our F-structure on M_i is equal to the dimension of the orbit $C(H)(0)$. We shall prove $\dim C(H)(0)\neq 0$. If 0 is not a fixed point of $C(H)$, there is nothing to show. We assume that there exists $\gamma\in C(H)\setminus\{1\}$ such that $\gamma(0)=0$. Take $\gamma_i\in C(H_i)$ such that $\lim\gamma_i=\gamma$. We have

(9-8) $$\lim_{i\to\infty} d(\gamma_i(0),\, 0)=0.$$

Let δ be an arbitrary small positive number. Then (9-8) and the fact that the action of H_i is free imply the existence of n_i such that

(9-9) $$\delta\geqq\lim_{i\to\infty} d(\gamma_i^{n_i}(0),\, 0)\neq 0.$$

We can take a subsequence $k(i)$ such that $\lim_{i\to\infty}\gamma_{k(i)}^{n_{k(i)}}$ converges to an element γ' of $C(H)$. Then by (9-9) we have

(9-10) $$\delta\geqq d(\gamma'(0),\, 0)\neq 0.$$

Since δ is arbitrary small, (9-10) implies $\dim(C(H)(0))\neq 0$.

Thus we have constructed a pure F-structure on M_i for a sufficiently large i. This contradicts our choice of M_i. Q.E.D.

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COLLAPSING RIEMANNIAN MANIFOLDS WHILE KEEPING THEIR CURVATURE BOUNDED. I.

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0. Introduction

Let Y^n be a complete connected riemannian manifold, and $p \in Y^n$. The injectivity radius, i_p, of the exponential map at p is defined to be the smallest r such that $\exp_p|\overline{B_r(p)}$ fails to be a diffeomorphism onto its image. The present paper is the first of two which are concerned with the situation in which the size of injectivity radius is "small" relative to the curvature.

In this part I, we show that if a smooth manifold X^n admits a certain topological structure called an *F-structure of positive rank*, then X^n also admits a family of metrics, g_δ, such that as $\delta \to 0$, i_p *converges uniformly to zero at all points*, p, but the curvature, K_α, stays bounded (independent of p and δ). Such a family of metrics is said to *collapse* with bounded curvature (by rescaling, one can assume $|K_\delta| \leqslant 1$).

In part II we prove a sort of strengthened converse to this collapsing result. A riemannian manifold Y^n is said to be *ε-collapsed* if $i_p < \varepsilon$ for all p. Intuitively, such a manifold appears to have dimension $< n$ if one examines it on a scale $\gg \varepsilon$. We show that in each dimension, there exists a *critical radius*, $\varepsilon(n)$, such that if Y^n is $\varepsilon(n)$-collapsed and $|K| \leqslant 1$, then Y^n admits an F-structure of positive rank. Thus, if Y^n admits a metric which is sufficiently collapsed, it actually admits a family of metrics which collapse with bounded curvature.

An F-structure on a space, X, is a natural generalization of a torus action. Different tori (possibly not all of the same dimension) act locally on finite covering spaces of subsets of X. These local actions satisfy a compatibility condition, which insures that X is partitioned into disjoint "orbits." The F-structure is said to have *positive rank* if all orbits are of positive dimension.

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The existence of an F-structure of positive rank is a definite constraint on the topology of a space. In the *compact* case, it implies, for example, that the Euler characteristic is zero (compare however Example 1.6 of [4]). This vanishing phenomenon does *not* carry over to Pontrjagin numbers, except in the presence of further hypotheses; see Example 1.9 (we will show elsewhere that the Pontrjagin numbers of X^{4l} vanish if it admits a so-called "pure structure of positive rank, with amenable holonomy"). However, there is a strong interaction between F-structures, characteristic numbers, and secondary geometric invariants; see [3], [5], [13].

The collapsing family of metrics g_δ, associated to an F-structure of *positive* rank is obtained roughly as follows. Start with a metric g which is *invariant* for the structure in the sense that the local torus actions are isometric. Then shrink g in certain directions tangent to the orbits. In some cases, it is also necessary to expand g in directions orthogonal to the orbits, in order to keep the curvature bounded. Thus, the diameter, $\operatorname{diam}(Y_n, g_\delta)$, and volume, $\operatorname{Vol}(Y_n, g_\delta)$, may go to infinity as (Y_n, g_δ) collapses; it may also happen that they stay bounded or converge to zero.

The following examples (although they are presented informally) should serve to give some feeling for the concepts mentioned so far; see §1 for the precise definition of "F-structure" (which is somewhat technical).

Example 0.1 (*The Klein bottle*). View the Klein bottle as the total space of a circle bundle $S^1 \to K^2 \xrightarrow{p} S^1$. For each interval, I, in the base space, there are two canonical fiber preserving circle actions on $p^{-1}(I)$, which differ by the automorphism $x \to x^{-1}$. If one of these local actions is continued around the base circle, the opposite action is obtained. As a consequence of this *holonomy* phenomenon, no global action exists. (See Example 1.2 for further discussion.)

Example 0.2 (*Graph manifolds*). Take a finite collection of surfaces, Σ_i^2, with $\partial\Sigma_i^2 = \bigcup_{j=1}^{N(i)} S_{i,j}^1$, a disjoint union of circles. The product manifolds, $\Sigma_i^2 \times S_i^1$, have boundary components which are tori, $S_{i,j}^1 \times S_i^1$. Form a manifold with empty boundary, Y^3, by identifying these tori in pairs by elements of $\mathrm{SL}(2, Z)$.

On each piece $\Sigma_i^2 \times S_i^1 \subset Y^3$, S^1 acts by rotation of the factor S_i^1. At boundary components which have been identified, the corresponding circle actions need not agree. But if not, they generate an action of a 2-torus, which extends both of them. Thus, in this example, the torus which acts locally is of dimension 2 near such identified boundary components and of dimension 1 elsewhere.

Example 0.3 (*Compact flat manifolds*). If X^n is compact and flat, by the Bieberbach Theorem there is a finite normal covering $\tilde{X}^n$, which is isometric to a torus. Since the action of this torus on itself is transitive, the induced orbit structure on X^n consists of a single orbit, X^n.

Example 0.4 (*Collapse by scaling*). If (Y_n, g) is a complete manifold with injectivity radius uniformly bounded from above, then the family $(Y_n, \delta^2 g)$ collapses (intuitively, to a single point, in the compact case). However, for this collapse, the sectional curvature does not remain bounded unless Y_n is flat.

In view of Example 0.4, from now on the word "collapse" will be taken to mean "collapse with bounded curvature."

The following is the most transparent and in a sense the most basic collapse with bounded curvature.

Example 0.5 (*Generalized warped products*). Start with a surface of revolution, M^2, obtained by revolving an arc in the upper half plane about the x-axis. Thus, M^2 is diffeomorphic to $S^1 \times I$. The obvious isometric circle action on M^2 lifts to an isometric R action on the infinite cyclic covering $\tilde{M}^2 = R \times I$. Let $\{\delta Z\} \subset R$ denote the subgroup generated by a translation of size δ. Then the family $\tilde{M}^2/\{\delta Z\}$ collapses, but the curvature remains unchanged (we have unrolled M^2 and then rolled it up more tightly).

To extend the above example to higher dimensions, take $\tilde{M}^{k+l} = X^k \times R^l$, with (generalized) warped product metric

$$g = g_1(x) + \sum_{i,j=1}^{l} a_{ij}(x)\, dy_i\, dy_j. \tag{0.1}$$

Then $M^{k+l} = X^k \times R^l/\delta Z^l$ collapses (to X^k).

Note that the orbits of the F-structure on M^{k+l} have *constant dimension*, l. In such cases (as above) the collapse can always be performed so that the *diameter remains bounded*. In particular, the volume goes to zero.

Example 0.2 (*continued*). The collapse associated to the F-structure on the graph manifold, Y^3, is particularly easy to describe if the identifications of the boundaries simply interchange the roles of the two circles. In this case, choose a "cusp-like" metric on Σ_i^2 which near the boundary is isometric to the product of an interval and a circle, S_δ^1, of length δ. The curvature and volume can be chosen bounded independent of the size of δ for such a metric.

Now form $\Sigma_i^2 \times S_\delta^1$ with the product metric, and identify corresponding boundary components. The resulting manifold, (Y^3, g_δ), has injectivity radius everywhere $= \delta$. In fact, $\mathrm{Vol}(Y^3, g_\delta) \leqslant c\delta$. However, the orbits of the F-structure are not of constant dimension and $\mathrm{diam}(Y^3, g_\delta) \to \infty$.

As far as we are aware, the first example of collapse (apart from warped products and scaling) was discovered by M. Berger in about 1962. He considered the collapse of the unit sphere S^3, obtained by shrinking the circles of the Hopf fibration. It is clear that the "limit" of this collapse should be S^2 (in fact, with a metric of curvature $\equiv 4$). The notion of the limit of a collapse can be made precise by introducing the concept of *Hausdorff limit*: see [9] and §2. Berger's interest in the collapse of S^3 stemmed from his observation that it provides a counterexample to a specific conjecture concerning a lower bound for the injectivity radius on odd dimensional manifolds of positive curvature.

Another significant collapse (with variable topology) was discovered in the context of manifolds of positive curvature by Aloff and Wallach [1]. They exhibited an infinite sequence of pairwise nonhomeomorphic, homogeneous 7-manifolds with uniformly pinched positive curvature. By the finiteness theorem in riemannian geometry (see [2], [4], [9], [11]) such a sequence must collapse.

The remainder of this paper is divided into five sections and one appendix as follows.

1. $\tilde{\Gamma}$-structures and F-structures
2. Pure polarized collapses with bounded diameter
3. Polarized volume collapses
4. Nonpolarized collapses
   (a) Introduction
   (b) Main computation
   (c) Construction of slice polarizations
   (d) Collapse
5. F-structures and complete metrics on open manifolds
   (a) Introduction
   (b) Construction of a complete metric, $\mathscr{g}_0$
   (c) Expansion of $\mathscr{g}_0$
   (d) Collapse of the expanded metric

Appendix: Pure polarized structures on essential manifolds.

In §1, we define and give examples of generalized group actions called $\tilde{\mathscr{g}}$-structures. Essentially, an F-structure is a $\tilde{\mathscr{g}}$-structure for which all the groups which act locally are tori. In §2, we consider the case of a *pure* structure. Basically, this means that a *single* connected group acts locally, up to automorphism, on a finite covering space. We assume, moreover, that *all orbits are of the same positive dimension*; compare Examples 0.1, 0.4, and 0.5. This second condition defines what is called a *pure polarized structure*. For such structures, by shrinking a compatible metric in the direction of the orbits while leaving it unchanged in the orthogonal directions, we obtain a collapse for which the diameter stays uniformly bounded.

In §3 we consider the polarization for which the groups which act locally are not all of the same dimension. In this case, we can collapse in such a way that $\mathrm{Vol}(Y_n, g_\delta) \to 0$, but $\mathrm{diam}(Y_n, g_\delta) \to \infty$; compare Example 0.2.

As already indicated, there exist manifolds, M_F^{4l}, admitting F-structures (which are in fact pure) of positive rank, but for which some characteristic number is nonzero. By the Chern-Weil theory, these manifolds admit no collapse with bounded curvature, such that $\mathrm{Vol}(M_F^{4l}, g_\delta) \to 0$. In particular, M_F^{4l} does not carry any polarized F-structure. However, in §4, we show that any F-structure of positive rank admits what we call a *slice polarization*. This can be used to collapse in such a way that the volume behavior is controlled by the geometry of the orbit structure. For example, the manifolds M_F^4 can be collapsed so that the volume stays bounded. But it can also happen that the volume goes to infinity *or to zero* (even though the slice polarization is not an honest polarization: compare [10]).

In §5 we consider open manifolds which carry an F-structure outside a compact set. On such manifolds, we obtain complete metrics of bounded curvature with properties analogous to those of the metrics constructed in §2–4.

In the Appendix we exhibit a class of manifolds with the property that if a pure F-structure exists, it must be polarized. As a consequence, many of these manifolds can be shown to admit no pure F-structure of positive rank, although they do not admit such structures which are not pure.

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1. $\tilde{\mathscr{G}}$-structures and F-structures

In this section, we discuss certain generalizations of the concept of a group action.

A *partial action*, A, of a topological group, G, on a Hausdorff space, X, is given by the following data.

(i) A neighborhood $\mathscr{D} \subset G \times X$ of $e \times X$, where $e \in G$ is the identity element. This $\mathscr{D}$ is called the *domain* (of definition) of the action.

(ii) A continuous map $A\colon \mathscr{D} \to X$, also, written $(g, x) \to gx$, such that $(g_1 g_2)x = g_1(g_2 x)$ whenever $(g_1, g_2 x)$ and $(g_1 g_2, x)$ lie in $\mathscr{D}$, and such that $ex = x$ for all $x \in X$.

Two partial actions $(A, \mathscr{D}_1)$, $(A_2, \mathscr{D}_2)$ are called (locally) *equivalent* if there is a domain $\mathscr{D} \subset \mathscr{D}_1 \subset \mathscr{D}_2$, containing $e \times X$, such that $A_1|\mathscr{D}_2 = A_2|\mathscr{D}_2$. A *local action*, $\{A\}$, is defined as the equivalence class of a partial action A of G on X. Every global action A defines an obvious local action A_{loc}. A local action which can be obtained in this way is called *complete*.

Remark 1.1. An elementary connectedness argument shows that A_{loc} determines A uniquely, in case G is connected.

In the smooth case, the category of local actions is equivalent to the category of *infinitesimal actions*: these are continuous homomorphisms of the Lie algebra of G to the Lie algebra of vector fields on X. For example, if $G \simeq R$, then a local action is given by a vector field on X, and completeness amounts to the integrability of the field.

From now on, we assume that G is *connected.*

A subset $X_0 \subset X$ is called (locally) $\{A\}$-*invariant* if for some representative $(A, \mathscr{D}) \in \{A\}$, one has $ga \in X_0$ for all $(g, a) \in \mathscr{D}$. Since the intersection of $\{A\}$-invariant sets is $\{A\}$-invariant, it follows that each point $x \in X$ is contained in a unique minimal $\{A\}$-invariant subset called the *orbit* $\mathscr{O} = \mathscr{O}_x \subset X$, and that the orbits partition the space X. Moreover, if A_{loc} is complete, the orbits of A_{loc} and A coincide.

A local action, $\{A\}$, on X can be restricted to any open subset, $U \subset X$, by taking an open subset $\mathscr{D}' \subset G \times X$, which contains $e \times U$ and such that $gx \in U$ for all $(g, x) \in \mathscr{D}'$ with $x \in U$. Furthermore, if $Y \to X$ is a *local homeomorphism*, then $\{A\}$ pulls back to a local action, $f^*\{A\}$, on Y, in a similar way.

Now consider a sheaf, $\mathscr{g}$, of connected topological groups over X. Let $g(U)$ denote the group of sections over U. An *action* of $\mathscr{g}$ on X is given by a local action of the group $\mathscr{g}(U)$ on U for every connected open set $U \subset X$, such that the structure homomorphisms $\mathscr{g}(U) \to \mathscr{g}(U')$ (for $U' \subset U$) agree with the restrictions of the local actions from U to U'.

A set S is called *invariant* if for all open sets U, the intersection $S \cap U$ is invariant for $\mathscr{g}(U)$. Again, X is partitioned into minimal invariant sets called *orbits*. A set which is the disjoint union of orbits is called *saturated*.

Example 1.1. In the smooth case, an action of $\mathscr{g}$ on X amounts to a homomorphism of the Lie algebra sheaf associated to $\mathscr{g}$ into the sheaf of germs of vector fields on X. As a specific example, let X be an affine flat manifold, infinitesimally (and hence locally) acted on by the Lie algebra sheaf of *parallel* vector fields.

Let G_x denote the stalk of $\mathscr{g}$ at x. If $f\colon Y \to X$ is a locally homeomorphic map, let $f^*(\mathscr{g})$ denote the pullback sheaf.

The following is a significant generalization of the concept of completeness introduced previously.

Definition 1.1. An action of a sheaf $\mathfrak{g}$ of connected groups on X is called *complete* if for all $x \in X$, there exists an open neighborhood $V(x)$ and a locally homeomorphic map π: $\tilde{V}(x) \to V(x)$ ($\tilde{V}(x)$ Hausdorff) such that

(i) If $\pi(\tilde{x}) = x$, then for any open neighborhood $W \subset \tilde{V}(x)$ of $\tilde{x}$, the structure homomorphism $\pi^*(\mathfrak{g})(W) \to G_{\tilde{x}} \stackrel{\text{def}}{=} G_x$ is an isomorphism.

(ii) The local action of $\pi^*(\mathfrak{g})$ on $\tilde{V}(x)$ is complete.

Example 1.1 (*continued*). For affine flat manifolds, this agrees with the usual definition of completeness.

Note that the orbits of $\pi^*(\mathfrak{g})$ on $\tilde{V}(x)$ project to orbits of $\mathfrak{g}$ on $V(x)$.

Suppose π: $\tilde{V}(x) \to V(x)$ is a normal covering. Then the group, Γ, of covering transformations of π: $\tilde{V}(x) \to V(x)$ has a natural (holonomy) action on $\pi^*(\mathfrak{g})$. It follows that there is a sheaf $\mathscr{S}$ on $\tilde{V}(x)$ such that the stalk of $\pi^*(\mathscr{S})$ at $\tilde{y} \in \tilde{V}(x)$ is the image of the structure homomorphism $\pi^*(\mathfrak{g})(\tilde{V}(x))$ and by Remark 1.1, for $\gamma \in \Gamma$, $g \in \pi^*(\mathfrak{g})(\tilde{V}(x))$, $\tilde{y} \in \tilde{V}(x)$, we have

$$\gamma(g\tilde{y}) = \gamma(g)\gamma(\tilde{y}). \tag{1.1}$$

Definition 1.2. A $\tilde{\mathfrak{g}}$-structure, $\mathscr{G}$, on X is a sheaf, $\mathfrak{g}$, of connected topological groups on X and a complete local action of $\mathfrak{g}$ on X such that the sets $V(x)$ can be chosen to satisfy the following conditions.

(i) π: $\tilde{V}(x) \to V(x)$ is a normal covering.

(ii) For all x, $V(x)$ is saturated.

(iii) For all $\mathcal{O}$, if $x, y \in \bar{\mathcal{O}}$, then $V(x) = V(y)$.

It follows from (iii) that $\mathfrak{g}|\bar{\mathcal{O}}$ is a *locally constant* sheaf, i.e. $\mathfrak{g}|\bar{\mathcal{O}}$ is locally isomorphic to the sheaf of locally constant maps of $\bar{\mathcal{O}}$ to the group G_x. Put otherwise, $\mathfrak{g}|\bar{\mathcal{O}}$ is a flat bundle such that each fiber is a group and the holonomy acts by automorphisms of the fiber. However, it need not be the case that $\mathfrak{g}$ is locally constant on some neighborhood of $\bar{\mathcal{O}}$, since the structure homomorphisms need not be injective; see Example 1.4 and Remark 1.2.

Definition 1.3. If $\mathfrak{g}$ is locally constant on $V(x)$ for all x, then $\mathscr{G}$ is called *pure*.

Suppose $\mathscr{G}$ is pure. Let $x \in X$, and fix $\tilde{x} \in \tilde{V}(x)$ with $\pi(\tilde{x}) = x$. Since $\mathfrak{g}(\tilde{V}(x)) = G_x$, it follows that $\tilde{V}(x) \xrightarrow{\tau} \tilde{V}^E(x) \to V(x)$, where $\tilde{V}^E(x)$ is the *holonomy covering* of $\mathfrak{g}|V(x)$ (with base point $\tau(\tilde{x})$). As a consequence of (1.1), the action of G_x descends to $\tilde{V}^E(x)$ and (1.1) continues to hold there.

For $x \in X$, let $(\tilde{X}^E, \tilde{x})$ denote the holonomy covering of the locally constant sheaf $\mathfrak{g}$ with canonical basepoint. Let π: $\tilde{X}^E \to X$ be the projection. Given $y \in X$ and a curve, c, from x to y, the space $(\tilde{V}^E(y), \tilde{y})$ is naturally identified with a component of $\pi^{-1}(V(y)) \subset (\tilde{V}^E, x)$. Also parallel translation along c includes an isomorphism $G_x \to G_y$. Thus, the action of G_y on $(\tilde{V}^E(y), \tilde{y})$

induces an action of G_x on the corresponding component of $\pi^{-1}(V(y))$ and it is immediate from (1.1) and the definition of $(\tilde{X}^E, \tilde{x})$ that these actions give rise to a global action of G_x on $(\tilde{X}^E, \tilde{x})$. Thus,

Proposition 1.1. *If a $\tilde{\mathscr{g}}$-structure is pure, then the local action of $\tilde{\mathscr{g}}(\tilde{X}^E, \tilde{x})$ is complete.*

Example 1.2 (*Flat bundles*). A basic example of a pure $\tilde{\mathscr{g}}$-structure is the following. Let E be the total space of a locally constant sheaf, $\mathscr{g}$, of connected groups, and ρ: $E \to x$ the projection. Then for $x \in X$ and $y \in \rho^{-1}(x)$, there is an obvious action of the stalk G_y of $\rho^*(\mathscr{g})$ on $\rho^{-1}(U)$, provided U is chosen so that $\mathscr{g}|U$ is trivial. In particular there is a pure $\tilde{\mathscr{g}}$-structure on E, where the sheaf which acts is $\rho^*(\mathscr{g})$.

Definition 1.4. A $\tilde{\mathscr{g}}$-structure is called an *F-structure* if for all x, the group G_x is isomorphic to a torus, and the sets $V(x)$ (of Definition 1.2) can be chosen so that the coverings $\tilde{V}(x)$ are finite.

Definition 1.5. If one can always choose $\tilde{V}(x) = V(x)$, then the F-structure is called a T-structure.

Example 1.3 (*Example* 0.3 *reformulated*). Let X^n be a compact flat riemannian manifold. By the Bieberbach Theorem, for each $x \in X$ the holonomy covering $(\tilde{X}^n, \tilde{x})$ has the natural structure of a torus, T_x^n. The torus T_x^n acts on itself by the left translation. Hence, it acts canonically on any $(\tilde{X}^n, \tilde{y})$ as well. The holonomy transformations act on T_x^n by conjugation. The set $\bigcup_x T_x^n$ has the natural structure of a locally constant sheaf $\mathscr{g}$ (with stalk T_x^n) and the action of T_x^n on any fixed $(\tilde{X}^n, \tilde{x}_0)$ induces the local action of $\mathscr{g}$ on X.

Example 1.4 (*Structure homomorphisms not injective*). Let X be the space formed as follows. Take $S^1 \times [0,1]$ and attach $S^1 \times 1$ to $S^1 \times 0$ by a covering map of degree 2. The image of $S^1 \times [1-\varepsilon, 1]$ in X is a Möbius band B_ε. The image $S^1 \times [0, \varepsilon]$ in X is a half open cylinder, C_ε. Moreover, $\overline{C}_\varepsilon \cap B_\varepsilon = S$, where S is the central circle in B_ε (and $\overline{C}_\varepsilon$ is the closure of C_ε).

The orbits, $\mathscr{O}$, in X will be the images of circles, $S^1 \times A$. For each connected open set $U \subset X$, put

$$T(U) = \{\bigcup \mathscr{O} | \mathscr{O} \cap U \leq \varnothing\}. \tag{1.2}$$

Let $\mathscr{g}$ be the sheaf associated to the presheaf which assigns to each U the identity component of the isometry group of $T(U)$. By definition, there is a complete action of $\mathscr{g}$ on X, which defines a T-structure.

Note that for all $T(U) \neq X$, $\mathscr{g}(U)$ is isomorphic to a circle. However, if, for example, $T(U_1) = B_\varepsilon \cup C_\varepsilon$ and $T(U_2) = C_\varepsilon$, then the restriction map $\mathscr{g}(U_1) \to \mathscr{g}(U_2)$ is a 2-fold covering. As a consequence the total space of $\mathscr{g}$ is not Hausdorff at points lying over S, and the local action of $\mathscr{g}$ is not locally isomorphic to a pure structure in a neighborhood of S.

Remark 1.2. Observe that since a nontrivial local isometry defined on a connected subset of a riemannian manifold is not equal to the identity on any nonempty open subset, examples like the one above do *not* occur for *effective* local actions of *compact* groups in the smooth category. From now on, *we will restrict attention to $\tilde{\mathscr{g}}$-structures of this type*. For such structures, the restriction maps $\mathscr{g}(U_1) \to \mathscr{g}(U_2)$ are *injective*.

If the action of $\mathscr{g}$ on X defines a $\tilde{\mathscr{g}}$-structure, $\mathscr{G}$, and $\mathscr{g}' \subset \mathscr{g}$ is a subsheaf, then $\mathscr{g}'$ defines a $\tilde{\mathscr{g}}$-structure, $\mathscr{G}'$, called a *substructure*. We write $\mathscr{G}' \subset \mathscr{G}$. Note that the stalks of $\mathscr{G}'$ are *not* required to be closed subgroups. The subsheaf whose stalks are the closures of those of $\mathscr{G}'$ is written $\overline{\mathscr{G}}'$, the *closure* of $\mathscr{G}'$.

For a $\tilde{\mathscr{g}}$-structure as in Remark 1.2 above, for each $x \in X$ the neighborhood $V(x)$ of Definition 1.2 can be chosen such that there is a (unique) *pure substructure*, $\mathscr{G}_\alpha$, of $\mathscr{G}|V(x)$, with stalk $G_{x,\alpha} = G_\alpha$.

The *rank* of $\mathscr{G}$ at x is defined as $\dim \mathscr{O}_x$. We say that $\mathscr{G}$ has *positive rank* if the rank is positive for all $x \in X$.

Let $\mathscr{G}' \subset \mathscr{G}$ (with $\mathscr{G}$ as above). Let $X = \bigcup U_\alpha$ be a *locally finite* covering by connected open sets U_α. For each α, let $\mathscr{G}'_\alpha \subset \mathscr{G}'$ be a pure substructure with stalk $\mathscr{G}'_{x,\alpha} \subseteq \mathscr{G}'_x$ at $x \in U_\alpha$.

Definition 1.6. The collection $\{(U_\alpha, \mathscr{G}'_\alpha)\}$ is called an *atlas* for $\mathscr{G}'$ if

(i) Each U_α is saturated for the orbits of $\overline{\mathscr{G}'}$.

(ii) For each x, there exists $U_\alpha \ni x$, such that $G'_{x,\alpha} = G'_x$.

Definition 1.7. A substructure $\mathscr{P} \subset \mathscr{G}$ is called a *polarization* if it has an atlas, $\mathscr{A}$, such that for all α, the rank of $\mathscr{G}'_\alpha$ is the same positive number at all $x \in \mathscr{G}'_\alpha$ (although rank $\mathscr{G}'_\alpha$ might depend on α).

A polarization is called *pure* if $\mathscr{G}'$ is a pure substructure (in which case it suffices to take a single $U_\alpha = X$). The notions of atlas and polarization play an important role in the collapsing constructions of §§2, 3, and 4.

If $\mathscr{G}$ has positive rank, one way to find a substructure, $\mathscr{G}' \subset \mathscr{G}$, of positive rank which possesses an atlas is the following. Take a locally finite open covering by sets $U_\alpha = V(x_\alpha)$ and pure a substructure $\mathscr{G}_\alpha$ on each U_α, such that $G_{x_\alpha,\alpha} = G_{x_\alpha}$ and the rank of $\mathscr{G}_\alpha$ is positive. Enlarge this covering by adding all nonempty intersections, $U_{(\alpha)} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ and assign to $U_{(\alpha)}$ the pure substructure whose stalk at $x \in U_{(\alpha)}$ is the smallest subgroup $G_{(\alpha)}$ containing $\bigcup_1^k G_{x,\alpha_j}$. Then $\{(U_{(\alpha)}, G(\alpha))\}$ is an atlas for the substructure $\mathscr{G}'$, determined by the condition $G'_x = G_{x,(\alpha)}$, where $U_{(\alpha)}$ is the intersection of all those U_α containing x. The rank of $\mathscr{G}'$ is positive and it is easy to see that in fact we can choose $\mathscr{G}'$ such that $\operatorname{rank} \mathscr{G}' = \operatorname{rank} \mathscr{G}$.

The following lemma is convenient for the constructions of §5 and provides a simple picture of structures which possess an atlas.

Lemma 1.2. *Let $\{(U_\alpha, \mathscr{G}_\alpha)\}$ be an atlas for $\mathscr{G}$ on (a possibly open) manifold X. Then there is an atlas $\{(\underline{U}_\alpha, \underline{\mathscr{G}}_\alpha)\}$ for $\mathscr{G}$ with the following properties*:

(1) *the sets $\underline{U}_\alpha$ have compact closure.*

(2) *If $x \in \underline{U}_{\alpha_1} \cap \cdots \cap \underline{U}_{\alpha_k}$, then (for some ordering) $G_{x,\alpha_1} \subseteq G_{x,\alpha_2} \subseteq \cdots \subseteq G_{x,\alpha_k}$.*

(3) *For all $\underline{U}_\alpha$ and all $x \in \underline{U}_\alpha$, there is at most one $\underline{U}_\beta$ $(\beta \neq \alpha)$ with $x \in \underline{U}_\beta$ and $G_{x,\alpha} = G_{x,\beta}$.*

Proof. (1) Clearly, we can assume $\{U_\alpha\}$ itself has this property.

(2) Let U_β, U_γ satisfy $x \in U_\beta \cap U_\gamma$ but neither $G_{x,\beta} \subseteq G_{x,\gamma}$ nor $G_{x,\gamma} \subseteq G_{x,\beta}$. Take saturated open sets $\hat{U}_\beta$, $\hat{U}_\gamma$ with $\hat{U}_\beta \cap \hat{U}_\gamma = \varnothing$ and $U_\beta \setminus \overline{U}_\gamma \subset \hat{U}_\beta \subset U_\beta$, $U_\gamma \setminus \overline{U}_\beta \subset \hat{U}_\gamma \subset U_\gamma$. Since, clearly, $G_{x,\beta} \neq G_x \neq G_{x,\gamma}$, it follows that $\hat{U}_\beta$, $\hat{U}_\gamma$ together with the remaining $\{U_\alpha\}$ cover X. We can now construct a covering $\{\underline{\underline{U}}_\alpha\}$ by induction, such that $\underline{\underline{U}}_\alpha \subset U_\alpha$, and if we put $\underline{\underline{\mathscr{G}}}_\alpha = \mathscr{G}_\alpha | \underline{\underline{U}}_\alpha$, then $\{\underline{\underline{U}}_\alpha, \underline{\underline{\mathscr{G}}}_\alpha\}$ satisfies (1) and (2).

(3) Let $\underline{\underline{U}}_{\alpha_1}, \underline{\underline{U}}_{\alpha_2}, \ldots$ be a maximal subcollection of $\{\underline{\underline{U}}_\alpha\}$ such that $\bigcup \underline{\underline{U}}_{\alpha_j}$ is connected and $G_{x,\alpha_i} = G_{x,\alpha_j}$ whenever $x \in \underline{\underline{U}}_{\alpha_i} \cap \underline{\underline{U}}_{\alpha_j}$. Put $\underline{U}_{\alpha_1} = \underline{\underline{U}}_{\alpha_1}$. Let, say, $\underline{\underline{U}}_{\alpha_2} \cdots \underline{\underline{U}}_{\alpha_{k_2}}$ be those $\underline{\underline{U}}_{\alpha_j}$ whose intersection with $\underline{\underline{U}}_{\alpha_1}$ is nonempty. Let $\underline{U}_{\alpha_2}, \ldots, \underline{U}_{\alpha_{l_2}}$ be the connected components of $\underline{\underline{U}}_{\alpha_2} \cup \cdots \cup \underline{\underline{U}}_{\alpha_{k_2}}$. By proceeding in this way and repeating the process for all subcollections as above, we obtain the required covering.

An atlas satisfying the properties of Lemma 1.2 is called *regular*.

Remark 1.3. Let $\mathscr{A}$ be a regular atlas for $\mathscr{G}$. Let $\{U'_\alpha\}$ be an open covering by saturated subsets, $U'_\alpha \subset U_\alpha$. Then by restricting $\mathscr{G}_\alpha$ to U'_α we obtain a regular atlas, $\mathscr{A}'$, for a substructure $\mathscr{G}' \subset \mathscr{G}$. We write $\mathscr{A}' \subset \mathscr{A}$.

Remark 1.4. If the atlas $\{(U_\alpha, \mathscr{G}_\alpha)\}$ in Lemma 1.2 is a polarization, then the regular atlas $\{(\underline{U}_\alpha, \mathscr{G}_\alpha)\}$ is a polarization as well.

Remark 1.5. By dropping (1) in Lemma 1.2, we can also drop (3) and strengthen (2) to read $G_{x,\alpha_1} \not\subseteq G_{x,\alpha_2} \not\subseteq G_{x,\alpha_k}$.

A riemannian metric g is called *invariant* for $\mathscr{G}$ if the local action of the sheaf g is isometric.

Lemma 1.3. *Let $\mathscr{A} = \{(U_\alpha, \mathscr{G}_\alpha)\}$ be a regular atlas for $\mathscr{G}$ and let $\mathscr{A}' = \{(U'_\alpha, \mathscr{G}_\alpha)\}$, where $\overline{U}'_\alpha \subset U_\alpha$. Suppose $\mathscr{G}$ has the property that all coverings $\tilde{V}(x) \to V(x)$ (in Definition* 1.2) *can be chosen finite. Then there is an invariant metric for $\mathscr{G}'$.*

Proof. The relation of Lemma 1.2(2) induces a natural partial ordering on the $\{U_\alpha\}$. Start with some maximal U_α and take a finite covering of $\overline{U}'_\alpha$ by sets of the form $V(x_1) \cdots V(x_k)$ with $\overline{V(x_j)} \subset U_\alpha$. Take any metric g_0 on $V(x_1)$, pull it back to $\tilde{V}(x_1)$, and average over the action of G_x and over the finite group of covering transformations. Extend the resulting metric to any smooth metric g_1 on $V_1(x) \cup V_2(x)$. Repeat the process for $g_1 | V(x_2)$. By continuing in this way we get a metric on $\bigcup U'_\alpha$ with U'_α maximal and in the same way we get the required invariant metric for $\mathscr{G}'$.

By using (invariant smoothings of) distance functions for an invariant metric, we can construct invariant smooth functions, say $f_\alpha \colon U'_\alpha \to [1/2, 1]$, with $\bigcup_\alpha f_\alpha^{-1}(1/2) = X$. Then a standard application of Sard's Theorem gives

Lemma 1.4. *For almost all $c_\alpha \in (1/2, 1)$ the sets $f_\alpha^{-1}(c_\alpha)$ are smooth closed codimension 1 submanifolds which intersect transversally (and define an atlas $\mathscr{A}'' \subset \mathscr{A}$).*

We can now verify the result on the vanishing of the Euler characteristic mentioned

Proposition 1.5. *Let X be a compact manifold which carries an F-structure of positive rank, then $\chi(X) = 0$.*

Proof. Let $\mathscr{A}$ be an atlas for a substructure of positive rank with $U_\alpha = V(x_\alpha)$. Let $\mathscr{A}'' \subset \mathscr{A}$ be as in Lemma 1.4. Then for every intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ there is a finite covering, say $\tilde{U}_{(\alpha)} \subset \tilde{V}_{(x_{\alpha_1})}$, on which a torus G_{x_α}, acts with no common fixed points. By a well-known argument almost all elements of $G_{x_{\alpha_1}}$ are fixed point free. Thus, by the Lefschetz fixed point theorem, $\chi(\tilde{U}_{(\alpha)}) = \chi(U_{(\alpha)}) = 0$. Then $\chi(X) = 0$ as well.

Remark 1.6. In Proposition 1.5 it is not essential that X is a manifold.

Here are some further examples of F-structures.

Example 1.5 (S^3 *and* R^4). View R^4 as $\mathbf{C}^2 = (z_1, z_2)$. There is an obvious T^2-action ($T^2 = (\theta_1, \theta_2)$) given by

$$(\theta_1, \theta_2) \cdot (z_1, z_2) = \left(e^{i\theta_1} z_1, e^{i\theta_2} z_2\right), \tag{1.3}$$

with orbits of dimensions 0, 1, 2. Since there exist orbits of dimension 0, the corresponding T-structure admits no polarization.

There is also an induced T-structure on the unit sphere, S^3. All orbits are of dimension 2, with the exception of the circles $S^3 \cap \{(z_1, 0)\}$ and $S^3 \cap \{(0, z_2)\}$. Any choice of 1-parameter subgroup, S^1_γ, with $0 < \theta_1/\theta_2 = \gamma < \infty$, gives rise to a pure polarization, $\mathscr{P}_\gamma$, for which all orbits are 1-dimensional.

We can define another T-structure on S^3 which is not pure by picking η, with $1/\sqrt{2} < \eta < 1$, and setting

$$U_j = \left\{(z_1, z_2) \in S^3 \middle| |z_j| < \eta\right\}, \qquad j = 1, 2. \tag{1.4}$$

For $x \in U_1 \setminus U_2$ ($U_2 \setminus U_1$) we let $G_x = S^1_{\gamma_1}$ ($G_x = S^1_{\gamma_2}$) and $V(x) = U_1$ ($V(x) = U_2$). For $x \in U_1 \cap U_2$, $G_x = T^2$ and $V(x) = U_1 \cap U_2$.

Note that for $\gamma \neq 1$, the orbits $S^3 \cap \{(z_1, 0)\}$ and $S^3 \cap \{(0, z_2)\}$ are never principle orbits of S^1_γ, i.e., their isotropy groups, while discrete, are not minimal.

Finally, observe that this example generalizes to higher dimensions.

Example 1.2 (*continued, solvemanifolds*). Let $A \in \mathrm{Sl}(2, Z)$ be an automorphism of the torus T^2 with two real distinct eigenvalues. Thus, if

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \tag{1.5}$$

then $|a + d| \geqslant 2$. The mapping torus, M^3, of A, is by definition the affine flat bundle $T^2 \to M^3 \to S^1$, with holonomy A (as is well known, M^3 is a solvemanifold).

As above, there is a pure T-structure on M^3 whose orbits are the fibers. This T-structure has a natural pure polarization (of rank 2). It also admits exactly two pure polarizations of rank 1, the orbits of which correspond to the eigen-directions of A (and are not closed).

Example 1.6 (*The flat bundle $\mathscr{E}_\theta^3$*). Let $R^2 \to \mathscr{E}_\theta^3 \to S^1$ denote the trivial R^2 bundle over S^1, equipped with the connection whose holonomy is given by rotation through an angle $2\pi\theta$. A point in $\mathscr{E}_\theta^3$ is denoted by (t, w) where $t \in R/Z$ $(= S^1)$ and $w \in R^2$. Then parallel translation v units along the base is given by

$$P(v)(t, w) = (t + v, R(v\theta)w), \tag{1.6)v}$$

where $R(v\theta)$ denotes rotation through an angle $2\pi v\theta$.

Observe that $\mathscr{E}_\theta^3$ carries the structure of a complete flat riemannian manifold whose isometry group $I(\mathscr{E}_\theta^3)$ is the torus, $S^1 \times S^1$, generated by

$$T(u)(t, w) = (t + u, w), \qquad R(v)(t, w) = (t, R(v)w). \tag{1.7}$$

The full group $I(\mathscr{E}_\theta^3)$ defines a pure T-structure which is of rank 2 everywhere except along the zero section of $\mathscr{E}_\theta^3$. Any 1-parameter subgroup other than $R(v)$ defines a pure polarization of rank 1.

Example 1.7 (*The space $\mathscr{M}^4 = \bigcup_\theta \mathscr{E}_\theta^3$*). Consider the family $[0,1] \times \mathscr{E}_\theta^3$ consisting of pairs $(\theta, \mathscr{E}_\theta^3)$. The spaces $\mathscr{E}_0^3$ and $\mathscr{E}_1^3$ are abstractly isometric. Their parallel translations are given by

$$P(v)(t, w) = \begin{cases} ((t + v), w), & \theta = 0, \\ (t + v, R(v)w), & \theta = 1. \end{cases} \tag{1.8}$$

The map

$$f(0, t, w) = (1, t, R(t)w) \tag{1.9}$$

provides an isometry between $\mathscr{E}_0^3$ and $\mathscr{E}_1^3$. It induces the isomorphism of isometry groups given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.10}$$

The space formed from $[0,1] \times \mathscr{E}_\theta^3$ by identifying $(0, E_0^3)$ with $(1, E_1^3)$ via f will be denoted by $\mathscr{M}^4$. The flat T^2-bundle (locally constant sheaf) over S^1 with holonomy given by (1.10) is a nilmanifold. Its pullback to $\mathscr{M}^4$ defines a pure T-structure on $\mathscr{M}^4$. As above, this structure has rank 2 everywhere except along the zero sections of the various $\mathscr{E}_\theta^3$, where it has rank 1. Moreover, unlike the matrix A in (1.5), the matrix in (1.10) has only a single eigenvector. It corresponds to the circle $R(v)$, the action of which fixes the zero sections of the $\mathscr{E}_\theta^3$. From this it is clear that *the pure T-structure on $\mathscr{M}^4$ has positive rank but admits no polarization*. In fact, it is the most basic example of an F-structure with this property. There are no such examples in dimension 3 and any example in dimension 4 contains an orbit, a neighborhood of which looks like (perhaps a finite covering of) this example.

Example 1.8 (*A nonpolarizable structure on $T^2 \times R^4$*). The space $\mathscr{M}^4$ in Example 1.7 can be regarded as the total space of the complex line bundle with first Chern number 1, over T^2. If we take the Whitney sum of this bundle with the bundle of Chern number -1, we obtain the trivial bundle with total space $T^2 \times R^4$. Now, by a simple modification of the previous example, we find a pure T-structure on $T^2 \times R^4$ which is of rank 2 except at $T^2 \times 0$ where it is of rank 1. Moreover, this structure admits no polarization.

Example 1.9 (*Pure T-structure on M_F^{4l} with $\sigma(M_F^{4l}) = 2$*). The previous example of a pure T-structure of positive rank which admits no polarizations can be sharpened. There exist closed manifolds which carry a pure T-structure of positive rank, but which have nonzero signature. Hence, as noted in §0, they admit no polarized T-structure whatsoever.

The following particularly nice family of such examples is due, essentially, to T. Januszkiewicz. To describe them. Let

$$T^{2l+1} = (e^{i\theta_1}, \cdots, e^{i\theta_{2l+1}}), \qquad D = (e^{i\theta}, \cdots, e^{i\theta}) \tag{1.11}$$

and let S_j denote the image of

$$(1, \cdots, e^{i\theta}, 1 \cdots) \tag{1.12}$$

in T^{2l+1}/D. Then T^{2l+1}/D acts on

$$\mathbf{C}P(2l) = (z_1, \cdots, z_{2l+1})/D, \qquad \sum |z_i|^2 = 1, \tag{1.13}$$

with $2l + 1$ fixed points,

$$p_j = \left(0, \cdots, \overset{j}{1}, 0, \cdots\right). \tag{1.14}$$

If we use the product structure

$$S^1 \times \cdots \times \hat{S}_j \times \cdots \times S_{2l+1} \tag{1.15}$$

on T^{2l+1}/D and identify the tangent space to $\mathbf{C}P(2l)$ at p_j, with

$$(z_1, \cdots, \hat{z}_j, \cdots, z_{2l+1}), \tag{1.16}$$

then T^{2l+1}/D acts by the standard representation of T^{2l}. Let D_j denote the diagonal of T^{2l+1}/D with respect to the product structure in (1.15). Then the action of S_j on the tangent space at p_k is given by

$$S_j(z_1,\cdots,\hat{z}_k,\cdots,z_{2l+1}) = D_k(\bar{z}_1,\cdots,\hat{z}_k,\cdots,\bar{z}_{2l+1}). \tag{1.17}$$

Now choose normal coordinate systems at the points p_j and from each of these delete a ball, B_j^{4l}, about the origin. Take two copies Σ_1^{4l}, Σ_2^{4l} of the resulting manifold with boundary and form a closed manifold, M_F^{4l}, as follows. Let

$$f_j\colon \{1,\cdots,2l+1\} \to \{0,1\} \tag{1.18}$$

be any function which takes the value 1 an *odd* number of times, $j = 1,\cdots,2l+1$. Put

$$F = (f_1,\cdots,f_{2l+1}). \tag{1.19}$$

To obtain M_F^{4l}, glue corresponding boundary components of Σ_1^{4l} and Σ_2^{4l} by the identifications

(1.20)
$$(z_1,\cdots,\hat{z}_j,\cdots,z_{2l+1}) \sim (z_1,\cdots,\bar{z}_{i_1(j)},\cdots,\hat{z}_j,\cdots,\bar{z}_{i_{t(j)}(j)},\cdots,z_{2l+1}),$$

where $i_1(j),\cdots,i_{t(j)}(j)$ are the integers at which f_j takes the value of 1. The torus action on Σ_1^{4l}, Σ_2^{4l} gives rise to a pure T-structure on M_F^{4l}. To describe the holonomy of the corresponding flat bundle, E, it suffices to consider loops $l_1,\cdots,l_{2l}$, where l_j passes from Σ_1^{4l} to Σ_2^{4l} through ∂B_{2l+1}^{4l} and returns to Σ_j^{4l} through ∂B_j^{4l}. Then using (1.12)–(1.17), it follows that the holonomy around l_j is given by the matrix

$$\begin{pmatrix} (-1)^{\tau_j(1)} & a_j(1) & & \\ \ddots & \vdots & & 0 \\ & (-1)^{\tau_j(j)} & & \\ 0 & \vdots & \ddots & 0 \\ & a_{2l}(1) & & (-1)^{\tau_j(l)} \end{pmatrix}, \tag{1.21}$$

where

$$\tau_j(i) = f_j(k) + f_{2l+1}(k), \tag{1.22}$$

and for $k \neq j$,

$$a_j(k) = \begin{cases} 0, & f_j(k) + f_j(2l+1) \equiv 0 \mod 2, \\ -2, & f_j(k) + f_j(2l+1) \equiv 1 \mod 2, \end{cases} \tag{1.23}$$

Finally, since the identifications on the boundary components are *orientation reversing*, if Σ_1^{4l}, Σ_2^{4l} are *both* given the orientation induced from $\mathbf{C}P(2l)$, then M_F^{4l} also acquires an orientation. Moreover, the signature, $\sigma(M_F^{4l})$, is given by

$$\sigma(M_F^{4l}) = 2\sigma(\Sigma^{4l}) = 2\{\sigma(\mathbf{C}P(2l)) - (2l+1)\sigma(B^{4l})\} \\ = 2\sigma(\mathbf{C}P(2l)) = 2. \tag{1.24}$$

2. Pure polarized collapses with bounded diameter

In this section, we discuss the collapse associated to a pure polarization, $\mathscr{P}$, of an F-structure, $\mathscr{F}$, on a manifold Y_n. Let

$$g = g' + h \tag{2.1}$$

be a metric which is invariant for $\mathscr{F}$; see Lemma 1.2. Here h vanishes on vectors tangent to the orbits of $\mathscr{P}$, and g' vanishes on vectors normal to these orbits. Put

$$g_\delta = \delta^2 g' + h. \tag{2.2}$$

Theorem 2.1. *As $\delta \to 0$, the family (Y_n, g_δ) collapses. Moreover,*

$$\operatorname{dist}_{\delta_1}(p,q) \leqslant \operatorname{dist}_{\delta_2}(p,q), \qquad \delta_1 < \delta_2, \tag{2.3}$$

and for each compact set $\overline{U}$, there is a constant $c(\overline{U})$[1] *such that*

$$\sup_{\overline{U},\delta\geqslant 1} |K_\delta| \leqslant c(\overline{U}). \tag{2.4}$$

Proof. Observe that (2.3) is obvious and the main point is, of course, (2.4). We claim that when the appropriate coordinates are introduced, (2.4) is obvious as well.

Let $p \in Y$. Take a basis of Killing fields, $\{X_i\}$, tangent to the orbits of $\mathscr{P}$ in a neighborhood of p.

Let N^{n-k} be a local transversal to the orbits $\{\mathscr{O}\}$ with $p \in N^{n-k}$. Choose local coordinates, $(y_1,\cdots,y_{n-k})$ on N^{n-k} with p at the origin. Since $[X_i, X_j] \equiv 0$, there is a unique coordinate system $(x_1,\cdots,x_k)$ on each orbit $\mathscr{O}$, with $\mathscr{O} \cap N^{n-k} = (0,\cdots,0)$ and $X_i = \partial/\partial x_i$. By projecting onto N^{n-k}, i.e. $\mathscr{O} \to \mathscr{O} \cap N^{n-k}$, we obtain coordinates $(x_1,\cdots,x_k,y_1,\cdots,y_{n-k})$ in a neighborhood $B_r(O) \times B_s(O)$ of p with, say, $\Sigma x_i^2 \leqslant r^2$, $\Sigma y_i^2 \leqslant s^2$.

In terms of these coordinates, the matrix $(g(x,y,\delta))$, representing the metric g_δ, can be calculated as follows. Note that translation in the direction of x_i preserves (coordinate fields and) inner products, since $\partial/\partial x_i$ is a Killing field.

[1] The notation $c(\cdot)$ will always mean a constant which depends only on the quantities within the parentheses.

Thus, if we put

$$\frac{\partial}{\partial y_i} = X_i + V_i, \tag{2.5}$$

where X_i is tangent to orbits and V_i is normal to orbits,

$$\langle X_i, V_j \rangle_1 = 0, \tag{2.6}$$

and the following matrices are independent of $x_1, \cdots, x_k$.

$$A(y_1, \cdots, y_{n-k}) = \left(\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_1 \right), \tag{2.7}$$

$$B(y_1, \cdots, y_{n-k}) = \left(\left\langle \frac{\partial}{\partial y_i}, X_j \right\rangle_1 \right), \tag{2.8}$$

$$C(y_1, \cdots, y_{n-k}) = \left(\langle X_i, X_j \rangle_1 \right), \tag{2.9}$$

$$D(y_1, \cdots, y_{n-k}) = \left(\langle V_i, V_j \rangle_1 \right). \tag{2.10}$$

Here $\langle\ ,\ \rangle_\delta$ denotes the inner product for g_δ. It follows that

$$\begin{aligned} (g(x, y, \delta)) &= \left(\begin{array}{c|c} \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g & \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta \\ \hline \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta & \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle_\delta \end{array} \right) \\ &= \left(\begin{array}{c|c} \delta^2 A(y) & \delta^2 B(y) \\ \hline \delta^2 B(y) & \delta^2 C(y) + D(y) \end{array} \right). \end{aligned} \tag{2.11}$$

As $\delta \to 0$, the matrix in (2.11) becomes singular. But if we make the change of coordinates

$$u_i = \delta x_i, \quad du_i = \delta dx_i, \quad \frac{\partial}{\partial u_i} = \frac{1}{\delta} \frac{\partial}{\partial x_i}, \tag{2.12}$$

then

$$\begin{aligned} (g(u, y, \delta)) &= \left(\begin{array}{c|c} \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle_\delta & \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta \\ \hline \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial y_k} \right\rangle_\delta & \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle_\delta \end{array} \right) \\ &= \left(\begin{array}{c|c} A(y) & \delta B(y) \\ \hline \delta B(y) & \delta^2 C(y) + D(y) \end{array} \right). \end{aligned} \tag{2.13}$$

The family $(g(u, y, \delta))$ can be regarded as being defined on all of $R^k_u \times B_s(0)$. As $\delta \to 0$, it converges smoothly to the *generalized warped product metric*

$$(g(u, y, 0)) = \left(\begin{array}{c|c} A(y) & 0 \\ \hline 0 & D(y) \end{array}\right) \tag{2.14}$$

on $R^k_u \times B_s(0) \subset R^k_u \times R^{n-k}_y$, where for each fixed y, the induced metric on R^k_u is *flat*: compare Example 0.5. Since

$$(g(x, y, \delta)) | B_r(0) \times B_s(0) \subset R^k_x \times R^{n-k}_y \tag{2.15}$$

is isometric to

$$(g(u, y, \delta)) | B_{\delta r}(0) \times B_s(0) \subset R^k_u \times R^{n-k}_y, \tag{2.16}$$

it is clear that $|K_\delta|$ is uniformly bounded independent of δ on compact subsets. This gives (2.4).

To see that (Y_n, g_δ) collapses, consider the closure $\bar{\mathscr{P}}$ of $\mathscr{P}$. The orbits $\{\bar{\mathscr{O}}\}$ of $\bar{\mathscr{P}}$ are compact flat manifolds. Since g_δ restricted to the normal space of any $\bar{\mathscr{O}}$ is independent of δ, it follows easily that the distance between any two such orbits is bounded below independent of δ. If Y_n is not complete, the same holds for $\inf_\delta \operatorname{dist}(\bar{\mathscr{O}}_q, \bar{Y}^n \setminus Y_n) = d_q$ (where $\bar{Y}^n$ is the completion of Y_n). In particular, the closed tubular neighborhood $T_r(\bar{\mathscr{O}}_q)$ is compact, independent of δ, for $r < d_q$. If $B_r(q, g_\delta)$ denotes the ball of radius r about q, with respect to g_δ, clearly

$$B_r(q, g_\delta) \subset T_r(\bar{\mathscr{O}}_q), \tag{2.17}$$

and by (2.2),

$$\lim_{\delta \to 0} \operatorname{Vol}_\delta(B_r(q, g_\delta)) = \lim_{\delta \to 0} \operatorname{Vol}_\delta(T_r(\mathscr{O}_q)) = 0. \tag{2.18}$$

Let $V(c, s)$ denote the volume of the ball of radius s on the sphere of curvature $c = c(T_r(\bar{\mathscr{O}}_q))$, the constant in (2.4). It follows that for any $s < r$, we have

$$i_q(g_\delta) < s, \tag{2.19}$$

if δ is so small that

$$\operatorname{Vol}(B_s(q, g_\delta)) < \operatorname{Vol}(B_r(q, g_\delta)) \leqslant V(c, s). \tag{2.20}$$

Thus, (Y_n, g_δ) collapses.

A metric space X is said to be the *Hausdorff limit* (as $\delta \to 0$) of the family of metric spaces X_δ, if for all $\varepsilon_1, \varepsilon_2$ there exists $\delta(\varepsilon_1, \varepsilon_2)$ such that for $\delta < \delta(\varepsilon_1, \varepsilon_2)$ there are ε_1 dense sets $\{p_i(\varepsilon_1, \varepsilon_2, \delta)\}$ in X_δ and $\{p_i(\varepsilon_1, \varepsilon_2)\}$ in X with

$$\begin{aligned} (1 + \varepsilon_2)^{-1} \overline{p_i(\varepsilon_1, \varepsilon_2), p_j(\varepsilon_1, \varepsilon_2)} &\leqslant \overline{p_i(\varepsilon_1, \varepsilon_2, \delta), p_j(\varepsilon_1, \varepsilon_2, \delta)} \\ &\leqslant (1 + \varepsilon_2) \overline{p_i(\varepsilon_1, \varepsilon_2), p_j(\varepsilon_1, \varepsilon_2)}; \end{aligned} \tag{2.21}$$

see [9] for further discussion. Clearly, the Hausdorff limit of the family $g(x, y, \delta)$ on $B_r(0) \times B_s(0) \subset R_x^k \times R_y^{n-k}$ is $B_s(0)$, equipped with the metric corresponding to $(D(y))$. By definition, this is the metric on N^{n-k} for which the length of a vector is the length with respect to g of its projection orthogonal to $\mathcal{O}$. Up to isometry, it is independent of the choice of transversal, and is the unique metric on the *local quotient space* defined by (pieces of) the orbits of $\mathscr{P}$, for which the projection is a riemannian submersion.

If the orbits $\{\mathcal{O}\}$ are not closed, the *global quotient spaces* $X/\mathscr{P}$ is not Hausdorff. But we can still look at the quotient space, $X/\overline{\mathscr{P}}$, for the orbits of $\overline{\mathscr{P}}$. Since $\mathcal{O}$ is dense in $\overline{\mathcal{O}}$ it follows from (2.2) that

$$\lim_{\delta \to 0} \operatorname{diam}_\delta\left(\overline{\mathcal{O}}_q\right) = 0. \tag{2.22}$$

Thus, the Hausdorff limit of a compact subset $\overline{U}$ of Y, which is saturated for $\overline{\mathscr{P}}$, is $\overline{U}/\overline{\mathscr{P}}$, with the obvious quotient metric. Of course, this is not a smooth manifold near exceptional orbits.

Example 1.4 (*continued*). The polarization $\mathscr{P}_\gamma$ defined by the subgroup S_γ is closed if and only if γ is rational. If $1 \neq \gamma = p/q$ is rational, the Hausdorff limit $S^3/\mathscr{P}_\gamma$ is the surface of revolution, obtained by revolving the curve

$$y = \frac{1}{2} \frac{\sin x \cos x}{\left(p^2 \cos^2 x + q^2 \sin^2 x\right)^{1/2}}, \tag{2.23}$$

$0 \leqslant x \leqslant \pi/2$, about the x-axis. Thus it is a topological S^2 with two non-smooth points.

For γ irrational, $S^3/\overline{\mathscr{P}}_\gamma$ is the interval $[0, \pi/2]$.

Example 2.1 (*Tori*). The pure polarizations $\mathscr{P}(E^k)$ of the canonical T-structure on the standard torus, T^n, are parametrized by subspaces, E^k, of R^n which pass through the origin. When T^n is collapsed along E^k, of course

$$\operatorname{Vol}_\delta(T^n) = \delta^k \operatorname{Vol}_1(T^n). \tag{2.24}$$

But if one looks at (T^n, g_δ) up to homothety (i.e. isometry and scaling) it is a classical fact that the family (T^n, g_δ) corresponds to the image in the moduli space $\mathrm{SO}(n, R)\setminus \mathrm{SL}(n, R)/\mathrm{Sl}(n, Z)$ of a geodesic which goes to infinity in $\mathrm{SO}(n, R) \setminus \mathrm{SL}(n, R)$.

For example, if $n = 2$ and $(\mathscr{P}, E_\gamma^1)$ corresponds to a line of slope $\gamma = p/q$, then (T^2, g_δ) goes to infinity in $H^2/\mathrm{SL}(2, Z)$. However, if γ is irrational, the (T^2, g_δ) makes an infinite sequence of excursions which carry it successively further towards infinity, followed by returns to a fixed compact set. The precise behavior is determined by the continued fraction expansion of γ. Thus, for γ rational, up to scaling, (T^2, g_δ) becomes arbitrarily thin as $\delta \to 0$. For γ irrational, there exist δ for which (T^2, g_δ) is arbitrarily thin. But there are also arbitrarily small δ for which it is fat.

Example 2.2 (*Almost flat manifolds; see* [6]). The simplest (but quite typical) almost flat manifolds arise very naturally in our context. Let $S^1 \to N^3 \to T^2$ be a circle bundle with connection over T^2. If this bundle is topologically nontrivial, then N^3 does *not* have the fundamental group of a flat manifold. In fact, N^3 is a quotient of the Heisenberg group, and as such, is a nilmanifold (the fiber S^1 corresponds to the center). If T^2 is given a metric, the connection induces a metric on N^3, for which rotation through the angle θ in the fibers is an isometry. Choose the metric on T^2 to be flat and note that all fibers have the same length. Then by (2.13) and (2.14), for the collapse (N^3, g_δ) along the fibers, g_δ converges locally to a *flat* metric. In fact, for the sequence $(N^3, \delta^2 g_{\delta^{2(1+\varepsilon)}})$ (where $\varepsilon > 0$), both the curvature *and* the diameter approach zero, so that the limit of this collapse is a *point*.

Remark 2.1. It is easy to see that the calculation of Theorem 2.1 can be generalized to the case in which the abelian Lie algebra of Killing fields is replaced by a nilpotent Lie algebra. The latter is collapsed as in Example 2.2, rather than by scaling.

3. Polarized volume collapses

For collapses associated to a polarization for which all orbits are not of the same dimension, we will need a slight generalization of the calculation of Theorem 2.1. Suppose that in (2.2) we replace δ by a function ρ which is constant on orbits;

$$\rho = \rho(y_1, \cdots, y_{n-k}) > 0. \tag{3.1}$$

We fix attention on the origin $(0,0)$ in (x, y)-space and make the change of coordinates,

$$u_i = \rho(0) x_i. \tag{3.2}$$

Now we obtain

$$(g(u, y, \rho)) = \left(\begin{array}{c|c} \dfrac{\rho^2(v)\, A(y)}{\rho^2(0)} & \dfrac{\rho^2(v)\, B(y)}{\rho(0)} \\ \hline \dfrac{\rho^2(v)\, B(y)}{\rho(0)} & \rho^2(y) C(y) + D(y) \end{array}\right). \tag{3.3}$$

It follows that for, say, $|\rho| \leqslant 1$,

$$|K_\rho| \leqslant c(A, B, C, D, |\rho'/\rho|, |\rho''/\rho|), \tag{3.4}$$

where ρ' and ρ'' denote typical first and second partials of ρ with respect to $y, \cdots, y_{n-k}$

Theorem 3.1. *Let Y^n be compact, let $\mathscr{P}$ be a polarization of an F-structure, $\mathscr{F}$, on Y^n and let g be an invariant metric. Then there exists a family of metrics, g_δ, which are invariant for the F-structure defined by $\bar{\mathscr{P}}$, such that for $\delta \leqslant 1/2$,*

(1) (Y^n, g_δ) *is $c\delta$-collapsed,*

(2) $\operatorname{diam}(Y^n, g_\delta) \leqslant c|\log \delta|$,

(3) $\operatorname{Vol}(Y^n, g_\delta) \leqslant c\delta^k|\log \delta|^n$,

(4) $|K_\delta| \leqslant c$.

Proof. Let $\{U_\alpha\}$ be as in Definition 1.7 and let $f_\delta\colon U_\alpha \to [1/2, 1]$ be smooth functions such that $f_\alpha \equiv 1$ near U_α and

$$\bigcup_\alpha f^{-1}(1/2) = Y^n. \tag{3.5}$$

As in Lemma 1.4, we can assume that f_α is constant on every orbit $\mathscr{O}$ of $\bar{\mathscr{P}}$. Set

$$\rho_\alpha = \delta^{\log f_\alpha/\log 1/2}. \tag{3.6}$$

The metric g_δ will depend on a choice of ordering, $U_1, U_2, \cdots$ of the $\{U_\alpha\}$ (although its essential properties are independent of this choice). We start with the metric $\log^2\delta \cdot g$ and put

$$\log^2\delta \cdot g = g_1' + h_1 \tag{3.7}$$

on U_1, where the decomposition is as in (2.2) and h_1 vanishes on the orbits of $\mathscr{G}_1$. We then define g_1 by

$$g_1 = \begin{cases} \rho_1^2 g_1' + h_1, & U_1, \\ \log^2\delta g, & Y \setminus U_1, \end{cases} \tag{3.8}$$

where ρ_1 is as in (3.6). Proceeding by induction, we put

$$g_j = g_{j+1}' + h_{j+1}, \tag{3.9}$$

where h_{j+1} vanishes on the orbits of $\mathscr{G}_{j+1}$ and define g_{j+1} by

$$g_{j+1} = \begin{cases} \rho_{j+1}^2 g_{j+1} + h_{j+1}, & U_{j+1}, \\ g_j, & Y \setminus U_{j+1}. \end{cases} \tag{3.10}$$

We claim that

$$g_\delta = g_N \tag{3.11}$$

has the required properties. Note that (2) and (3) are obvious, and that (1) follows as in the proof of Theorem 2.1.

To see (4), let $p \in Y^n$ and $G_{p,\alpha} = G_p$, the stalk of $\mathscr{P}$. Let $(\underline{x}_1 \cdots \underline{x}_l, \underline{y}_1 \cdots \underline{y}_{n-l})$ $(l \geqslant k = \operatorname{rank} \mathscr{P})$ be coordinates as in (2.11) above, for the metric g such that $p = (0, 0)$. Thus, the $\underline{x}$-coordinates are constant along some transversal to the orbit $\mathcal{O}$ through p ($l = \dim \mathcal{O}$) and the y-coordinates are constant along the orbits of $\mathscr{g}(\tilde{U}_\alpha) = G_p$. We will keep track of the effect at p of the changes of metric corresponding to $j = 1, \cdots, N$ in coordinate systems derived from $(\underline{x}, \underline{y})$.

Observe that for $j = 1, \cdots, N$ the functions ρ_j depend only on $\underline{y}$ (and δ) since they are constant on orbits. Moreover, in the coordinate system

$$(3.12) \qquad x_i = \underline{x}_i \cdot \log \delta, \qquad y_i = \underline{y}_i \cdot \log \delta,$$

the matrix representing the metric $\log^2 \delta \cdot g$ has bounded partial derivatives (of all orders) and the functions $|\rho_j'/\rho|$, $|\rho_j''/\rho|$ ($j = 1, \cdots, N$) are *bounded independent of* δ (as is immediate from (3.7)).

Finally, we need only consider the effect at p of the changes of metric corresponding to those j for which $p \in U_j$. For such j, the orbits of $\mathscr{G}_j$ are contained in those of $\mathscr{G}_\alpha$ on $U_\alpha \cap U_j$.

Let β be the first value of j for which $p \in U_\beta$. By making a linear change of coordinates we can suppose that the orbits of $\mathscr{G}_\beta$ are given by $x_{t+1} =$ const, $\cdots, x_l =$ const, $y_1 =$ const, $\cdots, y_{n-l} =$ const, near p. We then introduce new coordinates $(u_1, \cdots, u_t, x_{t+1}, \cdots, x_l, y_1, \cdots, y_{n-l})$ as in (3.3). Since the ρ_j depend only on $y_1 \cdots y_{n-l}$ they have the same expressions as before. Thus, (4) follows by proceeding by induction.

Remark 3.1. The initial step in Theorem 3.1 in which distances are expanded in *all* directions by a factor $|\log \delta|$ is not optional, i.e. $\mathrm{Vol}(Y_n, g_\delta)$ does not always approach zero as rapidly as possible as $\delta \to 0$. This loss of sharpness is not very serious in the present context since at best one could replace $\mathrm{Vol}(Y_n, \delta) \sim \delta^k |\log \delta|^n$ by $\mathrm{Vol}(Y_n, \delta) \sim \delta^k$; compare Example 0.2 (continued). However, in Example 4.2 and in §5 we proceed more carefully.

4. Nonpolarized collapses

(a) Introduction. To be able to collapse when no polarization exists, we must:

(i) Describe a structure (referred to, somewhat informally, as a slice polarization) which replaces that of a polarization and which exists in general.

(ii) Check that there is a collapsing procedure based on this structure.

We begin by illustrating (i) and (ii) in an example which was considered in §1. Then we do the calculation behind (ii). Next we explain how the structure of (i) is constructed in general. Finally, we describe the collapse.

Example 1.7 (*continued*). Let $Z^2 \subset \mathscr{M}^4$ denote the union of the zero sections of the flat bundles $\mathscr{E}_\theta^3$, i.e. the 2-torus on which the dimension of the orbits drops from 2 to 1. The T-structure has nilpotent holonomy (given by the matrix (1.10)) in the direction of the θ-circle in Z^2 (the circle transverse to the orbits). As a consequence, the sub-bundle defined by the isotropy subgroups H_p, $p \in Z^2$, has no complementary flat sub-bundle. Equivalently there is no 1-dimensional polarization near Z^2.

However, for each fixed θ, the restriction of our structure to a 3-dimensional slice, $\mathscr{E}_\theta^3 \subset \mathscr{M}^4$, has no holonomy, and hence admits a 1-dimensional polarization. For example, on each slice we can choose the 1-parameter subgroup of the isometry group of $\mathscr{E}_\theta^3$ induced by parallel translation of $\mathscr{E}_\theta^3$ (see (1.6)). This family of "slice polarizations" varies continuously with θ (the corresponding family of infinitesimal generators gives rise to a vector field V on $\mathscr{M}^4$, which is tangent to $\mathscr{E}_\theta^3$ for each fixed θ, and such that $V|\mathscr{E}_\theta^3$ is a Killing field).

If the metric is collapsed in the direction of V, the curvature does *not* remain bounded, because V deviates from being a Killing field through its dependence on θ. To obtain a collapse with bounded curvature, we must simultaneously expand the metric in the θ direction (at an equal rate). This has the effect of making the above deviation negligible.

(b) Main computation. The essential quantitative features of nonpolarized collapse are captured by the following 3-dimensional situation. To simplify notation, we will only write the computation explicitly in this case.

Let $R^3 = R^2 \times R$, where the third coordinate is denoted by z. Let g be a riemannian metric on R^3 and let V be a nonvanishing vector field such that

(1) V is tangent to the slices, $z = \text{const}$.

(2) The restriction of V to any slice is a Killing field.

(3) There is an abelian Lie algebra, $\mathscr{F}$, of Killing fields such that if $X \in \mathscr{F}$, then X is tangent to every slice, $z = z_0$. Moreover, for each z_0, there exists $X_{z_0} \in \mathscr{F}$, with $X_{z_0}|_{z=z_0} = V|_{z=z_0}$.

It suffices to consider, say, $z_0 = 0$. Choose a local coordinate system (x, y, z) with $\partial/\partial x = X_0$.

To begin with we observe that

$$\left[\frac{\partial}{\partial x}, V\right] \equiv 0, \tag{4.1}$$

where [,] is the Lie bracket. In fact, for each fixed z_0,

$$\left[\frac{\partial}{\partial x}, X_{z_0}\right] \equiv 0, \tag{4.2}$$

since $\partial/\partial x = X_{z_0} \in \mathscr{F}$. But $\partial/\partial x$, V, X_{z_0} are all tangent to the slice $z = z_0$. So for $z = z_0$, the brackets in (4.1) and (4.2) can be computed in this slice, and there $V = X_{z_0}$.

Let

$$\frac{\partial}{\partial z} = U + N \tag{4.3}$$

be the decomposition of $\partial/\partial z$ into components tangent and normal to $z = z_0$ respectively. Since $\partial/\partial x$ is a Killing field tangent to $z = z_0$ we have

$$0 \equiv \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right], \quad 0 \equiv \left[\frac{\partial}{\partial x}, U\right], \quad 0 \equiv \left[\frac{\partial}{\partial x}, N\right]. \tag{4.4}$$

Now let (b_1^α, b_1^β) denote the components of $\partial/\partial x$ with respect to an orthonormal basis adapted to the decomposition $\{V\}$, $\{V\}^\perp \cap \{\partial/\partial x, \partial/\partial y\}$, $\{\partial/\partial x, \partial/\partial y\}^\perp$. Let (b_2^α, b_2^β), $(b_3^\alpha, b_3^\beta, b_3^\gamma)$ be the corresponding component functions for $\partial/\partial y$, $\partial/\partial z$. Notice, that all seven of these functions *do not depend on* x. This follows from (4.1), (4.4) and the fact that $\partial/\partial x$ is a Killing field. For example,

$$\frac{\partial}{\partial x}(b_3^\alpha) = \frac{\partial}{\partial x}\left(\frac{\langle U, V\rangle}{\langle V, V\rangle}\right) = \left\langle\left[\frac{\partial}{\partial x}, U\right], V\right\rangle \frac{1}{\langle V, V\rangle} + \cdots = 0. \tag{4.5}$$

Finally, observe that by (3),

$$b_1^\beta(y, 0) = 0, \tag{4.6}$$

since $\partial/\partial x|_{z=0} = V|_{z=0}$.

Let $g_{ij}(y, z, \delta)$ denote the metric obtained by the following operations (as above the subscripts $i, j = 1, 2, 3$ correspond to the variables x, y, z respectively).

$(*)$ Multiply the metric $g_{ij}(y, z)$ by the factor δ^{-2} in the direction $\{\partial/\partial x, \partial/\partial y\}^\perp$, while leaving it unchanged on $\{\partial/\partial x, \partial/\partial y\}$.

$(**)$ Multiply the metric obtained in $(*)$ by a factor δ^2 in the direction of $\{V\}$, while leaving it unchanged on $\{V\}^\perp$.

We have

$$g_{11}(y, z, \delta) = \delta^2 (b_1^\alpha)^2 + \left(b_1^\beta\right)^2, \tag{4.7}$$

$$g_{12}(y, z, \delta) = \delta^2 b_1^\alpha b_2^\alpha + \left(b_1^\beta b_2^\beta\right), \tag{4.8}$$

$$g_{22}(y, z, \delta) = \delta^2 (b_2^\alpha)^2 + \left(b_2^\beta\right)^2, \tag{4.9}$$

$$g_{13}(y, z, \delta) = \delta^2 b_1^\alpha b_3^\alpha + b_1^\beta b_3^\beta, \tag{4.10}$$

$$g_{23}(y, z, \delta) = \delta^2 b_2^\alpha b_3^\alpha + b_2^\beta b_3^\beta, \tag{4.11}$$

$$g_{33}(y, z, \delta) = \delta^2 (b_3^\alpha)^2 + \left(b_3^\beta\right)^2 + \frac{1}{\delta^2}(b_3^\gamma)^2. \tag{4.12}$$

Make the change of variables $u = \delta x$, $w = z/\delta$. In terms of these new coordinates

$$g_{11}(y, \delta w, \delta) = (b_1^\alpha)^2 + \left(\frac{1}{\delta} b_1^\beta\right)^2, \tag{4.13}$$

$$g_{12}(y, \delta w, \delta) = \delta b_1^\alpha b_2^\alpha + \left(\frac{1}{\delta} b_1^\beta\right) b_2^\beta, \tag{4.14}$$

$$g_{22}(y, \delta w, \delta) = \delta^2 (b_2^\alpha)^2 + (b_2^\beta)^2, \tag{4.15}$$

$$g_{13}(y, \delta w, \delta) = \delta^2 b_1^\alpha b_3^\alpha + \delta b_1^\beta b_3^\beta, \tag{4.16}$$

$$g_{23}(y, \delta w, \delta) = \delta^3 b_2^\alpha b_3^\alpha + \delta b_2^\beta b_3^\beta, \tag{4.17}$$

$$g_{33}(y, \delta w, \delta) = \delta^4 (b_3^\alpha)^2 + \delta^2 (b_3^\beta)^2 + (b_3^\gamma)^2. \tag{4.18}$$

In view of (4.6),

$$\lim_{\delta \to 0} \frac{1}{\delta} b_1^\beta(y, \delta w) = \frac{\partial}{\partial w} b_1^\beta(y, 0) w. \tag{4.19}$$

Moreover

$$\lim_{\delta \to 0} \left(g_{ij}(y, 0, \delta)\right) = \begin{pmatrix} (b_1^\alpha)^2 & 0 & 0 \\ 0 & (b_2^\beta)^2 & 0 \\ 0 & 0 & (b_3^\gamma)^2 \end{pmatrix} \tag{4.20}$$

which is positive definite. Thus, it is clear that the curvature stays bounded as $\delta \to 0$.

The calculation just given can be generalized. Since the details are straightforward, we will merely state the results.

(A) First of all, instead of the coordinates x, y, z we can as well have several coordinates $x_1, \cdots, x_{n_1}$, $y_1, \cdots, y_{n_2}$, $z_1, \cdots, z_{n_3}$ (where $x_1, \cdots, x_{n_1}$ correspond to $V_1, \cdots, V_{n_1}$). Moreover, we can collapse only, say, $V_1, \cdots, V_{m_1}$ (and make the changes of coordinates $u_1 = \delta x_1 \cdots u_{m_1} = \delta_m x_m$, $w_1 = z_1/\delta \cdots w_{n_3} = z_{n_3}/\delta$). Finally, we can artificially treat a subset of $y_1 \cdots y_{n_2}$ as z-coordinates, even though this is not required in order to keep the curvature bounded.

(B) As in (3.1)–(3.4), δ can be replaced by a function $\rho(y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}, \delta)$. The curvature of the collapsed metric depends on $|\rho'/\rho|$, $|\rho''/\rho|$; compare (3.4).

(c) Construction of slice polarizations. We now explain how the "slice polarizations" which are described in the continuation of Example 1.7 (at the beginning of this section) are obtained in general, starting with the case of a pure structure, $\mathscr{F}$. For this we must consider the orbit stratification associated to $\mathscr{F}$ and construct an invariant metric on Y_n and inner products in the stalks, G_p, which are suitably compatible with this stratification.

Let $\mathscr{F}$ be a pure F-structure of positive rank on Y_n and let g be an invariant metric. There is a natural stratification of Y_n into maximal strata, Σ_i, such that rank $\mathscr{F} = i$ for $p \in \Sigma_i$. Since the groups G_p are abelian, it follows that the identity components, H_p^0, of isotropy groups, H_p, are invariant under parallel translation along any curve in Σ_i. (Recall that the structure sheaf of $\mathscr{F}$ can be regarded as a flat bundle.) Each Σ_i is totally geodesic for the metric g, since locally it can be viewed as the set of common zeros of a collection of Killing fields.

Let $q \in \Sigma_k$ and let $(\Sigma_k)_q$ denote the tangent plane to Σ_k at q. Let $p \in \Sigma_i$ and consider the collection of subspaces of Y_p of the form $\lim_{q \to p} (\Sigma_k)_q = Q_k$ $(k > i)$. Since all groups G_p are abelian, it follows that the Q_k are coordinate hyperplanes relative to some fixed orthogonal basis of Y_p. Moreover, $\{Q_k\}$ is invariant under parallel translation in the normal bundle $\nu(\Sigma_i)$.

Let Σ_{ε_i} denote the set of points of Σ_i at distance $> \varepsilon_i$ from $\partial\Sigma_i$. Let exp be the exponential map of the normal bundle $\nu(\Sigma_{\varepsilon_i})$. For r_i sufficiently small, exp restricted to the subset $S_{\varepsilon_i, r_i} = \{v \in \nu(\Sigma_{\varepsilon_i}) | \|v\| < r_i\}$ is a diffeomorphism onto a set $\Sigma_{\varepsilon_i, r_i}$. Let $\pi_i\colon \Sigma_{\varepsilon_i, r_i} \to \Sigma_{\varepsilon_i}$ denote the corresponding projection map.

Lemma 4.1. *The invariant metric g and numbers ε_i, r_i can be chosen such that*

(1) $\bigcup \Sigma_{\varepsilon_i, r_i} = Y$.

(2) *If $i_1 < i_2$, then $\pi_{i_1} = \pi_{i_1}\pi_{i_2}$ on $\Sigma_{\varepsilon_{i_1}, r_{i_1}} \cap \Sigma_{\varepsilon_{i_2}, r_{i_2}}$.*

Proof. Start with any invariant metric g_0. Choose $\varepsilon_1 = 0$ and r_1 so small that $\exp|S_{\varepsilon_1, r_1}$ is a diffeomorphism onto $\Sigma_{\varepsilon_1, r_1}$. There is a *natural metric* on S_{ε_1, r_1} which is *flat on the fibers*, for which the subspace orthogonal to the fibers is given by the connection on $\nu(\Sigma_1)$ and for which projection onto the zero section is a riemannain submersion. Push this metric down to a metric g_1 on $\Sigma_{\varepsilon_1, r_1}$ via exp. Note that g_1 is compatible with $\mathscr{F}|\Sigma_{\varepsilon_1, r_1}$ and hence that $\Sigma_i \cap \Sigma_{\varepsilon_1, r_1}$ is totally geodesic for g_1.

It follows easily from the construction that (2) is satisfied on $\Sigma_{\varepsilon_1, r_1}$. Moreover (using (2)) it follows that near $\Sigma_i \cap \Sigma_{\varepsilon_1, r_1}$, the pullback of g_1 via the exponential map of $\partial(\Sigma_i)$, actually coincides with the natural metric of $\nu(\Sigma_i)$.

Now we can proceed by induction. Extend g_1 to an invariant metric for $\mathscr{F}$ on all of Y_n, choose $r_2 \ll \varepsilon_2 \ll r_1$, and replace the metric $g_1|\Sigma_{\varepsilon_2, r_2}$ with the push down of the natural metric on S_{ε_2, r_2}. Let g_2 be the metric on $\Sigma_{\varepsilon_1, r_1} \cup \Sigma_{\varepsilon_2, r_2}$ so obtained. By what was noted above, g_2 coincides with g_1 on $\Sigma_{\varepsilon_1, r_1}$. By proceeding in this way, we obtain the required metric.

Put $U_i = \Sigma_{\varepsilon_i, r_i}$. Let $q \in U_i$ and let γ be the unique minimal geodesic from q to $\pi_i(q)$. Parallel translation along γ induces an isomorphism $G_q \to G_{\pi_i(q)}$, which we will also denote by π_i.

Note that at each point p, the metric g of Lemma 4.1 induces a natural inner product on the Lie algebra, g_p, of G_p. For this, we identify a Killing field X with $(X(p), \nabla X(p))$ (we assume G_p acts effectively; see Remark 1.2). The resulting inner product is invariant under the local action of G_q, but not under the maps π_i: $G_q \to G_{\pi_i(q)}$.

Lemma 4.2. *There exists an inner product $\langle\ ,\ \rangle_q$ on g_q which is invariant under the local action of G_q and under the projection π_i (for $q \in U_i$).*

Proof. On Σ_1, define $\langle\ ,\ \rangle_q$ to be the inner product above. Extend $\langle\ ,\ \rangle_q$ to U_1 by making it invariant under π_1. In view of 2) of Lemma 4.1, $\langle\ ,\ \rangle_q$ is invariant under the local action of G_q and under π_i on $U_1 \cup U_i$. Clearly, we can extend $\langle\ ,\ \rangle_q | U_1 \cap \Sigma_{\varepsilon_2}$ to an inner product on g_q, for all $q \in \Sigma_{\varepsilon_2}$, which is invariant under the local action G_q. Then extend to U_2 by composing with π_2. This is consistent with $\langle\ ,\ \rangle_q$ as defined previously on $U_1 \cap U_2$. By proceeding in this way, we construct $\langle\ ,\ \rangle_q$ on all of Y with the desired properties.

For $p \in S_{\varepsilon_i, r_i}$, we let K_p^i denote the connected (but not necessarily closed) subgroup whose Lie algebra is the orthogonal complement of that of the isotropy group, H_p. For $q \in U_i$, we put

$$K_q^i = \pi_1^{-1}\big(K_{\pi_i(q)}\big). \tag{4.21}$$

It follows from Lemmas 4.1 and 4.2 that the assignment $q \to K_q^i$ is invariant under the local action of G_q. Moreover, if $q \in U_i \cap U_j$, $i \leqslant j$, then

$$K_q^i \subseteq K_q^j. \tag{4.22}$$

Finally, if $q_1, q_2 \in U_i \cap U_j$, $i \leqslant j$, and $\pi_i(q_1) = \pi_i(q_2)$, then $K_{q_1}^i = K_{q_2}^j$.

(d) Collapse. We can now collapse Y_n by a straightforward variant of the procedure of §3. Choose functions f_i, ρ_i on U_i as in (3.5) and (3.6). Fix q and let $U_{i_1}, \cdots, U_{i_j}$, $i_q < \cdots < i_j$, denote the U_k with $q \in U_k$. Let $Z_{i_1} \subset \cdots \subset Z_{i_j}$ denote the subspaces of Y_q tangent to the orbits of $K_q^{i_1} \cdots K_q^{i_j}$ and $W_{i_j} \subset \cdots \subset W_{i_1}$ the tangent spaces to $\pi_1^{-1}(\mathcal{O}_{\pi_{i_1}}(q)), \cdots, \pi_i^{-1}(\mathcal{O}_{\pi_{i_j}}(q))$. Then

$$Z_{i_1} \subset \cdots \subset Z_{i_j} \subset W_{i_j} \subset \cdots \subset W_{i_1}. \tag{4.23}$$

Let g be as in Lemma 4.1 and put

$$\log^2 \delta g = g_1^1 + h_1 + k_1, \tag{4.24}$$

where the decomposition (4.24) corresponds to $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Set

$$g_1 = \begin{cases} \rho_1^2 g_1^1 + h_1 + \rho_1^{-2} k_1, & U_1, \\ \log^2 \delta g, & Y \setminus U_1. \end{cases} \tag{4.25}$$

Define g_{j+1} by induction:

$$g_{j+1} = \begin{cases} \rho_{j+1}^2 g_{j+1}^1 + h_{j+1} + \rho_{j+1}^{-2} k_{j+1}, & U_{j+1}, \\ g_j, & Y \setminus U_{j+1}, \end{cases} \tag{4.26}$$

where the decomposition corresponds to Z_{j+1}, $Z_{j+1}^{\perp} \cap W_{j+1}$, $W_{j+1}^{\perp}$.

We claim that g_n $(n = \dim Y)$ collapses with bounded curvature as $\delta \to 0$ (where ρ_j depends on δ as in (3.6)).

To see that the curvature remains bounded, let $U_{i_1} \cdots U_{i_j}$ be those U_{i_k} with $q \in U_{i_k}$, and choose local coordinates near q as follows. Let

$$m_i = \dim \Sigma_i - i. \tag{4.27}$$

Choose local coordinates functions $s_1, \cdots, s_{m_{i_1}}$ on Σ_{i_1}, which are constant on the orbits. Extend these to $U_{i_1} \cap \cdots \cap U_{i_j}$ by composing with π_{i_1}. Next choose $s_{m_i+1}, \cdots, s_{m_{i_2}}$ on Σ_2 so that $s_1, \cdots, s_{m_{i_2}}$ are coordinates transverse to the orbits on Σ_2. Extend these to $U_{i_1} \cap U_{i_2}$ by composing with π_{i_2} (recall $\pi_{i_1} = \pi_{i_1}\pi_{i_2}$). By proceeding in this way, we obtain $s_1, \cdots, s_{m_{i_j}}$. Extend $s_1, \cdots, s_{m_{i_j}}$ to a complete system of local coordinates transverse to the orbit of $K_q^{i_j}$ through q, by choosing additional functions, $t_1, \cdots, t_{n-i_j-m_{i_j}}$, which are constant on the orbits of $K_p^{i_j}$ (for p near q). Finally, choose $x_1, \cdots, x_{i_k}$, $i = 1, \cdots, m$, such that for fixed $t_1, \cdots, s_{m_{i_j}}$, the fields $\partial/\partial x_1, \cdots, \partial/\partial x_{i_k}$ are Killing fields generated by the action of $K_q^{i_k}$.

$$\begin{aligned} z_1 &= s_1 \\ &\;\vdots \\ z_{m_{i_1}} &= s_{m_{i_1}} \end{aligned} \tag{4.28}$$

$$\begin{aligned} y_1 &= s_{m_{i_1}+1} \\ &\;\vdots \\ y_{m_{i_j}-m_{i_1}} &= s_{m_{i_j}} \end{aligned} \tag{4.29}$$

$$\begin{aligned} y_{m_{i_j}-m_{i_1}+1} &= t_1 \\ &\;\vdots \\ y_{n-i_j-m_{i_1}} &= t_{n-i_j-m_{i_j}}. \end{aligned} \tag{4.30}$$

The effect of the change of metric corresponding to U_i in this coordinate system is to collapse only $x_1, \cdots, x_{i_1}$ while expanding all directions orthogonal to $z_1 = \text{const}, \cdots, z_{m_{i_1}} = \text{const}$. The change corresponding to U_{i_j} collapses all $x_1 \cdots x_{i_j}$ directions while expanding directions normal to $y_1 = \text{const}, \cdots, y_{m_{i_j}-m_{i_1}} = \text{const}$, as well as $z_1 = \text{const}, \cdots, z_{m_{i_1}} = \text{const}$ (compare (A) which follows (4.20) above). The change corresponding to U_{i_k}, $1 < k < j$, has an effect intermediate between the two above. Thus, by successively changing coordinates as in (B) above and observing, as in §3, that $|\rho'_{i_k}/\rho_{i_k}|$, $|\rho''_{i_k}/\rho_{i_k}|$ remain bounded in the new coordinate systems, we see that the curvature of g_n is bounded independent of δ.

To see that g_n collapses as $\delta \to 0$, for each $p \in Y$ choose $r = r(\mathcal{O}_p)$ such that the exponential map of the normal bundle to $\mathcal{O}_p$ is a diffeomorphism when restricted to vectors of length r. Take a finite covering of Y^n by tubular open neighborhoods $T_{r_i/2}(\mathcal{O}_{p_i})$. For $q \in T_{r_i/2(\mathcal{O}_{p_i})}$ it follows that $\operatorname{dist}(q, \partial T_{r_i}(\mathcal{O}_{p_i}) > c(i) > 0$ for some $c(i)$ independent of δ. But through every such q passes a curve of length $c_1(i)\delta$, which is not contractible in $T_{r_i}(\mathcal{O}_{p_i})$. For $c_1(i)\delta < c(i)/2$, this implies that there is a closed noncontractible geodesic loop on q of length $< c_Y(i)\delta$. Hence $i(q) < c_1(i)\delta$.

Finally, we note that if m_i is as in (4.27) and we put

$$\kappa = \min_{U_{i_1} \cap \cdots \cap U_{i_j} \neq \varnothing} \left(i_1 - m_{i_1}\right) + \cdots + \left(i_j - m_{i_j}\right), \tag{4.31}$$

then

$$\operatorname{Vol}(Y^n, g_n(\delta)) \leqslant c\delta^{\kappa}|\log\delta|^n. \tag{4.32}$$

In particular, for this method of collapsing (which we indicate how to sharpen in §5) the volume goes either to infinity or to zero. In fact,

$$\lim_{\delta \to 0} \operatorname{Vol}(Y^n, g_n(\delta)) = 0 \tag{4.33}$$

if and only if for all i,

$$i - m_i > 0. \tag{4.34}$$

or equivalently,

$$i > \tfrac{1}{2}\dim \Sigma_i. \tag{4.35}$$

The procedure just described has a straightforward generalization to structures which are not pure. For this, we choose a regular atlas, $\{U_\alpha\}$, for a substructure of positive rank. Over each U_α we have a pure substructure, G_α (see Remark 1.2), and as in Lemma 1.3 there is a natural partial ordering among the U_α. Note that if $G_{p,\beta} \subset G_{p,\alpha}$, then the orbit stratification for the local action of $G_{p,\alpha}$ refines that for $G_{p,\beta}$. From this we easily obtain the existence of a metric g and inner product $\langle\ ,\ \rangle_q$ on g_q, with properties which generalize in the obvious way those of Lemmas 4.1 and 4.2.

Now on each U_α we collapse as above, except that we modify the cut off functions, ρ_i^α, in such a way that $\rho_i^\alpha \equiv 1$ in a small neighborhood of ∂U_α. By performing these collapses successively, we collapse Y^n with bounded curvature. Thus, we have the following result.

Let Y^n admit an F-structure of positive rank. Let $\{U_\alpha\}$ be a covering as above. Let $m_i^\alpha = \dim \Sigma_i^\alpha$, where Σ_i^α is defined by the action of G_α and put

$$\kappa = \inf_\alpha \kappa_\alpha, \tag{4.36}$$

where κ_α is defined as in (4.31).

Theorem 4.1. *There exists a family of invariant metrics, g_δ, on Y^n such that for $\delta \leqslant 1/2$.*

(1) (Y^n, g_δ) *is $c\delta$-collapsed.*

(2) $\operatorname{diam}(Y^n, g_\delta) \leqslant c|\log \delta|$.

(3) $\operatorname{Vol}(Y^n, g_\delta) \leqslant c\delta^\kappa |\log \delta|^n$.

(4) $|K_\delta| \leqslant c$.

Example 4.1 (*Nonpolarized volume collapse*). Let $R^2 \to \mathscr{M}^4 \xrightarrow{\pi} S^1 \times S^1_\theta$ be the space of flat bundles considered in Example 1.7, where S^1 denotes the zero section of $\mathscr{E}^3(\theta)$ and S^1_θ the circle which parametrizes the $\eta^3(\theta)$. If $\mathscr{M}_1, \mathscr{M}_2$ are 2-copies of $\mathscr{M}$, we can form

$$R^4 \to \mathscr{M}_1 \times \mathscr{M}_2 \xrightarrow{\pi_1 \times \pi_2} S^1 \times S^1_{\theta_1} \times S^1 \times S^1_{\theta_2}.$$

Let

$$T^3 = \{(x_1, \theta, x_2, \theta)\} \subset S^1 \times S^1_{\theta_1} \times S^1 \times S^1_{\theta_2} \tag{4.37}$$

and let $\mathscr{N}^7 = (\pi_1 \times \pi_2)^{-1}(T^3)$. The T-structure on $\mathscr{M}$ gives rise to an obvious nonpolarizable T-structure on $\mathscr{N}^7$ with orbits of dimension 2, 3, 4. The corresponding strata satisfy $\dim \Sigma_2 = 3$, $\dim \Sigma_3 = 5$, $\dim \Sigma_4 = 7$. Thus, (4.35) holds. By regarding $R^4 \to \mathscr{N}^7 \to T^3$ and letting Y^7 denote the double of the corresponding disc bundle, we obtain a specific example of a compact manifold which can be volume collapsed by means of a nonpolarizable T-structure.

Remark 4.1. The above Y^7 actually does admit a polarized T-structure. But probably there exist manifolds which can be volume collapsed although they admit no such structure.

The following example indicates how the construction of Theorem 4.1 can be sharpened.

Example 4.2 (*Collapsing M_F^4 with bounded volume*). The manifolds M_F^4 of Example 1.9 have F-structures which are of rank 2, except on Σ_1, which is the union of three connected codimension 2 submanifolds. These are either tori or Klein bottles depending on the particular choice of F. It will suffice to collapse tubular neighborhoods of the components of Σ_1, such that the volume stays finite and near the boundary the collapse agrees with the standard collapse of a pure rank 2 structure.

Let M^2 be a component of Σ_1. We start with a metric on the normal bundle, $\nu(M^2)$, which is cylindrical on the fibers. That is, on each fiber, R^2, it is of the form $dr^2 + f^2(r)d\theta^2$, where $f(r) \equiv 1$ for $r \geqslant 1$.

Given $\delta > 0$, we can construct a δ-collapsed metric on the disk-bundle, $0 \leqslant r \leqslant 1$, by means of a slice polarization. Thus, we multiply the metric by δ^2 on the subspace, X, tangent to the 1-dimensional orbits of the slice polarization and multiply the metric by δ^{-2} on the subspace W, orthogonal to the slices.

Let V be the orthogonal complement of X in the tangent space to the orbit. We extend the collapse to the annular region $1 \leqslant r \leqslant |\log \delta|$ by multiplying the metric in the direction of V by a factor $\rho^2(r)$, where $|\rho'/\rho|$, $|\rho''/\rho|$ are bounded, $\rho \equiv 1$ near $r = 1$ and $\rho \equiv \delta$ near $r = |\log \delta|$. Observe that the volume of this region is bounded independent of δ. Moreover, near $r = |\log \delta|$ we have the standard collapse of a pure structure. However, the metric is still expanded by a factor $\rho^2(r - |\log \delta| + 1)$ on the subspace W and for different components M_1^2, M_2^2 the subspaces W_1, W_2 do not correspond. Thus, we extend the collapse to the region $|\log \delta| \leqslant r \leqslant 2|\log \delta| - 1$ by multiplying the metric by a factor $\delta^2(r - |\log \delta| + 1)$ on the subspace W. It is easy to see that the curvature remains bounded independent of δ as does the volume.

By gluing the metrics just constructed onto the standard δ-collapsed metric for the rank 2 polarization on the remaining piece of M_F^4, we obtain the required δ-collapsed metric on M_F^4, with curvature and volume bounded independent of δ.

5. F-structures and complete metrics on open manifolds

(a) Introduction. In this section, we consider an open manifold, Y^n, which carries an F-structure, $\mathscr{F}$, or polarization, $\mathscr{P}$, on the complement of some compact subset. We treat in detail the case of a polarization, showing that Y^n admits a complete metric, g_∞, such that $|K_{g_\infty}| \leqslant 1$, $\mathrm{Vol}(Y^n, g_\infty) < \infty$. The analogous result for F-structures is the existence of a complete metric, g_∞, such that $|K_{g_\infty}| \leqslant 1$ and the injectivity radius goes uniformly to zero as $p \to \infty$; i.e. the family $Y^n \setminus B_R(q)$ collapses as $R \to \infty$. The proof of this latter result will be omitted since the ingredients which are required (beyond those of §4) will be presented in proving the existence of metrics of finite volume.

It is necessary to refine the constructions of the previous sections at two points.

Point 1. The invariant metric constructed in Lemma 1.3 is not guaranteed to have additional nice properties such as completeness, bounded curvature, etc. in case the manifold is open. Thus, we must begin by showing the existence of such an invariant metric, g_0, for a subpolarization $\mathscr{P}' \subset \mathscr{P}$. Moreover, this g_0 can be chosen such that $\mathscr{P}'$, when measured with respect to g_0, has essentially the same kind of uniform local behavior as in the compact case. This is achieved by making the metric grow sufficiently fast at infinity, and there is no attempt to control the volume at this stage.

Point 2. Rather than passing from g_0 to $\log^2\delta \cdot g_0$ (as in (3.27) and (4.24)) we will make a sequence of changes which expand g_0 by a factor $\log^2\delta_\alpha$ in a single (radial) direction near the boundary of each U_α'. The construction is such that the numbers $|\log\delta_\alpha|$ can be selected independently. If $c_\alpha = \mathrm{Vol}(U_\alpha, g_0)$, we choose $\{|\log\delta_\alpha|\}$ so small that

$$\sum_{\alpha=1}^{\infty} c_\alpha\delta_\alpha \cdot |\log\delta_\alpha| < \infty. \tag{5.1}$$

Then, by proceeding as in §3, we obtain

$$\mathrm{Vol}(Y^n, g_\infty) < c\sum_{\alpha=1}^{\infty} c_\alpha\delta_\alpha \cdot |\log\delta_\alpha|, \tag{5.2}$$

where g_∞ is the required metric.

(b) Construction of a complete metric g_0. We can assume that the polarization $\mathscr{P}$ is regular and that the boundaries $\{\partial U_\alpha\}$ are smooth and interest transversally. Moreover, after modifying the invariant metric, we can assume that g is such that

(1) The exponential map on the normal bundle, $\nu(\partial U_\alpha)$, is a diffeomorphism when restricted to vectors of length $\leqslant 2\varepsilon_\alpha$.

(2) Let $\underline{r}_\alpha$ denote the distance function from ∂U_α. Then on $T_{2\varepsilon_\alpha}(\partial U_\alpha) \cap T_{2\varepsilon_\beta}(\partial U_\beta)$

$$\langle \mathrm{grad}_g \underline{r}_\alpha, \mathrm{grad}_g \underline{r} \rangle_g \equiv 0. \tag{5.3}$$

(3) The sets $\{U_\alpha \setminus \overline{T_{2\varepsilon}(\partial U_\alpha)}\}$ cover a neighborhood of infinity.

Let $\mathscr{P}' \subset \mathscr{P}$ denote the polarization defined by $\{U_\alpha'\}$, where $U_\alpha' = U_\alpha \setminus T_{\varepsilon_\alpha}(\partial U_\alpha)$.

The construction of the metric g_0 is based on the following lemma which is essentially a restatement of Lemma 5.4 and Theorem 5.5 of [3]. Unfortunately, the presentation of these results in [3] was somewhat garbled due to a confusion between the functions $\bar{k}$ and $1/\bar{k}$ below. For this reason, we will repeat some of the details here.

Lemma 5.1. *Let $k(p)$ be a locally bounded nonnegative function on a riemannian manifold, Y^n, with possibly incomplete metric g. Then there exists a smooth function, k^*, such that*

(1) $k \leqslant 3k^*$.

(2) *If* $g_0 = (k^*)^2 g = e^{2\log k^*} g$, *then g_0 is complete with curvature* $|K(g_0)| \leqslant 1$ *and injectivity radius* $i_p(g_0) \geqslant 1$ *for all p. Moreover,*

$$\left\| \operatorname{grad}_{g_0} \log k^* \right\|_{g_0} \leqslant c(n), \tag{5.4}$$

$$\left\| \operatorname{Hess}_{g_0} \log k^* \right\|_{g_0} \leqslant c(n). \tag{5.5}$$

(3) *If Y^n carries regular polarizations $\mathcal{P}' \subset \mathcal{P}$ and the metric g as above, then g_0 can be chosen invariant for $\mathcal{P}'$.*

Proof. By increasing the function k if necessary, we can assume

$$k(p) > \sup_{\tau \in \Lambda^2(TY_p)} \left| K_g(\tau) \right|^{1/2}, \tag{5.6}$$

$$k(p) \geqslant 1/i_p(g). \tag{5.7}$$

$$k(p) > 1/\overline{p, \infty}, \tag{5.8}$$

$$k \not\equiv 0, \tag{5.9}$$

where $\overline{p, \infty}$ denotes the supremum of the radii of open metric balls at p whose closure is compact.

Put

$$\bar{k}(p) = \inf\left\{ \frac{1}{R} \,\middle|\, \sup_{q \in B_R(p)} k(q) \leqslant 1/R \right\}. \tag{5.10}$$

It follows directly from (5.10) that if $\lambda \geqslant 0$ and $\overline{p, q} \leqslant \lambda/\bar{k}(p)$, then

$$\frac{1}{1+\lambda} \bar{k}(p) \leqslant \bar{k}(q). \tag{5.11}$$

Moreover, if $\lambda < 1$,

$$\bar{k}(q) \leqslant \frac{1}{1-\lambda} \bar{k}(p) \tag{5.12}$$

((5.11) and (5.12) replace Lemma 5.4 of [3]). The construction of k^* now proceeds as in the proof of Theorem 5.5 of [3], but with the following proviso: $\bar{k}$ is to be replaced by $1/\bar{k}$, except in the expression $\bar{k}^2 g$. This gives (1) and (2).

(3) If $\mathcal{P}' \subset \mathcal{P}$ as above, the function $\bar{k}(p)$ need not be invariant for $\mathcal{P}$ at points $p \in U_\alpha$ with

$$\bar{k}(p) \leqslant 1/\operatorname{dist}_g(p, \partial U_\alpha). \tag{5.13}$$

However, if we put

$$\eta(p) = \sup_{p \in U'_\alpha} \varepsilon_\alpha \tag{5.14}$$

(where $U'_\alpha = U_\alpha \setminus \overline{T_{\varepsilon_\alpha}(\partial U_\alpha)}$) and require in addition to (5.9)–(5.12) that, say,

$$k(p) \geqslant 10\eta(p), \tag{5.15}$$

then $\bar{k}(p)$ is invariant for $\mathscr{P}'$. If we now combine the argument of [3] with a standard averaging argument, (3) follows.

Let $p \in Y^n$ and let $(\underline{z}_1, \cdots, \underline{z}_n)$ be a local coordinate system with p at the origin. Suppose that on the $\underline{z}$-coordinate ball, $B_\varepsilon(p)$, the matrix $(g_{ij}(z))$ for the metric g satisfies, say

$$\tfrac{1}{2} \leqslant \det g_{ij}(\underline{z}) \leqslant 2, \tag{5.16}$$

$$\left| g'_{ij}(\underline{z}) \right| \leqslant \Omega, \tag{5.17}$$

$$\left| g''_{ij}(\underline{z}) \right| \leqslant \Omega^2. \tag{5.18}$$

We choose $k^* \geqslant \max(1/\varepsilon, \Omega)$ and make the change of variables

$$z_i = k^*(0)\underline{z}_i. \tag{5.19}$$

Then the metric $g_0 = (k^*)^2 g$ satisfies

$$\tfrac{1}{9} \leqslant \det(g_0)_{ij}(z) \leqslant 12. \tag{5.20}$$

$$\left| (g_0)'_{ij}(z) \right| \leqslant c(n), \tag{5.21}$$

$$\left| (g_0)''_{ij}(z) \right| \leqslant c(n) \tag{5.22}$$

on the z-coordinate ball $B_1(0)$ (see (5.4), (5.5), (5.11), (5.12) and (1) above).

Remark 5.1. By making the function k grow sufficiently rapidly we can find at each point a coordinate system satisfyiing (5.20)–(5.22) in which the basic computation, (2.11)–(2.14), will apply.

(c) Expansion of g_0. Let $\mathscr{P}'$ be as above and put $I_\alpha = U'_\alpha \cap \overline{T_\varepsilon(\partial U'_\alpha)}$, where as in (5.3), the tubular neighborhood is with respect to the metric g. If $p \in I_1 \cap \cdots \cap I_l$ we can introduce a local coordinate system, $(\underline{x}, \underline{y}, \underline{r})$, near p, as follows. As usual, the fields $\partial/\partial\underline{x}_1, \cdots, \partial/\partial\underline{x}_k$ are Killing fields spanning the orbit $\mathscr{O}_p$ (with respect to $\mathscr{P}'$). The functions $\underline{r}_1, \cdots, \underline{r}_l$ are as in (5.3). In view of (5.3) we can find additional coordinates $\underline{y}_1, \cdots, \underline{y}_t$, such that the matrix of g for these coordinates is of the form

$$\begin{pmatrix} A(\underline{y}, r) & 0 \\ 0 & I \end{pmatrix}, \tag{5.23}$$

where I is the identity matrix and $A(x, y, \underline{r})$ represents the inner product on the subspace spanned by $\{\partial/\partial\underline{x}_1, \cdots, \partial/\partial\underline{y}_t\}$. As in Remark 5.1, we can assume that for the coordinate system

$$(5.24) \qquad x_i = k^*(p)\underline{x}_i, \quad y_i = k^*(p)\underline{y}_i, \quad \underline{r}_i = k^*(p)\underline{r}_i,$$

the matrix

$$(5.25) \qquad (g_0(x, y, r)) = \frac{(k^*(y, r))^2}{(k^*(p))^2}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

satisfies (5.20)–(5.22). Here

$$(5.26) \qquad A = A(y/k^*(p), r/k^*(p))$$

and the r_i take values which include the interval $(0, 1)$.

In order to expand g_0, we choose functions h_d: $[0,1] \to [1, \infty)$, each $d \in R^+$, such that

(i) $h_d \equiv 1$ on fixed intervals $[0, \varepsilon]$, $[1 - \varepsilon, 1]$.

(ii) $\int_0^1 h_d = d$.

(iii) The derivatives of the function $1/h_d$ are uniformly bounded independent of d.

Now on each subset I_α, multiply the metric in the direction of grad r_α by the function $h^2_{d_\alpha}(r_\alpha)$, while leaving it unchanged in the orthogonal directions. Call the new metric g_{ex}. The constant d_α will be specified below.

To see the effect of this change of metric on $I_1 \cap \cdots \cap I_l$, we make the change of variables

$$(5.27) \qquad \int_0^{r_i} h_{d_i}(v)\,dv = s_i, \qquad h_{d_i}(r_i)\,dr_i = ds_i,$$

$$(5.28) \qquad \frac{1}{h_{d_i}(r_i)}\frac{\partial}{\partial r_i} = \frac{\partial}{\partial s_i}.$$

Then using (iii) and (5.28) it follows that

$$(5.29) \qquad (g_0(x, y, s)) = \frac{(k^*(y, r))^2}{(k^*(p))^2}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

satisfies (5.20)–(5.22) (for some $c(n)$ independent of $d_1, \cdots, d_l$). Moreover, the functions, s_i, take values which include the interval $(0, d_i)$.

(d) Collapse of the expanded metric. Now choose nonincreasing functions ρ_d: $[0, d] \to [0, 1]$ such that $\rho_d \equiv 1$ on $[0, \varepsilon_1]$ and $\rho_d \equiv e^{-d}$ near d. These can be chosen such that $|\rho_d'/\rho_d|$, $|\rho_d''/\rho_d|$ are bounded independent of d (compare

(3.4)). Finally, by choosing $\varepsilon_1 < \varepsilon$ (where ε is as in (i)) we can arrange that

$$|h_d(r(x)) \cdot \rho_d(s)| \leqslant 1, \tag{5.30}$$

where r and s are related as in (5.27).

The function $\rho_{d_\alpha}(s_\alpha)$, defined near $\partial U'_\alpha$, has an obvious extension to all of Y^n. Start with the metric g_{ex} and successively multiply the inner product on the subspace tangent to the orbits of each $\mathscr{G}'_\alpha$ by the function $\rho^2_{d_\alpha}$. Call the resulting metric g_∞. In view of (5.30), the volume form of g_∞ is pointwise smaller than that of g_0. Now choose $d_\alpha = |\log \delta_\alpha|$, where δ_α is as in (5.1). Then it follows from (3) of the subsection above that $\mathrm{Vol}(Y^n, g_\infty) < \infty$.

Finally, since $\mathscr{P}'$ is regular, each point is contained in at most n different sets U'_α. So by Remark 5.1 and the bounds for (5.29) it is clear that $|K_{g_\infty}|$ is uniformly bounded. Thus, we have

Theorem 5.2. (1) *If Y^n admits a polarization $\mathscr{P}$ on the complement of a compact subset, C, then Y^n admits a complete metric, g_∞, invariant for some $\mathscr{P}' \subset \mathscr{P}$, with $|K_{g_\infty}| \leqslant 1$ and $\mathrm{Vol}(Y^n, g_\infty) < \infty$.*

(2) *If C is empty, Y^n admits a family, $g_{\infty,\delta}$, of such complete metrics, with $|K_{g_{\infty,\delta}}| \leqslant 1$ and $\lim_{\delta\to 0}\mathrm{Vol}(Y^n, g_{\infty,\delta}) = 0$.*

In the same way we have

Theorem 5.3. (1) *If Y^n admits an F-structure, $\mathscr{F}$, of positive rank of the complement of a compact subset, C, then Y^n admits a complete metric g_∞, invariant for some $\mathscr{F}' \subset \mathscr{F}$, such that $|K_{g_\infty}| \leqslant 1$ and $i_p \to 0$ uniformly as $p \to \infty$.*

(2) *If C is empty, Y^n admits a family, $g_{\infty,\delta}$, of such metrics, such that $|K_{g_{\infty,\delta}}| \leqslant 1$ and $(Y^n, g_{\infty,\delta})$ collapses.*

Remark 5.2. Clearly, a sharper statement of Theorem 5.2 is possible; see Theorem 4.1 and Example 4.2.

Appendix: Pure polarized structures on essential manifolds

Let X^n be a closed oriented manifold and let $f\colon X \to K(\pi, 1)$ be the classifying map, where $\pi \simeq \pi_1(X^n)$. We call X^n *essential* if the fundamental class, $[X^n] \in H_n(X^n, R)$, satisfies $f_*([X^n]) \neq 0$ (compare [6], [7]).

Theorem A.1. *Let $\mathscr{F}$ be a pure F-structure on an essential manifold X^n, such that the group which acts locally is isomorphic to a k-torus, T^k. Then $\dim \mathscr{O}_p p = k$ for all $p \in X^n$. Moreover, there exists a free normal abelian subgroup, $A^k \subset \pi_1(V)$ of rank k, whose action on the higher homotopy groups $\pi_i(X^n)$ $(i \geqslant 2)$ is trivial.*

Corollary A.2. *The connected sum $X^n \# M^n$, where M^n is an arbitrary n-dimensional manifold which is not a homology sphere, admits no pure F-structure of rank $\geqslant 1$.*

Example A.1. If $n = 2l + 1$ is odd, and M^n admits a (possibly nonpure) T-structure which is of rank l on some open set, then $T^n \# M^n$ also admits a (nonpure) T-structure. In fact, let $p \in T^n$, $q \in M^n$ lie on principal orbits $\mathcal{O}_p$, $\mathcal{O}_q$ of rank l. Let $T_\varepsilon(\mathcal{O}_p)$, $T_\varepsilon(\mathcal{O}_q)$ denote the small (saturated) tubular neighborhoods of p, q. If we form $T^n \# M^n$ by removing balls of radius $\varepsilon/2$ about p and q, we can regard $T^n \setminus T_\varepsilon(\mathcal{O}_p)$ and $M^n \setminus T_\varepsilon(\mathcal{O}_q)$ as contained in $T^n \# M^n$. On these sets, the T-structure on $T^n \# M^n$ can be taken to coincide with the restrictions of the given structures on T^n and M^n (compare [12]).

Proof of Theorem A.1. By Proposition 1.1, $\pi^*(\tilde{X}^E) \cong T^k$ acts on $\tilde{X}^E$, the holonomy covering. Let $\tilde{x} \in \tilde{X}^E$ and consider the orbit map $T^k \times \tilde{x} \to \tilde{\mathcal{O}}_x$. We claim that it suffices to show that the induced homomorphism

$$\mathbf{Z}^k = \pi_1(T^k) \xrightarrow{i_*} \pi_1(\tilde{X}^E, \tilde{x}) \subset \pi_1(X, x) \tag{A.1}$$

is injective. To see this, note that if $\dim \mathcal{O}_x < k$ for some x, then $\ker i_*$ contains the image in $\pi_1(T^k)$ of $\pi_1(H_x)$, where H_x denotes the isotropy group of x. For the second assertion, we observe that it is well known and easy to see that $i_*(\pi_1(T^k)) \subset \pi_1(X, x)$ is central and acts trivially on $\pi_i(X, x)$, $i \geqslant 2$.

Let $T^l_* \subset T^k$ denote the unique sub-torus commensurable with $\ker i_*$. Then T^l_* defines a substructure, $\mathcal{F}^*$, with the following property. for each orbit, $\mathcal{O}^*_p$, of $\mathcal{F}_1^*$ there is a finite covering $\tilde{\mathcal{O}}^*_x \to \mathcal{O}^*_x$ such that the induced map $\pi_1(\tilde{\mathcal{O}}^*_x) \to \pi_1(X)$ is the zero map (see below for further details).

Suppose first that $\tilde{\mathcal{O}}^*_x = \mathcal{O}^*_x$ for all x, and so $\pi_1(\mathcal{O}^*_x) \to \pi_1(X)$ is the zero map. Then if $X/\mathcal{F}^*$ denotes the orbit space, it follows that the induced map $p_*\colon \pi_1(X) \to \pi_1(X/\mathcal{F}^*)$ is an isomorphism. In fact, since the inverse image, ω^*_x, of each point in $X/\mathcal{F}^*$ is connected, p_* is surjective. Moreover, $\ker p_*$ is spanned by the normal subgroup generated by $\bigcup_x [Im(\pi_1(\mathcal{O}^*_x)) \subset \pi_1(X)] = 0$.

Since homotopy classes of maps $f\colon X \to K(\pi_1)$ are in 1-1 correspondence with (conjugacy classes of) homomorphisms, $f_*\colon \pi_1(X) \to \pi_1(K(\pi_1)) = \pi$, it follows that f is homotopic to $\tilde{f} \circ p$ for some $\tilde{f}\colon X/\mathcal{F}^* \to K(\pi, 1)$. Since X is essential, we can assume that $(\tilde{f} \circ p)_*$ is not the zero map. But then $H_n(X^n/\mathcal{F}^*) \neq 0$, which is possible only if $\dim T^l_* = 0$. Thus $\ker i_* = 0$ and the theorem follows in this case.

If $\tilde{\mathcal{O}}^*_x \neq \tilde{\mathcal{O}}_x$ for some x, the idea is similar but requires some further technical elaboration. Let $U(x)$ be a small equivariant tubular neighborhood of x. By passing to a finite covering, $\tilde{U}'_x \to U_x$, we can assume that the lifted orbit, $\tilde{\mathcal{O}}^{*\prime}_x$, is induced by the action of T^l_* (see Definition 1.2). Note that T^l_* is only commensurable with $\ker i_*$ and that $\tilde{\mathcal{O}}_x^{*\prime}$ might be a multiple orbit in its stratum. Thus, $\pi_1(\tilde{\mathcal{O}}_x^{*\prime}) \to \pi_1(\mathcal{O}_x^*)$ need not be the zero map. But after passing to a finite covering, $\tilde{\mathcal{O}}_x^* \to \tilde{\mathcal{O}}_x^{*\prime}$, we can assume that is the case for $\pi_1(\tilde{\mathcal{O}}_x^*) \to \pi_1(\mathcal{O}_x^*)$. Moreover, if $\tilde{U}_x \to U_x$ denotes the corresponding covering of U_x, then the same holds for any $\tilde{\mathcal{O}}_y^* \subset \tilde{U}_x$, since the inclusion, $\tilde{\mathcal{O}}_x^* \to \tilde{U}_x$, is a homotopy equivalence.

In order to make use of the covering spaces, $\tilde{U}_x \to U_x$, we need the following lemma.

Lemma A.3. *Let Y^n be a closed manifold which is the union of open submanifolds $U_1^n \cdots U_m^n$, whose (smooth) boundaries $\{\partial U_Y^n\}$ intersect transversally. Let π_j: $\tilde{U}_j \to U_j$ be finite coverings. Then there exists an n-dimensional polyhedron $\overline{Y}^n$ and a continuous map g: $\overline{Y}^n \to Y^n$, such that*

(1) g_*: $H_*(\overline{Y}^n, Q) \to H_*(Y^n, Q)$ is surjective.

(2) If $C_j \subset U_j^n$ is closed, the map $g|g^{-1}(C_j) \subset Y^n$ factors through a map h: $g^{-1}(C_j) \to \tilde{U}_j$.

Proof. Let Z be an arbitrary topological space, $U \subset Z$ an open subset and π_1: $\tilde{U} \to U$ a finite covering map. Denote by $\overline{Z} = X \vdash \tilde{U}$ the set $(Z \setminus U) \cup \tilde{U}$, and by $\bar{\pi}$: $\overline{Z} \to Z$ the obvious map. Define the topology in $\overline{W}$ by the condition that $A \subset \overline{Z}$ is closed if and only if $\bar{\pi}(A) \subset Z$ is closed. It is easy to see that $\bar{\pi}$ is surjective on rational homology and that the covering π factors through a unique map $\tilde{U} \to Z$.

Now specialize to the case of a closed manifold $Z = Y = \bigcup_1^m U_j$ as above. Put $Y' = Y \vdash \tilde{U}_1$, $U_j' = \pi_1^{-1}(U_j)$, $j = 2, \cdots, m$, and let π_j': $\tilde{U}_j' \to U_j'$ be the covering maps induced by $\bar{\pi}_1$ from κ_j. Then take $Y'' = Y' \vdash \tilde{U}_2'$, $Y''' = Y'' \vdash \tilde{U}_2''$, etc. After m steps, we reach the required polyhedron $Y^{(m)} = \overline{Y}$.

To complete the proof of Theorem A.1, we consider some sufficiently fine open covering, $\{U_j\}$, of Y by *saturated* open subsets with smooth boundary such that $\{\partial U_j\}$ intersect transversally; compare Lemma 1.4. The local actions on $\tilde{U}_i$ induce corresponding actions on $\overline{Y}$. As in the special case considered above, we want to show that if f: $X \to K(\pi, 1)$, then up to homotopy, $f \circ g$: $\overline{X} - K(\pi, 1)$ lifts to $\tilde{f} \circ p \circ g$. But it follows as above, that if $(p \circ g)_*$: $\pi_1(\overline{X}) \to \pi_1(X/\mathscr{F}^*)$ and $(f \circ g)_*$: $\pi_1(\overline{X}) \to \pi_1(K(\pi, 1))$, then $\ker(f \circ g)_* \subset \ker(p \circ g)_*$. This implies that the desired lift exists and suffices to complete the proof.

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COLLAPSING RIEMANNIAN MANIFOLDS WHILE KEEPING THEIR CURVATURE BOUNDED. II

JEFF CHEEGER & MIKHAEL GROMOV

0. Introduction

This is the second of two papers concerned with the situation in which the injectivity radius at certain points of a riemannian manifold is "small" compared to the curvature.

In Part I [3], we introduced the concept of an F-*structure* of *positive rank*. This generalizes the notion of a torus action, for which all orbits have positive dimension. We showed that if a compact manifold, Y^n, admits an F-structure of positive rank, then it also admits a family of riemannian metrics, g_δ, whose sectional curvatures are uniformly bounded independent of δ and for which the injectivity radius, $i_y(g_\delta)$ goes uniformly to zero at all points $y \in Y^n$, as $\delta \to 0$. Such a sequence is said to *collapse with bounded curvature* (see Part I for variants and refinements of the above result).

In the present paper, we prove a kind of strengthened converse to the collapsing theorem. If $y \in Y^n$, let $|K(y)|$ denote the maximum of the absolute value of the sectional curvature over $\tau \in \Lambda^2(T_y(Y^n))$.

Theorem 0.1. *There exist constants $c_1(n), c_2(n) > 0$ such that if Y^n is a complete riemannian manifold, then $Y^n = Y_F^n \cup Y_G^n$, where*

(1) *Y_F^n is an open set which admits an F-structure of positive rank, whose orbits, $\mathscr{O}_y$, have diameter satisfying* $\operatorname{diam}(\mathscr{O}_y) \le c_1(n) i_y$,

(2) *for all $y \in Y_G^n$, there exists w in the ball $B_{i_y/c_2(n)}(y)$ with*

$$|K(w)|^{1/2} i_y \ge c_2(n). \tag{0.2}$$

Remark 0.3. For the F-structure we construct, the local actions almost preserve the metric. By applying Lemma 1.3 of [3], we can replace the metric on Y^n by a nearby metric which is invariant for the F-structure on Y_F^n.

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Remark 0.4. The set Y_F^n can be taken to be the interior of a submanifold with boundary.

Remark 0.5. The constants $c_1(n)$ and $c_2(n)$ can be estimated explicitly, although we do not do this here. But there is one point in our construction, Proposition 3.4, which is considerably easier to treat by a noneffective argument based on the compactness theorem in riemannian geometry [2], [13], [11], [17]. For completeness, we indicate a second proof of Proposition 3.4, which yields explicit constants.

Remark 0.6. If $|K(y)|$ is uniformly bounded, say $|K(y)| \leq 1$, then the set Y_G^n has *bounded geometry.* In this case, roughly speaking, by the compactness theorem, all geometrical and topological measurements of Y_G^n can be estimated in terms of its size. Thus, the thrust of Theorem 0.1 for the case of bounded curvature is that Y^n admits a decomposition into two pieces, on each of which there is a certain kind of control. Earlier versions of this decomposition were known to Margulis (unpublished) for manifolds of negative curvature, in which case they can be obtained much more directly by special arguments; see [18] for an exposition in the case of 3-manifolds of constant negative curvature.

Remark 0.7. The hypothesis of completeness in Theorem 0.1 is just a convenience since, for an arbitrary manifold, the same decomposition holds sufficiently far from $\overline{Y}^n \backslash Y^n$ (here $\overline{Y}^n$ is the completion of Y^n).

Remark 0.8. Although there is an essentially canonical set of choices for the F-structure on M_F (which are dictated by the local geometry) there is a certain ambiguity in the construction which cannot be entirely removed. In fact, if the F-structure were uniquely determined, it would vary continuously with the local geometry. Then, of necessity, it would always be *pure* (see Part I, §1). But this would contradict the results of Part I (see Theorems 4.1 and A.1).

By combining Theorem 0.1 with the main results of Part I [3] we obtain corollaries such as the following.

Corollary 0.9. (*Critical radius*) *If a compact manifold Y^n admits a metric which is sufficiently collapsed at all points* (*say* $|K(y)| \leq 1$, $i_y < c_2(n)$), *then Y^n admits a family of metrics which collapses with bounded curvature.*

The proof of Theorem 0.1 will be given in the remaining sections.

F-structures are discussed in §1.

An *F-structure*, $\mathscr{F}$, on U consists of a *sheaf*, $\mathscr{t}$, on U whose stalk, $\mathscr{t}_x$, at each point $x \in U$, is isomorphic to some torus and a *local action*, μ, of $\mathscr{t}$ on U, for which certain additional conditions are satisfied.

Suppose we are given a finite normal covering $\tilde{U}$ of U and a representation $\rho\colon \Gamma \to \mathrm{Sl}(k, \mathbb{Z})$ of the covering group Γ. Then ρ determines a flat T^k bundle, $\mathscr{f}$, over U. Given an action of the semidirect product, $\Gamma \times_\rho T^k$, on $\tilde{U}$, which extends the action of Γ, we obtain a local action μ of $\mathscr{f}$ on U. The pair $(\mathscr{f}, \mu)$ determines a so-called *elementary F-structure*, $\mathscr{F}$.

Typically, an F-structure is specified by a locally finite collection of open sets, $\{V_\alpha\}$, each of which carries an elementary F-structure, $\mathscr{F}_\alpha$. On nonempty intersections, $V_\alpha \cap V_\alpha$, we require that $\mathscr{f}_\alpha$ agrees with a sub-bundle of $\mathscr{f}_\beta$, or vice versa, that the corresponding local actions agree, and that $V_\alpha \cap V_\beta$ is saturated for the local action of the larger of $\mathscr{f}_\alpha$, $\mathscr{f}_\beta$. In this situation, $\mathscr{f} = \bigcup_\alpha \mathscr{f}_\alpha$.

There is a stability result for elementary structures which follows from a simple generalization of the stability theorem for compact group actions. As a consequence, a collection, $\{(V_\alpha, \mathscr{F}_\alpha)\}$ as above, for which the corresponding local actions on intersections only agree to a high degree of approximation, can be perturbed to one which determines an F-structure. This observation (see Lemma 1.5) provides the framework for the proof of Theorem 0.1. (Actually, Lemma 1.5 will be formulated in terms of the concept of *weak F-structure*, since this turns out to be more convenient for the application to the proof of Theorem 0.1; see §1 for details.)

In proving Theorem 0.1, first we find a covering of the sufficiently collapsed part of Y^n by a collection of sets which are the homeomorphic images of certain subsets of complete flat manifolds. The homeomorphisms are almost isometries. Then, we transfer to Y^n, certain elementary F-structures which are defined over these subsets. Finally, we fit together the transferred elementary F-structures, using Lemma 1.5.

The relevant discussion of elementary F-structures on complete flat manifolds is given in §2. First we describe a class of elementary F-structures of positive rank, which are carried by a noncontractible flat manifold, X^n; for these manifolds $|K(x)|^{1/2} \cdot i_x \equiv 0$. Each such structure is determined by a union of conjugacy classes, $\{\gamma_j\}$, of geodesic loops γ_j. The γ_j lie in the canonical normal abelian subgroup, $A \subset \pi_1(X^n)$, whose existence follows from the Bieberbach Theorem (and the Soul Theorem). In particular, a loop γ lies in A if the rotational angles of its holonomy are not too big.

Next we describe the elementary F-structures which are utilized in the proof of Theorem 0.1. Each of these is specified by a collection of loops at x which lie in A, with the following property. A loop γ is in the collection if and only if every loop at x with the same length and isomorphic holonomy transformation is also included. These collections are not necessarily invariant under conjugation by elements of $\pi_1(X^n)$ and, in general, the corresponding elementary F-structures are only defined over proper open subsets, $V \subset X^n$.

In §3, we show that if $|K(w)|^{1/2} \cdot i_y$ is sufficiently small for w near $y \in Y^n$, then we can find an open neighborhood U of y, a complete flat manifold $\mathbf{Y}^n$ and a quasi-isometry $f\colon U \to T_u(S^m)$. Here $T_u(S^m)$ is the u-tubular neighborhood of a soul, $S^m \subset \mathbf{Y}$. The quasi-isometry, f, is almost an isometry if $|K(w)|^{1/2} \cdot i_y$ is sufficiently small.

In §4, this approximation is regularized so that holonomies P_γ and P_γ of corresponding loops γ and γ in U and $T_u(S^m)$ are close if the loops are not too long.

With the results of §§3 and 4, we can transfer an elementary F-structure from a subset of $T_u(S^m)$ to a subset of U. Moreover, a structure so obtained has an approximate description in terms of geodesic loops of Y^n.

The proof of Theorem 0.1 is carried out in §5, by implementing Lemma 1.5.

If $y \in Y^n$ is a point such that $|K(w)|^{1/2} i_y$ is small for w near y, then there exist various local flat approximations to (Y^n, y) as in §§3 and 4. To each such point y, we assign a flat approximation $f_y\colon U_y \to T_{u_y}(S_y)$, a thin subset V_y, with $y \in V_y \subset U_y$, and an elementary F-structure, $\mathscr{F}_y$, as above, over V_y.

The main point is to make these choices such that on all intersections, $V_{y_1} \cap V_{y_2}$, either $\mathscr{f}_{y_1} \supseteq \mathscr{f}_{y_2}$ or vice versa. This condition is called property (F_1); compare the discussion above, of the contents of §1.

Since the corresponding local actions for both $\mathscr{f}_{y_1}$ and $\mathscr{f}_{y_2}$ have an approximate description in terms of geodesic loops of Y, these actions will be close if the maps f_{y_1} and f_{y_2} are sufficiently close to being isometries. In fact, were it not for the fact that $\{V_y\}$ has infinite multiplicity, $\{(V_y, \mathscr{F}_y)\}$ would actually satisfy the hypothesis of Lemma 1.5.

Thus, if we choose a locally finite subcollection, $\{V_{y_\alpha}\}$, with suitably bounded multiplicity, then the full hypothesis of Lemma 1.5 is satisfied for the collection $\{(V_{y_\alpha}, \mathscr{F}_\alpha)\}$ and we obtain a weak F-structure (of positive rank). Our particular method of selecting $\{(V_y, \mathscr{F}_y)\}$ (which guarantees that property (F_1) holds) will also enable us to conclude that our weak F-structure is actually an F-structure.

A more detailed outline of the argument is given at the beginning of §5.

In the Appendix to §2 we give some examples which show that the elementary F-structures discussed in §2 which are defined over all of X^n do not satisfy the hypothesis of §1, since the size of their orbits grows too rapidly at infinity.

Let us mention that by replacing the compactness theorem used in §3 by one proved recently by M. Anderson (see his preprint "Convergence and Rigidity of Manifolds under Ricci Curvature Bounds") the hypothesis of Theorem 0.1 can be replaced by the following assumptions: In (0.2), one can substitute "Ricci curvature" for "sectional curvature," provided one also assumes that for some sufficiently small constant, $c_3(n)$,

$$\int_{B_{i_y/c_3(n)}(y)} |R|^{n/2} < c_3(n). \tag{0.10}$$

Finally, we point out that K. Fukaya has obtained a number of remarkable results on collapsing in the case of bounded curvature and diameter; see [7]–[10]. His techniques are rather different from those employed here and in [4]. In recent joint work with Fukaya, a common generalization of a portion of his work and ours is obtained by combining the two approaches.

1. F-structures and their stability

Before beginning we recall an elementary fact which is used (sometimes without further mention) in this section and the next.

Let G be a connected topological group which acts on a space Z. Then this action lifts (necessarily uniquely) to the action of a covering group, $\tilde{G}$, on a covering space, $(\tilde{Z}, \tilde{z})$, if and only if

$$(\phi_z)_*(\pi_1(\tilde{G}, \tilde{e})) \subset \pi_1(\tilde{Z}, \tilde{z}) \subset \pi_1(Z, z),$$

where $\phi_z(g) \stackrel{\text{def}}{=} g(z)$.

Equivalently, let $\tilde{G}$, the universal covering of G, act on $\tilde{Z}$, the universal covering of Z. If $G = \tilde{G}/H$ and $Z = \tilde{Z}/\Gamma$ then the action of $\tilde{G}$ on $\tilde{Z}$ descends to an action of G on Z if and only if the action of G normalizes that of Γ and $H \subset \Gamma$.

For the convenience of the reader, we begin by reviewing some definitions from [3] (to which we refer for further details).

A *partial action*, A, of a topological group, G, on a Hausdorff space, X, is given by

(1) a neighborhood $\mathscr{D} \subset G \times X$ of $e \times X$, where e is the identity of G, and a continuous map $A: \mathscr{D} \to X$, also written $(g, x) \to gx$, such that

(2) $(g_1 g_2)x = g_1(g_2 x)$ whenever $(g_1 g_2, x)$ and $(g_1, g_2 x)$ lie in $\mathscr{D}$, and such that $ex = x$ for all x.

Two partial actions $(A_1, \mathscr{D}_1)$ and $(A_2, \mathscr{D}_2)$ are called equivalent if there is a neighborhood $\mathscr{D} \subset \mathscr{D}_1, \mathscr{D}_2$ containing $e \times X$, such that $A_1|\mathscr{D} = A_2|\mathscr{D}$. A *local* action, $\{A\}$, is an equivalence class of partial actions.

Assume G is connected.

A subset $X_0 \subset X$ is called $\{A\}$-invariant if for some (equivalently, any) representative we have $gx \in X_0$ for all $x \in X_0$ with $(g, x) \in \mathscr{D}$. It is easy to see that the X is partitioned into minimal invariant sets called *orbits*. Let $\mathscr{O}_x$ denote the orbit of X.

A local action can be restricted to any open set $U \subset X$ by restricting the domain, $\mathscr{D}$, of some representative to $\mathscr{D}' \supset e \times X$, such that $gx \in U$ for $(g, x) \in \mathscr{D}'$. Similarly a local action can be pulled back under a locally homeomorphic map.

Now consider a sheaf, $\mathscr{g}$, of connected topological groups over X. Let $\mathscr{g}(U)$ denote the group of sections over U. An *action* of $\mathscr{g}$ on X is a local action of $\mathscr{g}(U)$ on U, for every connected open set $U \subset X$, such that the structure homomorphisms $\mathscr{g}(U) \to \mathscr{g}(U')$ (for $U' \subset U$) commute with the restriction of local actions.

A set is *invariant* if its intersection with U is invariant for all U. Again, X is partitioned into minimal invariant subsets called *orbits*. A set is called *saturated* if it is a union of orbits. The *rank* of the action at $x \in X$ is the dimension of the orbit, $\mathscr{O}_x$. The action has *positive rank* if $\dim \mathscr{O}_x > 0$, for all $x \in X$.

An action of $\mathscr{g}$ is called *complete* if for all $x \in X$ there is an open neighborhood, $V(x)$, of x and a locally homeomorphic map, $\tilde{V}(x) \to V(x)$ ($\tilde{V}(x)$ Hausdorff), such that:

(1) If $\pi(\tilde{x}) = x$, then for any open neighborhood $W \subset \tilde{V}(x)$ of $\tilde{x}$, the structure homomorphism, $\pi^*(\mathscr{g})(W) \to \mathscr{g}_{\tilde{x}} \stackrel{\text{def}}{=} \mathscr{g}_x$ is an isomorphism.

(2) The local action of $\pi^*(\mathscr{g})$ comes from a global action of $\pi^*(\mathscr{g})(\tilde{V}(x)) = \mathscr{g}_{\tilde{x}}$.

Definition 1.1. A $\tilde{\mathscr{G}}$*-structure* on X is given by the complete action of a sheaf of connected topological groups, $\mathscr{g}$, on X, such that the neighborhood, $\tilde{V}(x)$, can be chosen to satisfy:

(1) $\pi: \tilde{V}(x) \to V(x)$ is a normal covering.

(2) For all x, $V(x)$ is saturated.
(3) For an orbit, $\mathscr{O}$, if $x, y \in \mathscr{O}$, then $V(x) = V(y)$.

Definition 1.2. A $\tilde{\mathscr{G}}$-structure is called an *F-structure* if

(1) For all x, the stalk, $\mathscr{g}_x$, is isomorphic to a torus.
(2) For all x, the normal covering, $\tilde{V}(x) \to V(x)$, can be chosen to be finite.

A structure satisfying (1) and (3) of Definition 1.2 (but not necessarily (2)) is called a *weak $\tilde{\mathscr{G}}$-structure.* A weak $\tilde{\mathscr{G}}$-structure which satisfies the additional conditions of Definition 1.2 is called a *weak F-structure.*

We emphasize that the existence of a weak F-structure of positive rank does *not* guarantee that we can perform the collapsing constructions of [3]. However, we will formulate Lemma 1.5 in terms of this concept, since this turns out to be convenient for the application to the proof of Theorem 0.1.

For the remainder of this section we restrict attention to F-structures (although everything we say generalizes to $\tilde{\mathscr{G}}$-structures).

Definition 1.3. An F-structure is called *elementary* if $\tilde{V}(x) \to V(x)$ can be chosen independent of x.

Note that in Definition 1.3, necessarily, we have $V(x) = X$. Also, as indicated in the introduction, the concept of elementary F-structure can be reformulated as follows.

Suppose we are given

(1) a (possibly disconnected) finite normal covering, $\tilde{X} \to X$, with covering group Γ,
(2) a representation, $\rho \colon \Gamma \to \mathrm{Aut}(t^k)$, for some torus T^k,
(3) an action of the semidirect product, $\Gamma \times_\rho T^k$, extending the action of $\gamma \subset \Gamma \times_\rho T^k$.

The above data determines an elementary F-structure, $\mathscr{F}$ on X, for which the sheaf, $\mathscr{f}$, is the associated flat bundle on X, with fiber isomorphic to T^k and holonomy representation isomorphic to ρ. The action of $T^k \subset \Gamma \times_\rho T^k$ on $\tilde{X}$ determines an obvious action of $\mathscr{g}$ on X.

For $\mathscr{F}$ as above, let $\mathscr{f}' \subset \mathscr{f}$ be a sub-bundle with fiber $T^{k'} \subset T^k$. Then the action of $\mathscr{f}$ restricts to an action of $\mathscr{f}'$. Moreover, the restriction of $\mathscr{f}'$ to any set U' which is saturated by the orbits of $\mathscr{f}'$ determines an elementary F-structure over U'.

Typically, an F-structure is determined by specifying the following data.

Let $\{V_\alpha\}$ be a locally finite collection of open subsets of X and, for each α, let $\mathscr{F}_\alpha = (\mathscr{f}_\alpha, \mu_\alpha)$ be an elementary F-structure over V_α. Assume that

(F_1) for all α, β, either $\mathscr{f}_\alpha | V_\alpha \cap V_\beta$ is a sub-bundle of $\mathscr{f}_\beta | V_\alpha \cap V_\beta$ or vice versa;

(F_2) in the former case, μ_α is obtained restricting μ_β and $V_\alpha \cap V_\beta$ is saturated for μ_β.

Note that in (F_1) above, we allow $\mathscr{f}_\alpha | V_\alpha \cap V_\beta$ to coincide with $\mathscr{f}_\beta | V_\alpha \cap V_\beta$.

Obviously, a collection, $\{V_\alpha, \mathscr{F}_\alpha\}$, satisfying (F_1) and (F_2) determines an F-structure, $\mathscr{F}$, over $\bigcup_\alpha V_\alpha$, for which the associated sheaf, $\mathscr{f}$, is $\bigcup_\alpha \mathscr{f}_\alpha$.

If we replace condition (F_2) by

$(F_2)^{\mathrm{w}}$ in the former case, μ_α is obtained by restricting μ_β and $V_\alpha \cap V_\beta$ is saturated for μ_α,

then a collection satisfying (F_1) and $(F_2)^{\mathrm{w}}$ determines a weak F-structure.

In the proof of Theorem 0.1, we will apply Lemma 1.5 to obtain a collection satisfying (F_1) and $(F_2)^{\mathrm{w}}$. But, it will turn out that two additional conditions ((F_3) and (F_4)) are satisfied. These guarantee that the weak F-structure is actually an F-structure.

(F_3) If $V_{\alpha_0}, \cdots, V_{\alpha_l}$ is any sequence such that, for $i = 0, \cdots, l-1$, $V_{\alpha_i} \cap V_{\alpha_{i+1}} \neq \varnothing$ and $\mathscr{f}_{\alpha_i}$ is *properly* contained in $\mathscr{f}_{\alpha_{i+1}}$ on $V_{\alpha_i} \cap V_{\alpha_{i+1}}$, then $\mathscr{f}_{\alpha_0}$ extends over $\bigcup_0^l V_{\alpha_i}$.

(F_4) If $V_{\beta_0}, \cdots, V_{\beta_{l'}}$ is a second such sequence and $V_{\alpha_l} \cap V_{\beta_{l'}} \neq \varnothing$ then the extensions of $\mathscr{f}_{\alpha_0}$, $\mathscr{f}_{\beta_0}$ to V_{α_l}, $V_{\beta_{l'}}$ satisfy $\mathscr{f}_{\alpha_0} \subseteq \mathscr{f}_{\beta_0}$ or vice versa on $V_{\alpha_l} \cap V_{\beta_{l'}}$.

Note that the extension of $\mathscr{f}_{\alpha_0}$, assumed to exist in (F_3), is necessarily unique.

Let $s(\alpha)$ denote those β for which there exists a sequence as in (F_3) with $\alpha = \alpha_0$ and $\beta = \alpha_l$. Put $W_\alpha = \bigcup_{\beta \in s(\alpha)} V_\beta$. Then if (F_3) and (F_4) hold, we claim that $\{(W_\alpha, \mathscr{f}_\alpha)\}$ satisfies (F_1) and (F_2). Hence $\{(W_\alpha, \mathscr{f}_\alpha)\}$, or, equivalently, $\{(V_\alpha, \mathscr{F}_\alpha)\}$, determines an F-structure.

Observe that the part of condition (F_2) which relates to the actions is automatic. Also, $W_{\alpha_0} \cap W_{\beta_0}$ is a union of sets, $V_{\alpha_l} \cap V_{\beta_{l'}}$ as in (F_4), and we can assume that $\mathscr{f}_{\alpha_l} = \mathscr{f}_{\beta_{l'}}$. For if, say, $\mathscr{f}_{\alpha_l}$ is properly contained in $\mathscr{f}_{\beta_{l'}}$, then $V_{\beta_{l'}} \subset W_{\alpha_0}$ and we can replace the sequence $V_{\alpha_0}, \cdots, V_{\alpha_l}$ by $v_{\alpha_0}, \cdots, V_{\alpha_l}, V_{\beta_{l'}}$. Thus, $V_{\alpha_l} \cap V_{\beta_{l'}}$ is saturated for $\mathscr{f}_{\alpha_l} = \mathscr{f}_{\beta_{l'}}$ and hence for $\mathscr{f}_{\alpha_0}$ and $\mathscr{f}_{\beta_0}$. Therefore, (F_2) holds. (F_1) is obvious from (F_4).

The main result of this section, Lemma 1.5, says essentially that if (F_1) is satisfied and $(F_2)^{\mathrm{w}}$ holds to a high degree of approximation, then the collection can be perturbed to one for which both (F_1) and $(F_2)^{\mathrm{w}}$ hold. This is a consequence of the stability theorem for compact group actions, in the form given in [14] (compare also [16]).

We begin by adapting their theorem to our context.

Let $V_j \subset X$ be open sets, $j = 1, 2$. Let $(V_j, \mathscr{f}_j, \mu_j)$ be an elementary F-structure such that μ_j is induced by an action of $\Gamma_j \times_{\rho_j} T^k$ on a normal covering space, $\tilde{V}_j \xrightarrow{\pi_j} V_j$. We suppose that $\mathscr{f}_1 | V_1 \cap V_2$ agrees with a subbundle, $\mathscr{f}_{12}$, of $\mathscr{f}_2 | V_1 \cap V_2$.

Let $T^k = S^1 \times \cdots \times S^1$ and let $d(g)$ denote the distance of $g \in T^k$ from the identity element, under the metric obtained by averaging the product metric under the holonomy of $\mathscr{f}_1 | V_1 \cap V_2$. Assume that V_j has a metric $\langle\ ,\ \rangle_j$, which is invariant for μ_j and let $V_j^\rho \subset V_j$ denote the set of points at distance $\geq \rho$ from ∂V_j for the metric $\langle\ ,\ \rangle_j$. Assume that the injectivity radius for $\langle\ ,\ \rangle_j$ is bounded below by $\frac{1}{2}$ and that the sectional curvature is bounded by 1 in absolute value. Finally, assume there is a $\frac{1}{2}$-quasi-isometry between $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ (see (3.3).

Let $x_0 \in V_1 \cap V_2$ and let $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2$ be representative partial actions for μ_1, μ_2 on some contractible neighborhood W of x_0. If $d(g)$ is sufficiently small, we define $\eta(g)\colon W \to X$ by $\eta(g) = \boldsymbol{\mu}_2(g^{-1})\boldsymbol{\mu}_1(g)$. We say that $(\mu_1, \mathscr{f}_1)$ $(\mu_2, \mathscr{f}_{12})$ are δ (C^1-close) on $V_1 \cap V_2$, if for all such x_0, g the map $\eta(g)$ is $d(g)\delta$ (C^1-close) to the inclusion, $W \hookrightarrow S$.

Let $\phi\colon V_1^\rho \to V_1$ be an imbedding which is ε (C^1-close) (in the sense of [14]) to the inclusion, with $\varepsilon < \frac{1}{2}$. Since the injectivity radius of the metric $\langle\ ,\ \rangle_1$ is $> \frac{1}{2}$, there is a natural identification of $(\phi^{-1})^*(\mathscr{f}_1 V_1^\rho)$ with $\mathscr{f}_1 | \phi(V_1^\rho)$. This identification is understood implicitly in (2) and (4) of Lemma 1.4 below.

Lemma 1.4. *For all $1 > \rho > 2\varepsilon > 0$, there exists $\delta = \delta(\rho, \varepsilon, N) > 0$ such that if $(\mu_1, \mathscr{f}_1)$ and $(\mu_2, \mathscr{f}_2)$ are δ (C^1-close), and the coverings $\tilde{V}_j \to V_j$ have order, $N_j \leq N$, then there exists an embedding, $\phi\colon V_1^\rho \to V_1$, with the following properties*:

(1) *ϕ is ε (C^1-close) to the inclusion $V_1^\rho \hookrightarrow V_1$ and $\phi(x) = x$ for $x \in V_1 \setminus V_2^{\rho/2}$.*

(2) *$(\mathscr{f}_1, \phi\mu_1\phi^{-1})$ agrees with $(\mathscr{f}_{12}, \mu_2)$ on $\phi(V_1) \cap V_2$.*

(3) *If for some $x \in \phi(V_1^\rho)$ and all g with $d(g)$ sufficiently small, we have $\eta(g)(x) = x$, then $x \in V_1^\rho$ and $\phi(x) = x$.*

(4) $(\phi(V_1^\rho), \mathscr{N}_1, \phi\mu_1\phi^{-1})$ *and* $(V_2^\rho, \mathscr{N}_2, \mu_2)$ *determine an F-structure over* $\phi(V_1^\rho) \cup V_2^\rho$.

Proof. Consider the subset $\mu_1(\pi_1^{-1}(V_1^{\rho/4} \cap V_2^{\rho/4})) \subset \tilde{V}$, the saturation of $\pi_1^{-1}(V_1^{\rho/4} \cap V_2^{\rho/4})$ by the action of T^k which lifts μ_1. By writing an arbitrary element $g \in T^k$ as $g \in h^m$, where h is sufficiently close to the identity, and then comparing with the local action of the lift of μ_2, we easily find that for δ sufficiently small,

$$\mu_1(\pi_1^{-1}(V_1^{\rho/4}) \cap V_1^{\rho/4}) \subset \pi_1^{-1}(V_1 \cap V_2).$$

We also obtain the corresponding statement with the roles of μ_1 and μ_2 reversed (for the action of $\mu_2(\mathscr{N}_{12})$).

Let $\widetilde{V_1 \cap V_2} \xrightarrow{\pi} V_1 \cap V_2$ be a common covering of $\pi_j^{-1}(V_1 \cap V_2)$. We can assume $V_1 \cap V_2$ is normal and of order $\mathbf{N} < N^2$. Put $\mathbf{N}/N_j = l_j$. The action of $T^k = \mathbb{R}^k/\mathbb{Z}^k$ on $\mu_j(\pi_j^{-1}(V_1^{\rho/4} \cap V_2^{\rho/4}))$ lifts to an action of $\mathbb{R}^k/l_j\mathbb{Z}^k$ on the inverse images of $\mu_j(\pi_j^{-1}(V_1^{\rho/4} \cap V_2^{\rho/4}))$ in $V_1 \cap V_2$. By composing with the homomorphisms $\tilde{T}^k = \mathbb{R}^k/l_1 l_2\mathbb{Z}^k \to \mathbb{R}^k/l_j\mathbb{Z}^k$, we obtain actions $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of the same torus on these inverse images (in general, these actions are noneffective). Let Γ_j and Γ denote the covering groups of $\pi_j^{-1}(V_1 \cap V_2)$ and $\widetilde{V_1 \cap V_2}$. By using the homomorphisms $\tilde{\rho}_j \colon \Gamma \to \Gamma_j \xrightarrow{\rho_j} \mathrm{Sl}(k, \mathbb{Z})$, we extend $\tilde{\mu}_j$ to an action of the semidirect product $\Gamma \times_{\tilde{\rho}_j} \tilde{T}^k$.

Since the order of the covering $\tilde{T}^k \to T^k$ is bounded (by N^2) it is clear that if μ_1 and μ_2 are C^1-close, then $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are C^1-close on the intersection of their domains (write $g = h^m$ as above).

If δ is sufficiently small, we can restrict the domains of the $\tilde{\mu}_j$ to obtain domains $\tilde{W}_j$ for $\tilde{\mu}_j$ such that

$$\pi^{-1}(V_1^\rho \cap V_2^\rho) \subset \tilde{W}_1 \subset \tilde{W}_2 \subset \pi^{-1}(V_1^{\rho/2} \cap V_2^{\rho/2})$$

and the boundaries of these sets are at mutual distance at least $\rho/24$ for $\langle\ ,\ \rangle_1$. Again, for δ sufficiently small, the argument of [14] gives an embedding, $\tilde{\psi} \colon \tilde{W}_1 \to \tilde{W}_2$, as C^1-close as we like to the inclusion, satisfying $\tilde{\psi}\tilde{\mu}_1 = \tilde{\mu}_2\tilde{\psi}$. Moreover, $\tilde{\psi}$ is the identity at points at which $\tilde{\mu}_1$ and $\tilde{\mu}_2$ agree locally.

Put $\pi(\tilde{W}_j) = W_j$. The embedding $\tilde{\psi}$ induces $\psi \colon W_1 \to W_2$, satisfying $\psi\mu_1 = \mu_2\psi$ with ψ as C^1-close to the inclusion as we like. Let U_1 be invariant for μ_1 and satisfy $V_1^\rho \cap V_2^\rho \subset U_1 \subset W_1$, with the boundaries of these sets at mutual distance at least $\rho/100$ for the metric $\langle\ ,\ \rangle_1$. By using the Isotopy Extension Theorem, we can find an embedding $\phi\colon W_1 \to W_1$, as C^1-close to ψ as we like, such that $\phi|U_1 = \psi|U_1$, ϕ is the identity near ∂W_1, and $\phi(x) = x$ if $\psi(x) = x$. Then we can extend ϕ to all of V_1 by making it the identity off W_1. Finally, we can assume that ϕ is close enough to the inclusion so that $\phi(V_1^\rho) \cap V_2^\rho \subset \phi(U_1)$. The resulting map satisfies (1)–(4). q.e.d.

Let $\{V_\alpha\}$ be a covering. Assume there are at most N_1 of those sets whose intersection with any fixed V_{α_0} is nonempty. Let $\mathscr{F}_\alpha = \{\mathscr{f}_\alpha, \mu_\alpha\}$ be a collection of elementary F-structures over the sets $\{V_\alpha\}$ such that condition (F_1) above holds. Assume that the orders of the coverings $\tilde{V}_\alpha \to V_\alpha$ are all $\le N_2$ and that the fibers of the $\mathscr{f}_\alpha$ all have dimension $\le N_3$. Finally, assume that each V_α carries an invariant metric for μ_α, with injectivity radius $\ge \frac{1}{2}$ and curvature ≤ 1 in absolute value and that these metrics are $\frac{1}{2}$-quasi-isometric on intersections.

In the following lemma we identify $(\phi_\alpha^{-1})^*(\mathscr{f}_\alpha|V_\alpha^\rho)$ with $\mathscr{f}_\alpha|\phi_\alpha(V_\alpha^\rho)$ as in (2) and (4) of Lemma 1.4.

Lemma 1.5. *For all $1 > \rho > 2\varepsilon > 0$, there exists $\delta = N_1 2^{N_1} \cdot \delta(\rho, \varepsilon, N_2, N_3) > 0$ such that if for all α, β (say) $\mathscr{f}_\alpha|V_\alpha \cap V_\beta$ agrees with $\mathscr{f}_{\alpha,\beta}|V_\alpha \cap V_\beta$ (where $\mathscr{f}_{\alpha,\beta} \subset \mathscr{f}_\beta$), and $(\mathscr{f}_\alpha, \mu_\alpha)$ and $(\mathscr{f}_{\alpha,\beta}, \mu_\beta)$ are δ (C^1-close), then there are embeddings $\phi_\alpha\colon V_\alpha^\rho \to V_\alpha$, with $\rho' \le \rho$, such that the following holds:*

(1) *For all α, the embedding ϕ_α is ε (C^1-close) to the inclusion $V_\alpha^{\rho'} \to V_\alpha$.*

(2) *The collection $\{(\phi_\alpha(V_\alpha^{\rho'}), \mathscr{f}_\alpha, \phi_\alpha\mu_\alpha\phi_\alpha^{-1})\}$ satisfies (F_1) and $(F_2)^{\mathrm{w}}$, and hence determines a weak F-structure over $\bigcup_\alpha \phi_\alpha(V_\alpha^{\rho'})$.*

Proof. Consider the collections $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_j)$ of indices such that $V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}$ is maximal with respect to the property of having nonempty intersection. Choose an enumeration, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$, of these. For each $\boldsymbol{\alpha}_j$, we can reorder the subscripts, $\alpha_k \in \boldsymbol{\alpha}_j$ such that on $V_{\alpha_0} \cap \cdots \cap V_{\alpha_j}$, we have $\mathscr{f}_{\alpha_1} \subseteq \mathscr{f}_{\alpha_2} \subseteq \cdots \subseteq \mathscr{f}_{\alpha_i}$.

Now we go through the $\boldsymbol{\alpha}_j$ in order and for each one we do the following. Order the pairs $(\alpha_k, \alpha_{k'})$ with $k < k'$ by $(\alpha_k, \alpha_{k'}) < (\alpha_l, \alpha_{l'})$ if $k' < l'$ or $k' = l'$ and $k < l$. Then run through these pairs in descending order. At each stage apply Lemma 1.4, with $\rho/(N_1 2^{N_1})$, $\varepsilon/(N_1 2^{N_1})$

in place of ρ, ε to the subsets $V^{\cdot}_{\alpha_k}$, $V^{\cdot}_{\alpha_{k'}}$ of V_{α_k}, $V_{\alpha_{k'}}$, which have produced possible previous applications of Lemma 1.4, at earlier stages of the process.

We claim that the above process produces a collection for which (1) and (2) hold.

To see this let $x \in \bigcup_\alpha \phi_\alpha(V_\alpha^{\rho'})$ and let $\alpha(x)$ be the set of those α with $x \in V_\alpha$. Let $\boldsymbol{\alpha}_{j_1} < \boldsymbol{\alpha}_{j_2} < \cdots$ where $j_1 < j_2 < \cdots$, be those $\boldsymbol{\alpha}_j$ which contain $\alpha(x)$ and put $\boldsymbol{\alpha}(x) = \boldsymbol{\alpha}_{j_1}$. By referring to (3) of Lemma 1.4 we see that if the actions on those $V^{\cdot}_\alpha$ with $\alpha \in \alpha(x)$ agree at the point x, after the stage of the process corresponding to $\boldsymbol{\alpha}(x)$ has been concluded, then they do not change during the remainder of the process.

It suffices to check that after this stage has been concluded, all of these actions agree at x. Recall that Lemma 1.4 is applied for each pair of subscripts α_k, $\alpha_{k'} \in \boldsymbol{\alpha}(x)$, with $k < k'$. Moreover, these pairs are considered in descending order and the action is changed only on a subset of $V^{\cdot}_{\alpha_k}$. Thus, we can assume that for some α_l with $l > k'$, the actions for the pairs (α_k, α_l) and $(\alpha_{k'}, \alpha_l)$ are compatible before the step corresponding to $(\alpha_k, \alpha_{k'})$ but the actions corresponding to (α_k, α_l) are not compatible after this step. However, by (3) of Lemma 1.4 (and induction) this does not happen. q.e.d.

2. Elementary F-structures on complete flat manifolds

(a) **Preliminaries; short loops.** Let M^n be a complete riemannian manifold. For c a curve in M^n, let $L[c]$ denote the length of c.

Given curves c_1 and c_2 with the same end points, we say that c_1 and c_2 are *short homotopic*, if they are homotopic keeping end points fixed, through curves of length at most $\max_j L[c_j]$.

Let $m \in M^n$. Let R_m be the largest number such that $\exp_m |B_{R_m}(0) \subset M^n_m$ is nonsingular. If c is closed with $c(0) = m$, $L[c] < R_m$, then c is short homotopic to a unique geodesic loop γ on m. Suppose, in particular, that $c = \gamma$ and that τ is a curve with $\tau(0) = m$. Let τ^s denote $\tau|[0, s]$. As long as the closed curve $\tau^s \cup \gamma \cup -\tau^s$, on $\tau(s)$, is homotopic to a geodesic loop γ_s on $\tau(s)$, with $L[\gamma_s] < R_{\tau(s)}$, then γ_s is unique. We say that γ_s is obtained from $\gamma_0 = \gamma$ by *sliding* along τ. The map, $\gamma_0 \to \gamma_s$, is compatible with the isomorphism between $\pi_1(X^n, \tau(0))$ and $\pi_1(X^n, \tau(s))$ induced by τ^s.

If γ_1 and γ_2 are geodesic loops on m with $L[\gamma_1]+L[\gamma_2]<R_m$, then $\gamma_1\cup\gamma_2$ is short homotopic to a unique geodesic loop, $\gamma_1*\gamma_2$. In particular, if $R_m=\infty$, then $\pi_1(M^n, m)$ is isomorphic to the group of geodesic loops on m with the product $*$. In this case, a loop at m gives rise to a collection of loops $\{\gamma\}_{m_1}$ at each point $m_1\in M^n$, each of which is free homotopic to γ. The collection $\{\gamma\}_{m_1}$ represents a conjugacy class in $\pi_1(M^n, m_1)$.

Let i_m denote the injectivity radius at m.

Lemma 2.1. *There is a constant $c(n)$ such that if the sectional curvature of M^n satisfies $|K|\le 1$ and $\Lambda i_m<\pi/2$ $(\Lambda>0)$, then there are at most $c(n)\Lambda^n$ geodesic loops on m of length $\le\Lambda i_m$.*

Proof. Each loop γ lifts to a segment of a ray, $\hat\gamma$, through the origin in M^n_m. Clearly, there exists $c(n)$ such that if there are more than Λ^n geodesics of length at most $\Lambda\cdot i_m$ then endpoints of some pair $\hat\gamma_1$, $\hat\gamma_2$ are at distance less than $2i_m(\Lambda i_m/\sinh\Lambda i_m)$. It follows that the loop which is short homotopic to $\gamma_1*\gamma_2^{-1}$ has length $<2i_m$. This is a contradiction.

(b) **Geometry of complete flat manifolds.** Let X^n be a complete flat manifold. Write $X^n=\overline{X}^l\times R^k$, isometrically, where $\overline{X}^l$ has no Euclidean factor. Then $\overline{X}^l$ contains a unique compact flat totally geodesic submanifold, S^m, the *soul*, such that $\overline{X}^l$ is isometric to the total space of the normal bundle $\nu(S^m)$ (see [3; 19, Theorem 3.3]). There the metric on $\nu(S^m)$ is induced by its natural flat connection.

Note that any tubular neighborhood $T_u(S^m)$ $(u\ge 0)$ is *totally convex*, i.e., any geodesic with endpoints in $T_u(S^m)$ lies in $T_u(S^m)$.

From now on we assume $k<n$, or, equivalently, $m>0$.

Let S^m be a soul of X^n and let $\tilde S^m\xrightarrow{\pi}S^m$ denote the holonomy covering of the compact flat manifold $\tilde S^m$. By Bieberbach's theorem, $\tilde S^m$ is isometric to a flat torus and $\tilde S^m\to S^m$ has order at most $\lambda(n)$, for some constant $\lambda(n)$ depending only on n $(\ge m)$. Since $S^m\hookrightarrow X^n$ is a homotopy equivalence, we can regard $\mathbb{Z}^k\simeq A=\pi_1(\tilde S^m)$ as a normal subgroup of $\pi_1(X^n)$. Clearly, A is independent of the particular choice S^m.

Let γ be a geodesic loop with orientation preserving holonomy, having all its rotational angles $<\pi/\lambda(n)$ in absolute value. We write $\mathrm{rot}(P_\gamma)<\pi/\lambda(n)$. In this case $\gamma\in A$. In fact, let τ be a minimal geodesic with $\tau(l)=\gamma(0)$ and $\tau(0)$ the point on S^m closest to $\gamma(0)$. By sliding γ along τ we obtain a geodesic loop $\gamma_s\subset T_s(S^m)$ at $\tau(s)$. In particular, $P_{\gamma_0}\simeq P_\gamma$ (since X^n is flat), $P_{\gamma_0}\subset S^m$, and the claim follows from Bieberbach's theorem.

Note that $L[\gamma_s]$ is given by the increasing function

$$L[\gamma_s] = (L^2[\gamma_0] + (2\sin\theta/2)^2 s^2)^{1/2}, \tag{2.2}$$

where P_{γ_0} rotates $\tau'(0)$ through an angle θ. This follows by an elementary argument after one lifts γ_0 to the universal covering space of X^n.

If $\gamma_0 \in A$, then γ_0 is automatically smooth closed since it lifts to a loop $\tilde{\gamma}_0$ contained in the torus $\tilde{S}^m$.

(c) **Elementary F-structures.** We will explain how a finite subset of A which is invariant under conjugation by elements of $\pi_1(X^n)$ and for which the corresponding holonomy transformations are orientation preserving, gives rise to an elementary F-structure. This construction depends on a suitable set of choices of logarithms for the holonomy transformations.

Let (w, e^B) represent an isometry of R^n, with translational part w. Put $w = w' + w''$, where $e^B(w') = w'$ and w'' is orthogonal to the $+1$-eigenspace of e^B. Let $(1-e^B)^{-1}w''$ denote the unique inverse image of w'' orthogonal to $\ker(1-e^B)$. Then the curve

$$t \to (tw' + (1-e^{Bt})(1-e^B)^{-1}w'', e^{Bt}) \tag{2.3}$$

is a 1-parameter subgroup passing through (w, e^B) at $t = 1$. The orbit, $\mathscr{O}$, of the origin, is the curve $t \to tw' + (1-e^{Bt})(1-e^B)^{-1}w''$. Let L be the length of the restriction of this curve to the interval $0 \le t \le 1$. An elementary computation shows that

$$\|w\| \le L \le \left[\|w'\|^2 + \left(\frac{\lambda}{2\sin\lambda/2}\right)\|w''\|^2\right]^{1/2}, \tag{2.4}$$

where λ is the largest eigenvalue of B which is not an integral multiple of $2\pi i$.

Let $\{(w_j, e^{B_j})\}$ be a collection of mutually commuting isometries, such that the $\{B_j\}$ are mutually commuting skew symmetric transformations with no eigenvalue of the form $2\pi ik$, for $k \neq 0$. By a trivial calculation, for all j, k, we have

$$(1 = e^{B_k})w_j = (1-e^{B_j})w_k, \tag{2.5}$$

$$(1-e^{B_k})w'_j = (1-e^{B_j})w'_k = 0. \tag{2.6}$$

It follows easily that the subgroups given by (2.4) are mutually commuting.

Conversely, let $\{g_j\}$ be mutually commuting elements of $\mathrm{SO}(n)$. Then we can find skew symmetric transformations, $\{B_j\}$, such that $e^{B_j} = g_j$, the $\{B_j\}$ are mutually commuting, and each B_j has no eigenvalue of the form $2\pi ik$ for $k \neq 0$. In particular, if $\mathrm{rot}(g_j) < \pi$, then the B_j are uniquely determined if we require $\|B_j\| < \pi$. In any case, given a mutually commuting set $\{(w_j, g_j)\}$, we can obtain mutually commuting 1-parameter subgroups as above.

Now assume that the $\{(w_j, e^{B_j})\}$ form a group $\Delta \simeq \mathbb{Z}^k$ of covering transformations of R^n. Given a finite subset $\{(w_j, e^{B_j})\}$, $j = 1, \cdots, N$, we obtain an action of the Cartesian product of the corresponding 1-parameter subgroups on R^n, which descends to a T^N action on R^n/Δ (see the discussion at the beginning of §1). This action need not be effective, but an effective action can be obtained by passing to a quotient of T^N.

Example 2.7. Let $(w, e^{B(\theta)})$ denote the isometry of R^3 such that w is a translation in the direction of a unit vector along the x-axis and $B(\theta)$ is given by the matrix

$$B(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \tag{2.8}$$

in the y, z-plane. The isometries $(w, e^{B(\theta)})$ and $(2w, e^{B(2\theta)})$ generate a group $\Lambda = \Delta \simeq \mathbb{Z}$ (we assume θ, $2\theta \not\equiv 0 \bmod 2\pi$). The construction above gives a noneffective T^2 action on R^3/Δ, inducing an effective action of T^1. If we use $B(2\theta - 2\pi)$ in place of $B(2\theta)$, we obtain an effective T^2 action. Note for $0 < \theta < \pi/2$, $|2\theta| < \pi$ while for $\pi/2 < \theta < \pi$, $|(2\theta - 2\pi)| < \pi$.

Now suppose that π_1 is a group of covering transformations and that $\Delta \simeq \mathbb{Z}^k$ is a normal subgroup of finite index $\leq \lambda(n)$. Suppose $\{(w_j, e^{B_j})\}$, $j = 1, \cdots, N$, is invariant under conjugation by elements of π_1. Then there is an induced representation $\rho\colon \pi_1/\Delta \to \mathrm{Aut}(T^N)$, which together with the action of T^n on R^n/Δ determines an elementary F-structure on R^n/π_1.

The F-structure just constructed can also be described in terms of geodesic loops on $X^n = R^n/\pi_1$. Identify $X^n_x \simeq \mathbb{R}^n$ with the universal covering space of X^n. Then the group of isometric covering transformations is isomorphic to the group of geodesic loops at x. The elément corresponding to a loop, γ, can be recovered as $(V_\gamma, P_{-\gamma})$; where V_γ denotes translation by $L[\gamma] \cdot \gamma'(0)$ and $-\gamma$ denotes γ transversed in the opposite sense.

A collection $\{\gamma_j\}_x$, $j = 1, \cdots, N$, of conjugacy class of loops, $\gamma_j \in A$, determines an elementary F-structure, $\mathscr{F}$, on X^n. In the sequel we are always concerned with the case $\mathrm{rot}(P_{\gamma_j}) < \pi/\lambda(n)$. Note that for any

$\gamma_j \in A$, the conjugacy class $\{\gamma_j\}_x$ contains at most $\lambda(n)$ loops. The fiber, $\mathscr{f}_{x_1}$, at an arbitrary point $x_1 \in X^n$ of the sheaf (flat T^N bundle) $\mathscr{f}$, associated to $\mathscr{F}$, can be identified with the Cartesian product of loops in $\{\gamma_j\}_{x_1}$.

We now describe a class of elementary F-structures which, in general, are defined only over proper subsets of X^n. These will be used in the construction of the F-structure of Theorem 0.1.

Let $[P_\gamma]$ denote the isomorphism class of P_γ.

Let $\gamma_1, \cdots, \gamma_N$ be loops at $x \in X^n$ which lie in A. Fix $\varepsilon > 0$. Assume that if $\gamma \in A$ and $\gamma \neq \gamma_i$ for any i, then for all i, at least one of the following holds:

$$|L[\gamma] - L[\gamma_i]| \geq \varepsilon \tag{2.9}$$

or

$$[P_\gamma] \neq [P_{\gamma_i}]. \tag{2.10}$$

Let $\mathscr{F}'$ be any elementary F-structure as above on X^n and let $T_\varepsilon(\mathscr{O}'_x)$ denote the open tubular neighborhood of the orbit, $\mathscr{O}'_x$, of radius ε.

Lemma 2.11. (1) *At each $x_1 \in T_{\varepsilon/4}(\mathscr{O}'_x)$ there are exactly N loops, $\hat{\gamma}$, which, for some i, satisfy*

$$|L[\hat{\gamma}] - L[\gamma_i]| \leq \varepsilon/2, \qquad [P_{\hat{\gamma}}] = [P_{\gamma_i}]. \tag{2.12}$$

(2) *The collection $\hat{\gamma}_1, \cdots, \hat{\gamma}_N$ of such loops is the collection obtained from $\gamma_1, \cdots, \gamma_N$ under homotopy in $T_{\varepsilon/4}(\mathscr{O}'_x)$; i.e., sliding a loop, γ_i, from x to x_1 along any curve $c \subset T_{\varepsilon/4}(\mathscr{O}'_x)$ gives a loop, $\hat{\gamma}_j$, for some j.*

Proof. Note first that sliding a loop, γ, does not change $[P_\gamma]$. Then, by an obvious continuity argument, (2) implies (1).

Since $A \subset \pi_1(X^n)$ is normal, the collection of loops at x lying in A can be obtained by sliding the collection of loops lying in A at x along any curve c. If $x_1 \in T_{\varepsilon/4}(\mathscr{O}'_x)$, there is a minimal geodesic σ of length $s < \varepsilon/4$ connecting x_1 to a point on $\mathscr{O}'_x$. Since sliding a loop along σ changes its length by at most $2s < \varepsilon/2$, it suffices to assume $x_1 \in \mathscr{O}'_x$ and to show that for some curve c, from x_1 to x_2, sliding loops of A along c leaves their lengths unchanged.

Let $\tilde{x} \in \tilde{X}$ be a lift of x, let $\tilde{\gamma}$ be a loop at $\tilde{x}$ lifting $\gamma \in A$, and let T^l be the torus corresponding to $\mathscr{F}'$, which acts on $\tilde{X}^n$. We can find a curve $g(t) \subset T^l$ with $g(0)$ the identity element and $g(1)\tilde{x} = \tilde{x}_1$ a lift of x_1. The curve $g(t)$ projects to a curve c from x to x_1 and by an obvious continuity argument, $g(1)(\tilde{\gamma})$ projects to the loop obtained by sliding γ along c. Since $g(1)$ is an isometry, our claim follows. q.e.d.

Let $\gamma_1, \cdots, \gamma_N, \gamma_{N+1}, \cdots, \gamma_{N'}$ be a collection of loops at x which lie in A and let $\mathscr{F}'$ be the elementary F-structure determined by the union of conjugacy classes, $\{\gamma_j\}_x$, $1 \le j \le N'$. Assume that $\gamma_1, \cdots, \gamma_N$ satisfy (2.9) and (2.10) above. Then Lemma 2.11 implies

Corollary 2.13. *The set $\gamma_1, \cdots, \gamma_N$ is invariant under conjugation in $\pi_1(T_{\varepsilon/4}(\mathscr{O}'_x))$ and hence defines an elementary F-structure, $\mathscr{F}$, over $T_{\varepsilon/4}(\mathscr{O}'_x)$.*

Let $\gamma \in A$ be a loop at x, with lift $\tilde{\gamma}$ at $\tilde{x}$. For the circle action on $\tilde{X}^n$ corresponding to $\tilde{\gamma}$, the orbit of $\tilde{x}$ (counted with multiplicities) is homotopic to $\tilde{\gamma}$ (see (2.3)). Since $L[\tilde{\gamma}] > 0$ is of shortest length in its homotopy class, the orbit of $\tilde{x}$ has positive length. Thus, the elementary F-structures constructed above all have *positive rank*.

Clearly, an orbit of any elementary structure as above lies at constant distance from any soul, S^m. The maximum size of an orbit is controlled by the upper bound in (2.4). If $\|B_j\| < \pi$ for all j, then the orbit in X^n corresponding to the jth circle in $T^N = S^1 \times \cdots \times S^1$, has length at most $\frac{\pi}{2} L[\gamma_j]$.

Remark 2.14. The injectivity radius need not be constant on orbits. However, in view of the obvious relation

$$i_x \le i_{\tilde{x}} \le \lambda(n) i_x, \tag{2.15}$$

the ratio of the maximum value of the injectivity radius to the minimum value, on an orbit, is bounded by $\lambda(n)$.

Appendix to §2: Growth of the injectivity radius

We claim that it is not possible to assign to each complete flat manifold, X^n, an elementary F-structure, $\mathscr{F}(X^n)$, of the type considered in §2, in such a way the ratio of the diameter of the orbit, $\operatorname{diam}(\mathscr{O}_x)$, to the injectivity radius, i_x, remains uniformly bounded as x and X^n vary.

Suppose first that the rotational angles of P_{γ_0} are all rational multiples of 2π, for some loop γ_0 on $z \in S^m$. Then

$$\gamma_0 \overbrace{* \cdots *}^{N} \gamma_0 \overset{\text{def}}{=} N\gamma_0$$

has trivial holonomy, for some smallest integer N. Let τ be a geodesic normal to S^m with $\tau(0) = z$. Let $N\gamma_s$ be the geodesic at $\tau(s)$ obtained by sliding $N\gamma_0$ along τ. Then $L[N\gamma_x] = L[N\sigma_0]$.

On the other hand, if σ_0 is any loop on z with $\sphericalangle(P_\gamma(\tau'(0)), \tau'(0)) = \theta > 0$ then $L[\sigma_s]$ grows linearly along τ (see (2.1)).

It follows that those elementary F-structures constructed in §2, for which the diameter of the orbits does not grow linearly in almost all directions, are precisely the ones generated by loops with trivial holonomy.

The following example is typical.

Example A.1. Let X_θ^3 be the total space of the flat 2-plane bundle over S^1 with holonomy θ. For each $\theta = \frac{p}{q}\pi$ (with $\frac{p}{q} < 1$ in lowest terms) there is an elementary F-structure with sublinear (actually constant) asymptotic growth with orbits, $\mathscr{O}_{\tau(s)}$, of length $qL[S^1] = 2i_{\tau(s)}$, for s large. Then however,

$$L[\mathscr{O}_{\tau(s)}]/i_{\tau(s)} \approx 2q \tag{A.2}$$

for s small. Here q can be taken arbitrarily large.

If X^n is such that there exists no geodesic loop with rational holonomy, then for all γ_0, the function $L[\gamma_s]$ grows linearly in almost all directions. Hence, the same holds for the orbits of any elementary F-structure arising from the construction of §2. But the injectivity radius itself always satisfies the following estimate (put $i_s = i_{\tau(s)}$).

Lemma A.3. *For say* $s > \frac{1}{3}i_0$,

$$i_s \leq c(n)[\operatorname{Vol}(S^m)]^{1/m+c} s^{c/m+c}, \tag{A.4}$$

where

$$c = [(n-m)/2]. \tag{A.5}$$

Proof. We can assume $i_{\tau(0)} = 1$. There are at least $c_1(n)r^m/\operatorname{Vol}(S^m)$ geodesic loops in S^m on $\tau(0)$ of length $\leq r$, where $r \geq i_{\tau(0)}$. At least one of these, σ, has $\operatorname{rot}(P_\sigma) \leq \varepsilon \cdot \pi$, if

$$c_1(n)\frac{r^m}{\operatorname{Vol}(S^m)} = \varepsilon^{-c}. \tag{A.6}$$

Then, by (2.2),

$$L[\sigma_s] \leq \left(r^2 + \left(2s \cdot \sin\frac{\varepsilon}{2}\pi\right)^2\right)^{1/2}. \tag{A.7}$$

Given s, choose r and ε, which satisfy (A.6) and

$$r = \varepsilon s. \tag{A.8}$$

Then

$$\begin{aligned} L[\sigma_s] &\leq (r^2 + (\varepsilon s)^2)^{1/2} \leq \sqrt{2}r \\ &= c(n)(\operatorname{Vol}(S^m))^{1/m+c} s^{c/m+c}. \end{aligned} \tag{A.9}$$

q.e.d.

Let $X^n = X^l \times R^{n-l}$. The isometry group of $\tilde{X}^l$ is generated by a collection of circle actions, one for each set of generators for $\pi_1(\tilde{S}^m) \simeq \mathbb{Z}^m$ and the orthogonal transformations of the normal bundle $\nu(S^m)$ (leaving S^m pointwise fixed) which centralize the holonomy group. The function $\tilde{i}_x$ is constant on orbits and the isometry group is transitive on fibers of $\nu(\tilde{S}^m)$.

Lemma A.10. *Let $\sigma(s)$ be a normal geodesic in X^n and put $i_{\sigma(s)} = i_s$. Then for all s*

$$i_s \leq i_0 + 2s, \tag{A.11}$$

and for say $s \geq \frac{1}{3} i_0$,

$$i_s \leq c(n) i_0^{1/c+1} s^{c/c+1} = c(n) i_0 (s/i_0)^{c/c+1}. \tag{A.12}$$

Proof. The estimate in (A.12) is clear. The proof of (A.13) is completely analogous to that of (A.4). We just restrict attention to multiples of a fixed loop.

3. Local approximation by complete noncontractible flat manifolds

Let Y^n be a complete riemannian manifold and let $y \in Y^n$. Set

$$v(y, R) \stackrel{\text{def}}{=} \sup_{B_{R \cdot i_y}(y)} |K(w)|^{1/2} i_y. \tag{3.1}$$

By Theorem 4.3 of [5] (see also [6]) it follows that

$$i_w \geq i_y \min(\pi/v(y, R), c(n)) e^{-(n-1)R \cdot v(y, R)}. \tag{3.2}$$

If U_1 and U_2 are riemannian manifolds and $f \colon U_1 \to U_2$ is a C^1-smooth quasi-isometry, let $M(f)$ denote the infimum of those ε such that if $V(y, \delta^{-1}) \leq \delta$,

$$e^{-\varepsilon} g_1 \leq f^*(g_2) \leq e^{\varepsilon} g_1. \tag{3.3}$$

The following proposition will allow us to transfer the elementary F-structures on complete noncontractible flat manifolds which were discussed in §2 to more general manifolds.

Proposition 3.4. *Given a continuous decreasing function $h \colon (0, \infty) \to (0, \infty)$ and $k > 0$, there exist $\delta = \delta(h, k, n)$, $R(h, k, n)$, such that if $v(y, \delta^{-1}) \leq \delta$, then there exists*

(i) *a complete flat manifold $\mathbf{Y}^n$ and a soul $S \subset \mathbf{Y}^n$,*

(ii) *a quasi-isometry, $f \colon U \to T_u(S^m)$, with $u < R(h, k, n) i_y$, and U an open neighborhood of y, such that*

(iii) $M(f) \le h(u/i_y)$,

(iv) $\max(i_y, \overline{f(y), S}, \operatorname{diam}(S)) < u/k$,

(v) $i_y = i_{f(y)}$.

Proof. Assume the contrary. Then (after possible rescaling) there are sequences (Y_j^n, y_j) such that $i_{y_j} = 1/v(y_j, j) \le 1/j$ and either there exists no f as above satisfying (iii) and (iv) or the smallest u for which there exists such an f is $\ge j$. By the compactness theorem in riemannian geometry, there is a pointed C^∞ manifold $(\mathbf{Y}^n, \mathbf{y})$ with a $C^{1,\alpha}$ riemannian metric (for all $\alpha > 1$) such that for some infinite subsequence (Y_{j_s}, y_{j_s}), and any r, the sequence of balls $B_r(y_{j_s})$ converges in the Lipschitz metric to $B_r(\mathbf{y})$. Clearly, $\mathbf{Y}^n$ is complete flat and noncontractible ($i_y = 1$). In particular its metric is C^∞. Since $i_y = 1$, $\overline{\mathbf{y}, S^m} < \infty$, $\operatorname{diam}(S^m) < \infty$ for some soul $S^m \subset \mathbf{Y}^n$, we obtain a contradiction.

Remark 3.5. Although the fact that h can be chosen to be an arbitrary decreasing function of r is of interest in describing the local geometry of the manifolds considered in Proposition 3.4, for the application to the proof of Theorem 0.1 it will suffice to choose h to be a sufficiently small constant.

Remark 3.6. Lipschitz convergence (i.e., (iii) above) is actually not strong enough for our purposes since we will want to compare holonomies around corresponding loops in Y^n, $\overline{Y}^n$ and not just their lengths. In fact the versions of the compactness theorem proved in [11] or [17] show that in harmonic coordinates the convergence of metric tensors actually takes place in the $C^{1,\alpha}$ topology. The compactness theorem as stated in [13] would also suffice. However, in order to emphasize the elementary nature of our result, we show in the next section, by a simple direct argument, that Lipschitz convergence implies C^1 convergence, in case the limit is flat. For this result we do not require a special coordinate system.

Example 3.7. Fix $\theta > 0$ and let E_θ^3 denote the complete flat manifold obtained by dividing R^3 by the group of isometries generated by the isometry $(w, e^{B(\theta)})$ of Example 2.7. Let S be the soul of E_θ^3. We will show directly that Proposition 3.4 holds for the family (E_θ^3, y), where y is a variable point in E_θ^3.

Observe that if γ_y is a shortest geodesic loop at y, then the holonomy, P_{γ_y}, converges to the identity transformation as $\overline{y, S} \to \infty$. This is an immediate consequence of the discussion of the Appendix to §2.

Let S_l^1 denote the circle of length l. Then $0 \times S_l^1$ is a soul of the riemannian product $R^2 \times S_l^1$. Fix $k > 0$. It follows easily from the observation above that for $\overline{y, S}$ sufficiently large there exists a neighborhood U_y of y and a quasi-isometry $f_y \colon U_y \to T_{2ki_y}(0 \times S^1_{2i_y})$, with $f(y) \in 0 \times S^1_{2i_y}$. Moreover, $M(f_y) \to 0$ as $\overline{y, S} \to \infty$.

Given a function h as in Proposition 3.4, choose Λ such that $M(f_y) < h(2k)$, if $\overline{y, S} > \Lambda$. For such points, the quasi-isometry, f_y, satisfies the conditions of Proposition 3.4 (with $u = 2ki_y$, $u/k = 2i_y$). Moreover, we can take $R(k, h, 3) = 2k$ for the subfamily consisting of the (E_θ^3, y) with $\overline{y, S} > \Lambda$.

The set of points for which $\overline{y, S} \le \Lambda$ is compact. Thus, for all these points, we can take f_y to be the identity map on a sufficiently large tubular neighborhood of S. Then we take $R(k, h, 3)$ for the whole family (E_θ^3, y) to be the larger of $2k$ and the radius of this tube.

In order to estimate explicitly the constants $c_1(n)$ and $c_2(n)$ in Theorem 0.1, it is necessary to give a proof of Proposition 3.4 which does not depend on an argument by contradiction. We now briefly outline such an argument; details will appear elsewhere.

(1) Rescale the metric on Y such that $i_y = 1$ and view $B_R(g)$ as the quotient of a ball on the tangent space by an isometric pseudo-group, Γ. In the spirit of [12] (see also [1]), we can imitate the proof of the Soul Theorem for flat manifolds, given in [19, Theorems 3.2.8 and 3.3.3]. In this way we obtain a group, $\mathbf{\Gamma}$, which acts isometrically in $\mathbb{R}^n$ and freely on a large ball about the origin. Moreover, $\mathbf{\Gamma}$ has an abelian subgroup, $A \simeq \mathbb{Z}^k$, of index $\le \lambda(n)$. Finally, Γ is isomorphic to a subpseudogroup of $\mathbf{\Gamma}$.

(2) By deforming the action of Γ slightly if necessary, we can assume that $\mathbf{\Gamma}$ acts freely on $\mathbb{R}^n$.

(3) By a generalization of the argument of Example 3.7, after making a second small deformation of the action of $\mathbf{\Gamma}$, we can assume that the bounds of (iv) of Proposition 3.4 hold for $\mathbf{R}^n/\mathbf{\Gamma}$.

(4) Finally we construct a quasi-isometry f between a slightly smaller ball $B_{R'}(y) \subset B_R(y)$ and a ball in $\mathbb{R}^n/\mathbf{\Gamma}$. Here we use the result of [15] to take care of the finite group $\mathbf{\Gamma}/A$.

4. Regularization of the approximation

Let $y \in Y^n$ and let $f \colon U \to T_u(S^m)$ be as in Proposition 3.4. Let H_f denote the Hessian of f.

Proposition 4.1. *The constants $\delta(h, k, n)$ and $R(h, k, n)$ can be chosen such that there exists f satisfying* (i)–(iv) *of Proposition* 3.4 *and the additional estimate*

$$\|H_f\| \leq h(u/i_y). \tag{4.2}$$

The idea of the proof is to regularize f by convolving with a suitable smoothing kernel. For an arbitrary map, this would only have the effect of making the Hessian bounded. But by using the fact that f maps U to a flat space with $M(f)$ small, it will follow that the Hessian of the regularized map is actually small.

Proof of Proposition 4.1. We can assume $i_y = 1$.

Let $\bar{\psi}(s)\colon [0, 1] \to [0, 1]$ be a C^∞ function such that $\bar{\psi} \equiv 1$ near $s = 0$ and $\bar{\psi} \equiv 0$ near $s = 1$. Put $\bar{\psi}_\lambda(s) = \bar{\psi}(s/\lambda)$. Let $w_1, w_2 \in Y^n$ and denote the distance from w_1 to w_2 by $\overline{w_1, w_2}$. Finally, let ω denote the volume form on Y^n. Put

$$\psi_\lambda(w_1, w_2) = \frac{\bar{\psi}_\lambda(\overline{w_1, w_2})}{\int_{B_2(w_1)} \bar{\psi}(w - 1, w_2)\omega}, \tag{4.3}$$

where the integration is with respect to w_2.

Choose $\delta = \delta(h_1, 2k, n)$ where $h_1 < \frac{1}{10}$ is to be determined later (see Proposition 3.5). If $v(y, \delta^{-1}) \leq \delta$, standard estimates give

$$\|d\psi_\lambda\| \leq c(\delta)\lambda^{-1}, \tag{4.4}$$

$$\|H_{\psi_\lambda}\| \leq c(\delta)\lambda^{-2}, \tag{4.5}$$

on $B_{\delta^{-1}-\lambda}(y)$.

Let $f\colon U \to T_u(S^m)$ be the map provided by Proposition 3.4. Lemma A.3 and Remark A.10 give a lower bound, i_0, for i_z on $T_u(S^m)$. If we choose

$$\lambda < \tfrac{1}{2} i_0, \tag{4.6}$$

then for all $y_1 \in U$, the range of $f|B_\lambda(y_1)$ is contained in a convex subset of a flat space. Hence,

$$f_\lambda = \int \psi_\lambda(w_1, w_2) f(w_2)\omega \tag{4.7}$$

is well defined.

Let $\mathbf{l}$ be a real valued, affine linear function, with $\|\mathbf{l}\| \le 1$ on $B_r(y) \subset T_u(S^m)$ where $r < \frac{1}{2}i_0$. Then

$$\|H_{f_\lambda}\| = \|H_{(\mathbf{l}\circ f)_\lambda}\|. \tag{4.8}$$

Up to a constant, any $\mathbf{l}$ as above can be written in the form

$$\mathbf{l} = \frac{\rho_{a_1}^2 - \rho_{a_2}^2 - \mathbf{d}^2}{2\mathbf{d}}, \tag{4.9}$$

where ρ_{a_j} is the distance function from $a_j \in T_u(S^m)$, and $\mathbf{d} = \overline{a_1, a_2} = \frac{1}{2}$. Let $f(y_j) = a_j$ and consider the function

$$l = \frac{\rho_{y_1}^2 - \rho_{y_2}^2 - \mathbf{d}^2}{2\mathbf{d}}. \tag{4.10}$$

Then by (4.4)(and (4.5)) l has differential everywhere close to 1, small Hessian and is uniformly close to $\mathbf{l}\circ f$. The explicit bounds depend on h_1. It suffices to estimate $H_{(\mathbf{l}\circ f - l)_\lambda}$. Since $\mathbf{l}\circ f - l$ is arbitrarily small for suitably small h_1, it is clear that given h, we can choose h_1 such that f_λ will satisfy (3.2). q.e.d.

Let $f\colon U \to T_u(S^m)$ be as in Propositions 3.4 and 4.1. Let $\gamma \subset U$ be a geodesic loop on y with $L[\gamma] < R_y$, where $\exp_y B_{R_y}(0) \subset Y_y^n$ is nonsingular. Let $\gamma \subset \mathbf{Y}^n$ be the unique geodesic loop which is short homotopic to $f(\gamma)$.

Corollary 4.11. *Put $h = h(u/i_y)$. Then*

$$e^{-h}L[\gamma] = L[\gamma] \le e^{h}L[\gamma], \tag{4.12}$$

$$\sphericalangle(\gamma'(0), df^{-1}(\gamma'(0)) \le \frac{c(n)\cdot L[\gamma]\cdot h}{i_y}, \tag{4.13}$$

$$\|P_\gamma - df^{-1}P_\gamma\, df\| < \frac{c(n)\cdot L[\gamma]\cdot h}{i_y}. \tag{4.14}$$

Proof. Relation (4.12) follows from the minimizing properties of γ, γ and (iii) Proposition 3.4. By using, in addition, (4.2), relations (4.13) and (4.14) also follow by straightforward arguments.

Suppose that for γ as above, $NL[\gamma] < R_y$. Let $N\gamma$ denote the unique geodesic loop which is short homotopic to the N-fold iterate of γ. Then we have

Corollary 4.15.

$$\|P_{N\gamma} - (P_\gamma)^N\| \le \frac{N \cdot c(n) \cdot L[\gamma] \cdot h}{i_y}. \tag{4.16}$$

Proof. This follows immediately from Corollary 4.11 and the fact that the holonomy of a curve depends only on its homotopy class in the flat case.

5. Construction of the F-structure

(a) **Outline of the construction.** In this section we prove our main result, Theorem 0.1, by using the results of §§2, 3, and 4 to implement Lemma 1.5.

Our basic strategy was sketched in §0. Given a complete riemannian manifold Y^n, let Y^n_δ denote the set of points at which $v(y, \delta^{-1}) < \delta$ (see (3.1)). To each $y \in Y^n_\delta$ (δ sufficiently small) we assign a set, V_y, containing y, and an elementary F-structure, $\mathscr{F}_y$, over V_y. This is done in such a way that $\{(V_y, \mathscr{F}_y)\}$ satisfies all the conditions of Lemma 1.5, apart from the bound, N_1, on the multiplicity. Then we extract a suitable locally finite subcover $\{V_{y_\alpha}\}$. The collection $\{(V_{y_\alpha}, \mathscr{F}_{y_\alpha})\}$ satisfies the hypothesis of Lemma 1.5 and leads to the desired F-structure.

In this subsection, we outline the steps involved in selecting $\{(V_y, \mathscr{F}_y)\}$ and $\{(V_{y_\alpha}, \mathscr{F}_{y_\alpha})\}$. Further details are given in subsections (b)–(g) (which correspond to Steps 1–6 below).

Step 1. To each point $y \in Y^n_\delta$ we assign a set of short geodesic loops $[\gamma_j]_y$, with $\operatorname{rot}(P_{\gamma_j}) < \pi/3\lambda(n)$ ($\lambda(n)$ as in §2). Our choice depends only on the lengths of the short loops at y and on the isomorphism classes of their holonomy transformations. Moreover, the following precursor of property (F_1) holds. If y_1, y_2 are sufficiently close, then $[\gamma_j]_{y_1}$ contains or is contained in $[\gamma_j]_{y_2}$. (As usual we identify loops at y_1 with loops at y_2 by sliding them along the unique minimal geodesic from y_1 to y_2.)

Step 2. Let $f_y \colon U_y \to T_{u_y}(S_y)$ be any map as provided by Proposition 3.4. The set of loops of $T_{u_y}(S_y)$ corresponding to $[\gamma_j]_y$ determines an elementary F-structure, $\boldsymbol{\mathscr{F}}_y$, over a neighborhood, $\mathbf{V}_y$ of $f_y(y)$, as in Corollary 2.13. The fiber of the corresponding elementary F-structure, $\mathscr{F}_y$, over $V_y = f_y^{-1}(\mathbf{V}_y)$ can be identified with the Cartesian product of the loops in $[\gamma_j]_y$. It follows that the collection $\{(V_y, \mathscr{F}_y)\}$ satisfies a weak version of property (F_1): If y_1 and y_2 are sufficiently close, either $\mathscr{f}_{y_2} \supseteq \mathscr{f}_{y_1}$, or vice versa, on $V_{y_1} \cap V_{y_2}$.

Step 3. Clearly, V_{y_1} and V_{y_2} can have nonempty intersection even if y_1 and y_2 are not close. But by using Lemma 2.9 and Remark 2.14, we find that property (F_1) holds for $\{(V_y, \mathscr{F}_y)\}$.

Step 4. On a set $V_{y_1} \cap V_{y_2}$ the closeness of corresponding local actions for $\mathscr{f}_{y_1}$ and $\mathscr{f}_{y_2}$ is determined by the deviation from isometry (in the C^2-topology) of the maps f_{y_1} and f_{y_2}. This is an immediate consequence of the description of elementary F-structures in terms of geodesic loops, for the flat case discussed in §2.

To apply Lemma 1.5 to a subcollection, $\{(V_{y_\alpha}, \mathscr{F}_{y_\alpha})\}$, these deviations must be small relative to the size of the V_{y_α} and the multiplicity, N_1, of $\{V_{y_\alpha}\}$.

Step 5. By a simple variant of a standard packing construction, we select a subcover, $\{V_{y_\alpha}\}$, with $\bigcup_\alpha V_{y_\alpha} \subset Y_\delta^n$, whose multiplicity, N_1, is bounded by $c(n)$.

Step 6. By the results of §4, the deviation from isometry (in the C^2-topology) of a map f_y is controlled by the function h of Proposition 3.4. In view of the bound of Step 5, it suffices to take $h(r) = \varepsilon(n)$, for $\varepsilon(n) > 0$ sufficiently small. Then the covering $\{V_{y_\alpha}\}$ satisfies the hypothesis of Lemma 1.5. The weak F-structure obtained by applying Lemma 1.5 is easily seen to have properties (F_3) and (F_4) of §1. Hence it is an F-structure.

(b) **Assigning short loops to points.** Our procedure for choosing the collections $[\gamma_j]_y$ is based on some trivial observations about sequences.

Let $b_1 \leq b_1 \leq \cdots \leq b_M$ be a nondecreasing sequence such that for some $c_1 < c_2$ and $N \leq M$

$$b_1 \leq c_1 < c_2 < b_{N+1}. \tag{5.1}$$

Clearly, there exists at least one index, $J \leq N$, such that

$$b_j + \frac{c_2 - c_1}{2N} \leq b_{J+1}, \tag{5.2}$$

$$b_J \leq \frac{c_1 + c_2}{2}. \tag{5.3}$$

Remark 5.4. The collection of all such J depends only on the subsequence, $b_1 \leq b_2 \leq \cdots \leq b_N$.

The following lemma is obvious.

Lemma 5.5. *Let $b_1' \leq b_2' \leq \cdots \leq b_M'$ be a second sequence and let π be a permutation of $\{1, \cdots, M\}$ such that for $j \leq M$,*

$$|b_j - b'_{\pi(j)}| < \frac{c_2 - c_1}{4N}. \tag{5.6}$$

Then if J *satisfies* (5.2), π *preserves the sets* $\{0, \cdots, J\}$ *and* $\{J + 1, \cdots, M\}$.

Choose a nondecreasing function $\phi\colon [0, \pi] \to [0, \infty)$, with

$$\phi|[0, \pi/6\lambda(n)] \equiv 1 \tag{5.7}$$

and

$$\phi|[\pi/3\lambda(n), \pi] \equiv 6(6\lambda(n))^{[n/2]}. \tag{5.8}$$

Define a function $a(\gamma)$ on loops at y by

$$a(\gamma) = \phi(\mathrm{rot}(P_\gamma)) \cdot L[\gamma]. \tag{5.9}$$

Clearly, we have $L[\gamma] \le a(\gamma)$.

Lemma 5.10. *For* $\delta \le \delta_0$ *sufficiently small,*

$$\min_\gamma a(\gamma) \le 2(6\lambda(n))^{[n/2]} \cdot i_y. \tag{5.11}$$

The inequality

$$L[\gamma] \le a(\gamma) \le 6(6\lambda(n))^{[n/2]} \cdot i_y \tag{5.12}$$

holds for at most $N = N(n)$ *loops. For all such loops*

$$\mathrm{rot}(P_\gamma) \le \frac{\pi}{3\lambda(n)}. \tag{5.13}$$

Proof. Let γ be a shortest loop at y. Thus, $L[\gamma] = 2i_y$,. By Corollary 4.15 and the standard packing argument, if δ_0 is sufficiently small, there exists k such that

$$L[k\gamma] \le 2ki_y \le 2(6\lambda(n))^{[n/2]}, \tag{5.14}$$

$$\mathrm{rot}(P_{k\gamma}) \le \frac{\pi}{3\lambda(n)}. \tag{5.15}$$

Lemma 2.1 implies (5.12), and (5.13) is clear from (5.8) and (5.9). q.e.d.

From now on, we assume $\delta \le \delta_0$ as above.

Let $y \in Y_\delta^n$ and let $\gamma_1, \gamma_2, \ldots$ be an ordering of the loops at y such that

$$a(\gamma_1) \le a(\gamma_2) \le \cdots. \tag{5.16}$$

It follows from Lemma 5.5 that there exists a smallest index, $J \le N = N(n)$, such that

$$a(\gamma_j) + \frac{c_2 - c_1}{2N} \le a(\gamma_{J+1}), \tag{5.17}$$

with $c_1 = 2(6\lambda(n))^{[n/2]}$ and $c_2 = 2c_1$.

Define $[\gamma_j]_y$ to be the set $\{\gamma_1, \cdots, \gamma_J\}$. Note that the ordering γ_1, γ_2, ... need not be uniquely determined if the numbers $\{a(\gamma_j)\}$ are not all distinct. However, the set $[\gamma_j]_y$ is independent of the choice of ordering. Also, by (5.13), for $\gamma_j \in [\gamma_j]_y$

$$\operatorname{rot}(P_{\gamma_j}) \leq \frac{\pi}{3\lambda(n)}. \tag{5.18}$$

Lemma 5.19. *There exists $0 < \varepsilon(n) < 1$ such that if $y_1, y_2 \in Y_\delta^n$ and $\overline{y_1, y_2} \leq \varepsilon(n) i_{y_1}$, then either $[\gamma_j]_{y_1} \supseteq [\gamma_j]_{y_2}$ or vice versa.*

Proof. Let $\overline{y_1, y_2} \leq \varepsilon i_{y_1}$. Let $\{\gamma_1^k \cdots \gamma_{T_k}^k\} = \mathscr{L}^k$, $k = 1, 2$, be the loops at y_k, with $h(\gamma_1^k) \leq \cdots \leq h(\gamma_{T_k}^k) \leq c_2$. By Remark 5.4, the sets $[\gamma_j]_{y_k}$ are determined by $\{a(\gamma_1^k, \cdots, a(\gamma_{T_k}^k)\}$ or by any larger subsets of $\{a(\gamma_1^k), \ldots\}$.

If $\varepsilon \leq \varepsilon(n)$, it is clear that by using (4.14), we can find subsets $\boldsymbol{\mathscr{L}}^k \supset \mathscr{L}^k$, which are identified with each other under the correspondence between loops at y_1 and y_2, and such that for $\gamma_j^k \in \boldsymbol{\mathscr{L}}^k$,

$$a(\gamma_j^k) \leq 3c_2 = 6(6\lambda(n))^{[n/2]}. \tag{5.20}$$

Let $b_1 \leq \cdots \leq b_M$ $(M \leq N)$ be the sequence obtained by arranging the numbers $\{a(\gamma_j^1)\}$, $\gamma_j^1 \in \boldsymbol{\mathscr{L}}^1$, in ascending order. Let $b_1' \leq \cdots \leq b_M'$ be obtained similarly from $\boldsymbol{\mathscr{L}}^2$. Let π be the permutation of $\{1, \cdots, M\}$ induced by the correspondence between $\boldsymbol{\mathscr{L}}^1$ and $\boldsymbol{\mathscr{L}}^2$. Our claim now is a direct consequence of Lemma 5.10.

(c) **Assigning elementary** F**-structures to points.** A map $f\colon U \to T_u(S^m)$ as in Proposition 3.4 is determined by a number $k > 0$ and a decreasing function $h(r)$. In what follows, it will suffice to choose h to be a sufficiently small constant, and to take

$$k = 18(6\lambda(n))^{[n/2]}. \tag{5.21}$$

For each $y \in Y_\delta^n$ ($\delta \leq \delta_0$ sufficiently small) we can, by Proposition 3.4, find a map $f_y\colon U_y \to T_{u_y}(S_y)$. Note that by our choice of k, each loop of $[\gamma_j]_y$ is contained in U. Let $[\gamma_j]_{f_y(y)}$ denote the collection of loops at $f_y(y)$ which are homotopic to the images of $[\gamma_j]_y$. By Corollary 2.13 $[\gamma_j]_y$ determines an elementary F-structure, $\boldsymbol{\mathscr{F}}_y$, over a neighborhood $\mathbf{V}_y \overset{\text{def}}{=} T_{r_y}(\mathscr{O}_{f_y(y)})$. Here we take

$$r_y = \frac{t}{6}(6\lambda(n))^{-[n/2]} \min_\gamma a(\gamma), \tag{5.22}$$

where the minimum is over all loops γ at y. The number $t < 1$ will be specified below. Note that $r_y \leq \frac{t}{3} i_y$.

Let $\mathscr{F}_y$ be the elementary F-structure over $V_y = f_y^{-1}(\mathbf{V}_y)$. The fiber $(\mathscr{f}_y)_y$ of $\mathscr{f}_y$ at y can be identified with the Cartesian product of the loops in $[\gamma_j]_y$. Thus, it follows from Lemma 5.19 that if $\overline{y_1, y_2} < 2(r_{y_1} + r_{y_2})$ and $t < \varepsilon(n)$ for $\varepsilon(n) > 0$ sufficiently small, then $\mathscr{f}_{y_1} \subseteq \mathscr{f}_{y_2}$ on $V_{y_1} \cap V_{y_2}$ or vice versa. This is the precursor of property (F_1) of §1.

(d) **Property** (F_1) **for** $\{(V_y, \mathscr{F}_y)\}$. By Lemma 2.9, the set of values which the function $a(\gamma)$ takes on loops of $A \subset \pi_1(T_{u_y}(S_y))$ is constant on orbits of the elementary F-structure $\boldsymbol{\mathscr{F}}_y$. Let $N = N(n)$ and let t be as in parts (b) and (c). For each point $y_1 \in V_y$, consider the set consisting of the N smallest values (counted with multiplicities) of the function a. Then if $\varepsilon(n)$ is sufficiently small, $t < \varepsilon(n)$, and f_y is sufficiently C^2-close to being an isometry, the above set of values is as close as we like to being independent of the point y_1. Now, the argument of part (c) shows that $\{(V_y, \mathscr{F}_y)\}$ has property (F_1).

(e) **Closeness of local actions.** The fiber of $\mathscr{f}_y$ at y can be identified with the Cartesian product of at most $N(n)$ loops (see (5.17)) of length bounded by (5.12). By (2.3), (2.4), Corollary 4.11, and (4.16) we can insure that the local actions of $\mathscr{f}_{y_1}$ and $\mathscr{f}_{y_2}$ are as C^1-close as we like on $V_{y_1} \cap V_{y_2}$, provided that h of Proposition 3.4 and δ_0 above are sufficiently small. (In measuring the closeness of local actions we rescale the metric so that, say, $i_{y_1} = 1$, to conform to the context of Lemma 1.5.)

The degree of closeness required in Lemma 1.5 depends on the maximum fiber dimension, N_3, on the maximum order, N_2, of a covering space associated to the elementary F-structure and on the multiplicity, N, of the covering. In our situation $N_3 \leq N(n)$ and $N_2 \leq \lambda(n)$. In part (f) below, we will extract a subcovering, $\{V_{y_\alpha}\}$ of $\{V_y\}$, with bounded multiplicity.

(f) **The subcover** $\{v_{y_\alpha}\}$. Let $q(y)$ denote the number of loops in $[\gamma_j]_y$ and let $Y^n_{\delta, q} \subset Y^n_\delta$ be the set of points, y, with $q(y) = q$.

Let q_0 be the largest value of q for which $Y^n_{\delta, q}$ is nonempty. Choose a maximal set of points from $Y^M_{\delta; q_0}$ such that

$$\overline{\mathscr{O}_{y_\alpha}, \mathscr{O}_{y_\beta}} \geq \tfrac{1}{2} \min(r_{y_\alpha}, r_{y_\beta}), \tag{5.23}$$

where r_{y_α} and r_{y_β} are as in (c) above. Then choose a maximal set of points from $Y^n_{\delta, q-1}$ such that (5.24) continues to hold for all points (in

$Y^n_{\delta, q_0} \cup Y^n_{\delta, q_0-1}$) selected so far. By proceeding in this way, we obtain a set of points $\{y_\alpha\}$. Clearly, for every point $y \in Y^n_\delta$ there exist y_α with $q(y_\alpha) \geq q(y)$ and

$$\overline{\mathscr{O}_y, \mathscr{O}_{y_\alpha}} \leq \tfrac{1}{2}\min(r_y, r_{y_\alpha}). \tag{5.24}$$

Since $q(y_\alpha) \geq q(y)$, it is clear that for δ sufficiently small, say

$$\overline{y, \mathscr{O}_{y_\alpha}} \leq \tfrac{2}{3}\min(r_y, r_{y_\alpha}) \tag{5.25}$$

(and the same holds for all points of $\mathscr{O}_y$). Thus, $\{V_{y_\alpha}\}$ covers Y^n_δ and in fact, $\{f^{-1}_{y_\alpha}(T_{3r_\alpha/4}(\mathscr{O}_{f_{y_\alpha}(y_\alpha)}))\}$ still covers.

Now, by using the standard packing argument as in [13, Theorem 5.3], the multiplicity of $\{V_{y_\alpha}\}$ can now be bounded by some $N_1(n)$.

(g) **Fitting together local F-structures.** The collection $\{(V_{y_\alpha}, \mathscr{F}_{y_\alpha})\}$ constructed in (f) above satisfies the hypothesis of Lemma 1.5. Thus, we obtain a weak F-structure, $\mathscr{F}$, on a set containing Y^n_δ, for $\delta < \delta_0(n)$ sufficiently small. Since the elementary F-structures, $\mathscr{F}_{y_\alpha}$, have positive rank, so does $\mathscr{F}$. The bound on the diameter of orbits (see (1) of Theorem 0.1) is also satisfied.

To see that the structure we have constructed is actually an F-structure, we observe that property (F_3) of §1 holds if t of (5.22) is sufficiently small. Note that the maximal length of a chain $V_{\alpha_0}, \cdots, V_{\alpha'_l}$, as in (F_3) is, of course, bounded by $N(n)$, the maximal dimension of the fiber. Now it is clear from Corollary 2.13 that if t in (5.22) is taken to be $1/4N(n)$ times the value dictated by our previous considerations, then (F_3) and (F_4) hold.

As mentioned in §2, the local actions might be noneffective for the structure just constructed, but this can be remedied by passing to a quotient.

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THE EXISTENCE OF POLARIZED F-STRUCTURES ON VOLUME COLLAPSED 4-MANIFOLDS

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Abstract

By the work of [CG2] and [CG3], the Riemannian manifolds whose injectivity radii are small everywhere relative to the bound on sectional curvature admit positive rank F-structures. We will prove, in dimension four, that if the volumes of the manifolds are also smaller than a positive constant, then the manifolds actually admit polarized F-structures. In particular, this result implies an affirmative answer to the Gromov's Gap conjecture on vanishing minimal volume in dimension four.

0. Introduction

Let M be a manifold. The *minimal volume* of M, $\mathrm{MinVol}(M)$, is defined to be the infimum of the volumes of all the complete Riemannian metrics on M with sectional curvature $|K| \leq 1$. The work about this invariant can be found in [Gr2], [Fu4] and [BCG]. Motivated by Thurston's result in [Th] on the volumes of all 3-dimensional hyperbolic manifolds, M. Gromov conjectured ([Gr2]):

GAP CONJECTURE FOR VOLUME 0.1. *There exists a real number $v_n > 0$ depending only on n such that if an n-dimensional manifold M satisfies* $\mathrm{MinVol}(M) < v_n$, *then* $\mathrm{MinVol}(M) = 0$.

The Gromov's Gap Conjecture for volume is one of several Gap conjectures concerning collapsing Riemannian manifolds. From now on, unless specified elsewhere, all Riemannian manifolds will be assumed to be complete and have bounded sectional curvature, say $|K| \leq 1$ after normalizing the metrics. Let $\alpha(g)$ denote one of the following geometric measures of a Riemannian metric g: the diameter $D(g)$, the supremum $I(g)$ of injectivity radii at all points, or the volume $V(g)$. We consider the Riemannian manifolds whose $\alpha(g)$ are *sufficiently collapsed*, i.e., $\alpha(g)$ is smaller than a small constant depending only on the dimension of M. We say M admits an $\alpha(g)$*–collapse* if there exists a family of metrics $\{g_\delta\}$ on M, $0 < \delta \leq 1$, such that the sequence $\{\alpha(g_\delta)\}$ converges to zero as $\delta \to 0$.

The Gap conjecture for sufficiently $\alpha(g)$-collapsed Riemannian manifolds is

GAP CONJECTURE 0.2. *There exists a constant α_n, depending only on n, such that if a complete n-manifold M has $\alpha(g) < \alpha_n$, then M admits a α-collapse.*

The affirmative answers are known for $\alpha(g) = D(g), I(g)$. Gromov's work [Gr1] showed that a *sufficiently diameter collapsed* closed manifold is an infro-nilmanifold. Note that an infro-nilmanifold does admit a diameter collapse. The result in [CG3] asserts that a *sufficiently injectivity radius collapsed* manifold admits a kind of topological structure, $\mathcal{F}$, called a *positive rank F-structure associated to g* (compare with [CFG]). Further, using such a positive rank F-structure one is able to construct an injectivity radius collapse ([CG2]). Roughly speaking, an F-structure on a manifold M can be thought of as a family of local torus actions on M satisfying certain consistency conditions on overlaps so that M is partitioned into *orbits* of the actions (see §1). An F-structure is said to have *positive rank* if every orbit has positive dimension. A *pure* F-structure means that its local torus groups all have the same dimension.

An F-structure is said to be *polarized* if all isotropy groups of the local tori actions are of finite order. Assuming a polarized F-structure, one is able to construct an invariant volume collapse ([CG2], [Fu2]). Since on a 3-manifold, any positive rank F-structure has a polarized substructure ([Ro1]), the Gromov's Gap conjecture for $n = 3$ is a consequence of the main results in [CG2] and [CG3]. In [Fu3], Fukaya partially verified Gromov's Gap Conjecture for volume for aspherical manifolds under the strong assumption that the constant v_n also depends on the diameter of the manifold. Recently, Buyalo ([Bu1], [Bu2]) verified the Gromov's Gap Conjecture for volume for non-positively curved 4-manifolds. Essentially, the main approach taken by both [Bu1], [Bu2] and [Fu] is to show that an associated F-structure is actually *polarized* in those special circumstances.

The main result of this paper is given by the following theorem.

THEOREM 0.3. *There exists a real number $v > 0$ such that if M is any a 4-manifold with* $\mathrm{MinVol}(M) < v$, *then M admits a polarized F-structure.*[1]

Combining Theorem 0.3 with a result in [CG2] (see Theorem 1.3),

[1] In [Ro2], it will be shown that the polarized F-structure can actually be chosen as a substructure of the associated F-structure constructed in [CG3].

Theorem 0.3 has as a corollary the Gromov's Gap conjecture for volume in dimension four.

COROLLARY 0.4. *There exists a real number $v > 0$ such that if a 4-manifold M satisfies* $\mathrm{MinVol}(M) < v$, *then* $\mathrm{MinVol}(M) = 0$.

Remark 0.5: In contrast to the special situations considered by [Bu1], [Bu2] and [Fu3], if no additional restrictions on the metric, g, are assumed, the associated F-structure may not be polarized no matter how small the volume of (M, g) (see Example 1.4). This implies that our polarization in Theorem 0.3 may not be achieved solely through the geometrical constructions given in [CG3], [CFG] and [Fu]. In particular, the arguments used by [Bu1], [Bu2] and [Fu] cannot apply in our situation.

Our approach to Theorem 0.3 is to study in detail the *singularities* of an associated F-structure $\mathcal{F}$ on a manifold with small volume. The *singular set* $Z(\mathcal{F})$ of an F-structure $\mathcal{F}$ is by definition the union of all singular orbits of local torus actions. If $Z(\mathcal{F}) = \emptyset$ (i.e., $\mathcal{F}$ is polarized), then a result in [CG2] implies Gromov's Gap conjecture. In general, a component of $Z(\mathcal{F})$ is a union of compact totally geodesic submanifolds (with respect to any invariant metric). In our four dimensional case, each component is either a two torus, a Klein bottle or a cylinder. If the manifold is compact, then the closure of a cylinder component is a compact cylinder with boundary. We emphasize that there are infinitely many possibilities for the topological structure in a tubular neighborhood of a singular component (see §2).

Roughly speaking, we find that if a complete 4-manifold has sufficiently small volume, then it turns out that the singular structure of an associated F-structure is very special so that one can modify $\mathcal{F}$, in a saturated neighborhood U $(\neq M)$ of $Z(\mathcal{F})$, to obtain a new polarized F-structure which coincides with $\mathcal{F}$ on $M - U$. (Note that it does not follow a priori, that if such a neighborhood exists, it can always be chosen to be a tubular neighborhood $T_\epsilon(Z(\mathcal{F}))$.) We emphasize that in dimension four most singular structures do not have such a property although they are compatible with metrics which are arbitrarily injectivity radius collapsed.

To explain the above, we first consider an arbitrary positive rank F-structure $\mathcal{F}$ (on a 4-manifold). Clearly, we can assume that each component Z_0 of $Z(\mathcal{F})$ is *irremovable*, that is, the restriction of $\mathcal{F}$ to $T_\epsilon(Z_0)$ of Z_0, $\mathcal{F}|T_\epsilon(Z_0)$, has no polarized substructure. A non-singular S^1-orbit of $\mathcal{F}$ is called *exceptional* if, the isotropy group of the local S^1-action around the orbit is non-trivial. Consider the closure of union of all exceptional S^1-orbits. Let $E(\mathcal{F})$ denote the union of those components of all exceptional S^1-orbits which have non-empty intersection with $Z(\mathcal{F})$. Let $W(\mathcal{F}) = Z(\mathcal{F}) \cup E(\mathcal{F})$, and let W_0 denote a component of W. We will classify all possibilities for the restriction $\mathcal{F}|T_\epsilon(W_0)$. For each W_0 with compact closure, the structure in $T_\epsilon(W_0)$ can be characterized by an integer-valued invariant, $k(W_0)$. As a result, we are able to conclude that for W_0 with compact closure, one can modify $\mathcal{F}$ in $T_\epsilon(W_0)$ to obtain a polarized F-structure if and only if $k(W_0) = 0$. Moreover, it turns out that if the closure of W_0 is not compact, then one can always modify $\mathcal{F}$ in $T_\epsilon(W_0)$ to obtain a polarized F-structure.

Now suppose that $\mathcal{F}$ is the F-structure associated to a sufficiently injectivity radius collapsed metric. The integer invariant $k(W_0)$ does not directly relate to the volume of the metric. As one consequence of the compatibility of the metric and the structure ([CG3]), there is a second topological invariant, the *residue*, which will relate the structure of $\mathcal{F}|T_\epsilon(W_0)$ to the volume ([Ya]). The residue of a singular component, $\mathrm{Res}(Z_0)$, is a topological invariant of the structure, $\mathcal{F}|T_\epsilon(Z_0)$. The residue of a saturated open subset U is defined by the sum of the residues of the singularities contained in U. We will show that the residue of $T_\epsilon(W_0)$ determines the previous invariant of W_0, and $\mathrm{Res}(W_0) \in 1/2\mathbb{Z}$. In particular, we can modify $\mathcal{F}$ in $T_\epsilon(W_0)$ to obtain a polarized structure if and only if $|\mathrm{Res}(T_\epsilon(W_0))| < 1/2$ (in which case $\mathrm{Res}(W_0) = 0$). Here the fact that the set we consider is of form $T_\epsilon(W_0)$ is crucial since there are examples of saturated subsets whose residues can be arbitrarily small (or even zero) in absolute value but the structures on them cannot be modified to obtain a polarized structure (see Example 3.4).

Next, we will show that if a saturated compact manifold U with boundary satisfies the conditions: the second fundamental form of ∂U and the second fundamental form of $\mathcal{F}$-orbits are bound by a constant, each orbit of the restriction $\mathcal{F}|\partial U$ has *normal injectivity radius* (see **f** in §4) lower bounded by a universal constant, and the volume of U is smaller than a constant depending on C and ρ, then $|\mathrm{Res}(U)| < 1/2$. In the proof of this inequality, we will use the Atiyah-Patodi-Singer theorem and results in [CG2] and [Ro1] on limiting eta-invariants. The recent results in [CFG] and [CG4] enter to guarantee the existence of such neighborhood around each W_0.

By way of further explanation as to why we consider the sets W_0, we mention the following. We first point out that if a component Z_0 is removable, then $\mathrm{Res}(Z_0) = 0$. However, the converse, while it is true, cannot be seen directly; it requires additional work which is based on the main result of this paper. In any case, since $|\mathrm{Res}(Z_0)|$ can take arbitrarily small non-zero values, we cannot restrict attention to such individual singular component in proving Gromov's Gap conjecture.

This paper is organized as follows: In §1, we will recall some necessary notation about F-structures and the main results in [CG2] and [CG3].

In §2, we will prove a classification result for components of $T_\epsilon(W(\mathcal{F}))$.

As a result, we obtain a criterion for $\mathcal{F}$ to be modified in $T_\epsilon(W(\mathcal{F}))$ to obtain a polarized structure.

In §3, we will briefly recall the residue theory associated to F-structure in [Ya]. From the residue formula in [Ro1] we then obtain the explicit residue for $T_\epsilon(W_0)$, and restate the criterion of §2 in terms of the residue.

Finally, in §4, we will establish an inequality in which the absolute value of the residue bounds from below the volume of an invariant metric on M. This easily suffices to complete the proof of Theorem 0.3.

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1. Preliminaries

In this section, we will recall the notion of F-structure and the main results in [CG2], [CG3] (compare with [CFG]) which will be used in this paper.

a. F-structures on collapsed Riemannian manifolds.

Given a small number $\epsilon > 0$. For any Riemannian manifold M, there is a natural way to decompose M into the so-called *ϵ-thin part* M_ϵ and *ϵ-thick part* $M(\epsilon)$, where M_ϵ consists of all the points of M whose injective radius are smaller than ϵ and $M(\epsilon) = M - M_\epsilon$. The basic problem about the thin-part of Riemannian manifolds is to understand the interplay of the collapsing geometry and topology of underlying manifolds. This study was begun in [Gr1], and it has attracted much attention since then. Some fundamental results were obtained in [CG2], [CG3], [CFG] and [Fu1]–[Fu4]. There, it is found that the topological aspects of a sufficiently injectivity radius collapsed metric are, to a large extent, captured by the local isometric structure of a nearby metric. This structure is called an F-structure by Cheeger and Gromov ([CG2], [CG3]).

On a (complete) flat manifold, one can always obtain a collapse by scaling the metric by small positive numbers δ, $0 < \delta \leq 1$. An F-structure $\mathcal{F}$ on a manifold M is a kind of combination of local flat structures. More precisely, an F-structure on a manifold is a collection of triples, $\mathcal{F} = \{(\widetilde{U}_\alpha, U_\alpha, T^{k_\alpha})\}$, where $\{U_\alpha\}$ is a local finite cover (each U_α is also called a *chart* of $\mathcal{F}$) of M and $\pi_\alpha : \widetilde{U}_\alpha \to U_\alpha$ is a finite normal covering on which T^{k_α} acts which extends the deck transformations on $\widetilde{U}_\alpha$ (thus, the T^{k_α}-orbits is invariant under the deck transformation). The local torus actions satisfy the following consistency condition: if $U_\alpha \cap U_\beta \neq \emptyset$, then $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\pi_\beta^{-1}(U_\beta \cap U_\beta)$ have a common covering space on which the lifting of T^{k_α} is a subgroup of the lifting of T^{k_β} or vice versa.

The consistency condition implies that M is partitioned into orbits. $\mathcal{F}$ is said to have positive rank if every orbit of $\mathcal{F}$ has positive dimension. The singularity $Z(\mathcal{F})$ of $\mathcal{F}$ is the union of all singular orbits of local torus actions. In particular, if every isotropy subgroup of the local action is finite, $\mathcal{F}$ is said to be polarized. If the local tori act (almost) as isometries, we say $\mathcal{F}$ is *(almost) compatible* with the metric, or equivalently, that the metric is *(almost) invariant* with respect to $\mathcal{F}$. An F-structure is said to be pure if the local torus groups have the same dimension. A *polarization* of a positive rank F-structure is a collection of connected subgroups of the local torus groups such that the dimension of each subgroup is equal to the dimension of its orbits. If all the subgroups are compact, we call this polarization a *polarized substructure*.

There is another equivalent definition for F-structures which is useful in residue theory ([Ya]). If M is equipped with an invariant metric, the Lie algebra of the local torus group determines the local Killing vector fields on each chart U of $\mathcal{F}$. The consistency condition means these local Killing fields commute on overlaps (see [CG3], [Ya]). The fundamental result concerning the existence of a sufficiently injective radius collapsed metric is the following.

THEOREM 1.1 ([CG2], [CG3], [CFG]). *There exists a constant $i_n > 0$ depending only on n such that for all $0 < \epsilon \leq i_n$, if a complete n-manifold M with sectional curvature $|K| \leq 1$ has injectivity radii smaller than ϵ at all points, then M admits a positive rank F-structure $\mathcal{F}_\epsilon$ and an invariant metric g_ϵ such that*

(1.1.1) *There exist a constant $r_n > 0$ and a positive integer k_n depending only on n such that for all $p \in M$ there is a chart $(\widetilde{U}_i, U_i, T^{k_i}, \phi_i)$, such that $B_p(r_n) \subset U_i$, diam$(\mathcal{O}_x) \leq \epsilon$ and $\widetilde{U}_i \to U_i$ is at most a k_n-fold cover.*

(1.1.2) $e^{-\epsilon} g_\epsilon \leq g \leq e^{\epsilon} g_\epsilon, ||\nabla - \nabla^\epsilon|| < \epsilon, ||\nabla^k R^\epsilon|| \leq C(n, k, \epsilon)$, *where ∇ and ∇^ϵ are the Levi-Civita connections of g and g_ϵ respectively.*

Conversely, suppose a manifold admits a positive rank F-structure. Then it admits an invariant injectivity radius collapse.

We will call the constant i_n the *critical injectivity radius* of dimension n, and the positive rank F-structure corresponding to $\epsilon = i_n$, $\mathcal{F}_a$, an *associated F-structure*.

Remark 1.2: The first part of Theorem 1.1 was generalized considerably in [CFG] recently. There, an associated F-structure is realized as a substructure of some so-called *N-structure*. Basically, an orbit of a N-structure is an infro-nilmanifold whose center forms the orbit of the associated F-structure.

The basic difference between these two structures is that the orbits of the N-structure absorb all *collapsed directions* of the metric while the orbits of F-structure only point the most collapsed directions proportional to the injectivity radius (see §4 and [CFG] for more explanation).

We emphasize that the invariant injectivity radius collapse as in the second part of Theorem 1.1 may not be a volume collapse (see §0) or may not even have finite volume. In order to obtain an invariant volume collapse, certain properties of the F-structure are required.

THEOREM 1.3 ([CG2]). *Let M be a manifold. Suppose M admits a polarized F-structure. Then M admits an invariant volume collapse.*

Note that in [CG2], a condition on an F-structure weaker than the existence of a polarized F-structure was found which also yields a volume collapse. However, it seems very hard to find the assumptions on a collapsed metric which will guarantee that the associated F-structure satisfies this condition.

Because of Theorem 1.3, it is natural to ask whether for a collapsed metric with sufficiently small volume, an associated F-structure is actually polarized (compare [Bu1], [Bu2] and [Fu4]). The following simple example shows the answer is negative.

EXAMPLE 1.4: Take a one-sphere S^1 and a two sphere S^2, and set $M = S^2 \times S^1$. We will construct a volume collapse on M, $g_\delta, \delta \to 0$, such that for any small δ, the associated F-structure on M has singularity.

Let H be the S^1 subgroup of $SO(3)$ defined by

$$H = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \quad 0 \le t \le 2\pi \right\} .$$

Equipped with the product metric of the standard metrics on S^2 and S^1, $T^2 = H \times S^1$ acts as isometries on M by H acting on the first factor and by multiplication on the second factor. Take a dense R^1-subgroup R_θ of T^2 and split the metric into $g = g_0 \oplus g_1$, where g_0 is the restriction of g to the orbits of R_θ^1 and g_1 is its orthogonal component. We then construct a continuous collapse: $g_\delta = \delta g_0 \oplus g_1$. Clearly, $\mathrm{Vol}(M, g_\delta) \to 0$ as $\delta \to 0$, and the limiting space is a closed interval because of the density of R_θ^1 in T^2. Consequently, for all sufficiently small δ, the associated F-structures $\mathcal{F}_\delta$ to g_δ is of pure T^2-structure with singular set $Z(\mathcal{F}_\delta)$ consisting of the S^1-factors at the two poles of S^2.

Finally, we point out that in Example 1.4, $\mathcal{F}_\delta$ actually contains many obvious polarized substructures.

2. The Singularities of Positive Rank F-structures In Dimension Four

From now on, unless specified otherwise, M will be a 4-manifold and $\mathcal{F}$ will be an arbitrary positive rank F-structure on M with singularity $Z(\mathcal{F})$.

For a S^1-action on an open subset of M, a S^1-orbit is called *exceptional* if its isotropy group is finite non-trivial. Consider a non-singular S^1-orbit $\mathcal{O}_x$ of $\mathcal{F}$. Let $\pi_\alpha : \widetilde{U}_\alpha \to U_\alpha, x \in U_\alpha$ be any rank-one chart containing $\mathcal{O}_x$ (the *rank* of a chart is defined to be the dimension of the torus group). We call $\mathcal{O}_x$ an *exceptional orbit* of $\mathcal{F}$ if $\mathcal{O}_x$ is the projection of an exceptional orbit of the S^1-action on $\widetilde{U}_\alpha$. It is easy to check that this definition is independent of the choice of charts involved. As before, $E(\mathcal{F})$ will denote the union of the components of all exceptional orbits which have non-empty intersection with the closure of $Z(\mathcal{F})$, and $W(\mathcal{F}) = Z(\mathcal{F}) \cup E(\mathcal{F})$. The explanation for introducing $W(\mathcal{F})$ was given in §0. Further explanation will be given at the end of **b.**

In this section, we will study $W(\mathcal{F})$ and classify all possible structures on $T_\epsilon(W_0)$, where W_0 represents a component of $W(\mathcal{F})$ (Theorem 2.7). As a result, we will obtain a necessary and sufficient condition for modifying $\mathcal{F}$ in $T_\epsilon(W(\mathcal{F}))$ to obtain a polarized structure (Corollary 2.10).

a. The removable singularities.

We will first rule out a kind of trivial singularity. A component Z_0 of $Z(\mathcal{F})$ is said to be *removable* if $\mathcal{F}|T_\epsilon(Z_0)$ contains a polarized substructure. Otherwise, we call Z_0 *irremovable*. Clearly, if all components of $Z(\mathcal{F})$ are removable, then $\mathcal{F}$ contains a polarized substructure. In this part, we will first show that an irremovable singularity is contained in charts of rank two. Then, we will describe the possible structures of removable singularities.

First, we observe a simple fact.

LEMMA 2.1. *Assume M admits an effective T^3-action without fixed points. Then a singular orbit (if any) is an isolated two torus whose tubular neighborhood is homeomorphic to $D^2 \times T^2$.*

Proof: Assume $\mathcal{O}_x$ is a singular orbit through $x \in M$. We can choose a slice S_x of $\mathcal{O}_x$ at x such that S_x is homeomorphic to a unit ball in $\mathbf{R}^k$, where $k = 4 - \dim(\mathcal{O}_x)$, $\dim(\mathcal{O}_x) = 1$ or 2 (see [Br]). We first rule out the case $\dim(\mathcal{O}_x) = 1$. In this case, the identity component of the isotropy group at x is a two torus which acts effectively on S_x (a ball in $\mathbf{R}^3$); this is impossible. Further, for $\dim(\mathcal{O}_x) = 2$, because $G_x \simeq S^1$ has only single fixed point in $S_x \simeq D^2$, $\mathcal{O}_x$ is some isolated singular orbit. □

COROLLARY 2.2. *For $x \in M$, let $\mathcal{O}_x$ be a singular orbit of $\mathcal{F}$. If $\mathcal{O}_x$ is contained in a chart of rank three $(\widetilde{U}_\alpha, U_\alpha, T^3)$, then $\mathcal{O}_x$ is isolated and* *removable. In particular, any positive rank pure T^3-structure on M contains a polarized substructure.*

Proof: Assume $\mathcal{O}_x$ is in a chart, $(\widetilde{U}_\alpha, U_\alpha, T^3)$. By Lemma 2.1, $\mathcal{O}_x$ is some isolated singular orbit and is either a two torus T^2 or a Klein bottle K^2. In either case, let $\mathcal{O}_{\tilde{x}}$ be the singular orbit of the T^3-action in $\widetilde{U}_\alpha$, $\pi(\mathcal{O}_{\tilde{x}}) = \mathcal{O}_x$. From Lemma 2.1, $T_\epsilon(\mathcal{O}_{\tilde{x}}) \simeq D^2 \times \mathcal{O}_{\tilde{x}}$ for small ϵ. Clearly, we can remove the singularity $\mathcal{O}_x$ by replacing T^3 by a T^2-subgroup which acts freely on $\widetilde{U}_\alpha$ and is preserved by the deck transformations acting on $\widetilde{U}_\alpha$. □

We now consider a removable singular component Z_0 which is contained in some rank-two charts. Clearly, Z_0 is either a two torus (or a Klein bottle) or a cylinder. An embedded cylinder is called *finite* if its closure is a compact subset. Otherwise, we call it an *infinite cylinder*.

If Z_0 is a two torus, $T_\epsilon(Z_0)$ is homeomorphic to a bundle over a circle with fiber a solid torus. Since Z_0 is removable, we then conclude that the fiber bundle structure is trivial (compare with Structure I below).

Similarly, if Z_0 is an infinite cylinder, then $T_\epsilon(Z_0)$ is homeomorphic to a bundle over a line or semi-line with fiber a solid torus.

If Z_0 is a finite cylinder, the boundary of $T_\epsilon(Z_0)$ is actually a lens space, on which the restriction $\mathcal{F}|U$ is a mixed F-structure. This mixed structure on $\partial T_\epsilon(Z_0)$ can be expressed by $\{(D^2 \times S^1, S^1), (I \times T^2, T^2), (D^2 \times S^1, S^1)\}$, where $\partial T_\epsilon(Z_0) = D^2 \times S^1 \cup D^2 \times S^1$ and $I \times T^2 = D^2 \times S^1 \cap D^2 \times S^1$. Clearly, $\mathcal{F}|T_\epsilon(Z_0)$ has a polarized substructure if and only if the two S^1-actions agree on $I \times T^2$.

b. The three models of singularities.

In view of the preceding discussion, we will assume $\mathcal{F}$ has only irremovable singularities, and thus each singular component is either a two torus (or a Klein bottle) or a (finite or infinite) cylinder which is contained in rank-two charts. Note that, if necessary, we can always replace $\mathcal{F}$ by a substructure all of whose singular components are irremovable. This reduction is necessary to achieve a classification result for $T_\epsilon(W(\mathcal{F}))$ (see Remark 2.6).

We will first describe three different models of singularities. Then, we will look at all possible substructures of the three models. It turns out that these singular structures actually cover all the possibilities in dimension four.

STRUCTURE I. This singular structure appeared in Example 1.9 of [CG2].

Take $D^2 \times S^1$ with multi-polar coordinates (r, θ_1, θ_2), and form the product $(D^2 \times S^1) \times [0, 1]$. Then, identify the two ends, $(D \times S^1) \times \{0\}$ with

$(D \times S^1) \times \{1\}$, by the map $(r, \theta_1, \theta_2) \to (r, \phi_k(\theta_1, \theta_2))$, where

$$\phi_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : (\partial D \times S^1) \times \{0\} \to (\partial D \times S^1) \times \{1\} \ , \qquad (2.3)$$

and k is an integer. We use $U_{1,k}$ to denote the resulting compact manifold with boundary. From the construction, we see that $U_{1,k}$ is actually the total space of a bundle over S^1 with fiber a solid torus. Thus, the boundary $\partial U_{1,k}$ is a 3-nilpotent manifold with characteristic matrix ϕ_k. Note that $U_{1,k}$ has a pure T^2-structure $\mathcal{F}_{1,k}$ which is given by the obvious local T^2 action on fibers. The singular set $Z(\mathcal{F}_{1,k})$ is a two torus which is the union of the singular orbit of T^2-action on fibers.

To see that $\mathcal{F}_{1,k}$ $(k \neq 0)$ cannot be modified in $T_\epsilon(Z(\mathcal{F}_{1,k}))$ to obtain a polarization, we will recall Example 1.9 in [CG2]. There, for each $k \neq 0$, a closed orientable 4-manifold, M_k, with non-zero signature was found which admits a positive rank F-structure $\mathcal{F}$ such that $\mathcal{F}|T_\epsilon(Z_0)$ is isomorphic to $(U_{1,k}, \mathcal{F}_{1,k})$, where Z_0 is a component of $Z(\mathcal{F})$. By Theorem 1.3 and Hirzebruch's signature theorem, we then conclude that M_k does not admit any polarized F-structure. This implies that one cannot modify $\mathcal{F}_{1,k}$ in $T_\epsilon(Z(\mathcal{F}_{1,k}))$ to obtain a polarization for $k \neq 0$. On the other hand, it is obvious that $\mathcal{F}_{1,0}$ contains polarized substructures.

STRUCTURE II. This model is similar to Structure I except the singularity is a finite cylinder.

To begin with, we take two copies, Y_1 and Y_2, of $N \times S^1$, where $N \simeq D^2 \times S^1$ is a solid torus. We will glue Y_1 and Y_2 together in the following way: First, fix any $p_i \in \partial N_i$, and let $T_\epsilon(p_i \times S^1)$ denote the ϵ-tubular neighborhood of $p_i \times S^1$ in $\partial(Y_i \times S^1)$, $T_\epsilon(p_i \times S^1) = D^2_\epsilon \times S^1$. Then, we take $D^2_\epsilon \times S^1 \times [0,1]$, and identify its one end, $(D^2_\epsilon \times S^1) \times \{0\}$, with $T_\epsilon(p_1 \times S^1)$ and glue the other end, $D^2_\epsilon \times S^1 \times \{1\}$, with $T_\epsilon(p_2 \times S^1)$ via the map ϕ_k as in (2.3). Clearly, the result carries an obvious mixed F-structure, $\mathcal{F}_{2,k}$, which is the S^1-rotation on Y_i, and the pure T^2 elsewhere. Note that the singular set, $Z(\mathcal{F}_{2,k})$, is a finite cylinder. We will use $U_{2,k}$ to denote a tubular neighborhood $Z(\mathcal{F}_{2,k})$.

It turns out that if $\mathcal{F}_{2,k}$ can be modified in $T_\epsilon(Z(\mathcal{F}_{2,k}))$ to obtain a polarization, then the same conclusion holds for $\mathcal{F}_{1,k}$. Therefore, $\mathcal{F}_{2,k}$ can be modified in $T_\epsilon(Z(\mathcal{F}_{2,k}))$ to a polarized structure if and only if $k = 0$.

Remark 2.4: Note that in the construction of Structure II, the choice for Y_i does not play any significant role. In fact, the same construction works if N is replaced by any compact 3-manifold whose boundary is a closed surface of genus > 1.

STRUCTURE III. Let $U_3 \simeq (D^2 \times S^1) \times J$, where J is an interval, $J \simeq [0,1)$ or $J \simeq (0,1)$. Let $\mathcal{F}_3$ be the pure T^2-structure on U_3 given by the obvious T^2-action of $D^2 \times S^1$. The singular set $Z_3 \simeq S^1 \times J$. Clearly, Z_3 is a removable singularity.

Next, we will describe substructures of $\mathcal{F}_{i,k}$ $(i = 1, 2)$ and $\mathcal{F}_3$.

EXAMPLE 2.5: Let $(U, \mathcal{F})$ represent $(U_{i,k}, \mathcal{F}_{i,k})$ or $(U_3, \mathcal{F}_3)$. Consider the natural projection onto the orbit space, $\pi : U \to X \simeq U/\mathcal{F}$. Thus, the subset $Z/\mathcal{F} \simeq S^1$, $[0,1]$ and J corresponding to Structures I, II and III respectively. Given any locally finite open cover $\{V_j\}$ for $Z/\mathcal{F}$ and any family of S^1-actions $\{S^1, \psi_j\}$ on $D^2 \times S^1$ without fixed points, $\{(\pi^{-1}(V_j), S^1, \psi_j)\}$ determines a mixed substructure $\mathcal{F}'$ of $\mathcal{F}|\pi^{-1}(Z/\mathcal{F})$ with T^2-orbits on $V_i \cap V_j \neq \emptyset$. Note that $\mathcal{F}'$ is actually a substructure of $\mathcal{F}$ for $U = U_{1,k}$ or U_3. As for $U = U_{2,k}$, $\{(\pi_i^{-1}(V_j), S^1, \psi_j)\}$ together with the S^1-rotations on Y_1 and Y_2, determine a substructure of $\mathcal{F}_{2,k}$ on $U_{2,k}$. Clearly, in each case, $Z(\mathcal{F}')$ is a disjoint union of several finite cylinders embedded in $Z(\mathcal{F})$.

It is easy to see that any substructures of $\mathcal{F}_{i,k}$ $(i = 1, 2)$ and $\mathcal{F}_3$ can be constructed as in the above because any S^1-action on $D^2 \times S^1 \times I$, I an interval, is actually a S^1-action on $D^2 \times S^1$.

We now further explain the motivation for introducing $W(\mathcal{F})$ (compare §0). Let $(U, \mathcal{F})$ represent one of the models of Structure I, II and III. As seen earlier, we are able to modify $\mathcal{F}$ to obtain a polarization if and only if it is Structure III, or it is Structure I or II and $k = 0$. Clearly, the same is true for any substructure $\mathcal{F}'$ of $\mathcal{F}$ in Example 2.5. To be precise, we will replace $\mathcal{F}'|T_\epsilon(W(\mathcal{F}'))$ (not $\mathcal{F}'|T_\epsilon(Z(\mathcal{F}'))$!) by a S^1-action without fixed points and keep $\mathcal{F}'$ elsewhere. Note that for $\mathcal{F} = \mathcal{F}_{2,0}$, the S^1-action is actually uniquely determined. We emphasize that the conclusion that $\mathcal{F}'$ can be modified in a neighborhood to obtain a polarization cannot be achieved if the relation that $\mathcal{F}'$ is a substructure of $\mathcal{F}$ is not being explored.

We now consider an arbitrary positive rank F-structure $\mathcal{F}$. In view of the above, in order to obtain a necessary and sufficient condition for modifying $\mathcal{F}$, we have to divide the components of $Z(\mathcal{F})$ into groups in such a way that each group is embedded in the singularity of some model, as was done in Example 2.5. Thus, starting with a single singular component, we must be able to determine the whole collection of the singular components in the same group. In fact, two components, Z_1, Z_2, of $Z(\mathcal{F})$ are in the same group if and only if $T_\epsilon(Z_1)$ and $T_\epsilon(Z_2)$ lie in a component of $T_\epsilon(W(\mathcal{F}))$. For instance, as in Example 2.5, if all components of $Z(\mathcal{F}')$ are irremovable, then $W(\mathcal{F}') = Z(\mathcal{F})$ $(k \neq 0)$.

Remark 2.6: As in the above, if we allow $Z(\mathcal{F}')$ to have removable components, then $W(\mathcal{F}') \subsetneq Z(\mathcal{F})$.

c. The singular structures in dimension four.

In this part, we will prove the following classification result.

THEOREM 2.7. *Let M be a 4-manifold (without boundary), and let $\mathcal{F}$ be a positive rank F-structure on M. Then for every component W_0 of $W(\mathcal{F})$, $\mathcal{F}|T_\epsilon(W_0)$ is isomorphic to a substructure of some $(U_{i,k}, \mathcal{F}_{i,k})$ $(i = 1, 2)$ or $(U_3, \mathcal{F})$.*

Theorem 2.7 allows us to associate to each component W_0 of $W(\mathcal{F})$ an integer-valued invariant, $s(W_0)$, called the s-invariant of W_0: if $T_\epsilon(W_0)$ is homeomorphic to $U_{i,k}$, then $s(W_0) = k$, otherwise $s(W_0) = 0$. A component W_0 is called *essential* if $s(W_0) \neq 0$, and all other components of $W(\mathcal{F})$ are *inessential.*

We will first show that each component of $E(\mathcal{F})$ is either a two torus (or a Klein bottle) or a cylinder. This is actually a consequence of a result of [Fi1] and [Fi2].

LEMMA 2.8 ([Fi1], [Fi2]). *Let M be a 4-manifold on which S^1 acts without fixed points. Then, the orbit space M/S^1 is 3-manifold, and*

(2.8.1) *If M is closed, then each component of E is an embedding two torus on which the exceptional invariants are the same.*

(2.8.2) *If M is compact with $\partial M = N$, then a component of E is either an embedded two torus or a cylinder whose boundary is in N.*

LEMMA 2.9. *Let M be a 4-manifold, let $\mathcal{F}$ be a positive rank F-structure on M. Then,*

(2.9.1) *A component of $E(\mathcal{F})$ is homeomorphic to one of the following: a two torus (or Klein bottle), a finite cylinder or an infinite cylinder.*

(2.9.2) *If a component of $E(\mathcal{F})$ is a finite cylinder, then it has non-empty intersection with the closure of $Z(\mathcal{F})$.*

(2.9.3) *Let Z_0 be any irremovable singular component which is a finite cylinder. If Z_0 has empty intersection with $E(\mathcal{F})$, then $T_\epsilon(Z_0)$ is homeomorphic to $U_{2,k}$ for some $k \neq 0$.*

Proof: It is easy to see that (2.9.1) follows from Lemma 2.8.

Let E_0 be an exceptional component which is a finite cylinder, and let S_0 denote a boundary circle. Take any rank-one chart, U, which contains S_0. Let E_0' be the exceptional component in U which contains S_0. Clearly, $E_0 \subsetneq E_0'$. Take any exceptional orbit, S_0', in $E_0' - E_0$ (note that we may take S_0' to be one end of the finite cylinder E_0'). If (2.9.2) is false, then S_0' is in a T^2-orbit of $\mathcal{F}$, say $\mathcal{O}$, such that $\mathcal{F}|T_\epsilon(\mathcal{O})$ is polarized pure T^2. Since S_0' is in $\mathcal{O}$ and since S_0' is exceptional, we then conclude that $\mathcal{O}$ is an exceptional T^2-orbit. Since in dimension four any exceptional T^2-orbit is isolated, we then conclude that S_0' is an isolated exceptional S^1-orbit; a contradiction.

Consider $T_\epsilon(Z_0)$, where Z_0 is a finite cylinder. Clearly, $T_\epsilon(Z_0)$ is formed by gluing two solid tori along their boundaries. Take two rank-one charts, U_i $(i = 1, 2)$ which contain the boundary circles of Z_0 respectively. By the assumption in (2.9.3), the S^1-actions on $T_\epsilon(Z_0)$ have no exceptional orbits around the boundary circles of Z_0, and this implies the gluing map is defined in (2.3). Since Z_0 is irremovable, we then conclude $k \neq 0$. Finally, we can find an embedded Y_i in U_i (compare with Remark 2.4). □

Proof of Theorem 2.7: Let W_0 be a component of $W(\mathcal{F})$. Since each singular component is either a two torus (or a Klein bottle) or a cylinder, from (2.9.1), we immediately see that W_0 is homeomorphic to either a two torus (or a Klein bottle), a finite cylinder or an infinite cylinder.

First, if W_0 is a two torus or an infinite cylinder, then clearly $T_\epsilon(W_0)$ is homeomorphic to a model of Structure I or III. Assume W_0 is an finite cylinder. We observe that (2.9.2) implies that the boundary circles of W_0 are not exceptional S^1-orbits. Thus, replacing $\mathcal{F}|T_\epsilon(W_0)$ by the obvious pure T^2-structure, we then reduce to the situation as in (2.9.3). Therefore, we can conclude that $T_\epsilon(W_0)$ is homeomorphic to $U_{2,k}$ for some k. □

Theorem 2.7 has the following important consequence.

COROLLARY 2.10. *Let M be a 4-manifold, and let $\mathcal{F}$ be a positive rank F-structure on M. Then, $\mathcal{F}$ can be modified in $T_\epsilon(W(\mathcal{F}))$ to obtain a polarized structure if and only if all components of $W(\mathcal{F})$ are inessential.*

3. A Residue Criterion

The purpose of this section is to restate Corollary 2.10 in terms of the residue of $T_\epsilon(W_0)$; which will be used in §4 to prove Theorem 0.3. In the following, we will first briefly review the residue theory in [Ya]. Then, we will compute the residue of $T_\epsilon(W_0)$.

d. The residue theory associated to F-structures.

Let M be a compact orientable $2n$-manifold with boundary N. A given metric g on N and a $2n$-characteristic form $P(\Omega)$ determine a geometric invariant of N, the so-called *secondary geometric invariant*, defined by

$$SP(N,g) = \int_{(M,\tilde{g})} P(\widetilde{\Omega}) \qquad \text{mod } \mathbb{Z}$$

where $\tilde{g}$ is any extension of g to M which is the product metric near N. The secondary geometric P-invariant $SP(N,g)$ depends only on the metric of N and P. This invariant were studied in [CS], [ChS], [APS1] and [APS2] etc.

Assume M admits a *Killing vector field* X without zero points in N. Then $SP(N,g)$ can be made into a topological invariant as follows: First, by the standard transgression formula in [Bo] and [BC], there is $(2n-1)$-form ξ_X defined on $M - \bigcup_i Z_i$ such that $P(\widetilde{\Omega}) = d\xi_X$, where Z_i are components of zeros of X. Then, by applying the Stokes theorem, we obtain

$$\int_{M-\bigcup_i T_\epsilon(Z_i)} P(\widetilde{\Omega}) - \int_N \xi_X = \sum_{Z_i} \int_{\partial T_\epsilon(Z_i)} \xi_X \ . \tag{3.1}$$

Taking limit in (3.1) as $\epsilon \to 0$, the integrals on the left side of

$$\int_M P(\widetilde{\Omega}) - \int_N \xi_X = \sum_{Z_i} \left(\lim_{\epsilon\to 0} \int_{\partial T_\epsilon(Z_i)} \xi_X \right) = \sum_{Z_i} \mathrm{Res}_P(Z_i) \ . \tag{3.2}$$

becomes a topological invariant depending only on P, N and X, called a *secondary topological invariant*. The chain of differential forms, $P(\widetilde{\Omega}) - \xi_X$, is known as the *Bott-form*.

Formula (3.2) was generalized in [Ya] to the situation where a single Killing vector field on M is replaced by an F-structure on M such that $\mathcal{F}|N$ is polarized. If M is equipped with an invariant metric, then the local Killing vector fields over charts of $\mathcal{F}$, which are generated by the Lie algebras of local tori actions, forms a sheaf of the local Killing vector fields over M (see [CG2], [CG3] and [Ya]). First, associated to an arbitrary F-structure $\mathcal{F}$ on M, M has a natural stratified decomposition into compact submanifolds with corners, $\{M_\alpha\}$, such that each M_α is contained in a chart of $\mathcal{F}$ (this stratification is called subordinate to $\mathcal{F}$). Using any invariant metric g, [Ya] constructed the *generalized Bott-form*, $\widetilde{P}(\Omega)$, which is a chain in the *de Rham complex* of the stratification subordinate to $\mathcal{F}$ (see [FGG]), and showed that the value of $\widetilde{P}(\Omega)$ on the complex, $M = \bigcup_\alpha M_\alpha$, is a topological invariant of M depending only on $\mathcal{F}|N$ and P. We will denote this invariant by $P[M,\mathcal{F}]$. Moreover, there is also a transgression formula, $\widetilde{P}(\Omega) = d\xi_{\mathcal{F}}$, on $M - Z(\mathcal{F}), Z(\mathcal{F}) = \bigcup_i Z_i$. Therefore, by the generalized Stokes theorem for de Rham complex,

$$P[M,\mathcal{F}] = -\sum_i \mathrm{Res}_P(\xi_{\mathcal{F}}, Z_i) \ , \tag{3.3}$$

where $\mathrm{Res}_P(\xi_{\mathcal{F}}, Z_i)$ is the residue of $\xi_{\mathcal{F}}$ at Z_i.

We now explain how to use the collapsing technique in [CG2] to express $P[M,\mathcal{F}]$ as an integral (see [Ya]). First, put $M_\infty = M \cup (N \times [0,\infty))$, and parallel extend $\mathcal{F}|N$ to a polarized F-structure on $N \times [0,\infty)$. Then, using the collapsing technique in [CG2] one can construct an invariant metric g on M_∞ with finite volume satisfying

(3.4.1) $|K_{g_\infty}| \leq \Lambda_1$ for some constant Λ_1

(3.4.2) $\mathrm{Vol}(\partial M_i) \to 0$ as $i \to \infty$, where $M_i = M \cup (N \times [0,i])$;

(3.4.3) $\|II(\partial M_i)\| \leq \Lambda_2$, where $II(\partial M_i)$ is the second fundamental form of ∂M_i.

It turns out that conditions (3.4.1)–(3.4.3) imply $\lim_{i\to\infty} \int_{\partial M_i} \xi_{\mathcal{F}} = 0$ as $i \to \infty$ (see [Ya]). Thus, by taking the limit, $i \to \infty$, to the equation $P[M,\mathcal{F}] = P[M_i,\mathcal{F}_i] = \int_{M_i} P(\Omega) - \int_{\partial M_i} \xi_{\mathcal{F}}$, we then obtain

$$P[M,\mathcal{F}] = \int_{M_\infty} P(\Omega) \ . \tag{3.5}$$

Formula (3.5) has the following consequence: If $\mathcal{F}'$ is a substructure of $\mathcal{F}$ such that $\mathcal{F}' = \mathcal{F}$ on N, then $P[M,\mathcal{F}] = P[M,\mathcal{F}']$. Combining with (3.3), this yields,

$$\sum_{Z_i \subset Z(\mathcal{F})} \mathrm{Res}_P(Z_i) = \sum_{Z_j \subset Z(\mathcal{F}')} \mathrm{Res}_P(Z_j) \tag{3.6}$$

Note that equation (3.6), in turn, implies that the requirement $\mathcal{F}'|N = \mathcal{F}|N$ is superfluous, that is, $P[M,\mathcal{F}] = P[M,\mathcal{F}']$ holds even $\mathcal{F}'|N$ is a substructure of $\mathcal{F}|N$.

Finally, we define the residue of an arbitrary saturated subset U with $\partial U \cap Z(\mathcal{F}) = \emptyset$ by $\mathrm{Res}_P(U) = \sum_{Z_i \subset U} Res_P(Z_i)$.

e. A residue criterion.

Based on the result in §2, we will compute the explicit value for the residue, $\mathrm{Res}(T_\epsilon(W_0))$, where W_0 is a component of $W(\mathcal{F})$ with compact closure.

LEMMA 3.7. (3.7.1) *If $T_\epsilon(W_0)$ is homeomorphic to $U_{1,k}$, then* $\mathrm{Res}(T_\epsilon(W_0)) = k$.

(3.7.2) *If a double cover of $T_\epsilon(W_0)$ is homeomorphic to $U_{1,k}$, then* $\mathrm{Res}(T_\epsilon(W_0)) = k/2$

(3.7.3) *If $T_\epsilon(W_0) \simeq U_{2,k}$, then* $\mathrm{Res}(T_\epsilon(W_0)) = k$.

Proof: In this proof, all Riemannian metrics are assumed to be invariant. We claim it suffices to compute $\mathrm{Res}(Z(\mathcal{F}_{i,k}))$, $i = 1,2$. By Theorem 2.7, the restriction $\mathcal{F}|T_\epsilon(W_0)$ is isomorphic to a substructure of $\mathcal{F}_{i,k}$ on $U_{i,k}$ (up to a double cover). According to the discussion in **d**, we then conclude that if $T_\epsilon(W_0) \simeq U_{i,k}$, then $\mathrm{Res}(T_\epsilon(W_0)) = \mathrm{Res}(Z(\mathcal{F}_{i,k}))$.

We first compute $\mathrm{Res}(Z(\mathcal{F}_{1,k}))$. Let X be the Killing vector field generated by the (unique) S^1-action on $U_{1,k}$ which is a pure S^1-substructure

$\mathcal{F}'_{1,k}$ of $\mathcal{F}_{1,k}$. Note that the set of zeros of X coincides with $Z(\mathcal{F}_{1,k})$. Applying the standard transgression formula in [Bo] and [BC], we can write $P_1(\Omega) = -d\xi_X$, on $U_{1,k} - T_\epsilon(Z(\mathcal{F}_{1,k}))$, and derive, $\mathrm{Res}(Z(\mathcal{F}_{1,k})) = \lim_{\epsilon\to 0}\int_{\partial T_\epsilon(Z(\mathcal{F}_{1,k}))}\xi_X = k$. This is because the *Euler number* of $U_{1,k}$, which is a D^2-bundle over T^2, is k with suitable orientation.

Assume a double cover of $T_\epsilon(W_0)$ is homeomorphic to $U_{1,k}$. Let π : $U_{1,k} \to V_{1,k}$ denote the covering map, $V_{1,k} = T_\epsilon(W_0)$. Put $U_{1,k,\infty} = U_{1,k}\cup(\partial U_{1,k}\times[0,\infty))$ and $V_{1,k,\infty} = V_{1,k}\cup(\partial V_{1,k}\times[0,\infty))$, and extend π to a covering map $\tilde\pi : U_{1,k,\infty} \to V_{1,k,\infty}$. As seen in **d**, we then obtain an invariant metric g_∞ on $V_{1,k,\infty}$ satisfying (3.4.1)–(3.4.3). Clearly, the pullback metric $\tilde\pi^*(g_\infty)$ shares the same properties. From (3.3), we then derive $\mathrm{Res}(Z(\mathcal{F}_{1,k})) = \int_{U_{1,k,\infty}} P_1(\widetilde{\Omega}) = 2\int_{V_{1,k,\infty}} P_1(\Omega) = 2\,\mathrm{Res}(T_\epsilon(W_{1,k}))$. Therefore, $\mathrm{Res}(T_\epsilon(W_{1,k})) = 1/2\ \mathrm{Res}(Z(\mathcal{F}_{1,k})) = k/2$.

Finally, we will compute $\mathrm{Res}(Z(\mathcal{F}_{2,k}))$. Note that $\mathcal{F}_{2,k}$ does not have any pure S^1-polarization if $k \neq 0$. The computation will follow the process in [Ya]; and the reader is referred to [Ya] for more detail.

We first choose a stratification of $U_{2,k}$ subordinated to $\mathcal{F}_{2,k}$ as $\{M_1, M_2, M_{12}\}$, where $M_1 = Y_1$, $M_2 = (D^2_\epsilon\times S^1\times[0,1])\cup_{\phi_k} Y_2$ and $M_{12} = \bar M_1\cap\bar M_2 \simeq D^2_\epsilon\times S^1\times\{0\}$. Note that each M_i is an open manifold and $\bar M_i$ is the compact manifold with corners. Let X_1 be the Killing vector fields on Y_1 generated by S^1-rotation, and let X_2 be the Killing vector field on M_2 which extends the S^1-rotation on Y_2. The singular set of X_1 and X_2, Z_{12}, consists of the points at which X_1 and X_2 are linearly dependent. Clearly, Z_{12} coincides with $Z(\mathcal{F}_{2,k})$.

Since X_i has no zeros in M_i, $P_1(\Omega) = -d\xi_i$ on M_i, $i = 1,2$. By employing a generalized transgress formula in [Ya], we can write $\xi_1 - \xi_2 = -d\xi_{12}$ on $M_{12} - T_\epsilon(Z_{12})$, where ξ_{12} is 2-form on $M_{12} - T_\epsilon(Z_{12})$. The residue $\mathrm{Res}(Z(\mathcal{F}_{2,k}))$ is defined to be $\lim_{\delta\to 0}\int_{\partial T_\delta(Z_{12})}\xi_{12}$. To evaluate this limit, we choose a multipolar coordinate, (r,θ_1,θ_2) for $D^2_\epsilon\times S^1$, and write $X_1 = \partial/\partial\theta_2$, $X_2 = \partial/\partial\theta_1 + k\partial/\partial\theta_2$. A straightforward computation shows that

$$\mathrm{Res}(Z(\mathcal{F}_{2,k})) = \lim_{\delta\to 0}\int_{\partial T_\delta(Z_{12})}\xi_{12} = k/1 - 0/1 = k\ .$$

□

Using Lemma 3.1, we are able to restate Corollary 2.10 as

PROPOSITION 3.8. *Under the same assumption as in Theorem 2.8, we can modify $\mathcal{F}$ on $T_\epsilon(W(\mathcal{F}))$ to obtain a polarized structure if and only if for every finite component W_0 of $W(\mathcal{F})$, $\mathrm{Res}(T_\epsilon(W_0)) < 1/2$ (in which case, $\mathrm{Res}(T_\epsilon(W_0)) = 0$).*

Finally, we will give an example showing that Proposition 3.2 fails when applied to an arbitrary saturated subset.

EXAMPLE 3.9: (i) Let N be a solid torus, and let $U = N \times (0,1)$. Given two S^1-actions, $(N, S_1^1), (N, S_2^1)$ on N, we define a mixed F-structure $\mathcal{F}$ on U, $\mathcal{F} = \{(N \times (0, 2/3), S^1), (N \times (1/3, 1), S_2^1)\}$. Fix an orientation for N, any S^1 on N is determined by the so-called *Seifert invariant*, (p, q), $0 < q < p$, up to an isomorphism (see [Or]). Assume the Seifert invariants of the two S^1-actions are (p_i, q_i), $i = 1, 2$. Then, by the same process as in the proof of (3.7.3) we then find that $\mathrm{Res}(U) = q_1/p_1 - q_2/p_2$. (Note that the orientation on U is the product of the orientation of N and the standard orientation of $[0, 1]$.)

(ii) Let M be the closed orientable 4-manifold which is the double of $U_{1,k}$ ($k \neq 0$), $M = U_{1,k} \cup U_{1,k}$, and let $\mathcal{F}$ be the pure T^2-structure on M. Note that $\mathrm{Res}(M) = \mathrm{Res}(U_{1,k}) - \mathrm{Res}(U_{1,k}) = 0$. However, as seen earlier, $\mathcal{F}$ cannot be modified around $Z(\mathcal{F})$ to obtain a polarized substructure.

4. The Residues and Volume

In this section, we will relate the residue to the volume. More precisely, we will show that if a complete 4-manifold M with $|K| \le 1$ has volume smaller than a constant $v > 0$, then all components of $W(\mathcal{F}_a)$ are inessential, where $\mathcal{F}_a$ is the associated F-structure (see §1). By Corollary 2.9, this implies Theorem 0.3.

We will first prove an inequality concerning complete $4n$-manifolds which admit compatible F-structures. This inequality asserts that if a saturated subset U satisfies certain geometrical conditions, then $|\mathrm{Res}_L(U)|$ bounds from below the volume of U.

Secondly, we will show that in dimension four, one can always find a tubular neighborhood around each component W_0 of $W(\mathcal{F})$ which meets these conditions. We emphasize that the Atiyah-Patodi-Singer index formula ([APS1], [APS2]) and the recent results in [CG1], [CFG] and [CG4] play a crucial role in the proof.

f. A basic inequality.

Let M be a complete manifold with $|K| \le 1$. Assume M admits a compatible F-structure $\mathcal{F}$. Let $\mathcal{O}$ be an $\mathcal{F}$-orbit. For $x \in \mathcal{O}$, let $T_x(\mathcal{O})$ denote the tangent space of $\mathcal{O}$ and $T_x^{\perp}(\mathcal{O})$ be the orthogonal complement of $T_x(\mathcal{O})$ in $T_x(M)$. The *normal injectivity radius of* $\mathcal{O}$, denoted by $\rho(\mathcal{O})$, is the largest r such that the exponential map, $\exp_y : T_x^{\perp}(\mathcal{O}) \to M$ is an embedding into M when restricted in an open ball of radius r. (Note that since the metric is invariant, the normal injectivity radius of $\mathcal{O}$ is independent of $x \in \mathcal{O}$). The *normal injectivity radius of* $\mathcal{F}$, $\rho(\mathcal{F})$, is defined to be the infimum of the normal injectivity radii of all orbits of $\mathcal{F}$.

We now assume $\dim(M) = 4n$, and let $P_L(\Omega)$ be the Hirzebruch L-polynomial in the curvature form. In this following discussion, C_i, Λ and ρ denote constants, and $C_j(\ )$ means the constant depends on numbers in the parentheses.

THEOREM 4.1. *Let $M^{4n}, \mathcal{F}$ and $P_L(\Omega)$ be as the above, let U^{4n} be a compact saturated $4n$-submanifold with smooth boundary such that $Z(\mathcal{F}) \cap \partial U^{4n}$ is empty. Assume the following conditions:*

$$\|II(\partial U^4)\| \leq \Lambda \ ; \tag{4.1.1}$$

$$\|II(\mathcal{O}_x)\| \leq \Lambda \ , \ \text{for all } x \in \partial U^4 \ ; \tag{4.1.2}$$

$$\rho(\mathcal{F}) \geq \rho > 0 \ . \tag{4.1.3}$$

Then, there is a constant $C(n, \Lambda, \rho) > 0$ such that

$$|\operatorname{Res}_L(U^4)| \leq C(n, \Lambda, \rho)(\operatorname{Vol}(U^{4n}) + \operatorname{Vol}(\partial U^{4n})) \ , \tag{4.1.4}$$

where the residue is with respect to Hirzebruch's L-class.

We first prove a lemma. Let M be a compact manifold with boundary N, let $\mathcal{F}$ be an F-structure on M which is polarized on N. Put $M_\infty = M \cup (N^7 \times \mathbb{R}^+)$, and parallel extend the polarized F-structure $\mathcal{F}|N$ to $N \times \mathbb{R}^+$.

LEMMA 4.2. *Let M, N and $\mathcal{F}$ be as in the above. Given any invariant metric g on N which satisfies $|K| \leq 1$, (4.1.2) and (4.1.3), there exists a complete metric on M_∞, g_∞ such that:*

(4.2.1) *the restriction of g_∞ to N coincides with g;*
(4.2.2) $\operatorname{Vol}(N \times \mathbb{R}^+) \leq C_1 \operatorname{Vol}(N)$,
(4.2.3) $|K_{g_\infty}| \leq C'(n, \Lambda, \rho)$;
(4.2.4) $|II(N)| \leq 1$.

Proof: We shall first construct a metric g' on $N \times \mathbb{R}^+$. Then we shall obtain g_∞ by extending g' to M.

From the proof of Theorem 1.3 in [CG2], we see that using the polarized F-structure $\mathcal{F}$ on N one is able to construct a continuous invariant volume collapse $\{g_\delta\}_{0<\delta\leq 1}$ on N, with $g_1 = g$, which satisfies

(4.2.5) $|K_{g_\delta}| \leq C_2(g)$;
(4.2.6) $\operatorname{Vol}(N, g_\delta) \leq C_2 \operatorname{Vol}(N) e^{k \ln \delta} |\ln \delta|^k$

where k is the *rank of* $\mathcal{F}$ (the rank of $\mathcal{F}$ is defined to be the smallest dimension of all orbits.) We define a Riemannian metric on $N \times \mathbb{R}^+$ by $g' = g_{e^{-r}} \oplus dr^2$. Then we extend g' to M such that $|K| \leq 1$ and $|II(N)| \leq 1$.

We now inspect (4.2.2) and (4.2.3). Let V_r denote the volume of $N \times \{r\}$. From (4.2.6), we then derive (4.2.2) by

$$\begin{aligned}\mathrm{Vol}(N \times \mathbf{R}^+) = \int_0^\infty V_r dr &\le \int_0^\infty C_2 \,\mathrm{Vol}(N) e^{-kr} r^3 dr \\ &< n! C_2 \,\mathrm{Vol}(N) = C_1 \,\mathrm{Vol}(N) \ .\end{aligned}$$

Note that by our definition for g', it is easy to see that (4.2.3) is equivalent to $|K_{g_\delta}| \le C_2(n, \Lambda, \rho)$. However, as indicated by (4.2.5), the bound on $|K_{g_\delta}|$ may depend on the initial metric (also see Example 4.4). We shall show that conditions (4.1.2) and (4.1.3) enter to guarantee a bound as in (4.2.3). This can be seen by carefully inspecting the proof of Theorem 1.3 in [CG2]. We now explain this in more detail.

Recall the construction of g_δ in [CG2]. The idea is to construct collapse on each chart, and match them up properly on overlaps of the charts. We first check the collapsed sequence on a chart.

For any $x \in N$, choose a chart of $\mathcal{F}|N$, $(\widetilde{U}_\alpha, U_\alpha, T^{k_\alpha})$. First, by (1.1.1) in Theorem 1.1, we can assume $B_x(\rho) \subset U_\alpha$, where ρ is given by (4.1.3). Since $\mathcal{F}$ is polarized and compatible with the metric, by passing to a finite cover, we may assume that T^{k_α} acts freely on $\widetilde{U}_\alpha$ as isometries. Thus, the orbit space $X_\alpha = \widetilde{U}_\alpha / T^{k_\alpha}$ is a Riemannian manifold with the quotient metric, and the projection $\pi_\alpha : \widetilde{U}_\alpha \to X_\alpha$ is a Riemannian submersion. The collapse formed on U_α, g_δ, is multiplying the metric on the vertical component by δ^2 while keeping the metric on the horizontal space (see [CG3]).

From the computation in [Gro], the bound on K_{g_δ} depends on the bound in (4.1.2) and bounds on K_g and K_α. By O'Neil's formula [O], the sectional curvature on X_α is always greater than the sectional curvature of the horizontal space in U_α. Here the problem is to bound K_α from above. By O'Neil's formula for horizontal section, it suffices to show that O'Neil's tensor, A, has a bound depending on (4.1.2) and (4.1.3) (see [O]).

Now given any unit horizontal vector fields Y_1 and Y_2 such that $g(Y_1, Y_2) = 0$ and given any unit vertical vector field T. Since $g([Y_1, T], Y_2) = 0$, we then derive

$$g(T, A_{Y_1} Y_2) = g(T, 1/2[Y_1, Y_2]) = g(T, \nabla_{Y_1} Y_2) = -g(\nabla_{Y_1} T, Y_2).$$

By the above, the proof reduces to bound $\|\nabla_{Y_p} T\|$ in terms of Λ and ρ.

We first fix a point p and let $\gamma : [0, r] \to M$ be the minimal geodesic tangent to $Y = Y_1$ at p, where $r = \rho/2$. Assume the parameter is the arc length. We can assume that there is a S^1-subgroup of T^{k_α} such that T is the velocity vector field. Note that in the above set up, we have $[Y, T] = 0$. By the compatibility with the metric, the S^1-action on γ generates a family of geodesics. Hence, T is actually a Jacobi field on γ. Put $\nabla_{Y_p} T = T'(0)$.

We first observe that the bound in (4.1.2) implies that $e^{-2\Lambda t}\|T(0)\| \leq \|T(t)\| \leq e^{2\Lambda t}\|T(0)\|$. This can be seen by integrate $g(T,T)'/g(T,T) = 2g(\nabla_Y T, T)/g(T,T) = -2g(\nabla_T Y, T)/g(T,T) = 2g(Y, \nabla_T T)/g(T,T) = 2II(T,T)$ on $[0,t]$. In particular, $\|T(r)\| \leq e^{2\Lambda r}$.

Let T_1 be a Jacobi field on γ such that $T_1(0) = T(0)$ and $T_1(r) = 0$. Then, the Rauch-estimate implies that the norm of $d\exp_p rT(0) = T_1'(0)$ is bounded. Put $T_2 = T - T_1$, a Jacobi field on γ. Since $T_2(0) = 0$ and $T_2(r) = T(r)$, we then have that $d\exp_p rT_2'(0) = T(r)$. Hence,

$$d\exp_p T'(0) = \frac{T(r) + d\exp_p T_1'(0)}{r} .$$

By the above discussion, the right-hand side of the above equation is bounded by a constant depending on Λ and ρ. By the Rauch-estimate, we then conclude that $\|T'(0)\| \leq C(\Lambda, \rho)$.

We now explain how the above analysis goes through the situation where U_α has non-empty overlap with charts of different ranks. From the proof of Theorem 1.3 in [CG3], we see, in this case, that the collapse on the overlap is modified by expanding the metric properly at a rate $|\ln \delta|$. Since (4.1.2) and (4.1.3) remain for all δ, we then conclude (4.2.3). □

COROLLARY 4.3. *Under the same assumption as in Lemma 4.2, assume M is also an orientable $4n$-manifold. Then,*

$$|\operatorname{Res}_L(M)| \leq |\sigma(M) - \eta(N)| + C_2(n, \Lambda, \rho)\operatorname{Vol}(N) . \tag{4.3.1}$$

Proof: Put $M_\infty = M \cup (N \times \mathbb{R}^+)$ and let g_∞ be as Lemma 4.2. From (4.2.3), we have $\|P_L(\Omega)\| \leq C_3(\Lambda, \rho)$. By (3.5) and (3.6), we derive

$$\begin{aligned} |\operatorname{Res}_L(M)| &= |\int_{M_\infty} P_L(\Omega)| \qquad (4.3.2)\\ &\leq |\int_M P_L(\Omega)| + \int_{N\times\mathbb{R}^+} \|P_L(\Omega)\| \\ &\leq |\int_M P_L(\Omega)| + C_3(n,\Lambda,\rho)\operatorname{Vol}(N) \end{aligned}$$

By (4.2.4), we can write $|II_\sigma(N)| \leq C_3 \operatorname{Vol}(N)$. Thus, applying the Atiyah-Patodi-Singer index formula to the first term, we obtain

$$\begin{aligned} |\int_M P_L(\Omega)| &\leq |\sigma(M) - \eta(N)| + |II_\sigma(N)| \qquad (4.3.3)\\ &\leq |\sigma(M) - \eta(N)| + C_3 \operatorname{Vol}(N) . \end{aligned}$$

Combining (4.3.2) and (4.3.3) we then see (4.3.1). □

Proof of Theorem 4.1: Let U^{4n} be as in Theorem 4.1. For the sake of distinction, we use g_0 to denote the original metric on U^{4n}, and use g_1 to denote the metric on U_∞ as in Lemma 4.2 constructed from $g_0|\partial U^{4n}$. Clearly, $g_0 = g_1$ on ∂U^{4n}. By Corollary 4.3, conditions (4.1.2) and (4.1.3) yield

$$|\operatorname{Res}_L(U^{4n})| \le |\sigma(U^{4n}) - \eta(\partial U^{4n}, g_1)| + C_3(\Lambda, \rho)\operatorname{Vol}(\partial U^{4n}, g_1) \qquad (4.4.1)$$

Applying the Atiyah-Patodi-Singer index formula to $\int_{U^{4n}} P_L(\Omega_0)$, we see

$$\sigma(U^{4n}) - \eta(\partial U^{4n}, g_0) = \int_{U^{4n}} P_L(\Omega_0) - II_\sigma(\partial U^{4n}, g_0)\ . \qquad (4.4.2)$$

Since $g_0 = g_1$ on ∂U^{4n}, plugging (4.4.2) into (4.4.1) yields

$$|\operatorname{Res}_L(U^{4n})| \le \left| \int_{U^{4n}} P_L(\Omega_0) \right| + |II_\sigma(\partial U^{4n}, g_0)| + C_3(\Lambda, \rho)\operatorname{Vol}(\partial U^{4n}, g_0)\ . \qquad (4.4.3)$$

Since $|K| \le 1$, we see $\|P_L(\Omega_0)\| \le C$ and thus $|\int_{U^{4n}} P_L(\Omega_0)| \le C\operatorname{Vol}(U^{4n}, g_0)$. By (4.1.1), we can have $|II_\sigma(\partial U^{4n}, g_0)| \le C(\Lambda)\operatorname{Vol}(\partial U^{4n}, g_0)$. Finally, substituting these two inequalities into (4.4.3) we then obtain (4.1.4).

□

Next, we give an example showing that condition (4.1.3) is necessary.

EXAMPLE 4.4: Let $N_1 = S^2 \times S^1$, and let $\mathcal{F}$ be the T^2-action on N_1. Let $\{g_\delta\}$ be the invariant volume collapse on N_1 as in Example 1.4. For each $\delta > 0$, let $\tilde g_\delta$ be the product metric of g_δ and the unit circle.

Choose any S^1-subgroup, T^1, of $\mathcal{F}$ which has no fixed-points on N_1. Clearly, for all $\delta > 0$, the second fundamental form of T^1-orbits is independent of δ (this can be seen by lifting the T^1-orbits to the universal covering space of N_1). For each fixed δ, we use $g_{\delta,\epsilon}$ to denote the invariant volume collapse formed by multiplying the metric on the subspace tangent to the T^1-orbit by ϵ^2 while keeping the metric g_δ on its orthogonal complement. The limit space is a product of the metric space, X_δ, with the unit circle, where X_δ is a rugby ball with singularity at two poles. Clearly, X_δ is getting thinner and thinner as $\delta \to 0$, and therefore the absolute value of the sectional curvature goes to infinity as $\delta \to 0$.

Note that $\rho(T^1, g_\delta) \to 0$ as $\delta \to 0$.

g. The proof of Theorem 0.3.

In this part, M^4 will be a sufficiently injectivity radius collapsed 4-manifold, $\mathcal{F}_a$ will be an associated F-structure on M^4. By Theorem 1.1, we can choose the collapsed metric compatible with $\mathcal{F}_a$.

THEOREM 4.5. *Let M^4 and $\mathcal{F}_a$ be as the above. Then there exist constants $\Lambda > 0$ and $\rho > 0$, such that each component W_0 of $W(\mathcal{F}_a)$ has a neighborhood U_0, $U_0 \cap W(\mathcal{F}_a) = W_0$, which satisfies* (4.1.1)–(4.1.3) *and* $\operatorname{Vol}(\partial U_0) \le C(\rho)\operatorname{Vol}(U_0)$.

We will first give a proof for Theorem 0.3 by assuming Theorem 4.5; the proof of Theorem 4.5 will occupy the rest of this paper.

Proof of Theorem 0.3: The constant v is determined as follows: Let $v_4 > 0$ be a sufficiently small number so that $\operatorname{Vol}(M^4) < v_4$ implies that for all $x \in M^4$, the injectivity radius of M^4 is smaller than i_4, the critical injectivity radius in dimension four (see §1). Choose U_0 as in Theorem 4.5, and let $\Lambda > 0$, $\rho > 0$ and $C(\rho)$ be the constants in Theorem 4.5. From Theorem 4.1, we obtain (4.1.4) with a constant $C(\Lambda, \rho)$. Put $v = (1/2)\min\{v_4, 1/[C(\Lambda,\rho)(1 + C(\rho))]\}$.

Assume $\operatorname{Vol}(M^4) < v$. Since $v < v_4$, we can assume the associated F-structure $\mathcal{F}_a$ on M. By Corollary 2.10, it suffices to show each component with compact closure W_0 of $W(\mathcal{F})$ is inessential.

From (4.1.4) and the choice for v, we then derive

$$\begin{aligned}|\operatorname{Res}_L(U)| &\le C(\Lambda,\rho)(\operatorname{Vol}(U_0) + \operatorname{Vol}(\partial U_0))\\ &\le C(\Lambda,\rho)(\operatorname{Vol}(U_0) + C(\rho)\operatorname{Vol}(U_0))\\ &\le C(\Lambda,\rho)(1 + C(\rho))\operatorname{Vol}(M^4) < 1/2\ .\end{aligned}$$

Since $U_0 \cap W(\mathcal{F}) = W_0$ (Theorem 4.5), $|\operatorname{Res}_L(T_\epsilon(W_0))| = |\operatorname{Res}_L(U_0)| < 1/2$. Finally, by Proposition 3.8 we then conclude that W_0 is inessential. □

h. F-structures and N-structures in dimension four.

We observe the assertion in Theorem 4.5 that $\rho(\mathcal{F}_a|\partial U_0) \ge \rho > 0$ amounts to saying that the $\mathcal{F}_a$-orbits absorb all collapsed directions of the metric; roughly speaking, this means that for all $x \in M$, the homotopy classes of all short geodesic loops at x are contained in $\pi_1(\mathcal{O}_x, x)$. According to [CG3], the orbit $\mathcal{O}_x$ points to the *most* collapsed directions at x, that is, the directions in which the geodesic loops have length proportional to the injectivity radius at x by a constant ([CG3]). A priori, the normal injectivity radius of an associated F-structure may not have a uniform lower bound.

In [CFG], the theory of F-structure was generalized to N-structure (compare Remark 1.2). The definition for N-structure is similar to the definition of F-structure (see §1) except in a chart $(\widetilde{U}_\alpha, U_\alpha, N^{k_\alpha})$, N^{k_α} is a nilpotent group of dimension k_α and $\pi_\alpha : \widetilde{U}_\alpha \to U$ is allowed to be an infinite cover. In [CFG], it was showed that on a sufficiently injectivity radius collapsed manifold, given all small $\epsilon > 0$, one can construct a N-structure, $\mathcal{N}_\epsilon$, which actually absorb all collapsed directions (compare with Theorem 1.1). In particular, the F-structure $\mathcal{F}_\epsilon$ is a (commutative) substructure of $\mathcal{N}_\epsilon$. We denote by $\mathcal{N}_a$ an *associated N-structure* with $\epsilon = i_n$, the critical injectivity radius in dimension n.

The reason we use F-structure instead of N-structure in this paper is that we can apply the residue theory in [Ya] while a N-structure is not suitable for this purpose. The following lemma is crucial to prove Theorem 4.1.

LEMMA 4.6. *Let M^4 be as in Theorem 4.5, let $\mathcal{N}$ and $\mathcal{F}$ be the associated N-structure and F-structure on M respectively. Then there is a constant $r > 0$ such that the restriction $\mathcal{N}|T_r(W_0) = \mathcal{F}|T_r(W_0)$.*

Remark 4.7: By Lemma 4.6, we can assume the $\mathcal{F}_a$-orbits in $T_r(W_0)$ absorb all collapsed directions. Since $\mathcal{F}_a$ is constructed with a fixed collapsing scale, from [CFG] (see page 365) we have the following easy consequences: (i) there exists a constant $\rho_0 > 0$ such that each S^1-orbit in $T_r(W_0)$ has normal injectivity radius $\geq \rho_0$. In particular, $T_\rho(W_0) \simeq U_{i,k}$ or U_3; (ii) the second fundamental forms of all $\mathcal{F}$-orbits in $T_r(W_0) - T_{\rho_0}(W_0)$ are bounded by $\Lambda(\rho_0)$; (iii) $\rho(\mathcal{F}|\partial T\rho_0(W_0)) \geq \rho(\rho_0) > 0$ for some constant ρ depending on ρ_0.

The proof of Lemma 4.6 requires certain preparation, and we will first give a proof for Theorem 4.5 by assuming Lemma 4.6. In the proof of Theorem 4.5, we also need a result in [CG4].

THEOREM 4.8 ([CG4]). *Let M^n be a complete n-manifold with bounded sectional curvature $|K| \leq 1$. Assume M^n admits an F-structure compatible with the metric. Then, given a saturated subset $X \subset M^n$, and a real number $0 < \epsilon \leq 1$, there is a saturated submanifold U^n with smooth boundary ∂U^n such that for some constant C_n depending on n,*

$$X \subset U^n \subset T_\epsilon(X) , \tag{4.8.1}$$

$$\mathrm{Vol}(\partial U^n) \leq C_n \,\mathrm{Vol}(T_\epsilon(X))\epsilon^{-1} , \tag{4.8.2}$$

$$||II(\partial U^n)|| \leq C_n \epsilon^{-1} , \tag{4.8.3}$$

Proof of Theorem 4.5: Let $\mathcal{N}_a$ and $\mathcal{F}_a$ denote the associated N-structure and the associated F-structure on M^4 respectively. By Lemma 4.6, there is a constant $r > 0$ such that $\mathcal{N}_a = \mathcal{F}_a$ on $T_r(W_0)$. From Remark 4.7, we then obtain a constant, ρ_0, such that $T_\rho(W_0)$ is homeomorphic to $U_{i,k}$ or U_3. Without loss of generality, we can assume $r/2 > \rho_0$. Clearly, $T_{\rho_0}(W_0) \cap W(\mathcal{F}) = \emptyset$. Finally, applying Theorem 4.8 to W_0 with $\epsilon = \rho_0/2$ we then obtain the desired neighborhood U_0. (Note that in our circumstances, it is not hard to see that U_0 is homeomorphic to $T_{\rho_0} W_0$.) Finally, we see that (4.1.1) is from (4.8.3), and (4.1.2) and (4.1.3) are from Remark 4.7. □

i. The limiting η-invariants associated to volume collapses.

As a preparation for the proof of Lemma 4.6, we will briefly recall the two different kinds of the limiting eta-invariants in [CG2]. Note that the inequality (4.10) and residue formula (4.11.2) below will be used in the proof of Lemma 4.6.

Let N be a $(4n-1)$-dimensional closed orientable manifold. Assume N admits a volume collapse, $\{g_i\}$. Let $\eta(N, g_i)$ denote the eta-invariant of Atiyah-Patodi-Singer (see [APS1], [APS2]). In [CG1], it was found that if $\{g_i\}$ satisfies certain conditions, then the limit, $\lim_{i\to\infty} \eta(N, g_i)$, exists and has topological significance.

Let $\pi : \widetilde{N} \to N$ be the universal covering. A metric g on N is said to have *bounded covering geometry* (briefly, BCG) if the pullback metric on $\widetilde{N}$ has injectivity radius ≥ 1. A volume collapse $\{g_i\}$ is said to have BCG if $\pi^*(g_i)$ has BCG for all i.

THEOREM 4.9 ([CG1]). *Let N be an orientable closed $(4n-1)$-manifold. Assume N admits a volume collapse $\{g_i\}$ with BCG. Then, the limit, $\eta^*(N) = \lim_{i\to\infty} \eta(N, g_i)$, exists and is independent of the volume collapse.*

A topological interpretation of $\eta^*(N)$ in 3-dimensional case can be found in [Ro1]. The proof of Theorem 4.8 in [CG1] is to show that for any metric on N with BCG

$$|\eta^*(N) - \eta(N, g)| \leq C_n \operatorname{Vol}(N, g) . \tag{4.10}$$

We now consider the other limiting eta-invariant. Assume N admits a polarized F-structure, $\mathcal{F}$. By Theorem 1.3, N admits invariant volume collapses.

THEOREM 4.11 ([CG1]). *Let N and $\mathcal{F}$ be as above. Then, for any invariant volume collapse as constructed in Theorem 1.3, then the limit, $\lim_{i\to\infty} \eta(N, g_i)$, exists, and is independent of the invariant volume collapse.*

We will call the limiting eta-invariants as in Theorem 4.11 the *limiting eta-invariant associated to $\mathcal{F}$*, and denoted it by $\eta(N, \mathcal{F})$. Notice that $\eta(N, \mathcal{F})$ does depend on $\mathcal{F}$, that is, N may admit two different polarized F-structures, $\mathcal{F}_1, \mathcal{F}_2$, such that $\eta(N, \mathcal{F}_1) \neq \eta(N, \mathcal{F}_2)$ (see [Ro1]).

Next, we will derive a residue formula for $\eta(N,\mathcal{F})$. For this purpose, we assume there is a compact orientable $4n$-manifold M, $\partial M = N$ and $\mathcal{F}$ extends to an F-structure on M (note that the extension may have singularity).

As in §3, we form $M_\infty = M \cup (N \times \mathbf{R}^+)$, and parallel extend $\mathcal{F}$ to M_∞, and then construct an invariant metric on M_∞, g_∞, which satisfies (4.2.1)–(4.2.3). Let $P_L(\Omega)$ be the Hirzebruch's signature form in the L-polynomial, P_L. Combining (3.5) and (3.6), we see $\int_{M_\infty} P_L(\Omega) = \sum_i \mathrm{Res}_L(Z_i)$.

From our construction for g_∞, it is easy to see that $M_i = M \cup (N \times [0,i])$ forms an invariant good chopping. Thus, we apply the Atiyah-Patodi-Singer index formula to M_i, and obtain

$$\int_{M_i} P_L(\Omega) = \sigma(M_i) - \eta(\partial M_i, g_i) - II_\sigma(\partial M_i) \ . \tag{4.11.1}$$

Moreover, $\sigma(M_i) = \sigma(M)$, $\lim_{i\to\infty} \eta(\partial M_i) = \eta(N,\mathcal{F})$ and $\lim_{i\to\infty} II_\sigma(\partial M_i) = 0$; the second equation follows from Theorem 4.11, and the third equation is due to the fact M_i is a good chopping. Finally, taking the limit to (4.11.1) as $i \to \infty$, combining with (3.3) we then conclude

$$\sigma(M) - \eta(N,\mathcal{F}) = \sum_i \mathrm{Res}_L(Z_i) \ . \tag{4.11.2}$$

The above two limiting eta-invariants coincides if N admits an *injective* F-structure. A polarized F-structure is said to be injective if the fundamental group of $\mathcal{F}$-orbit injects into the fundamental group of the manifold. In [Ro1], we showed that if $\mathcal{F}$ is injective, then the invariant volume collapse in Theorem 1.3 has BCG. In this case, $\eta(N,\mathcal{F}) = \eta^*(N)$.

LEMMA 4.12. (4.12.1) *$\partial U_{1,k}$ admits an invariant volume $\{g_\delta\}$ collapse with BCG and $\rho(\mathcal{F}_{1,k}, g_i) \geq 1$.*
(4.12.2) *$\sigma(U_{1,k}) - \eta_{(2)}(\partial U_{1,k}) = k/3$.*

Proof: Note that $\mathcal{F}_{1,k}|\partial U_{1,k}$ is injective. By Theorem 2.5 in [Ro1] we then conclude (4.12.1).

By Theorem 4.8 and Theorem 4.10, from (4.12.1) we have $\eta^*(N) = \lim_{i\to\infty} \eta(N,g_i) = \eta(N,\mathcal{F}_{ik})$. So, $\sigma(U_{1,k}) - \eta_{(2)}(\partial U_{1,k}) = \sigma(U_{1,k}) - \eta(\partial U_{1,k}, \mathcal{F}_{1,k})$. By Lemma 3.7, $\sigma(U_{1,k}) - \eta(\partial U_{1,k}, \mathcal{F}_{1,k}) = \mathrm{Res}_L(Z(\mathcal{F}_{1,k})) = 1/3\,\mathrm{Res}(Z(\mathcal{F}_{1,k})) = k/3$ because $P_L = 1/3 P_1$. □

Proof of Lemma 4.6: First, as seen in **g**, we obtain two compatible structures on M, the associated N-structure $\mathcal{N}_a$ and its a substructure, the associated F-structure $\mathcal{F}_a$.

We will proceed by contradiction. Assume there is a component W_0 of $W(\mathcal{F}_a)$ such that $T_\epsilon(W_0)$ contains nil-orbits for some sufficiently small $\epsilon > 0$. In this case, it is not hard to see that $T_\epsilon(W_0) \simeq U_{1,k}$ for some $k \neq 0$, and thus W_0 actually is an isolated singular orbit. By Theorem 1.7 in [CFG], we can choose a chart of $\mathcal{N}$, $(\widetilde{U}, U, N^3)$, such that $T_{r_0}(W_0) \subset U$ for some universal constant $r_0 > 0$. (Note that W_0 is an (singular) orbit of $\mathcal{N}$.) Also, since W_0 is an isolated singular orbit, the normal injectivity radius of W_0 satisfies $\rho(W_0) \geq \rho_1 > 0$, where ρ_1 is a constant (compare Remark 4.7). Without losing generality, we assume $r_0 \geq \rho_1$. Moreover, the second fundamental form of $\partial T_{\rho_1}(W_0)$ is bounded by a number $C(\rho_1)$.

We now apply the Atiyah-Patodi-Singer index formula to $T_{\rho_1}(W_0)$, and derive

$$|\sigma(T_{\rho_1}(W_0)) - \eta(\partial T_{\rho_1}(W_0))| \leq \left| \int_{\partial T_{\rho_1}(W_0)} P_L(\Omega) \right| + |II_\sigma(\partial T_{\rho_1}(W_0))| \,. \tag{4.13.1}$$

We claim that the right hand side can be made arbitrarily small, provided the metric is sufficiently injectivity radius collapsed. This is clear for the integral term because $|K| \leq 1$ and the volume $\mathrm{Vol}(T_{\rho_1}(W_0)) \simeq \mathrm{diam}(W_0)\rho_1$ can be made arbitrarily small. Since $|II(\partial T_{\rho_1}(\mathcal{O}))| \leq C(\rho_1)$ and $\mathrm{Vol}(\partial T_{\rho_1}(W_0))$ is as small as we like, provided $\mathrm{diam}(\partial T_{\rho_1}(W_0))$ is sufficiently small, $II_\sigma(\partial T_{\rho_1}(W_0))$ can be make sufficiently small.

On the other hand, since $T_{\rho_1}(W_0) \simeq U_{1,k}$ for some $k \neq 0$, by Lemma 4.12 and Theorem 4.10 we derive

$$\begin{aligned} \frac{1}{3} \leq \frac{|k|}{3} &= |\sigma(T_{\rho_1}(W_0)) - \eta_{(2)}(\partial T_{\rho_1}(W_0))| \qquad (4.13.2) \\ &\leq |\sigma(T_{\rho_1}(W_0)) - \eta(\partial T_{\rho_1}(W_0))| + |\eta_{(2)}(\partial T_{\rho_1}(W_0)) - \eta(\partial T_{\rho_1}(W_0))| \,. \end{aligned}$$

From (4.13.1) and the discussion, the first term on the right-hand side of (4.13.2) is sufficiently small, provided the metric is sufficiently collapsed. We will derive a contradiction by showing the second term on the right-hand side of (4.13.2) is also small. From (4.9), it suffices to check that the induced metric on $\partial T_{\rho_1}(W_0)$ has bounded covering geometry by a universal constant. This condition is satisfied because $\partial T_{\rho_1}(W_0)$ is a single $\mathcal{N}$-orbit which is injective. □

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EXISTENCE OF POLARIZED F-STRUCTURES ON COLLAPSED MANIFOLDS WITH BOUNDED CURVATURE AND DIAMETER

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Abstract

We study the class of collapsed Riemannian n-manifolds with bounded sectional curvature and diameter. Our main result asserts that there is a constant, $\delta(n,d) > 0$, such that if a compact n-manifold has bounded curvature, $|K_{M^n}| \leq 1$, bounded diameter, $\operatorname{diam}(M^n) \leq d$ and sufficiently small volume, $\operatorname{Vol}(M^n) \leq \delta(n,d)$, then it admits a mixed polarized F-structure. As a consequence, $\inf_g \operatorname{Vol}(M^n, g) = 0$, where the infimum is taken over all metrics with $|K_{(M^n,g)}| \leq 1$. This assertion can be viewed as a weakened version of Gromov's "critical volume" conjecture.

0. Introduction

We will begin by briefly recalling the notion of F-structure and some relevant related concepts; for further details, see [CG1], [CG2], [CR] and sections 1–3 below.

An *F-structure*, $\mathcal{F}$, on a manifold, M^n, is a kind of generalized torus action. Specifically, it is a sheaf of Lie algebras, together with a homomorphism of this sheaf onto a sheaf of abelian Lie algebras of vector fields, $e_{\mathcal{F}}$, for which a certain additional condition is satisfied. In the sequel, only the image sheaf $e_{\mathcal{F}}$ plays a role.

Let f denote a subsheaf of $e_{\mathcal{F}}$ and f_x its stalk at x. The additional condition on $e_{\mathcal{F}}$ is the following. For all $x \in M^n$, there exists an open neighborhood, $U(x)$, and a subsheaf, $f(x)$, of $e_{\mathcal{F}}|U(x)$, such that $f(x)_x = (e_{\mathcal{F}})_x$ and such that for some finite normal covering space, $\pi : \tilde{U}(x) \to U(x)$, the lifted Lie algebra sheaf, $\tilde{f}(x)$, is a constant sheaf, which is isomorphic to the infinitesimal generators of the effective action of a torus, $T^{k(x)}$, on $\tilde{U}(x)$.

If all stalks, $(e_{\mathcal{F}})_x$, of the sheaf, $e_{\mathcal{F}}$, have the same dimension, $k(x) = k$, the structure is called *pure*. Otherwise, it is called *mixed.*

If for all x, one can choose $U(x)$ and $\tilde{U}(x)$, such that $\tilde{U}(x) = U(x)$, then the F-structure is called a *T-structure*. In this case, $e_{\mathcal{F}}$ is actually the Lie algebra sheaf of a *sheaf of tori*, $\mathcal{E}_{\mathcal{F}}$. If in addition, $\mathcal{F}$ is a pure structure,

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then the sheaf, $\mathcal{E}_{\mathcal{F}}$, can be described alternatively as a flat torus bundle with holonomy in $SL(k, \mathbb{Z})$.

If M^n is simply connected, a pure F-structure is actually a T-structure for which the bundle, $\mathcal{E}_{\mathcal{F}}$, has trivial holonomy. Thus, in the simply connected case, modulo a choice of isomorphism of some fiber with the standard torus, a pure F-structure is just an ordinary torus action.

A *substructure* is defined by a subsheaf of $e_{\mathcal{F}}$, for which the action generated by each $\tilde{f}(x)$ is isomorphic to a torus action, i.e. the orbits are closed.

The action on each $\tilde{U}(x)$ of its covering group, preserves the orbits of the action generated by $\tilde{f}(x)$. Hence, the open set, $U(x)$, is partitioned into the projections of these orbits. Clearly, the projected orbit through a point, x, is independent of the choice of neighborhood, $U(x)$. It is denoted, $\mathcal{O}_x$, the *orbit* of x. It follows that M^n is the disjoint union of orbits, $\mathcal{O}_x$. Every such orbit is diffeomorphic to a compact flat Riemannian manifold, by a diffeomorphism which is unique up to affine equivalence of the flat manifold.

The *rank* of the structure is the dimension of the orbit, $\mathcal{O}_x$, of smallest dimension. An orbit, $\mathcal{O}_x$, is called *singular* if $\dim \mathcal{O}_x < k(x)$. The *singular set* S, is by definition, the union of the singular orbits. As with a group action the set S, has a canonical "coarse" stratification into strata, S_i. By definition, S_i consists of all orbits of dimension i. Note that S_i may contain exceptional orbits which are multiply covered.

If S is empty, the structure is said to be *polarized.*

A Riemannian metric, g, on M^n is called *invariant* for $\mathcal{F}$, if $e_{\mathcal{F}}$ is actually a sheaf of Killing fields of g. Every F-structure admits invariant metrics whose sectional curvatures satisfy the normalization, $|K| \leq 1$.

For additional background on the relation between F-structures and collapsed Riemannian manifolds with bounded curvature, see [CG1,2], [CR], [F1–4], [G1,2], [R1–3].

We now specialize to the situation which is the focus of this paper.

Let M^n be a compact Riemannian manifold, with bounded sectional curvature, say $|K_{M^n}| \leq 1$. By [CFG], [F1–4], there exists a constant $\epsilon(n,d) > 0$, such that if in addition, $\mathrm{diam}(M^n) \leq d$ and $\mathrm{Vol}(M^n) \leq \epsilon(n,d)$, then M^n admits a pure F-structure, $\mathcal{F}$, of positive rank, for which a metric, g', close to the given one is invariant. After multiplying g' by a suitable constant (close to 1) we can assume that g' satisfies

$$|K_{(M^n,g')}| \leq 1 \ , \quad \mathrm{diam}(M^n, g') \leq d' \ , \quad \mathrm{Vol}(M^n, g') \leq \epsilon(n, d') \ .$$

Moreover, we can assume that for the metric, g', there are definite bounds on the higher covariant derivatives of the curvature tensor.

Our main result, Theorem 0.1, asserts that pure F-structures which arise in this way enjoy a significant property which is not shared by pure F-structures in general. Such an F-structure will be called a *sufficiently collapsible pure F-structure.*

Theorem 0.1. *There exists $\delta(n,d) > 0$, such that if M_n satisfies $|K_{M^n}| \leq 1$, $\operatorname{diam}(M^n) \leq d$ and $\operatorname{Vol}(M^n) \leq \delta(n,d)$, then the associated sufficiently collapsible F-structure $\mathcal{F}$ admits a polarized substructure.*

For M^n simply connected, a pure F-structure is (up to choice of isomorphism) a torus action. If such a structure has positive rank, it follows that any 1-dimensional subgroup (with closed orbits) which does not intersect any nontrivial isotropy group defines a polarized substructure. Thus, in Theorem 0.1, implicitly our concern is with the nonsimply connected case.

Typically, the polarized substructure constructed in Theorem 0.1 will be mixed. In this connection, note that by Example 6.4 of [CR], there exist pure structures satisfying the assumptions of Theorem 0.1 (for fixed d and arbitrarily small δ) which admit no *pure* polarized substructure.

Gromov defined the *Minimal Volume* of a compact manifold by

$$\operatorname{Min}\operatorname{Vol}(M^n) = \inf_g \operatorname{Vol}(M^n, g) \ ,$$

where the infimum is taken over all metrics, with bounded sectional curvature, $|K_{(M^n,g)}| \leq 1$; see [G2]. He conjectured the existence of a "gap" or "critical volume", i.e. there exists $\delta(n) > 0$ such that $\operatorname{Min}\operatorname{Vol}(M^n) < \delta(n)$ implies $\operatorname{Min}\operatorname{Vol}(M^n) = 0$.

By the collapsing construction of [CG1], the existence of a polarized F-structure on M^n implies $\operatorname{Min}\operatorname{Vol}(M^n) = 0$. Thus, Theorem 0.1 implies the following weakened version of Gromov's conjecture.

Theorem 0.2. *There exists $\delta(n,d) > 0$ such that if M^n admits a metric with*

$$|K_{M^n}| \leq 1 \ , \quad \operatorname{diam}(M^n) \leq d \ , \quad \operatorname{Vol}(M^n) \leq \delta(n,d) \ ,$$

then Min Vol $(M^n) = 0$.

It might seem natural to try to replace the conclusion, $\operatorname{Min}\operatorname{Vol}(M^n) = 0$, in Theorem 0.2, with the stronger assertion that M^n collapses with bounded curvature and diameter. However, Example 6.4 of [CR] indicates that this could well be false in general.

By the finiteness theorem of [C], for all $v > 0$, there are only finitely many diffeomorphism types of manifolds satisfying $|K_{M^n}| \leq 1$, $\operatorname{diam}(M^n) \leq d$, for which in addition, $\operatorname{Vol}(M^n) \geq v$; see also [Pe]. Hence, we obtain

COROLLARY 0.3. *For all $n, d > 0$, there are only a finite number of diffeomorphism classes of manifolds of nonvanishing minimal volume, which admit a metric with* $|K_{M^n}| \leq 1$, $\mathrm{diam}(M^n) \leq d$.

Corollary 0.3 implies that there is a sense in which "most" manifolds with $|K_{M^n}| \leq 1$ have minimal volume zero. Indeed, according to [G1] for all $n \geq 3$, $d > 0$, there exist infinitely many manifolds admitting a metric with $|K_{M^n}| \leq 1$ and $\mathrm{diam}(M^n) \leq d$. Moreover, it follows from the construction of [CR], Example 6.4, that given $n \geq 4$, there exists an increasing sequence, $d_i \to \infty$, such that for all i, there are infinitely many manifolds admitting a metric with $|K_{M^n}| \leq 1$, $\mathrm{diam}(M^n) \leq d_{i+1}$, which admit no metric with $|K_{M^n}| \leq 1$, $\mathrm{diam}(M^n) \leq d_i$.

If M^{2k} has some real characteristic number nonzero, then by Chern-Weil theory, there is a definite positive lower bound on $\mathrm{Min\,Vol}(M^{2k})$; [C]. In [CG1], examples of pure positive rank F-structures on compact $4k$-manifolds with nonvanishing Pontrjagin numbers are given (the first such example was due to T. Januszkiewcz). These examples show that in order to obtain the existence of a polarized substructure, *some* additional geometric hypothesis on the pure F-structure is required.

It is possible however, that the bound on the diameter assumed in Theorem 0.1 is actually unnecessary and that a polarized substructure exists whenever $|K| \leq 1$, $\mathrm{Vol}(M^n) \leq \delta(n)$, a sufficiently small positive constant. Presently, this is known to hold for $n = 2$ ([C]), $n = 3$ ([CG1,2]) and $n = 4$ ([Bu1,2], [R1,2]); but compare Example 4.1 of [CG1]. If indeed, the bound on diameter is unnecessary, then by the collapsing construction of [CG1], the "critical volume" conjecture holds; in particular, it holds for $n \leq 4$.

We now briefly describe the contents of the remaining 5 sections of the paper.

As is explained in section 1, the proof of Theorem 0.1 will be carried out by working on the frame bundle, FM^n. In section 1, we also introduce a property of arbitrary pure F-structures and a property of pure F-structures which satisfy the geometric assumptions of Theorem 0.1. These two properties play a crucial role in the proof.

In section 2, we prove Theorem 0.1 modulo the above mentioned two properties.

In section 3, we establish the property of arbitrary pure F-structures; see Theorem 3.2. It concerns a certain canonical (mixed) substructure defined in a neighborhood of the singular set, S. This substructure, which is generated by the *kernels* of the local torus actions, turns out to be an F-structure of an extremely special type.

In section 4, we establish the property of pure F-structures which are compatible with sufficiently collapsed metrics; see Theorem 4.1. Namely, over each stratum, S_i, there exists a pure polarized substructure, $\mathcal{P}_i$.

In section 5, we give a generalization of Theorem 0.1 to the case in which only a bound on the diameter of each component, $S_{i,j}$, of S is assumed (rather than on the diameter of M^n itself).

1. Outline of The Proof

In this section we give an indication of the proof of Theorem 0.1. Thus, unless we make explicit mention to the contrary, we will assume here that our structure, $\mathcal{F}$, is a sufficiently collapsible pure F-structure, equipped with an invariant metric.

Our discussion is simplified considerably by working on the frame bundle, FM^n, rather than on M^n itself; compare [F1–4]. Although this necessitates our making all constructions $O(n)$-equivariant, in practice, for natural constructions, $O(n)$-equivariance turns out to be automatic. For instance, a pure substructure defined over an $O(n)$-invariant subset of FM^n is always $O(n)$-equivariant; see [CR, Remark 0.1].

The advantage of working on FM^n lies in the fact that the canonical lift to FM^n of an F-structure is actually a T-structure, $\mathcal{T}$, of a particularly simple type – namely, one for which the local actions are free. (The lift is defined via the differentials of the local torus actions.) In particular, given a pure F-structure on M^n, we can regard FM^n as the total space of an $O(n)$-invariant torus bundle, whose structural group lies in the group of affine automorphisms of the torus, T^k. Note that this group satisfies the exact sequence,

$$e \to T^k \to Aff(T^k) \to SL(k, \mathbb{Z}) \to e\ .$$

Before proceeding, we point out that the existence of pure F-structures of positive rank on sufficiently collapsed manifolds with bounded curvature and diameter was actually proved by working on the frame bundle; see [F1–4] and [CFG]; see also [CR] for further discussion.

In constructing a polarized substructure, it is clear that we can restrict attention to a neighborhood of the singular set, S; outside such a neighborhood, our polarized structure will be chosen to coincide with $\mathcal{F}$ itself.

Let D denote the inverse image of S in FM^n. Observe that D consists of those points for which the corresponding torus-fibre and $O(n)$-fibre intersect in a subset of positive dimension. We denote by D_i, the inverse image of S_i in FM^n.

On each stratum, D_i, we define the *isotropy substructure*, $\mathcal{I}_i$, to be the unique maximal substructure, whose projection to M^n has rank zero. The orbits of this structure are just the components of the intersections of torus-fibres and $O(n)$-fibres.

An $O(n)$-equivariant substructure, $\hat{\mathcal{T}}$, on FM^n, descends to a polarized substructure on M^n, if and only if on each D_i, it is *transversal* to $\mathcal{I}_i$, i.e. on each D_i, the intersection of an orbit of $\hat{\mathcal{T}}$ and an orbit of $\mathcal{I}_i$ consists of a finite set of points. Equivalently, $\mathcal{E}_{\hat{\mathcal{T}}} \cap \mathcal{E}_{I_i} = \mathcal{E}_0$, where $\mathcal{E}_0$ denotes the trivial subsheaf whose stalk at any point is the subgroup consisting of the identity element. A substructure of $\mathcal{T}$ with this property will be called *nondegenerate*.

Let $1 \gg r_1 \gg r_2 \gg \cdots > 0$. Let $\eta > 0$.

Put $H_i(\eta) = T_{\eta r_i}(D_i) \setminus \bigcup_{l<i} T_{\frac{1}{2}r_l}(D_l)$, where $T_r(\ \ )$ denotes the r-tubular neighborhood. We can assume that the sequence, $\{r_i\}$, decreases so rapidly that if $\eta \leq 3$, then for every point, p, of $H_i(\eta)$, there is a *unique* point of S_i closest to p. Note that for $i \neq j$, the intersection, $H_i(\eta) \cap H_j(\eta)$, can be nonempty and might not be connected.

We also put $H_i' = H_i(1) \setminus \bigcup_{i>l} H_l(2)$ and note that $H_i' \subset H_i(1)$ and $H_i' \cap H_j' = \emptyset$, for all distinct i, j.

Our $O(n)$-equivariant nondegenerate substructure of $\mathcal{T}$ will be constructed on $\bigcup_i H_i(1)$. A priori, it is not clear why there should exist such a substructure over even a single $H_i(1)$. However, using our geometric hypothesis, we will show the following; see Theorem 4.1.

Property of sufficiently collapsible pure F-structures. On each $H_i(1)$, there exists a pure nondegenerate substructure, $\mathcal{P}_i$, of $\mathcal{F}$.

The existence of a pure nondegenerate substructure on each set, $H_i(1)$, is the only consequence of our geometric assumptions which is used in the proof. Indeed, we have the following refinement of Theorem 0.1.

Theorem 0.1′. *Let $\mathcal{F}$ be an arbitrary pure F-structure on M^n. If for all i, there is a pure nondegenerate substructure, $\mathcal{P}_i$, on $H_i(1)$, then there exists a canonical mixed polarized substructure, whose lift to the frame bundle, $\mathcal{P}$, satisfies $\mathcal{P}|H_i' = \mathcal{P}_i$.*

The sense in which the substructure, $\mathcal{P}$, is canonical will be made clear in the proof of Theorem 0.1′.

To construct an $O(n)$-equivariant nondegenerate substructure on $\bigcup_i H_i(1)$, whose restriction to each H_i' coincides with $\mathcal{P}_i$, we will introduce a certain auxiliary substructure, $\mathcal{I}$, defined on $\bigcup_i H_i(2)$.

Since $D_i \cap H_i(2)$ is a deformation retract of $H_i(2)$, it follows that $\mathcal{I}_i|D_i \cap H_i(2)$ extends naturally to a pure substructure, $\mathcal{I}_i$, on $H_i(2)$. The collection $\{(H_i(2), \mathcal{I}_i)\}$ determines a mixed structure, $\mathcal{I}$, on $\bigcup_i H_i(2)$, whose orbit at a point, x, is the orbit of $\mathcal{I}_{i_0}$, where i_0 is the maximal i, for which $x \in H_i(2)$.

Clearly, on H_i', a pure substructure $\hat{\mathcal{T}} \subset \mathcal{T}$ is nondegenerate if and only if it is transversal to $\mathcal{I}|H_i'$. On the other hand, we claim that $\mathcal{I}|\big(\bigcup_i H_i(1) \setminus \bigcup_i H_i'\big)$ has a *canonical mixed nondegenerate substructure*, $\mathcal{C}$. As will be explained in section 2, the nondegenerate substructure, on $\bigcup_i H_i(1)$ which we are seeking, is obtained by suitably combining a portion of $\mathcal{C}$ with a collection of substructures derived from the nondegenerate substructures, $\{\mathcal{P}_i|H_i(1)\}$.

The existence of $\mathcal{C}$ is a direct consequence of the following property of arbitrary pure F-structures; see Theorem 3.2.

Property of arbitrary pure F-structures. There exists a *canonical inner product* on the Lie algebra, $(e_{\mathcal{I}})_x$, of each stalk of the sheaf, $e_{\mathcal{I}}$, such that the pointwise inner product of two local sections of the sheaf, $e_{\mathcal{I}}$, is a constant function. Moreover, if a subspace of $(e_{\mathcal{I}})_x$ exponentiates to a closed subgroup, then so does its orthogonal complement.

We close this section by mentioning that the arguments used in establishing the above mentioned property of sufficiently collapsible pure F-structures are related to those of [CR], where collapsed manifolds with bounded diameter and bounded covering geometry are studied. Here instead, we exploit *local bounded covering geometry*; see [CFG, Theorem 1.7] and section 4.

2. Proof of Theorem 0.1 Modulo Two Properties of Pure F-structures

Let $\mathcal{F}$ denote a pure F-structure on M^n with invariant metric and let $\mathcal{T}$ denote the lifted T-structure on FM^n.

In the proofs of Theorems 0.1, 0.1′, we will use the following procedure for constructing equivariant mixed substructures of $\mathcal{T}$.

Let $\{Z_\alpha\}$ be a covering of FM^n by $O(n)$-invariant sets. Assume that over each Z_α, we are given a pure substructure, $\mathcal{L}_\alpha$. Clearly, there is a unique smallest mixed substructure, $\mathcal{L}$, such that for all α, $\mathcal{L}_\alpha$ is a substructure of $\mathcal{L}|Z_\alpha$. Moreover, for any $\{\alpha\} = \{\alpha_1, \cdots, \alpha_i\}$ the restriction of $\mathcal{L}$ to $Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_i} \setminus \cup_{\alpha_j \notin \{\alpha\}} Z_{\alpha_j}$, is the smallest pure structure containing the restrictions of $\mathcal{L}_{\alpha_1}, \cdots, \mathcal{L}_{\alpha_i}$, to this set.

Now assume that $\mathcal{F}$ has nonempty singular set, S, with coarse stratification, $S_1, \ldots, S_k$. Put $D_i = \pi^{-1}(S_i)$. Let $H_1(\eta), \ldots, H_k(\eta)$ be defined as in section 1.

The proof of Theorem 0.1 consists of three steps: First, we construct a special invariant open cover for $\bigcup_i H_i(1)$. Then (as above) we assign to each open set of this cover, a pure substructure of $\mathcal{T}$. Finally, we verify that on every nonempty multiple intersection, the assigned pure substructures generate a nondegenerate pure substructure (i.e. one which is transversal to the isotropy substructure on the intersection).

a. An invariant open cover. For $1 \le i \le k$, put

$$A_i = H_i(1) \backslash \overline{\bigcup_{i>\ell} H_\ell\left(\frac{3}{2}\right)} .$$

For any $1 \le j < i \le k$, define

$$B_{i,j} = \left(H_i(1) \backslash \overline{\bigcup_{i>\ell>j} H_\ell\left(\frac{3}{2}\right)} \right) \cap H_j(2) .$$

Note that since $H_i(\eta)$ is invariant, so are A_i and $B_{i,j}$. Formally, A_i behaves like $B_{i,-1}$, although for this to be correct, we must define, $H_{-1}(2) = M^n$.

LEMMA 2.1.

(2.1.1) $\quad H_i' \subset A_i \subset H_i(1)$.

(2.1.2) $\quad H_i(1) = \left(\bigcup_{i>\ell} B_{i,\ell}\right) \cup A_i$.

(2.1.3) $\quad$ If $B_{i,j} \cap B_{i',j'} \neq \emptyset$ and $i > i'$ then $j \ge i'$.

(2.1.4) $\quad$ If $B_{i,j} \cap A_{i'} \neq \emptyset$, then $i' = i$ or $j \ge i'$.

(2.1.5) $\quad A_i \cap A_{i'} = \emptyset$, for $i \neq i'$.

Proof. Since (2.1.1), (2.1.3), (2.1.4) and (2.1.5) can be seen directly from the definition, we will only check (2.1.2). Put $A_{i,j} = H_i(1) \backslash \overline{\bigcup_{i>\ell\ge j} H_\ell(\frac{3}{2})}$. Then $A_i = A_{i,\ell}$, where ℓ is the smallest index such that D_ℓ is nonempty. It is easily checked that for $i-1 > j$, one has $B_{i,j} \cup A_{i,j} = A_{i,j+1}$ and $B_{i,i-1} \cup A_{i,i-1} = H_i(1)$. By an obvious inductive argument, the claim follows. □

As a consequence of Lemma 2.1, every nonempty intersection of a subcollection of $\{B_{i,j}\} \cup \{A_i\}$ can be written in one of the following forms:

(2.1.6) $X = B_{i_1,j_1} \cap \cdots \cap B_{i_1,j_{k_1}} \cap B_{i_2,l_1} \cap \ldots \cap B_{i_2,l_{k_2}} \cap \cdots \cap B_{i_r,m_1} \cap \cdots \cap B_{i_r,m_{k_r}}$, where $i_1 > j_1 > \cdots > j_{k_1} \ge i_2 > l_1 > \cdots > l_{k_2} \ge \cdots \ge i_r > m_1 > \cdots > m_{k_r}$.

(2.1.7) $X \cap A_i$, where X is as in (2.1.6) and either $i = i_r$ or $m_{k_r} \ge i$.

(2.1.8) A_i, for some i.

b. Assignment of pure structures. Assume that on each $H_i(1)$, there is a pure nondegenerate substructure, $\mathcal{P}_i$, of $\mathcal{T}|H_i(1)$; compare Theorem 4.1.

On each nonempty intersection, $H_i(1) \cap H_j(1)$, where $i > j$, there is a canonical substructure, $\mathcal{I}_{i,j} \subset \mathcal{I}_j$, such that $\mathcal{I}_{i,j}$ is transversal to $\mathcal{I}_i$. By definition, the Lie algebra of a stalk of $\mathcal{I}_{i,j}$ is the orthogonal complement of the Lie algebra of $\mathcal{I}_i$ in the Lie algebra of $\mathcal{I}_j$, with respect to the inner product described in the property of arbitrary pure F-structures stated in section 1; see Theorem 3.2. Thus, if $H_i(1) \cap \left(\bigcap_{s=1}^{\ell} H_{j_s}(1)\right) \neq \emptyset$ (where $i > j_1 > j_2 > \cdots > j_\ell$) then on this set, $\mathcal{I}_{i,j_1} \subset \cdots \subset \mathcal{I}_{i,j_\ell}$.

We now assign to each element of the collection $\{B_{i,j}\} \cup \{A_i\}$, a pure nondegenerate substructure as follows.

(2.2.1) To each A_i, assign the nondegenerate substructure $\mathcal{P}_i|A_i$ (note that $A_i \subset H_1(1)$).

(2.2.2) To each $B_{i,j}$, assign a pure substructure, $\mathcal{P}_{i,j}$, where $\mathcal{P}_{i,j} = \mathcal{P}_i \cap \mathcal{I}_j$, provided this substructure is nontrivial, and $\mathcal{P}_{i,j} = \mathcal{I}_{i,j}|\,B_{i,j}$ otherwise.

Observe that a pure substructure on $B_{i,j}$ is nondegenerate if and only if it is transversal to $\mathcal{I}_j|B_{i,j}$. From the above definition, it is clear that $\mathcal{P}_{i,j}$ is nondegenerate.

As explained at the beginning of this section, the collection, $\{(A_i, \mathcal{P}_i|A_i)\} \cup \{(B_{i,j}, \mathcal{P}_{i,j})\}$, generates a substructure, $\mathcal{P}$, of $\mathcal{T}|\bigcup_i H_1(1)$. Clearly, $\mathcal{P}|H_i'(1) = \mathcal{P}_i$. In the next subsection we will show that the substructure, $\mathcal{P}$, is nondegenerate.

c. Nondegeneracy on multiple intersections. The remainder of the proof of Theorem 0.1 uses only elementary linear algebra.

LEMMA 2.3. *Assume $B_{i,j_1} \cap \cdots \cap B_{i,j_\ell}$ is nonempty, where $j_1 > \cdots > j_\ell$. Then on this subset the pure substructure generated by $P_{i,j_1}, \ldots, P_{i,j_\ell}$ is nondegenerate. If in addition, $B_{i,j_1} \cap \cdots \cap B_{i,j_\ell} \cap A_{i'}$ is nonempty, where $i' = i$ or $j_k \geq i'$, then on this subset, the pure substructure generated by $\mathcal{P}_{i,j_1}, \ldots, \mathcal{P}_{i,j_\ell}, \mathcal{P}_{i'}$ is nondegenerate.*

Proof. Since $\mathcal{I}_{j_1} \subset \cdots \subset \mathcal{I}_{j_\ell}$ either $\mathcal{P}_i \cap \mathcal{I}_{j_1} \neq \emptyset$ or for some j_t, we have $\mathcal{P}_i \cap \mathcal{I}_s = \emptyset$, for $s = j_1, \ldots, j_t$, where j_t is the last such index. We will assume that the latter alternative holds, since the argument in the former case is entirely similar to the one that follows. For the same reason, we can assume $j_t < j_\ell$.

The substructures assigned to $B_{i,j_1}, \ldots, B_{i,j_t}$ are $\mathcal{I}_{i,j_1}, \ldots, \mathcal{I}_{i,j_t}$, respectively. The substructures assigned to $B_{i,j_t+1}, \ldots, B_{i,j_\ell}$, are $\mathcal{P}_i \cap \mathcal{I}_{j_t+1}, \ldots, \mathcal{P}_i \cap \mathcal{I}_{j_\ell}$, respectively. Thus, on $B_{i,j_1} \cap \cdots \cap B_{i,j_k}$ the pure substructure generated by $\mathcal{P}_{i,j_1}, \ldots, \mathcal{P}_{i,j_\ell}$, is actually generated by $\mathcal{I}_{i,j_t}$ and $\mathcal{P}_i \cap \mathcal{I}_{j_\ell}$. Moreover, $\mathcal{I}_{i,j_t}$ is transversal to $\mathcal{I}_i$, $\mathcal{I}_{i,j_t} \subset \mathcal{I}_{j_t}$, and $\mathcal{P}_i \cap \mathcal{I}_{j_\ell}$ is transversal to $\mathcal{I}_{j_t}$. To verify the first assertion of Lemma 2.3, it suffices to check that the substructure generated by $\mathcal{I}_{i,j_t}$ and $\mathcal{P}_i \cap \mathcal{I}_{j_\ell}$ is transversal to $\mathcal{I}_i$. In view of the above, this (pointwise) condition follows by elementary linear algebra.

We now verify the second assertion. If $i' = i$, our substructure is generated by $\mathcal{I}_{i,j_t}$ and $\mathcal{P}_i$, where $\mathcal{P}_i$ is transversal to $\mathcal{I}_{j_t} \supset \mathcal{I}_{i,j_t}$. On the other hand, if $i_k > i'$, our substructure is generated by $\mathcal{I}_{i,j_t}, \mathcal{P}_i \cap \mathcal{I}_{j_\ell}$ and $\mathcal{P}_{i'}$, where $\mathcal{P}_{i'}$ is transversal to $\mathcal{I}_{j_\ell}$. As above, in either case, the assertion follows. □

Proof of Theorem 0.1′. Given the characterization of the substructure, $\mathcal{L}$, generated by a collection, $\{(Z_\alpha, \mathcal{L}_\alpha)\}$, which was stated at the beginning of this section, it suffices to check that over each nonempty intersection of sets taken from a subcollection of $\{B_{i,j}\} \cup \{A_i\}$, the substructure generated by the relevant subset of $\{\mathcal{P}_{i,j}\} \cup \{\mathcal{P}_i\}$ is nondegenerate. But in view of the description of the possible nonempty intersections given in (2.1.6)–(2.1.8), the nondegeneracy follows by repeated application of Lemma 2.3 (and the elementary linear algebra facts, employed in its proof). □

Proof of Theorem 0.1. As explained in section 1, Theorem 0.1 follows directly from Theorem 0.1′ and the property of sufficiently collapsible F-structures stated in that section (i.e. Theorem 4.1). □

REMARK 2.4: Consider the lifted $\mathcal{T}$-structure associated to an *arbitrary* F-structure. As above, it follows that the collection, $\{(B_{i,j}, \mathcal{I}_{i,j})\}$ generates a canonical nondegenerate substructure, $\{\mathcal{C}\}$, over $\bigcup B_{i,j}$. Moreover, it is easy to check that $\bigcup B_{i,j} = \bigcup_i H_i(1) \setminus \bigcup_i H_i'$.

3. A Property of Arbitrary Pure F-structures

In this section we prove the property of arbitrary pure F-structures stated in section 1. Thus, throughout this section, we will consider an arbitrary pure F-structure, $\mathcal{F}$, on M^n, with nonempty singular set. We assume that the Riemannian metric on M^n is invariant, so that $\mathcal{F}$ lifts to an $O(n)$-invariant pure polarized T-structure, $\mathcal{T}$, on the frame bundle, $\pi : FM^n \to M^n$.

The inner products on stalks, $(e_\mathcal{I})_x$, arise from the isotropy representations of the local actions of the stalks of $\mathcal{E}_\mathcal{T}$ on finite covering spaces of neighborhoods in the base. For completeness, we will describe these local actions, in the process supplying further details of the description of F-structures given at the beginning of the introduction.

Let F denote the torus fibre of the T-structure, $\mathcal{T}$, and let $Aff_0(F)$ denote the identity component of the group of affine automorphisms, $Aff(F)$, of F. Recall that a choice of affine isomorphism, $F \simeq T^k$, induces an isomorphism, $Aff_0(F) \simeq T^k$, where k is the rank of $\mathcal{F}$.

Let $G(F) \subset O(n)$ denote the subgroup which preserves F under the natural action of $O(n)$ on FM^n. Thus, $G(F) = \{e\}$, the trivial subgroup, unless $F \subset D$. In particular there is a faithful representation, $\tau : G(F) \to$

$Aff(F)$. Let $G_0(F) \subset G(F)$ denote the identity component. Then $\tau : G_0(F) \to Aff_0(F)$.

Fix $\epsilon_1 > 0$ such that for every point $y \in T_{\epsilon_1}(F)$, the ϵ_1-tubular neighborhood of F, there is a unique point, $x \in F$, closest to y. Fix $\epsilon_2, \delta > 0$, so small that every component of $T_\delta(G(F))$ intersects a unique component of $G(F)$, in addition, $g(T_{\epsilon_2}(F)) \subset T_{\epsilon_1}(F)$ and finally, if $g(T_{\epsilon_2}(F)) \cap T_{\epsilon_2}(F) \neq \emptyset$, then $g \in T_\delta(G(F))$.

The action of $Aff_0(F)$ extends canonically to a torus-fibre preserving action on $T_{\epsilon_1}(F)$; see [CR, Section 2]. Moreover, for $g \in T_\delta(G_0(F))$, the automorphism in $Aff(F)$ defined by, $A \to g^{-1}Ag$, is continuously deformable to the identity and hence is trivial. In particular the action of elements of $T_\delta(G_0(F))$ commutes with the action of $Aff_0(F)$ on $T_{\epsilon_2}(F)$.

Put $W = T_{\epsilon_2}(F)$. Then W is a disjoint union of equivalence classes, where $y_1 \sim y_2$ if and only if $y_2 = gy_1$, with $g \in T_\delta(G(F))$. Moreover, $\pi(W)$ can be identified with the corresponding quotient space with its natural topology. Similarly, the equivalence relation $y_1 \sim y_2$ if and only if $y_2 = gy_1$, with $g \in T_\delta(G_0(F))$, can be identified with a finite normal covering space, $\tilde{\pi} : \widetilde{\pi(W)} \to \pi(W)$, with covering group, the group of components of $G(F)$.

Since the action of each element of $Aff_0(F)$ commutes with that of each element of $T_\delta(G_0(F))$, it follows that there is a canonical action of $Aff_0(F)$ on $\widetilde{\pi(W)}$.

Note the action of an element of $Aff_0(F)$ need not commute with that of an element of $T_\delta(G(F))$. Thus, $Aff_0(F)$ need not act naturally on $\pi(W)$ itself. Equivalently, an F-structure need not be a T-structure (nor in particular, is a flat manifold necessarily a torus).

Clearly, the isotropy group of any point of $\tilde{\pi}^{-1}(\pi(F)) \subset \widetilde{\pi(W)}$ is $\tau(G(F)) \subset Aff(F)$.

If $x \in F$, then by definition, the stalk of $\mathcal{E}_{\mathcal{T}}$ at x is $Aff_0(F)$. We have $x \in D_i$, for some i, if and only if $\dim G(F) > 0$. Let $x \in D$. By definition, $\tau(G_0(F))$ is the stalk of the subsheaf, $\mathcal{E}_{I_i}$, of $\mathcal{E}_{\mathcal{T}}$. Thus, there is a natural (faithful) *isotropy representation* of $(\mathcal{E}_{\mathcal{I}_i})_x$ on the tangent space, $W_{\tilde{\pi}(x)}$, for any $x \in F$. The *lifted isotropy representation*, ρ, acts on the quotient of the tangent space, W_x, by the tangent space to the $O(n)$-orbit, $O(n)_x$. Let $x_\ell \to x$, where $\{x_\ell\} \subset D_i$, $x \in D_j$ and $i > j$. Then the limit of the isotropy representations, $\lim_{\ell\to\infty} \rho(\mathcal{E}_{\mathcal{I}_i})_{x_\ell}$, is the restriction of the representation, $\rho((\mathcal{E}_{\mathcal{I}_j})_x)$, to the limit subgroup, $\lim_{\ell\to\infty}(\mathcal{E}_{\mathcal{I}_i})_{x_\ell} \subset (\mathcal{E}_{\mathcal{I}_j})_x$.

Let ρ_* denote the representation of Lie algebras induced by ρ. Since a torus is compact, the symmetric bilinear form,

$$\langle\langle A, B\rangle\rangle = -\tfrac{1}{2} tr\big(\rho_*(A)\rho_*(B)\big) \ ,$$

defines a *canonical inner product* on the Lie algebra, $(e_{\mathcal{I}})_x$, of the stalk, $(e_{\mathcal{I}})_x$, of $e_{\mathcal{I}}$ at $x \in D$. Recall that up to isomorphism, representations of a compact Lie group are isolated. Moreover, the bilinear form, $-\frac{1}{2}tr(\rho_*(A)\rho_*(B))$ is invariant under isomorphism. Thus, it follows that the inner product of two local sections of the sheaf, $e_{\mathcal{I}}$, is a constant function. Note that local sections of the *sheaf*, $e_{\mathcal{I}}$, can be described equivalently as local sections of the corresponding vector bundles which are parallel with respect to the canonical flat connection.

Observe that by the above discussion, if $x_\ell \to x$, where $x_\ell \in D_i$, $x \in D_j$ and $i > j$, then:

(3.1) The sequence of canonical inner products on Lie algebras, $(e_{\mathcal{I}_i})_{x_\ell}$, converges to an inner product on the limit Lie algebra, $\lim_{\ell\to\infty}(e_{\mathcal{I}_i})_{x_\ell} \subset (e_{\mathcal{I}_j})_x$. Moreover, the limiting inner product, coincides with the restriction to $\lim_{\ell\to\infty}(e_{\mathcal{I}_i})_{x_\ell}$, of the canonical inner product on $(e_{\mathcal{I}_j})_x$.

Recall that $\mathcal{I}$ is the substructure of $\mathcal{T}$ defined on $\bigcup_i H_i(2)$ by the collection, $\{(H_i(2), \mathcal{I}_i)\}$.

Now we can state the main result of this section.

Theorem 3.2. *For all i, there is a canonical pointwise inner product on stalks of $e_{\mathcal{I}_i}$ such that the inner product of two local sections is a constant function. Moreover, if $H_i(2) \cap H_j(2) \neq \emptyset$, where $i > j$, then:*

(3.2.1) *The canonical inner product on $e_{\mathcal{I}_i}|H_i(2) \cap H_j(2)$ coincides with the restriction of the canonical inner product on $e_{\mathcal{I}_j}|H_i(2) \cap H_j(2)$. In particular, the collection of inner products on the various $e_{\mathcal{I}_i}$, $i = 1, 2, \ldots$, defines an inner product on $e_{\mathcal{I}}$.*

(3.2.2) *There is a pure substructure, $\mathcal{I}_{i,j}$, of $\mathcal{I}_j|H_i(2)\cap H_j(2)$ such that each stalk $(e_{\mathcal{I}_{i,j}})_x$, is the orthogonal complement of $(e_{\mathcal{I}_i})_x$ in $(e_{\mathcal{I}_j})_x$.*

Proof. Clearly, the inner product on Lie algebras of stalks of $\mathcal{I}_i$, initially, defined over D_i, extends naturally over $H_i(2)$. As a consequence of the consistency condition implied by (3.1), it follows that if $x \in H_i(2) \cap H_j(2)$, where $i > j$, then the inner product on $(e_{\mathcal{I}_i})_x$, obtained by restricting the canonical inner product on $(e_{\mathcal{I}_j})_x$, coincides with the canonical inner product on $(e_{\mathcal{I}_i})_x$. This gives (3.2.1).

To verify (3.2.2), it suffices to consider an orthogonal representation of the standard k-torus, $T^k = S^1 \times \cdots \times S^1$. Let e_i denote the vector in the Lie algebra of T^k such that the i-th circle factor is the 1-parameter subgroup generated by e_i, and $\exp 2\pi e_i$ is the identity element. Subtori of T^k are in 1–1 correspondence with subspaces of $\mathbb{R}^k$, which admit a basis, $v_1, \ldots, v_j$, where $v_j = \sum_i a_{i,j}e_i$, and $a_{i,j}$ is rational, for all i, j. Thus, by elementary linear algebra, an inner product, $\langle\ ,\ \rangle$ satisfies that $\langle e_i, e_\ell\rangle$ is rational, for all i, ℓ, if and only if it has the property that the orthogonal complement of a subspace which exponentiates to a subtorus always exponentiates to a subtorus.

For any representation, ρ, of T^k, there is a decomposition,

$$\mathbb{R}^n = L_1 \oplus \cdots \oplus L_r \oplus K \ ,$$

into ρ-invariant subspaces, where each L_j is 2-dimensional and $\rho(T^k)$ acts trivially on K. On L_j, we have $\rho(\exp te/[\exp 2\pi e_i]) = R_{m_{i,\ell}t}$, where $m_{i,\ell} \in \mathbb{Z}$ and R_s denotes rotation by s. From this, it follows immediately that the inner product, $\langle\langle A, B\rangle\rangle = -\frac{1}{2}tr(\rho_*(A)\rho_*(B))$, has the above mentioned rationality property. □

4. A Property of Sufficiently Collapsible Pure F-structures

In this section, we will prove the property of sufficiently collapsible pure F-structures which was stated in section 1.

Theorem 4.1. *Let the assumptions be as in Theorem 0.1. If $\mathcal{F}$ is a sufficiently collapsible pure F-structure, with lifted structure, $\mathcal{T}$, then for all i, $\mathcal{T}|D_i$ has a pure nondegenerate substructure.*

First we will recall from [CR], geometric conditions which guarantee the existence of a nondegenerate pure substructure on the frame bundle over a subset of M^n. In [CR], the assumptions were such that this subset could be taken to be M^n itself. Here, we will show that these conditions are actually satisfied when restricted to each set, D_i.

a. A criterion for the existence of transversal substructures. Let $p : E \to B$ be a fiber bundle with fiber a torus, T^k, and structural group $Aff(T^k)$. Assume that E is equipped with an invariant metric, for the local action described in section 3. In particular, the projection, p, is a Riemann submersion.

Recall that a *subfibration* of $p : E \to B$ is a fibration, $p_1 : E \to B_1$, such that each fiber of p_1 is a totally geodesic submanifold of a fiber of p. Let p_2 be another subfibration of p. We say that p_2 is *transversal* to p_1 if the fiber of the latter is tranversal to that of the former at each point (cf. [CR]).

Theorem 4.2 [CR]. *There exists a constant, $\epsilon(n, d, \Lambda, \rho) > 0$, such that the following conditions imply the existence of a subfibration of p transversal to p_1,*

(4.2.1) $\mathrm{diam}(E) \leq d$,

(4.2.2) *the second fundamental form of each p-fiber satisfies* $\|II(F)\| \leq \Lambda$,

(4.2.3) *the injectivity radius of each p_1-fiber is greater than ρ,*

(4.2.4) *the diameter of every p-fiber satisfies,* $\mathrm{diam}(F) < \epsilon(n, d, \Lambda, \rho)$.

Now let M^n be as in Theorem 4.1, with the lifted T-structure, $\mathcal{T}$, on FM^n and a degenerate set D. Let $\tilde{f} : FM^n \to B_{\tilde{f}}$ be the projection to the orbit space of the bundle, $F \to FM^n \to B_{\tilde{f}}$ defined by $\mathcal{T}$. Then $\tilde{f}_i : D_i \to B_i$, the restriction of $\tilde{f}$ to D_i, is also an $O(n)$-invariant torus bundle. Moreover, the substructure, $\mathcal{I}_i$, of D_i gives rise to an $O(n)$-invariant subfibration, $p_i : D_i \to B_{p_i}$.

In view of Theorem 4.2, the following proposition implies Theorem 4.1.

PROPOSITION 4.3. *Let the assumptions be as in Theorem 4.1. Then, there exist constants, $h(n,d), \Lambda(n)$ and $\rho(n)$, such that for all i, the following hold.*
(4.3.1) *The second fundamental form of each $\tilde{f}_i$-fiber satisfies $|II(\tilde{f}_i^{-1}(x))| \leq \Lambda(n)$,*
(4.3.2) $\mathrm{diam}(D_i) \leq h(n,d)$,
(4.3.3) *the injectivity radius of each p_i-fiber is greater than $\rho_0(n)$.*

b. Proof of (4.3.1). By [CFG], there exists a constant, $\Lambda(n)$, such that the $O(n)$-invariant fibration, $\tilde{f} : FM^n \to B_{\tilde{f}}$ satisfies (4.3.1). Hence, $\tilde{f}_i : D_i \to B_i$ satisfies (4.3.1).

c. Proof of (4.3.2). As in section 1, we have $S_i = \pi(D_i)$, where S_i is a singular stratum of $S = \pi(D)$. There is a universal constant, C, such that

$$C^{-1} \cdot \mathrm{diam}(S_i) \leq \mathrm{diam}(D_i) \leq C \cdot \mathrm{diam}(S_i) \ .$$

By the above discussion, (4.3.2) is equivalent to

LEMMA 4.4. *Let the assumptions be as in Proposition 4.3. There exists a constant, $h(n,d) > 0$, depending on n and d such that each singular stratum, S_i, has diameter $\leq h(n,d)$.*

Proof. We argue by contradiction. Assume that there is a sequence of n-manifolds, $\{M_j^n\}$, which satisfy the assumptions of Theorem 4.1 and such that the invariant pure structure on M_j^n has a singular stratum, $S_{i_j}(M_j^n)$, with $\mathrm{diam}(S_{i_j}(M_j^n)) > j$.

As mentioned in the introduction, we can assume that the metric on FM_j^n has a uniform bound on the covariant derivative of the curvature tensor (see section 0 and [CFG]). Then, by Gromov's precompactness theorem, after passing to a subsequence, we can assume that $\{M_j^n\}$ converges to a metric space, B, and the sequence of the frame bundles, $\{FM_j^n\}$, converges to a Riemannian manifold $\tilde{B}$ (of lower dimension) such that for j sufficiently large the following diagram commutes (compare [F2]).

$$\begin{array}{ccc} FM_j^n & \xrightarrow{\tilde{\eta}_j} & \tilde{B} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ M_j^n & \xrightarrow{\eta_j} & B \end{array}$$

Here $\tilde{\eta}_j : FM_j^n \to \tilde{B}$ is an $O(n)$-invariant fibration with fiber affine isomorphic to a nilmanifold, and affine structural group; see [CFG] and compare section 1. In the language of [CFG], η_j defines a nilpotent Killing structure on M_j^n. The $O(n)$-invariance implies that $\tilde{B}$ admits an isometric $O(n)$-action such that $B = \tilde{B}/O(n)$ and the fibration $\tilde{\eta}_j$ descends to a singular fibration projection, $\eta_j : M_j^n \to B$. It follows from Proposition A1.14 of [CFG] that the $O(n)$-action on $\tilde{B}$ is effective. The centers of the nilpotent fibers form an $O(n)$-invariant torus bundle. This is the structure which was described in section 1 (see [CR]).

Note that the singular set of the nilpotent Killing structure coincides with that of the canonical F-structure; see [CR].

Let $\{Z_i\}$ denote the collection of all singular strata of the $O(n)$-action on $\tilde{B}$. Then the above commutative diagram implies that $\{\tilde{\pi}(Z_i)\}$ is the collection of images under the projection, η_j, of all singular strata of the nilpotent Killing structure on M_j^n. Thus, $\{f_j^{-1}(\tilde{\pi}(Z_i))\}$ is the collection of all singular strata of the nilpotent Killing structure on M_j^n. By the above discussion, $\{f_j^{-1}(\tilde{\pi}(Z_i))\}$ is the collection of all singular strata of the canonical pure F-structure on M_j^n.

Since $\tilde{\pi}(Z_i)$ has a definite diameter, the diameter of $f_j^{-1}(\tilde{\pi}(Z_i))$ is bounded for all j. Since there are only finitely many singular strata for the $O(n)$-action on $\tilde{B}$ (see [B]), we conclude that the diameters of all $f_j^{-1}(\tilde{\pi}(Z_i))$ are uniformly bounded; a contradiction. □

d. Proof of (4.3.3). Let M^n be as in Proposition 4.1 and let $\mathcal{F}$ be a sufficiently collapsible pure F-structure on M^n.

LEMMA 4.5. *There exists $c(n,r) > 0$, such that for all $p \in M^n$, there exists $q \in B_r(p) \setminus S$, such that the second fundamental form of $\mathcal{O}_q$ satisfies $\|II(\mathcal{O}_q)\| \le c(n,r)$.*

Proof. It follows from Theorem 1.7 of [CFG] (*local bounded covering geometry*) that the norm of the second fundamental form of a nonsingular orbit of $\mathcal{F}$ can be bounded above in terms of its distance from the singular set S. Thus, it suffices to show that each ball of radius r contains a nonsingular orbit lying at a definite distance (depending only on n and r) from S. This can be seen by an argument by contradiction analogous to the proof of

Lemma 4.4. In this connection, recall that the $O(n)$-action on $\tilde{B}$ is effective. Thus, the set of nonsingular orbits is dense. □

For a subset U of M^n, we use $\pi_U : \tilde{U} \to U$ to denote the universal covering space of U equipped with the pullback metric.

LEMMA 4.6 [CFG]. *There exists a constant, $\rho(n) > 0$, such that for any $p \in M^n$, there is an invariant open subset, U, containing the ball, $B_{2\rho(n)}(p)$, and each point in $\pi_U^{-1}(B_{\rho(n)}(p))$ has injectivity radius $\geq \rho(n)$.*

Note that Lemma 4.6 is a version of *local bounded covering geometry* which suffices for our present purposes (for the full statement, see [CFG, Theorem 1.7]).

Proof of (4.3.3). Let $x \in D_i$. Put $\pi(x) = p$. For $\rho(n)$ as in Lemma 4.6, and $r = \rho(n)$, let q be as in Lemma 4.5. Clearly, there exists $y \in \pi^{-1}(q)$ and a minimal geodesic, γ, with $\gamma(0) = x$, $\gamma(1) = y$, such that $\pi(\gamma) \subset B_{\rho(n)}(p)$.

By light abuse of notation, let $\mathcal{O}^{\mathcal{I}_i}_{\gamma(t)}$ denote the orbit through $\gamma(t)$, of the parallel translate along γ, of the stalk, $(\mathcal{E}_{\mathcal{I}_i})_x$. Here the parallel translation is with respect to the canonical connection on $\mathcal{E}_{\mathcal{I}}$, viewed as a flat bundle. By Lemma 4.6, $\pi(\mathcal{O}^{\mathcal{I}_i}_{\gamma(1)}) = \pi(\mathcal{O}^{\mathcal{I}_i}_y)$ has second fundamental form bounded in norm by $c(n, r)$. Moreover, for U as in Lemma 4.6, the family, $\pi(\mathcal{O}^{\mathcal{I}_i}_{\gamma(t)})$, provides a contraction in U, of $\pi(\mathcal{O}^{\mathcal{I}_i}_y)$ to point x. Let $\hat{y} \in \pi_U{}^{-1}(y)$ and let $\hat{\mathcal{O}}^{\mathcal{I}_i}_{\hat{y}}$ denote the component of $\pi_U^{-1}(\pi(\mathcal{O}^{\mathcal{I}_i}_y))$ through $\hat{y}$. Then $\pi_U|\hat{\mathcal{O}}^{\mathcal{I}_i}_{\hat{y}}$ is a homeomorphism. Thus, for the pull back metric, $\operatorname{inj\,rad}(\hat{\mathcal{O}}^{\mathcal{I}_i}_{\hat{y}}) = \operatorname{inj\,rad}(\pi(\mathcal{O}^{\mathcal{I}_i}_y))$. Since also $\|II(\hat{\mathcal{O}}^{\mathcal{I}_i}_{\hat{y}})\| = \|II(\pi(\mathcal{O}^{\mathcal{I}_i}_y))\| \leq c(n, \rho(n))$, it follows from Lemma 4.6 that $\operatorname{inj\,rad}(\hat{\mathcal{O}}^{\mathcal{I}_i}_{\hat{y}}) \geq \rho_0(n)$. The fact that $\pi : FM^n \to M^n$ is a Riemannian submersion, easily implies $\operatorname{inj\,rad}(\mathcal{O}^{\mathcal{I}_i}_y) \geq \rho_0(n)$ as well.

By (4.3.1) and (4.3.2) metrics on orbits of $\mathcal{I}_i$ are quasi-isometric, with the constant depending on n and d. Hence, it follows from the above that $\operatorname{inj\,rad}(\mathcal{O}^{\mathcal{I}_i}_x)$ has a lower bound depending only on n and d. □

5. A Generalization of Theorem 0.1

In this section, we will give a generalization of Theorem 0.1; see Theorem 5.2.

DEFINITION 5.1: Let $\mathcal{F}$ be a (possibly mixed) F-structure. A singular component, S_i, of $\mathcal{F}$ is called *essential* if $\mathcal{F}$ has no polarized substructure in any neighborhood of S_i.

By definition, an F-structure has a polarized substructure if and only all singular components are nonessential. Examples of positive rank F-structures with essential singularities were mentioned in the introduction.

Let M^n be a complete manifold with $|K_{M^n}| \le 1$. Recall that for all sufficiently small $\epsilon > 0$, there is a natural decomposition, $M^n = \mathcal{B}(\epsilon) \cup \mathcal{C}(\epsilon)$, where $\mathcal{B}(\epsilon)$ consists of points at which the injectivity radii are not less than ϵ and $\mathcal{C}(\epsilon)$ is the complement. If $M^n = \mathcal{C}(\epsilon)$, then M^n is called *ϵ-collapsed.*

The main result in [CFG] asserts that there is a constant, $\epsilon(n) > 0$, such that (after a slight adjustment of its boundary) $\mathcal{C}(\epsilon(n))$ admits a (possibly mixed) positive rank F-structure, $\mathcal{F}$, which is *almost* compatible with the metric. We will also call $\mathcal{F}$ the *associated F-structure.*

The following result can be viewed as a generalization of Theorem 0.1.

Theorem 5.2. *For all $d > 0$, there exists a constant, $0 < \epsilon(n,d) < \epsilon(n)$, such that the following holds. If M^n is an $\epsilon(n,d)$-collapsed complete manifold with $|K_{M^n}| \le 1$ such that the associated F-structure on M^n has essential singular components, then all such components have diameter $\ge d$.*

Note that the injectivity radius collapsed metric in Theorem 5.2 need not be *volume collapsed*, i.e. the volume need not be small and could be infinite.

COROLLARY 5.3. *Let M^n be a complete manifold with $|K|\le 1$ and $\mathrm{Vol}(M^n) < \infty$. Suppose that for the associated F-structure, $\mathcal{F}$, on $\mathcal{C}(\epsilon(n))$, all singular components have diameter $\le d$. Then, there is a constant, $0 < \epsilon(n,d) < \epsilon(n)$, such that $\mathcal{F}|\mathcal{C}(\epsilon(n,d))$ has a polarized substructure.*

Note that Corollary 5.3 means that $\mathcal{F}$ has a polarized substructure near infinity.

REMARK 5.4: Theorem 5.2 provides a geometric constraint on essential singular components. Nonessential singular components can have arbitrarily small diameter; see Example 5.7.

REMARK 5.5: Recall that given a positive rank F-structure, $\mathcal{F}$, there exists a family of invariant metrics with $|K| \le 1$ and injectivity radii uniformly converging to zero ([CG1]). An F-structure associated to each sufficiently collapsed metric is actually a substructure of $\mathcal{F}$. If, in addition, one assumes that $\mathcal{F}$ has essential singularities, then such an F-structure will have an essential singular component ([CG1]). (Note that by definition, any substructure of an F-structure with essential singularities has essential singularities).

Assume that M^n is ϵ-collapsed with $0 < \epsilon < \epsilon(n)$. Consider an associated F-structure, $\mathcal{F}$, on M^n. Note that $\mathcal{F}$ need not be a pure F-structure (see Example 0.1 of [CFG]). However, we have

LEMMA 5.6. *For all $d > 0$, there is a constant, $0 < \epsilon(n,d) < \epsilon(n)$, such that if $M^n = \mathcal{C}(\epsilon(n,d))$, then for all $x \in M^n$, the restriction of $\mathcal{F}$ to a subset containing $B_d(x)$ has a pure positive rank substructure.*

Proof. The proof is based on an observation concerning the construction of sufficiently collapsible F-structures in [CFG].

Fix any $d > 0$. It follows from section 5 of [CFG], there is a constant, $0 < \epsilon(n,d) < \epsilon(n)$, depending only on n and d such that if $M^n = \mathcal{C}(\epsilon)$, $\epsilon \leq \epsilon(n,d)$, then for all $x \in M^n$, a subset containing $B_d(x)$ admits a *pure* positive rank F-structure, say $\mathcal{F}_{x,d}$, such that all orbits have diameter less than ϵ.

If, in addition, we choose $\epsilon(n,d) \ll \epsilon(n)$, then $\mathcal{F}_{x,d}$ is actually a pure substructure of the associated F-structure, $\mathcal{F}$, on M^n. This can be seen from the construction of $\mathcal{F}$ in [CFG]. □

Now the proof of Theorem 5.2 follows easily from Lemma 5.6 and Theorem 0.1.

We conclude this paper with an example mentioned in Remark 5.4.

EXAMPLE 5.7: Consider the standard T^2-action on $S^2 \times S^1$. Using a standard method (see [CG1]), we will construct a (continuous) sequence of invariant metrics, g_ϵ, with $|K_{g_\epsilon}| \leq 1$ such that $(S^2 \times S^1, g_\epsilon)$ converges to a closed interval ($\epsilon \to 0$) in the Gromov-Hausdorff topology (see [GLP]). Clearly, the F-structure associated to any sufficiently collapsed metric coincides with the T^2-action. Observe that the length of each of the two singular circle orbits (each one is a non-essential singular component) goes to zero as $\epsilon \to 0$.

Take a one parameter subgroup, R, of T^2 such that the closure of R is T^2 and take a T^2-invariant metric, g, on $S^2 \times S^1$. At each point, write $g = g_R \oplus g_R^\perp$, where g_R is the restriction of g to the subspace tangent to the R-orbit and $g_R^\perp$ is the orthogonal compliment. Then $g_\epsilon = \epsilon^2 g_R \oplus g_R^\perp$, $0 < \epsilon \leq 1$.

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A BOUNDARY OF THE SET OF THE RIEMANNIAN MANIFOLDS WITH BOUNDED CURVATURES AND DIAMETERS

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Dedicated to Professor Itiro Tamura on his sixtieth birthday

0. Introduction

In [12], Gromov introduced a metric (Hausdorff distance) on the class of all metric spaces. There, he proved the precompactness of the set consisting of the isometry classes of Riemannian manifolds with bounded curvatures and diameters. In this paper we shall study the structure of the closure of this set.

Definition 0.1. For a natural number n and $D \in (0, \infty]$, we let $\mathscr{M}(n, D)$ denote the set consisting of all isometry classes of compact Riemannian manifolds M such that

(0.2.1) the dimension of M is equal to n,

(0.2.2) the diameter of M is smaller than D,

(0.2.3) the sectional curvature of M is smaller than 1 and greater than -1.

The following problem is fundamental in the study of the Hausdorff distance on $\mathscr{M}(n, D)$.

Problem 0.3. (A) Determine the closure of $\mathscr{M}(n, D)$ with respect to the Hausdorff distance. (Hereafter $\mathscr{CM}(n, D)$ denotes the closure.)

(B) Let X_i $(i = 1, 2, \cdots)$ be a sequence of elements of $\mathscr{CM}(n, D)$. Suppose X_i converges to a metric space X with respect to the Hausdorff distance. Then, describe the relation between the topological structures of X_i and X.

Our main result on Problem 0.3(A) is Theorem 0.5 and those on Problem 0.3(B) are Theorems 0.12 and 10.1.

First we deal with Problem 0.3(A). Let $\mathscr{PM}_n$ denote the set of all pointed compact Riemannian manifolds (M, p) satisfying (0.2.1) and (0.2.3), and $\mathscr{CPM}_n$ the closure of $\mathscr{PM}_n$ with respect to the pointed Hausdorff distance (see 1.6). If $M \in \mathscr{CM}(n, D)$ then $(M, p) \in \mathscr{CPM}_n$ for each $p \in M$. We let $M(n, D, \mu)$ denote the set of the elements of $\mathscr{M}(n, D)$ whose injectivity radii

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are greater than μ. Put

$$\mathrm{Int}(\mathscr{M}(n,D)) = \bigcup_{\mu>0} \mathscr{CM}(n,D,\mu),$$
$$\partial\mathscr{M}(n,D) = \mathscr{CM}(n,D) - \mathrm{Int}(\mathscr{M}(n,D)).$$

$\mathrm{Int}(\mathscr{PM}_n)$ and $\partial\mathscr{PM}_n$ are defined similarly.

Gromov, in [12], proved that the elements of $\mathrm{Int}(\mathscr{PM}_n)$ are manifolds. In general, elements of $\partial\mathscr{PM}_n$ have singularities. Several examples of elements of $\partial\mathscr{PM}_n$ can be constructed with help from torus actions and more generally from F-structures (see [3], [18]). One of the main theorems of this paper asserts that every element of $\mathscr{CPM}_n$ is locally of this type. To state it, we need a definition.

Definition 0.4. We say elements (X,p_0) and X of $\mathscr{CPM}_n$ and $\mathscr{CM}(n,\infty)$ are *smooth* if they satisfy the following:

For each point p of X, there exist a neighborhood U of p in X, a compact Lie group G_p and a faithful representation of G_p into the orthogonal group, $\mathrm{O}(n)$, such that the identity component of G_p is isomorphic to a torus and that U is homeomorphic to V/G_p for some neighborhood V of 0 in $\mathbf{R}^m$. Furthermore there exists a G_p-invariant smooth Riemannian metric g on V such that U is isometric to $(V/G_p,\bar{g})$, where $\bar{g}$ denotes the quotient metric.

Theorem 0.5. *Smooth elements are dense in $\mathscr{CPM}_n$ with respect to the pointed Lipschitz distance. In particular, every element of $\mathscr{CPM}_n$ is homeomorphic to a smooth one.*

Theorem 0.5 gives us complete information on the local topological structure of the elements of $\mathscr{CPM}_n$. Our result on global structure is not yet complete.

Theorem 0.6. *Let $X \in \mathscr{CPM}_n$. Then there exists a Riemannian manifold M on which $\mathrm{O}(n)$ acts as isometries such that the following holds.*

(0.7.1) *X is isometric to $M/\mathrm{O}(n)$. (Let $P\colon M \to X$ be the projection.)*

(0.7.2) *For each point p of X the group $\{g \in \mathrm{O}(n) \mid g(p) = p\}$ is isomorphic to G_p, where G_p is as in Definition* 0.4.

By virtue of Theorem 0.5, the Hausdorff dimension of each element of $\mathscr{CPM}_n$ is an integer. Inspecting this fact, we define stratifications on $\mathscr{CPM}_n$ and $\mathscr{CM}(n,D)$ as follows.

Definition 0.8.

$$\Xi\mathscr{M}_k(n,D) = \{X \in \mathscr{CM}(n,D) \mid (\text{Hausdorff dimension of } X) \le n-k\},$$
$$\Xi\mathscr{PM}_{n,k} = \{(X,p) \in \mathscr{CPM}_n \mid (\text{Hausdorff dimension of } X) \le n-k\}.$$

[12, 8.39] implies $\Xi\mathscr{M}_1(n,D) = \partial\mathscr{M}(n,D)$.

Our next result concerns the metric structure of the smooth elements of $\mathscr{CPM}_n$. Let (X, p_0) be a smooth element of $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$. Then X has a stratification $X = S_0(X) \supset S_1(X) \supset \cdots \supset S_k(X)$ such that $S_i(X) - S_{i+1}(X)$ is a $(k-i)$-dimensional smooth Riemannian manifold. In the case when X is not necessarily smooth, we define a stratification on X using that of a smooth one and the Lipschitz homeomorphism given by Theorem 0.5. [7, Example 1.13] or [16] shows that we cannot obtain an upper bound of the sectional curvatures of $S_i(X) - S_{i+1}(X)$ while X moves on $\mathscr{CPM}_n$. But we have the following.

Theorem 0.9. *Let (X_i, p_i) be a sequence of smooth elements of $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$ and (X, p_0) a pointed metric space. Assume that (X_i, p_i) converges to (X, p_0) in the sense of the pointed Hausdorff distance. Then X is contained in $\Xi\mathscr{PM}_{n,k+1}$ if one of the following two conditions is satisfied.*

(0.10.1) *There exist a positive c and a positive integer j such that*

(0.10.1.a) *$p_i \in S_j(X_i)$ and $d(p_i, S_{j+1}(X_i)) \geq c$, and*

(0.10.1.b) *the sectional curvatures of $S_j(X_i) - S_{j+1}(X_i)$ at p_i are unbounded.*

(0.10.2.a) *p_i satisfies* (0.10.1.a) *and*

(0.10.2.b) *the injectivity radius of $S_j(X_i) - S_{j+1}(S_i)$ at p_i converges to* 0 *when i tends to infinity.*

Furthermore, in the case when (0.10.1) *holds, we have $p_0 \in S_1(X)$.*

Theorems 0.5 and 0.9, combined with [9], [19] or [12, 8.28], imply the following.

Corollary 0.11. *Let (X, p_0) be a (not necessarily smooth) element of $\mathscr{CPM}_n$. Then $S_k(X) - S_{k+1}(X)$ is a Riemannian manifold with continuous metric tensor and $C^{1,\alpha}$-distance function, where α is an arbitrary number contained in* $[0, 1)$.

Next, we shall describe our results from Problem 0.3(B). In the case when $X_i \in \mathrm{Int}(\mathscr{M}(n, D))$ we have the following:

Theorem 0.12. *Let $M_i \in \mathrm{Int}(\mathscr{M}(n, D))$ and $X \in \mathscr{CM}(n, D)$. Suppose $\lim_{i\to\infty} d_{\mathrm{H}}(M_i, X) = 0$. Then, for each sufficiently large i, there exists a differentiable map $f\colon M_i \to X$ satisfying the following.*

(0.13.1) *For each j, the restriction of f to $f^{-1}(S_j(X) - S_{j+1}(X))$ is a fiber bundle whose fiber is diffeomorphic to an infranilmanifold.*

(0.13.2) *Let $p_0 \in X - S_1(X)$, $p \in X$, $F = f^{-1}(p - 0)$ and G_p be the group given in Definition* 0.4. *Then G_p acts freely on F and $f^{-1}(p)$ is diffeomorphic to the quotient space F/G_p.*

More precise informations on the map f and on its relation to the metric structures of X and M_i are in §10. In the case when $X_i \in \partial\mathscr{M}(n, D)$, we can prove a similar result. But, since the result is a bit complicated, we do not state it here (see §10), and restrict ourselves to the following simple case.

Theorem 0.14. $\Xi\mathscr{PM}_{n,k} - \Xi\mathscr{PM}_{n,k+1}$ *is complete with respect to the pointed Lipschitz distance. The pointed Hausdorff distance and the pointed Lipschitz distance define the same topology on it.*

In the case when $k = 0$, Theorem 0.14 follows from the results of [12].

In the course of the proof of Theorem 0.12, we shall prove the following finiteness theorem.

Theorem 0.15. *For each n and $D < \infty$, there exists a finite set Σ of manifolds whose dimensions are not greater than $n + (n-1)(n-2)/2$ and which satisfy the following. For each element M of $M(n, D)$, there exists a smooth map f from the bundle of orthonormal frames of M to an element of Σ, such that f is a fiber bundle with an infranilmanifold fiber.*

The following result is a direct consequence of Theorem 0.15.

Corollary 0.16. $\sup\{\sum_i \operatorname{rank}(H_i(M; K)) \mid M \in M(n, D), K\colon \text{field}\}$ *is finite for each $D < \infty$ and n.*

By a different method, M. Gromov proved in [11] the same conclusion without assuming that sectional curvature is less than or equal to 1.

The organization of this paper is as follows. In Chapter I, we shall prove Theorem 0.5. In §2, we take an element (X, p_0) of $\mathscr{CPM}_n$ and prove that, to verify Theorem 0.5, it suffices to show that X is smooth if (X, p_0) is a limit of pointed Riemannian manifolds (M_i, p_i), the derivatives of whose curvatures are uniformly bounded. In §3, we shall represent a neighborhood of each point of X as the quotient B/G of a Riemannian manifold B by a smooth action of a Lie group germ G. For this purpose, we shall pull back the metrics of M_i to their tangent spaces $T_{p_i}(M_i)$, following [12, 8.33–8.36], and represent neighborhoods of p_i as the quotient spaces B/Γ_i. Taking the limit, we obtain B and G. In §4, we shall prove that G is nilpotent. The proof of Theorem 0.5 is completed in §5.

Chapter II is devoted to the study of Problem 0.3(B). In §6, we shall introduce the set $\mathscr{FPM}_n$ consisting of the frame bundles of the elements of $\mathscr{PM}_n$, and shall prove that the smooth elements of the closure $\mathscr{CFPM}_n$ are Riemannian manifolds. In §7, we shall give an estimate on the sectional curvatures of the smooth elements of $\mathscr{CFPM}_n$. In §8, we shall prove Theorem 0.15. In §9, we shall prove an equivariant version of the result of [6], which is used in §10 to prove our results on Problem 0.3(B). The proof of Theorems 0.6 and 0.9 is also in §10.

In §1, we gather several notations used in this paper. The reader can skip this section and return there when §1 is explicitly quoted.

Some of the results of this paper were announced without proof in [7]. There we also gave several examples and open problems. See also [3], [4], [5], [6], and [18] for related results, and [8] for an application.

The author would like to thank the referee who pointed out an error in the first version of this paper.

1. Notation and preliminary considerations

In this section, X and Y denote metric spaces, $p_0 \in X$, $q_0 \in Y$, and M denotes a Riemannian manifold.

Notation 1.1. We put

$$B_D(p_0, X) = \{p \in X \mid d(p_0, p) < D\},$$
$$B(D) = B_D(0, \mathbf{R}^n), \qquad B = B(1).$$

Notation 1.2. Let $C(X,Y)$ denote the set of continuous maps from X to Y. We define a metric d on $C(X,Y)$ by

$$d(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

Notation 1.3. Set

$$FM = \{(V_1, \cdots, V_n) \mid (V_1, \cdots, V_n) \text{ is an orthonormal base of the tangent space of a point of } M\}.$$

We define a metric on FM as follows. Let $\pi\colon FM \to M$ be the natural projection. The fiber of π is identified with the orthogonal group $\mathrm{O}(n)$. Fix a canonical metric on $\mathrm{O}(n)$. For each $q \in FM$, using the Levi-Civita connection, the tangent space $T_q(FM)$ is decomposed into the vertical subspace $T_q(\pi^{-1}\pi(q))$, and the horizontal subspace H_q. We define a metric on $T_q(\pi^{-1}\pi(q))$ using the canonical metric on $\mathrm{O}(n)$ and on H_q so that $d\pi\colon H_q \to T_{\pi(q)}(M)$ is an isometry. Also, we let the horizontal and the vertical subspaces be orthogonal. Thus we obtain a metric on FM. The group $\mathrm{O}(n)$ acts as isometries on FM, and the quotient space $FM/\mathrm{O}(n)$ with the quotient metric is isometric to M.

Notation 1.4. Let γ be a selfisometry of M. Assume that $p \in M$ and that $d(p, \gamma(p))$ is smaller than the injectivity radius of M at p. Let $l\colon [0, t_0] \to M$ denote the minimal geodesic connecting p with $\gamma(p)$. (We assume that l has unit speed.) Let $P\colon T_{\gamma(p)}(M) \to T_p(M)$ denote the parallel transformation along l. We set

$$t_p(\gamma) = t_0 \cdot \dot{l}(0),$$
$$r_p(\gamma)\colon T_p(M) \to T_p(M)\colon V \mapsto P(d\gamma(V)),$$
$$m_p(\gamma)\colon T_p(M) \to T_p(M)\colon V \mapsto P(d\gamma(V)) + t_p(\gamma),$$
$$\|r_p(\gamma)\| = \text{the supremum of the angles between } V \text{ and } r_p(\gamma)(V),$$
$$\|m_p(\gamma)\| = \|r_p(\gamma)\| + \|t_p(\gamma)\|.$$

Notation 1.5. We put

$$\mathscr{M}(n, D \mid C) = \{M \mid M \text{ satisfies (0.2.1), (0.2.2) and the sectional curvature of } M \text{ is smaller than } C \text{ and greater than } -C\}.$$
$$\mathscr{PM}'_n(C) = \{(M,p) \mid M \in \mathscr{M}(n, \infty \mid C)\}.$$

(We do not assume that the elements of $\mathscr{PM}'_n(C)$ are compact.)

Definition 1.6. We recall the definition of the ε-Hausdorff approximation and its pointed version. A (not necessarily continuous) map $f\colon X \to Y$ [resp. $(X,p_0) \to (Y,q_0)$] is said to be an *ε-Hausdorff approximation* [resp. *ε-pointed Hausdorff approximation*] if

(1.7.1) The ε-neighborhood of $f(X)$ contains Y [resp. $B_{1/\varepsilon}(q_0, Y)$].

(1.7.2) For each two elements x, y of X [resp. $B_{1/\varepsilon}(p_0, X)$] we have

$$|d(x,y) - d(f(x), f(y))| < \varepsilon.$$

We define the *Hausdorff distance* [resp. *pointed Hausdorff distance*] $d_{\mathrm{H}}(X,Y)$ [resp. $d_{\mathrm{H}}((X,p_0),(Y,q_0))$] to be the infimum of the positive numbers ε such that there exist ε-Hausdorff approximations [resp. ε-pointed Hausdorff approximations] from X to Y and from Y to X [resp. from (X,p_0) to (Y,q_0) and from (Y,q_0) to (X,p_0)].

Notation 1.8. We let $d_{\mathrm{L}}(X,Y)$ and $d_{\mathrm{L}}((X,p_0),(Y,q_0))$ denote the Lipschitz distance and the equivariant Lipschitz distance, which is defined in [12, Chapitre 3A].

Definition 1.9. Next, we need equivariant versions of the notion of the Hausdorff distance. Let G and H be groups acting as isometries on X and Y respectively. A pair of maps (f,φ), $f\colon (X,p_0) \to (Y,q_0)$, $\varphi\colon G \to H$, is said to be an *ε-pointed equivariant Hausdorff approximation* if the following hold.

(1.10.1) f is an ε-pointed Hausdorff approximation.

(1.10.2) For each $g \in G$ and $x \in X$, we have

$$d(\varphi(g)(f(x)), f(g(x))) < \varepsilon$$

if x and $g(x)$ are contained in $B_{1/\varepsilon}(p_0, X)$, and if $f(x)$, $f(g(x))$ and $\varphi(g)(f(x))$ are contained in $B_{1/\varepsilon}(q_0, Y)$.

Let the *pointed equivariant Hausdorff distance*, $d_{\mathrm{e.H.}}((X, G, p_0), (Y, H, q_0))$, denote the infimum of the numbers ε such that there exist ε-pointed equivariant Hausdorff approximations from (X, G, p_0) to (Y, H, q_0) and from (Y, H, q_0) to (X, G, p_0). The nonpointed version is defined similarly. The equivariant Hausdorff distance defined here is equivalent to that of [5]. Therefore, [5, Theorem 2.1] implies the following:

Lemma 1.11. *If*

$$\lim_{i\to\infty} d_{\mathrm{e.H.}}((X, G, p_0), (Y, H, q_0)) = 0,$$

then

$$\lim_{i\to\infty} d_{\mathrm{H}}((X/G, \bar{p}_0),\ (Y/H, \bar{q}_0)) = 0.$$

Definition 1.12. Suppose that a group G acts on X and Y as isometries. We say a map f from X to Y is an *ε-G-Hausdorff approximation* if $(f, \text{identity})\colon (X, G) \to (Y, G)$ is an ε-equivariant Hausdorff approximation. We define the *G-Hausdorff distance*, $d_{\mathrm{G}\text{-}H}(X, Y)$, to be the infimum of the positive numbers ε such that there exist ε- G-Hausdorff approximations from X to Y and from Y to X.

Lemma 1.13. *Let $\mathscr{M}(n, D; G)$ denote the set of pairs (M, χ) of Riemannian manifolds M contained in $\mathscr{M}(n, D)$ and an isometric action χ of G on M. If $D < \infty$, then $\mathscr{M}(n, D; G)$ is precompact with respect to the G-Hausdorff distance.*

We omit the proof, which is an easier half of the argument presented in [5, §3].

CHAPTER 1

SINGULARITIES OF THE ELEMENTS OF THE BOUNDARY

2. Reduction to the case when the differentials of the curvatures are bounded

First we recall the following result. (The symbol d_{L} is as in 1.8.)

Theorem 2.1 (*Bemelmans, Min-Oo & Ruh* [1]). *For each positive number ε and Riemannian manifold $M \in \mathscr{M}(n, \infty)$, there exists a Riemannian manifold $M' \in \mathscr{M}(n, \infty)$ such that*

$$d_{\mathrm{L}}(M, M') < \varepsilon, \tag{2.2.1}$$

$$\|\nabla^k R(M')\| < C(n, k, \varepsilon). \tag{2.2.2}$$

Here the symbol $R(M')$ denotes the curvature tensor, $\|\ \|$ the C^0-norm, and $C(n, k, \varepsilon)$ a positive number depending only on n, k and ε.

Secondly we need the following. (The symbol d_{H} is defined in 1.6.)

Lemma 2.3. *Let X_i, Y_i, X, Y be metric spaces, all of whose bounded subsets are relatively compact. Suppose that*

$$\lim_{i\to\infty} d_{\mathrm{H}}(X_i, X) = 0, \qquad \lim_{i\to\infty} d_{\mathrm{H}}(Y_i, Y) = 0,$$

and that $d_{\mathrm{L}}(X_i, Y_i) \le \varepsilon$. Then we have $d_L(X, Y) \le \varepsilon$.

Proof. We may assume $d_{\mathrm{H}}(X_i, X) < 1/i$ and $d_{\mathrm{H}}(Y_i, Y) < 1/i$. Then there exist $(1/i)$-Hausdorff approximations $\varphi_i \colon X \to X_i$, $\psi_i \colon Y_i \to Y$. On the other hand, since $d_{\mathrm{L}}(X_i, Y_i) \le \varepsilon$, there exist homeomorphisms $f_i \colon X_i \to Y_i$ satisfying

$$e^{-\varepsilon} \le d(f_i(x), f_i(y))/d(x, y) \le e^{\varepsilon} \tag{2.4}$$

for each x, $y \in X_i$.

Next, take a dense countable subset X_0 of X. By a standard diagonal procedure, we may assume, by taking a subsequence if necessary, that $\psi_i f_i \varphi_i(x)$ converges for each $x \in X_0$. Let $f'(x)$ be the limit. Then formulas (1.7.2) and (2.4) imply

$$e^{-\varepsilon} \le d(f'(x), f'(y))/d(x, y) \le e^{\varepsilon} \tag{2.5}$$

for each x, $y \in X_0$. Therefore f' can be extended to a homeomorphism $f \colon X \to Y$ satisfying (2.5). The required inequality $d_{\mathrm{L}}(X, Y) \le \varepsilon$ follows. q.e.d.

Now we start the proof of Theorem 0.5. Let (X, p_0) be an arbitrary element of $\mathscr{CRM}_n$. Then there exists a sequence (M_i', p_i') of elements of $\mathscr{RM}_n$ such that $\lim_{i\to\infty} d_{\mathrm{H}}((X, p_0), (M_i', p_i)) = 0$. Hence, Theorem 2.1 implies that, for each positive number ε, there exists $(M_i(\varepsilon), p_i(\varepsilon)) \in \mathscr{RM}_n$ such that $d_{\mathrm{L}}((M_i(\varepsilon), p_i(\varepsilon)), (M_i', p_i')) < \varepsilon$ and

$$\|\nabla^k R(M_i(\varepsilon))\| < C(n, k, \varepsilon). \tag{2.6}$$

Since $\mathscr{CRM}_n$ is compact [12, 5.3], we may assume, by taking a subsequence if necessary, that $(M_i(\varepsilon), p_i(\varepsilon))$ converges to a metric space $(X(\varepsilon), p_0(\varepsilon))$ with respect to the Hausdorff distance. Then Lemma 2.3 implies $d_{\mathrm{L}}(X, X(\varepsilon)) \le \varepsilon$. Thus, we see that to prove Theorem 0.5 it suffices to show that $X(\varepsilon)$ is a smooth element of $\mathscr{CRM}_n$. The proof of this fact occupies the rest of this chapter. Hereafter we shall write (M_i, p_i) and (X, p_0) instead of $(M_i(\varepsilon), p_i(\varepsilon))$ and $(X(\varepsilon), p_0(\varepsilon))$, for simplicity.

3. Construction of the Lie group germ

Some part of the argument of this and the next sections overlaps with that of [12, 8.30–8.36 and 8.48–8.51]. But, since the argument here is a bit delicate and since the author cannot understand some part of the argument there, he will not omit the overlapped part.

By changing a base point, we see that it suffices to show that a neighborhood of p is smooth. We may assume that $d_{\mathrm{H}}((X,p_0),(M_i,p_i)) < 1/i$. Let $\varphi_i\colon (X,p_0) \to (M_i,p_i)$ denote a $(1/i)$-Hausdorff approximation and $f_i\colon \mathbf{R}^n \to M_i$ the composition of a linear isometry $\mathbf{R}^n \to T_{p_i}(M_i)$ and the exponential map $T_{p_i}(M_i) \to M_i$. By Rauch's comparison theorem (see [15, Chapter VIII, Theorem 4.1]), the map f_i is of maximal rank on the unit ball B (see 1.1). Let g_i $(= g_{i;j,k})\colon B \to \mathbf{R}^{n^2}$ be the Riemannian metric tensor induced by f_i from that of M_i. Formula (2.6) implies that

$$\left\| \frac{\partial^l g_{i;j,k}}{\partial x_{m_1}\partial x_{m_2}\cdots\partial x_{m_l}} \right\| < C_l.$$

It follows that we may assume, by taking a subsequence if necessary, that g_i converges to a C^∞-metric tensor g_0. Hereafter we let d_i $(i = 0,1,2,\cdots)$ denote the distance function associated to g_i and d the ordinary Euclidean distance.

First, we shall construct a local group G of isometries such that a neighborhood of p_0 in X is isometric to U/G for a neighborhood U of 0 in B. The fundamental definitions on local groups are presented in [20, §23D,$\cdots$,N]. There the notion of an action of a local group on a pointed topological space is not defined. But we omit the definition, since it can be defined in an obvious way.

Now, we define the local group G_i as

$$G_i = \{\gamma \in C(B(1/2),B) \mid f_i\gamma = f_i\},$$

where $C(A,B)$ is as in 1.2. The local group structure on G_i is defined as follows: for $\gamma_1,\gamma_2,\gamma_3 \in G_i$, we put $\gamma_1\gamma_2 = \gamma_3$ if the composition $\gamma_1\gamma_2$ is well defined and coincides with γ_3 in a neighborhood of 0. Next, for $p \in B(1/2)$ and $\varepsilon > 0$, we put

$$G_i(p,\varepsilon) = \{f \in G_i \mid d(f(p),p) < \varepsilon\}.$$

Second, we shall take the limit of G_i. Put

$$L = \{f \in C(B(1/2),B) \mid 1/2 \le d_0(f(x),f(y))/d_0(x,y) \le 2 \quad \text{for each } x,y \in B(1/2)\}.$$

Ascoli-Arzela's theorem implies that L is compact. It is well known that the set of closed subsets of a given compact set is compact with respect to the (usual) Hausdorff distance. Therefore, by taking a subsequence if necessary, we may assume that G_i converges to a closed subset G of L. We can define a local group structure on G by a method similar to that for G_i.

Remark that when a local group H acts as isometries on a pointed metric space (Y, p), the isometry type of a neighborhood of $(p \bmod H)$ in the quotient space Y/H is well defined (see [20, §23J]). We shall let this "local metric space" be denoted by $(Y, p)/H$. In our case, $(B(1/2, 0), 0)/G_i$ is isometric to $B_{1/2}(p_i, M_i)$. (Furthermore, in our case, the 1/2-neighborhood of $(0 \bmod G_i)$ is well defined.) This fact, combined with Lemma 1.11, implies that $(B(1/2, 0), 0)/G$ is isometric to $B_{1/2}(p_0, X)$. Let $\pi \colon B(1/2) \to B_{1/2}(p_0, X)$ and $\pi_i \colon B(1/2) \to B_{1/2}(p_i, M_i)$ denote the natural projections.

Third, we shall prove that our local group G is a Lie group germ. This fact follows from the following:

Lemma 3.1. *Suppose a local group G acts effectively on a pointed Riemannian manifold (M, p) as isometries. Assume that G is closed in $C(B_{D/2}(p, M),\ B_D(p, M))$. Then G is locally isomorphic to a Lie group and its action on (M, p) is smooth.*

Proof. This lemma seems to be known by the experts. But, since it seems that this fact is not proved in the literature, the proof will be given below. Let $\mathfrak{g}'$ be the set of all vector fields ξ such that the following condition holds.

Condition 3.2. There exists a smooth map $\varphi \colon (-\varepsilon, \varepsilon) \to G$ satisfying the following. (Since G is contained in a Frechet manifold $C(B_{D/2}(p, M), B_D(p, M))$, the smoothness of a map from $(-\varepsilon, \varepsilon)$ to G is well defined.)

$$\varphi(0) = \text{identity}, \tag{3.2.1}$$

$$\left.\frac{D\varphi(t)(p)}{dt}\right|_{t=0} = \xi(p). \tag{3.2.2}$$

Now since

$$\left.\frac{D\varphi_1(t)\varphi_2(t)}{dt}\right|_{t=0} = \left.\frac{D\varphi_1(t)}{dt}\right|_{t=0} + \left.\frac{D\varphi_2(t)}{dt}\right|_{t=0}$$

and since

$$\left.\frac{D}{dt^2}(\varphi_1(t)\varphi_2(t)\varphi_1^{-1}(t)\varphi_2^{-1}(t))\right|_{t=0} = \left[\left.\frac{D\varphi_1(t)}{dt}\right|_{t=0}, \left.\frac{D\varphi_2(t)}{dt}\right|_{t=0}\right],$$

it follows that $\mathfrak{g}'$ is a Lie algebra. Let G' be the local set consisting of all one-parameter groups of transformations associated with the elements of $\mathfrak{g}'$. Using the fact that $\mathfrak{g}'$ is a Lie algebra, we can prove easily that G' is a Lie group germ.

Sublemma 3.3. *G' is a sub-local group of G.*

Proof. Suppose that $\xi \in \mathfrak{g}'$ and that $\varphi \colon (-\varepsilon, \varepsilon) \to G$ satisfies Condition 3.2. Let Φ_t denote the one-parameter group of transformations associated with ξ. We shall prove that $\Phi_{t_0} \in G$ for small t_0. Put $\gamma_n = (\varphi(t_0/n))^n$.

Using (3.2.2), we can prove $\lim_{n\to\infty}\gamma_n = \Phi_{t_0}$. On the other hand, since G is closed, it follows that $\Phi_{t_0}\in G$. q.e.d.

Now, to prove Lemma 3.1, it suffices to show the following:

Sublemma 3.4. *G' contains a neighborhood of the identity of G.*

Proof. Suppose that the sublemma is false. Then there exists a sequence of elements γ_i of $G - G'$ which converges to the identity. Here we need a simple trick to make the action of G free. Let FM be as in 1.3. The action of G can be lifted to a free isometric action on FM. Take an element q of FM. Now, by replacing elements γ_i if necessary, we may assume the following:

(3.5) The minimal geodesic l_i connecting q with $\gamma_i(q)$ is perpendicular to the orbit $G'(q)$.

Now, since γ_i converges to the identity map, we may assume, by taking a subsequence if necessary, that there exists a strictly increasing sequence n_i of positive integers such that $\gamma_i^{n_i}$ converges to a nontrivial element γ. Then, fact (3.5) implies that $\gamma\notin G$. On the other hand we have

Assertion 3.6. $\gamma\in G'$.

Proof. For $t\in[0,1]$, we put $\varphi_t = \lim_{i\to\infty}\gamma_i^{[tm_i]}$, where $[c]$ denotes the maximum integer not greater than c. It is easy to see that φ_t is well defined and is a one-parameter group of transformations. It is also easy to see that $\varphi_1 = \gamma$ and $\varphi_t\in G$. Therefore $\gamma\in G'$ as desired. q.e.d.

This is a contradiction. The proof of Sublemma 3.4 is now complete.

4. Nilpotency of the local group G

Lemma 4.1. *The Lie algebra $\mathfrak{g}$ of G is nilpotent.*

Proof. Take a small neighborhood W of the identity in L such that $\|m_p(\gamma)\| < 0.49$ holds for each element γ of $W\cap G$ and $p\in B(1/2)$ (see 1.4 and 1.1). Now Lemma 4.1 follows from the following:

Lemma 4.1. *There exists a neighborhood W' of the identity in W such that the n-hold commutators of the elements of $G_i\cap W'$ are well defined in G and vanish.*

Remark 4.3. This corresponds to [12, 8.50]. In order to prove this lemma following the line described there, we have to overcome the difficulty pointed out in [2, Remark 3.1.6]. But the author cannot do this directly. Instead, we shall use the result of [6], and proceed as follows.

Proof of Lemma 4.2. By the result of §3, we see that there exists a point p in each neighborhood of 0 in B such that $\{\gamma\in g \mid \gamma(p) = p\} = \{1\}$. Hence, a neighborhood V of $\pi(p)$ in $B_{1/2}(p_0, X)$ is a Riemannian manifold. Therefore, by the main theorem of [6], we conclude that, for each sufficiently large i, there exists a fiber bundle $f_i\colon U_i\to V$ from a neighborhood U_i of $\pi_i(p)$ in M_i to V, such that the fiber of f_i is an infranilmanifold. Furthermore, §5 of [6] implies that there exists a positive number ε independent of i such that $G_i(p,\varepsilon)$ is a sub-local group of the fundamental group of the fiber of f_i. (Remark that $G_i(p,\varepsilon)$ coincides with what is called a local fundamental pseudogroup at the beginning of [6, §5].) Moreover, by virtue of the inequality $\|m_p(\gamma)\| < 0.49$, we see that the fundamental group of the fiber of f_i itself is nilpotent, without taking a finite covering (see the argument in [2, Chapter 3]). Hence every n-hold commutator of elements of $G_i(p,\varepsilon)$ vanishes.

On the other hand, it is easy to see that there exists W' such that

$$G_i(p,\varepsilon) \supset W' \cap G_i$$

for every i. This completes the proof.

5. The proof of Theorem 0.5

Let $\mathfrak{g}$ denote the Lie algebra of G and, for $p \in B(1/2)$, put

$$\mathfrak{h}_p = \{\xi \in \mathfrak{g} \mid \xi(p) = 0\}.$$

Lemma 5.1. *$\mathfrak{h}_p$ is contained in the center of $\mathfrak{g}$.*

Proof. (The following argument was suggested to the author by Hisayosi Matumoto.) Let $\xi \in \mathfrak{h}_p$. Since the closure of the one-parameter group of transformations associated with ξ is compact, it follows that the adjoint representation $\mathfrak{g} \to \mathfrak{g}$, $\eta \mapsto [\eta, \xi]$ is semisimple. Therefore, if ξ is not contained in the center, there exists $\eta \in \mathfrak{g} \otimes \mathbf{C}$ such that $[\eta, \xi] = \alpha\eta$ and $\alpha \neq 0$. But, then the Lie subalgebra $\mathbf{C}\xi \oplus \mathbf{C}\eta$ is not nilpotent. This is a contradiction. q.e.d.

The function which carries p to $\dim \mathfrak{h}_p$ is uppersemicontinuous. Hence, there exists a positive number C such that, for each element p of $B(C)$,

$$\dim \mathfrak{h}_p \leq \dim \mathfrak{h}_0. \tag{5.2}$$

Lemma 5.3. *$\mathfrak{h}_p \subseteqq \mathfrak{h}_0$ for each element p of $B(C/6)$.*

Proof. The proof is by contradiction. Take $\xi \in \mathfrak{h}_p - \mathfrak{h}_0$. Let φ_t be the one-parameter group of transformations associated with ξ. Since the closure $\{\varphi_t \mid t \in \mathbf{R}\}$ is compact, we may assume, by replacing ξ if necessary, that φ_1 is the identity. Put

$$A = \{q \in B(1/2) \mid \eta(q) = 0 \text{ for each } \eta \in \mathfrak{h}_0\}.$$

A is totally geodesic because all elements of $\mathfrak{g}$ are Killing vector fields. Since $p \in B(C/6)$ and since $\varphi_t(p) = p$, it follows that

$$d(\varphi_t(0), 0) \leq C/3. \tag{5.4}$$

On the other hand, since $\mathfrak{h}_0$ is contained in the center, we have $\varphi_t(0) \in A$. Now, define a φ_t-invariant function f on $B(C) \cap A$ by

$$f(q) = \int_0^1 d(\varphi_t(0), q)\, dt.$$

Since A is totally geodesic and since $C \leq 1$, it follows that f is a strictly convex function. On the other hand, formula (5.4) implies that

$$f(q) \geq 2C/3 \quad \text{for } q \in \partial B(C), \qquad f(0) \leq C/3.$$

Therefore, f has a unique minimum q_0 on $A \cap B(C)$. Then $\varphi_t(q_0) = q_0$. It follows that $\xi \in \mathfrak{h}_{q_0}$. On the other hand, $\mathfrak{h}_{q_0} \supset \mathfrak{h}_0$. Thus, we conclude $\dim \mathfrak{h}_{q_0} > \dim \mathfrak{h}_0$. This contradicts (5.2). q.e.d.

For a point p of $B(1/2)$, we put

$$H_p = \{\gamma \in G \mid \gamma(p) = p\}$$

and let H'_p denote the component of the identity of H_p.

Lemma 5.5. *There exists a positive number C' such that $H_p \subseteqq H_0$ for each point p of $B(C'/6)$.*

Proof. For a point p of A, put $\chi(p) = \#(H_p/H'_p)$. It is easy to see that $\chi(p)$ is uppersemicontinuous on A. Then there exists a positive number C' such that for each element p of $B(C') \cap A$, we have $\chi(p) \leq \chi(0)$. Now, we shall prove by contradiction that this number C' has the required property. Suppose that $p \in B(C'/6)$ and $\gamma \in H_p - H_0$. Lemma 5.4 and the compactness of H_p imply that there exists a positive integer m such that γ^m is contained in H_0. Put

$$A' = \{p \in B(C') \mid \gamma(p) = p \text{ for each } \gamma \in H_0\}.$$

Define $f' \colon A' \to \mathbf{R}$ by

$$f'(x) = \sum_{i=1}^{m} d(\gamma^i(x), x).$$

f' is γ-invariant, since $\gamma^m(x) = x$. Hence, as in the proof of Lemma 5.4, we can find $q \in B(C') \cap A'$ such that $\gamma(q) = q$. Therefore $H_q \supset H_0 \cup \{\gamma\}$. It follows that $\chi(q) > \chi(0)$. This is a contradiction. q.e.d.

Lemma 5.1 implies that H'_0 is a torus. Hence $(B(C'/6), 0)/H'_0$ is smooth. Since H_0 is compact, H_0/H'_0 is a finite group. Therefore, $(B(C'/6), 0)/H_0$ is also smooth. Furthermore, using Lemma 5.5, we can prove that H_0 is normalized by G_0. Therefore, G_0H_0/H_0 acts on $(B(C'/6), 0)/H_0$. Then Lemma 5.5 immediately implies that the action of $G_0 \cdot H_0/H_0$ on $B(C'/6)/H_0$ is free. It follows that $(B(C'/6), 0)/H_0G_0$ is smooth. Next, we need the following:

Lemma 5.6. *There exists D such that $G(0, D)$ is contained in H_0G_0.*

Proof. Suppose that there exists a sequence γ_i of elements of G such that $\gamma_i \in G(0, 1/i) - H_0G_0$. By taking a subsequence if necessary, we may assume that γ_i converges to an element γ. Then $\gamma(0) = 0$. Therefore $\gamma \in H$. On the other hand, $\lim_{i\to\infty} \gamma^{-1}\gamma_i = 1$. Hence $\gamma^{-1}\gamma_i \in G_0$ for sufficiently large i. Therefore, $\gamma_i \in H_0G_0$. This is a contradiction. q.e.d.

Lemma 5.6 implies that $B_D(p, X)$ is isometic to $(B(D), 0)/H_0G_0$. This completes the proof of Theorem 0.5.

CHAPTER 2

GENERALIZED FIBER BUNDLE THEOREM

6. A compactification of the set of frame bundles

In this chapter, we deal with Problem 0.3(B). One of the difficulties of this problem lies in the fact that the metric space X there is not necessarily a manifold. To avoid this difficulty, we consider the frame bundles. We put

$$\mathscr{FM}(n, D) = \{FM \mid M \in \mathscr{M}(n, D)\},$$
$$\mathscr{FPM}_n = \{(FM, p) \mid M \in M(n, \infty)\}.$$

(The Riemannian manifold FM is defined in 1.3.) Let $\mathscr{CFM}(n, D)$ and $\mathscr{CFPM}_n$ denote the closures of $\mathscr{FM}(n, D)$ and $\mathscr{FPM}_n$ with respect to the Hausdorff distance and the pointed Hausdorff distance respectively. By virtue of the results presented in [17], there exist positive numbers $C_1(n)$ and $C_2(n)$ depending only on n such that

$$\mathscr{FM}(n, D) \subset \mathscr{M}(n + (n-1)(n-2)/2, D + C_1(n) \mid C_2(n))$$

and $\mathscr{FPM}_n \subset \mathscr{PM}_n(C_2(n))$ (see 1-5). It follows that $\mathscr{CFM}(n, D)$ and $\mathscr{CFPM}_n$ are compact. Now, the main result of this and the next sections is the following:

Theorem 6.1. *There exists a positive constant $C_3(n)$ depending only on n such that the intersection of $\mathscr{CFPM}_n$ with*

$$\bigcup_{k=0}^{n+(n-1)(n-2)/2} \mathscr{PM}_k(C_3(n))$$

is dense in $\mathscr{CFPM}_n$ with respect to the pointed Lipschitz distance.

Proof. Let (X, q_0) be an arbitrary element of $\mathscr{CFPM}_n$. Take a sequence of elements (FM_i, q_i) of $\mathscr{FPM}_n$ such that $\lim_{i\to\infty} d_{\mathrm{H}}((FM_i, q_i), (X, q_0)) = 0$. Let $\pi_i \colon FM_i \to M_i$ denote the natural projection. Put $p_i = \pi_i(q_i)$. By an argument similar to one in §2, we may assume, by taking a subsequence if necessary, that

$$\|\nabla^k R(M_i)\| \le C_k.$$

In this section, we shall prove that, in that case, X is a Riemannian manifold. And, in the next section, we shall give an estimate on the sectional curvature of X. It suffices to show this in a neighborhood of q_0.

First remark that we may assume, by taking a subsequence if necessary, that (M_i, p_i) converges to a pointed metric space (Y, p_0) with respect to the pointed Hausdorff distance. We may assume that $d_{\mathrm{H}}((M_i, p_i), (Y, p_0)) < 1/i$ and $d_{\mathrm{H}}((FM_i, q_i), (X, q_0)) < 1/i$. Let $\psi_i \colon (X, q_0) \to (FM_i, q_i)$ and $\varphi_i \colon (Y, p_0) \to (M_i, p_i)$ be $(1/i)$-pointed Hausdorff approximations.

Next, we recall the argument of §3. There we defined pairs $((B(1/2), g_i), G_i)$ and $((B(1/2), g_0), G)$ such that $B(1/2)/G_i$ and $B(1/2)/G$ are isometric to $B_{1/2}(p_i, M_i)$ and $B_{1/2}(p_0, X)$ respectively and that G is locally isomorphic to a Lie group.

Now, we can lift the isometric actions of G_i and G on $(B(1/2), g_i)$ and $(B(1/2), g_0)$ to those on $(FB(1/2), \tilde{g}_i)$ and $(FB(1/2), \tilde{g}_0)$ respectively, where $\tilde{g}_i$ and $\tilde{g}_0$ denote the Riemannian metric defined in 1.3. Since the action of G on $B(1/2)$ is isometric, it follows that the action of G on $FB(1/2)$ is free. Hence $FB(1/2)/G$ is a Riemannian manifold.

On the other hand, it is easy to see that

$$\lim_{i\to\infty} d_{\mathrm{e.H.}}(((FB(1/2), \tilde{g}_i), G_i, 0), ((FB(1/2), \tilde{g}_0), G, 0)) = 0.$$

(The symbol $d_{\mathrm{e.H.}}$ is defined in 1.9.) Hence, Lemma 1.11 implies that

$$\lim_{i\to\infty} d_{\mathrm{H}}(FB(1/2)/G_i, FB(1/2)/G) = 0.$$

On the other hand, it is easy to see that $FB(1/2)/G_i$ is isometric to a neighborhood of q_i in FM_i. Therefore $FB(1/2)/G$ is isometric to a neighborhood of q_0 in X. Thus X is a Riemannian manifold, as required.

7. An estimate on sectional curvatures

We begin by proving a lemma.

Notation 7.1. Let G be a local group of isometries acting freely on a pointed Riemannian manifold (M, p). We put

$$(r/t)_p(G) = \sup\{\|r_p(g)\|/d(g(p), p) \mid g \in G, g \ne 1, \quad r_p(g) \text{ is well defined}\}.$$

(The symbol $r_p(g)$ is defined in 1.4.)

Lemma 7.2. *Suppose that the sectional curvature of M is not greater than a and not smaller than b. Then the sectional curvature of M/G at $P(p)$ is not greater than $a + 6((r/t)_p(G))^2$ and not smaller than b, where $P\colon M \to M/G$ denotes the natural projection.*

Proof. Put $q = P(p)$. Let λ be an arbitrary plane contained in $T_q(M/G)$. Take the plane Λ in $T_p(M)$ such that $dP(\Lambda) = \pi$ and Λ is perpendicular to the orbit $G(p)$. Let K_Λ and K_λ denote the sectional curvatures. For $\xi \in \Lambda$ and $t \in \mathbf{R}$, we see easily that

$$P(\exp(t\xi)) = \exp(t(dP(\xi))). \tag{7.3}$$

Now, let $i\colon S^1 \to \Lambda$ be the isometry onto the unit sphere. Recall the following formula.

$$\int_0^t l(\exp(s\cdot i))\,ds = \pi t^2 - \pi K_\Lambda t^4/12 + O(t^5), \tag{7.4}$$

where $l(\exp(t\cdot i))$ denotes the length of the loop, $\theta \mapsto \exp(t\cdot i(\theta))$. Similarly, using (7.3), we see that

$$\int_0^t l(P(\exp(s\cdot i)))\,ds = \pi t^2 - \pi K_\lambda t^4/12 + O(t^5). \tag{7.5}$$

Now, let $\varphi(\theta_0, t)$ denote the angle between

$$\left.\frac{D\exp(t\cdot i(\theta))}{d\theta}\right|_{\theta=\theta_0} \quad \text{and} \quad T_{\exp(t\cdot i(\theta_0))}(G(\exp(t\cdot i(\theta_0)))).$$

Then, it is easy to see that

$$1 \geq \frac{l(P(\exp(t\cdot i)))}{l(\exp(t\cdot i))} \geq \inf\{\sin(\varphi(\theta,t)) \mid \theta \in S^1\}. \tag{7.6}$$

On the other hand, by the definition of $(r/l)_p(G)$, we have

$$\limsup_{t\to 0} \frac{1}{t^2}[1 - \inf\{\sin\varphi(\theta,t) \mid \theta \in S^1\}] \leq \frac{((r/l)_p(G))^2}{2}. \tag{7.7}$$

Now, by (7.4), (7.6) and (7.7), we have

$$\begin{aligned} &\pi t^2 - \pi t^4 K_\Lambda/12 + O(t^5) \\ &\geq \int_0^t l(P(\exp(s\cdot i)))\,ds \\ &\geq \pi t^2 - \pi t^4 K_\Lambda/12 - \pi t^4((r/l)_p(G))^2/2 - O(t^5). \end{aligned}$$

From this formula and formula (7.5), the lemma follows immediately. q.e.d.

Next we shall prove the following:

Lemma 7.8. *Let (M_i, p_i) be a sequence of elements of $\mathscr{CRM}_n$ converging to a smooth element (X, p_0) of $\mathscr{CRM}_n$. Suppose that the sectional curvatures* *of M_i at p_i are unbounded. Then the dimension of the group G_{p_0} in Definition 0.4 is positive.*

Proof. Let $(M_{i,j}, p_{i,j})$ be elements of $\mathscr{PM}_n$ such that

$$d_{\mathrm{H}}((M_{i,j}, p_{i,j}), (M_i, p_i)) < 1/j.$$

As in §2, we may assume $\|\nabla^k R(M_{i,j})\| < C_k$. Hence, by the method of §3, we can construct metrics $g_{i,j}$, g_i, g_0 on B and local groups $G_{i,j}$, G_i, G consisting of isometries of $(B(1/2), g_{i,j})$, $(B(1/2), g_i)$, $(B(1/2), g_0)$, such that the quotient spaces $B(1/2)/G_{i,j}$, $B(1/2)/G_i$, $B(1/2)/G$ are isometric to neighborhoods of $p_{i,j}$, p_i, p_0, respectively. Then, Lemma 7.2 implies that the sectional curvatures of M_i at p_i are not smaller than -1 and not greater than $1 + 6 \cdot ((r/t)_0(G_i))^2$. Therefore, by assumption, we see that the numbers $(r/t)_0(G_i)$ are unbounded. Hence, by taking a subsequence if necessary, we may assume that there exists a sequence $\gamma_i \in G_i$ such that $\lim_{i\to\infty} \|r_0(\gamma_i)\|/d(0, \gamma_i(0)) = \infty$. It follows that we can find a sequence of integers n_i such that $\lim_{i\to\infty} d(\gamma_i^{n_i}(0), 0) = 0$, $\lim_{i\to\infty} r_0(\gamma_i^{n_i}) = A$, and that $\lim_{i\to\infty} n_i = \infty$, where $A \in \mathrm{O}(n)$ is a nontrivial element. Now for each number t contained in [0,1], we put $\eta_t = \lim_{i\to\infty} \gamma_i^{[tn_i]}$. Then, $\eta_t \in G$, $\eta_{t_1}\eta_{t_2} = \eta_{t_1+t_2}$, $\eta_1 \neq 1$ and $\eta_t(0) = 0$. Therefore, the dimension of G_p $(= \{g \in G \mid g(0) = 0\})$ is positive. q.e.d.

Now, Theorem 6.1 follows immediately from Lemma 7.8 and the fact that the elements of $\mathscr{CPFM}_n$ are manifolds, which was proved in §6.

8. The proof of Theorem 0.15

We begin by proving a lemma. Put

$$\mathscr{CFM}_k(n, D) = \{M \in \mathscr{CFM}(n, D) \mid \dim M \leq n + (n-1)(n-2)/2 - k\},$$

$$\mathscr{CFPM}_{n,k} = \{(M, p_0) \in \mathscr{CFPM}_n \mid \dim M \leq n + (n-1)(n-2)/2 - k\}.$$

Lemma 8.1. *For each ε there exists a positive number $\mu(\varepsilon, n)$ such that if a smooth pointed Riemannian manifold $(M, p_0) \in \mathscr{CFPM}_{n,k}$ satisfies $d_H((M, p_0), \mathscr{CFPM}_{n,k+1}) > \varepsilon$, then the injectivity radius of M at p_0 is greater than μ.*

Proof. The proof is by contradiction. Assume that a sequence of pointed Riemannian manifolds $(M_i, p_i) \in \mathscr{CFPM}_{n,k}$ satisfies $d_{\mathrm{H}}((M_i, p_i), \mathscr{CFPM}_{n,k+1}) > \varepsilon$ and that the injectivity radius of M_i at p_i is smaller than $1/i$. By virtue of the compactness of $\mathscr{CFPM}_n$, we may assume, by taking a subsequence if necessary, that (M_i, p_i) converges to an element (X, p_0) of $\mathscr{CFPM}_n$. Then, since the absolute values of sectional curvatures of M_i are bounded, [12, 8.39] implies that the Hausdorff dimension of X is strictly smaller than that of M_i. But, since $d_{\mathrm{H}}((M_i, p_i), \mathscr{CFPM}_{n,k+1}) > \varepsilon$, it follows that $X \notin \mathscr{CFPM}_{n,k+1}$. This is a contradiction.

Proposition 8.2. *There exist positive numbers $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ depending only on n such that the following holds.*

Suppose

$$X \in \mathscr{CFM}_k(n, D), Y \in \mathscr{CFM}(n, D) - \mathscr{CFM}_k(n, D),$$

and

$$d_{\mathrm{H}}(X, \mathscr{CFM}_{k+1}(n, D)) > \varepsilon_{k+1}.$$

Assume, furthermore, that $d_{\mathrm{H}}(X, Y) < \varepsilon_k$.

Then, there exists a map $f\colon X \to Y$ satisfying the following:

(8.3.1) *f is a fiber bundle with an infranilmanifold fiber.*

(8.3.2) *f is an almost Riemannian submersion. Namely, if $\xi \in T_p(M)$ is perpendicular to a fiber of f, then we have*

$$e^{-\tau(d_{\mathrm{H}}(X,Y))} < \|df(\xi)\|/\|\xi\| < e^{\tau(d_{\mathrm{H}}(X,Y))},$$

where $\tau(c)$ is a positive number depending only on c, n and D and satisfying $\lim_{c\to 0} \tau(c) = 0$.

Proof. This is an easy consequence of Theorem 6.1, Lemma 8.1 and the main theorem of [6].

Proof of Theorem 0.15. Define the subsets $\mathscr{U}_k$ of $\mathscr{CFPM}_k(n, D)$ by a downward induction on k as follows.

$$\mathscr{U}_{n+(n-1)(n-2)/2} = \mathscr{CFM}_{n+(n-1)(n-2)/2}(n, D),$$

$$\mathscr{U}_k = \mathscr{CFM}_k(n, D) - \bigcup_{i>k} \{X \in \mathscr{CFM}_k(n, D) \mid d_{\mathrm{H}}(X, \mathscr{U}_i) < \varepsilon_i\}.$$

(Remark that $\mathscr{CFM}_k(n, D)$ is empty for $k > n+(n-1)(n-2)/2$.) Then Lemma 8.1 implies that there exists a positive number μ such that the injectivity radii of the elements of $\bigcup \mathscr{U}_k$ are greater than μ. This fact, combined with Theorem 6.1, the compactness of $\mathscr{U}_k$ and [12, 8.25], implies that there exists a finite set Σ of manifolds such that every element of $\bigcup \mathscr{U}_k$ is diffeomorphic to an element of Σ.

Now, let M be an arbitrary element of $FM(n, D)$. Then, by the definition of $\mathscr{U}_k$, we see that either FM is contained in $\mathscr{U}_0$ or there exist k and $X \in \mathscr{CFM}_k$ such that $d_{\mathrm{H}}(FM, X) < \varepsilon_k$ and $d_{\mathrm{H}}(X, \mathscr{CFM}_{k+1}) > \varepsilon_{k+1}$. In the former case, FM is diffeomorphic to an element of Σ. In the later case, Proposition 8.2 implies that there exists a map $f\colon FM \to X$ satisfying conditions (8.3.1) and (8.3.2), and that X is diffeomorphic to an element of Σ. The proof of Theorem 0.15 is now complete.

9. Equivariant fiber bundle theorem

To deduce Theorem 0.12 from Theorem 6.1, we need the following equivariant version of the result of [6]. (The symbol $d_{G\text{-H}}$ is defined in 1.12.)

Theorem 9.1. *Let G be a locally compact group and let n, μ be positive numbers. Then there exists a positive number $\varepsilon(n, \mu)$ depending only on n and μ and satisfying the following.*

Suppose M, N are Riemannian manifolds on which G acts as isometries. Assume $d_{G\text{-H}}(M, N) < \varepsilon$, $M \in \mathscr{M}(n_1, \infty)$, $N \in \mathscr{M}(n_2, \infty, \mu)$, n_1, $n_2 \leq n$. Then there exists a G-map $f\colon M \to N$ satisfying (8.3.1) *and* (8.3.2).

Proof. There are two methods to prove this result. The first one is to construct f using the result of [6] and to make it a G-map using the center of mass technique (see [13]). The second one is the combination of the methods of [6] and [5, §7]. Here we shall give a proof following the second line. By assumption, we have an ε-G-Hausdorff approximation $\varphi'\colon M \to N$ (see 1.6). We can modify this map and we can assume that φ is a measurable map.

Secondly we use a Hilbert space version of the technique of [12], [14] or [6, §1]. Let $h\colon \mathbf{R} \to [0, 1]$ be a function satisfying [6, Condition (1.3)]. And let $L^2(N)$ denote the Hilbert space consisting of all L^2-functions on N. The group G acts on $L^2(N)$ in an obvious way. Define $f_N\colon N \to L^2(N)$ and $f'_M\colon M \to L^2(N)$, $f_M\colon M \to L^2(N)$, by

$$(f_N(p))(q) = h(d(p, q)),$$

$$(f'_M(p))(q) = h\left(\int_{x \in B_\varepsilon(\varphi(q), M)} d(p, x)\, dx / \mathrm{Vol}(B_\varepsilon(\varphi(q), M))\right),$$

$$f_M(p)(q) = \int_{g \in G} f'_M(g(p))(g(q))\mu_G(g),$$

where μ_G denotes the Haar measure. Then, by a method similar to [6], we can prove the following.

(9.2.1) f_N is an embedding.

(9.2.2) Put

$$B_C(Nf_N(N)) = \{(p, u) \in \text{ the normal bundle of } f_N(N) \mid \|u\| < C\}.$$

Then the restriction of the exponential map to $B_C(Nf_N(N))$ is a diffeomorphism, where C is a positive number depending only on n and μ.

(9.2.3) f_M is of C^1-class.

(9.2.4) The image of f_M is contained in the 6ε-neighborhood of $f_N(N)$.

(9.2.5) f_M is transversal to the fibers of the normal bundle of $f_N(N)$. (Here we identify the tubular neighborhood to the normal bundle.)

(9.2.6) f_M and f_N are G-maps.

Now, we put $f = f_N^{-1} \circ \pi \circ \mathrm{Exp}^{-1} \circ f_M$. Facts (9.2.2) and (9.2.4) imply that f is well defined. Fact (9.2.3) implies that f is of C^1-class. Fact (9.2.6) implies that f is a G-map. Fact (9.2.5) implies that f is a fiber bundle. The rest of the proof is similar to [6, §§4 and 5], and hence is omitted. The proof of Theorem 9.1 is now complete.

10. The proof of Theorem 0.12

Our result from Problem 0.3(B) in the case when X is general is the following.

Theorem 10.1. *Let X_i be a sequence of elements of $\mathscr{CM}(n, D)$. Suppose X_i converges to a metric space X with respect to the Hausdorff distance. Then, for sufficiently large i, there exist a map $f\colon X_i \to X$, metric spaces Y_i and Y on which $\mathrm{O}(n)$ acts as isometries and an $\mathrm{O}(n)$-map $\tilde{f}\colon Y_i \to Y$, such that the following holds.*

(10.2.1) *X_i and X are isometric to $Y_i/\mathrm{O}(n)$ and $Y/\mathrm{O}(n)$, respectively. (We let $\pi_i\colon Y_i \to X_i$, $\pi\colon Y \to X$ denote natural projections.)*

(10.2.2) *Y_i and Y are Riemannian manifolds with continuous metric tensors and $C^{1,\alpha}$-distance function.*

(10.2.3) *$\tilde{f}$ satisfies conditions* (8.3.1) *and* (8.3.2).

(10.2.4) *Let $p_i \in Y_i$, $p \in Y$. Then $\{g \in \mathrm{O}(n) \mid g(p) = p\}$ is isomorphic to $G_{\pi(p)}$ (which is defined in* 0.4*), and similarly for p_i.*

(10.2.5) $f \circ \pi_i = \pi \circ f$.

Theorems 0.12 and 0.14 are direct consequences of Theorem 10.1. Theorem 0.7 follows immediately from Theorem 10.1, Lemma 7.8 and [12, 8.39].

Proof of Theorem 10.1. Take $\mathscr{M}_{i,j} \in \mathscr{M}(n, D)$ satisfying $d_{\mathrm{H}}(M_{i,j}, X_i) < 1/j$. Lemma 1.13 implies that, by taking a subsequence if necessary, we may assume that

$$d_{\mathrm{O}(n)\text{-}\mathrm{H}}(FM_{i,j}, FM_{i',j'}) < 1/\min(j, j') + 1/\min(i, i').$$

Therefore, there exist $Y_i, Y \in \mathscr{FM}(n, D)$ on which $\mathrm{O}(n)$ acts as isometries such that

$$(10.3) \qquad d_{\mathrm{O}(n)\text{-}\mathrm{H}}(FM_{i,j}, Y_i) < 1/j, \qquad d_{\mathrm{O}(n)\text{-}\mathrm{H}}(Y_i, Y) < 1/i.$$

Theorem 6.1, combined with [9], implies that Y_i and Y satisfy (10.2.2). Inequality (10.3), combined with Lemma 1.11, implies (10.2.1). Theorem 9.1 implies that there exists an $\mathrm{O}(n)$-map $\tilde{f}\colon Y_i \to Y$ satisfying (10.2.3). Hence, there exists $f\colon X_i \to X$ satisfying (10.2.5). It is easy to verify (10.2.4). The proof of Theorem 10.1 is now complete.

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NILPOTENT STRUCTURES AND INVARIANT METRICS ON COLLAPSED MANIFOLDS

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CONTENTS

I. INTRODUCTION

0. BACKGROUND

Let M^n be a complete Riemannian manifold of bounded curvature, say $|K| \leq 1$. Given a small number, $\varepsilon > 0$, we put $M^n = \mathscr{B}^n(\varepsilon) \cup \mathscr{C}^n(\varepsilon)$, where $\mathscr{B}^n(\varepsilon)$ consists of those points at which the injectivity radius of the exponential map is $\geq \varepsilon$. The complementary set, $\mathscr{C}^n(\varepsilon)$ is called the *ε-collapsed* part of M^n.

If $x \in \mathscr{B}^n(\varepsilon)$, $r \leq \varepsilon$, then the metric ball $B_x(r)$ is quasi-isometric, with small distortion, to the flat ball $B_0(r)$ in the Euclidean space, R^n. After slightly

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adjusting the boundary of $\mathscr{B}^n(\varepsilon)$, we obtain a set whose quasi-isometry type is determined up to a finite number of possibilities by the ratio, $\operatorname{dia}(\mathscr{B}^n(\varepsilon))/\varepsilon$, where $\operatorname{dia}(\mathscr{B}^n(\varepsilon))$ denotes the diameter of $\mathscr{B}^n(\varepsilon)$. (Compare [C, GLP, GW, P]).

In this paper, we are concerned with what can be said about the ε-collapsed part, $\mathscr{C}^n(\varepsilon)$, for $\varepsilon = \varepsilon(n)$ a suitably small constant depending only on n. Roughly speaking, our main results show that the essential features of the local geometry are encoded in the symmetry structure of a nearby metric. More precisely, any metric of bounded curvature on M^n can be closely approximated by one that admits a sheaf of nilpotent Lie algebras of local Killing vector fields that point in all sufficiently collapsed directions of $C^n(\varepsilon)$. This sheaf is called the *nilpotent Killing structure.*

A second sheaf of nilpotent Lie algebras of vector fields, called the *nilpotent collapsing structure* will be discussed elsewhere. It plays a crucial role in constructions, which collapse away all sufficiently collapsed directions in the manifold (while keeping its curvature bounded). The fact that *two different* sheaves arise simply reflects the distinction between right and left invariant vector fields on a nilpotent Lie group (compare Example 1.6 and the discussion preceding it).

The first nontrivial example of a collapsing sequence of Riemannian manifolds was pointed out by Marcel Berger in about 1962. Berger started with the Hopf fibration, $S^1 \to S^3 \to S^2$, where S^3 carries its standard metric. He observed that if one multiplies the lengths of the fibres by ε, while leaving the metric in the orthogonal directions unchanged, then the sectional curvature stays bounded independent of ε, as $\varepsilon \to 0$. But as $\varepsilon \to 0$, S^3 more and more closely resembles S^2 (equipped with a metric of constant curvature 4). In the process, the injectivity radius converges to zero everywhere.

The first theorem on collapse characterizes "almost flat manifolds" [G1]. These manifolds, X^n, have bounded curvature, say $|K| \leq 1$, and are collapsed in the strongest sense possible. Namely, the diameter satisfies, $\operatorname{dia}(X^n) \leq \varepsilon(n)$. The theorem asserts that a finite normal covering space, $\widetilde{X}^n$, is diffeomorphic to a *nilmanifold*, $\Lambda\backslash N$.

Subsequently, by employing additional analytic arguments, Ruh proved that X^n itself is *infranil* [R]. This means that the covering group of $\widetilde{X}^n \to X$ acts by affine transformations with respect to the canonical flat affine connection on the tangent bundle of $\widetilde{X}^n$. Otherwise, put X^n is diffeomorphic to $\Lambda\backslash N$, where the covering group, Λ, acts by affine transformations, with respect to the canonical connection on N and the image of the holonomy homomorphism is finite. By the *canonical connection*, we mean the one for which all left invariant vector fields are parallel. An important by-product of Ruh's proof is the statement that the diffeomorphism between X^n and $\Lambda\backslash N$ can be chosen *canonically*, given the geometry of X^n *and a choice of base point*, $x \in X^n$. Moreover, in this case, a canonical left invariant metric on N that is actually invariant under Λ can also be chosen.

It is easy to see that although most infranil manifolds admit no flat metric, any such manifold admits a *sequence* of metrics with $|K| \leq 1$, for which the diameter becomes arbitrarily small; see [G1]. Thus, the above-mentioned results imply the existence of a *critical diameter*; if X^n admits a metric with $|K| \leq 1$ and $\mathrm{dia}(X^n) \leq \varepsilon(n)$, then it admits a sequence of metrics with $|K| \leq 1$ and $\mathrm{dia}(X^n) \to 0$.

The case of infranil manifolds illustrates a basic point. Collapse can take place *simultaneously on several different length scales* and not just on the scale of the injectivity radius. Indeed, the simplest nonflat nilmanifolds (with almost flat metrics) can be viewed as the total spaces of a nontrivial circle bundles, whose base spaces are isometric products of two circles of length ε and whose fibres have length ε^2. This kind of inhomogeneous scaling is actually *essential*, in order for the curvature to remain bounded as $\varepsilon \to 0$.

The ideas on almost flat manifolds were extended along two rather different lines, in order to study the collapsing phenomenon in greater generality. The goal of the present paper is to combine these two approaches.

In [CG3, CG4], generalizing the concept of a group action, the notion of an *action of a sheaf of groups* was introduced. An F-structure is an action of a sheaf of tori for which certain additional regularity conditions hold ("F" stands for "flat"). As in the case of a group action, an action of a sheaf of groups induces a partition of the underlying space into *orbits*. The main result of [CG4] asserts the existence of an F-structure of *positive rank* (i.e., all orbits have positive dimension) on the sufficiently collapsed part of a manifold with $|K| \leq 1$. Here, *no* assumption is made concerning the *size* of the manifold, which might even be *infinite*. In this generality, the dimension of the stalk of the F-structure is not always locally constant. If not, the structure is called *mixed*; if so it is called *pure*.

The infinitesimal generator of the local action of an F-structure is a sheaf of abelian Lie algebras of vector fields, which can be regarded as Killing fields for some Riemannian metric. For the F-structure constructed in [CG4], this metric can actually be chosen close to the original one. The Killing fields themselves point only in the "shortest" collapsed directions. As a consequence, this F-structure describes the *local geometry* of the collapsed region only on its smallest length scale, that of the injectivity radius. This accounts for the abelian (as opposed to nilpotent) character of the structure.

The existence of an F-structure of positive rank does impose a *global* constraint on the *topology* of the underlying space. For example, it implies that the Euler characteristic vanishes [CG3].

Example 0.1. The need to consider mixed F-structures in cases where diameter is not bounded is illustrated by the metric

$$dr^2 + e^{-(R+r)} d\theta_1^2 + e^{-(R-r)} d\theta_2^2,$$

on the set $(-R, R) \times S^1 \times S^1$, $(R >> 0)$. By counting the number of collapsed directions, it becomes clear that in this example, the tori that act locally near the ends are one-dimensional, while near the middle, a two-dimensional torus acts. Note however, that there is no completely canonical way of choosing the precise set of points at which the transition takes place.

There is also a converse to the existence theorem for the F-structure. Namely, associated to every F-structure of positive rank are sequences of metrics with $|K| \leq 1$, containing ones that are arbitrarily collapsed, for which the action of the structure is isometric (see [CG3]). This leads to the existence of a "critical injectivity radius," which is analogous to the notion of "critical diameter" as mentioned above.

In the approach due to the second author, the starting point is to consider a manifold, M^n, with $|K| \leq 1$, which, as in the Berger example above, "to the naked eye," closely resembles a lower-dimensional manifold Y^m. Technically speaking, one requires that M^n is sufficiently close to Y^m in the *Hausdorff distance* (see [GLP]). The manifold Y^m is assumed to have *bounded geometry* but its diameter need *not* be finite. The conclusion is that there is a fibration $Z^{n-m} \to M^n \to Y^m$ whose fibre, Z^{n-m}, is an infranil manifold. In case Y^m is a point, the assertion reduces to the theorem on almost flat manifolds (see [F1] and §2 of the present paper for details).

Although the context of this fibration theorem might at first seem rather special, it turns out that its *equivariant* generalization gives strong information on the structure of *arbitrary* collapsed regions of *bounded diameter.* The reason is as follows. Suppose that for a given manifold, both the curvature tensor *and its covariant derivative* are bounded (the assumption concerning the covariant derivative is actually not a serious one, since by results of [BMR], [Shi], and [A], an arbitrary metric can be approximated by one for which this holds). Then the frame bundle, FM^n, equipped with its natural metric, has bounded curvature as well. If U^n is a region that is *sufficiently collapsed relative to the size of its diameter*, one can show that there exists Y^m as in the fibration theorem, such that the frame bundle, FU^n, is sufficiently Hausdorff close to Y^m. Moreover, in this case, the fibre, Z, of the fibration, $Z \to FU^n \to Y^m$, is actually a nilmanifold (and not just infranil). (Ultimately, both assertions can be traced to the fact that an isometry of the base space, which fixes a point of the frame bundle, is the identity map.) The fibration, $Z^{\frac{n(n-1)}{2}+n-m} \to FU^n \to Y^m$, can be chosen to be *equivariant* with respect to the action of $O(n)$ on FU^n. As a consequence, a partition into infranilmanifolds, in general not all of the same dimension, is induced on U^n. These "orbits" contain *all* collapsed directions, and so determine all possible length scales on which collapse takes place. The flat orbits of the F-structure can be thought of as lying inside these nilpotent ones (in fact, they lie inside the pieces corresponding to the center of the nilpotent group); see [F3].

The fibration theorem is sharpened in another direction in [F2]. There, Ruh's theorem is used to obtain a smooth family of affine flat structures on the fibres. In fact, by Malcev's rigidity theorem, these are all affine equivalent to some fixed $\Lambda\backslash N$; see Theorem 3.7 and Proposition 3.8. As a consequence, the structural group of the fibration reduces to the group of affine automorphisms of $\Lambda\backslash N$. The existence of such a reduction is a necessary and sufficient condition for the total space of a fibration with fibre diffeomorphic to $\Lambda\backslash N$ to collapse to the base space keeping curvature bounded. Thus the theorem on the "critical diameter" generalizes to the fibration setting.

We point out that the result of [F2] is obtained without removing the dependence on the base point in Ruh's construction. As a consequence, the construction of [F2] is not G-equivariant. This point, which is important for the present paper, is dealt with in §3.

We refer to [CG3, CG4, F1–F3] for further background and examples.

1. Statement of main results and outline of their proof

As already mentioned, the goal of the present paper is to synthesize the two approaches to collapse that were described in the previous section. Thus, without assuming a bound on diameter, we will construct a nilpotent structure, in general of mixed type, which is nontrivial on sufficiently collapsed regions. The structure incorporates a description of the local geometry on a fixed scale and not just on the scale of the injectivity radius. It is called the *nilpotent Killing structure.* We will show that its action is isometric for a metric *close to the original one.*

As mentioned earlier, there is also a second structure, called the *nilpotent collapsing structure.* Although its orbits are the same as those of the nilpotent Killing structure, its construction requires a small amount of additional work. This, together with a description of its role in collapsing will be provided elsewhere; see, however, Example 1.6 and compare [F2].

The existence of a metric whose symmetry structure encodes the essential features of the geometry can be made precise without reference to sheaves. However, the compatibility between this metric and the sheaf structure imposes a consistency condition on the local symmetries at neighboring points, which captures the purely topological aspect of the discussion.

Let (M, g) be a Riemannian manifold. Let $V \subset M$ be open and $\pi: \widetilde{V} \to V$, a normal covering with covering group, Λ.

(1.1.1) Assume that there exists a Lie group, $H \supset \Lambda$, with finitely many components, and an *isometric* action of H on $\widetilde{V}$, extending that of Λ, such that

(1.1.2) H is generated by Λ and its identity component, N,

(1.1.3) N is nilpotent.

A Riemannian manifold (M, g) is called *(ρ, k)-round* at $p \in M$, if there exist V, $\widetilde{V}$, H satisfying (1.1.1)–(1.1.3) and the following additional conditions:

(1.1.4) V contains the metric ball, $B_p(\rho)$, of radius ρ centered at p.

(1.1.5) The injectivity radius at all points of $\widetilde{V}$ is $> \rho$.

(1.1.6) $\sharp(H/N) = \sharp(\Lambda/\Lambda \cap N) \leq k$.

A metric, g, is called *(ρ, k)-round* if it is (ρ, k)-round at p, for all p.

Modulo the choice of (ρ, k), V has a normal covering space with bounded geometry and a covering group that is almost nilpotent. By (1.1.5), *if the injectivity radius at p is $< \rho/k$, then the metric, g, has nontrivial local symmetries near p; i.e., the orbit, $H(\tilde{p})$, of $\tilde{p} \in \pi^{-1}(p)$, under H, has positive dimension.*

If (M, g) is (ρ, k)-round, it follows that the projected orbit, $\pi(H(\tilde{p}))$, contains those sufficiently collapsed directions corresponding to short geodesic loops that are homotopically nontrivial in V. The (ρ, k)-round metrics constructed in this paper actually have a stronger property. Namely, the orbits (have small diameter and) actually contain *all* sufficiently collapsed directions (again modulo the choice of k). One way of formulating this more precisely is to say that the orbit space has (ρ, k)-bounded geometry in a suitable sense. (See Definition 8.4 for the concept of (ρ, k)-bounded geometry of the orbit space and Remark 8.10; see also Appendix 1.)

Example 1.2. Let G be a connected Lie group and $\tilde{g}$ be a left invariant metric. Then, for each discrete subgroup Λ of G, the quotient metric, g, on the quotient space $M = \Lambda\backslash G$ is $(\rho, 1)$-round, where ρ depends on $\tilde{g}$ but is independent of Λ. This is a restatement of Zassenhaus's theorem, which asserts that every discrete subgroup of G generated by small elements is contained in a nilpotent subgroup of G. (See [GLP, 8.44].) More generally, put $\widetilde{M} = G/K$, for some compact subgroup K of G. Let $\tilde{g}$ be a G-invariant metric on $\widetilde{M}$. Take a discrete subgroup Λ of G acting freely on $\widetilde{M}$. Then, by Zassenhaus's theorem, we conclude that the quotient metric g on $M = \Lambda\backslash\widetilde{M}$ is (ρ, k)-round, where ρ, k are independent of Λ.

Let ∇^g denote the Levi Civita connection of g.
Our first main result is

Theorem 1.3 (Symmetrization). *For all $\varepsilon > 0$ and $n \in \mathbb{Z}_+$, there exists $\rho > 0$ and $k \in \mathbb{Z}_+$ such that if (M^n, g) is a complete Riemannian manifold with $|K| \leq 1$, then there is a (ρ, k)-round metric, g_ε, with*

(1.3.1) $e^{-\varepsilon} g < g_\varepsilon < e^\varepsilon g$,
(1.3.2) $|\nabla^g - \nabla^{g_\varepsilon}| < \varepsilon$,
(1.3.3) $|(\nabla^{g_\varepsilon})^i R_{g_\varepsilon}| < c(n, i, \varepsilon)$.

One might ask whether Theorem 1.3 can be strengthened to the assertion that in all instances there exists a (ρ, k)-round metric, g_ε, such that either $\rho \geq \rho(n, \varepsilon)$, $k \leq k(n)$ or $\rho \geq \rho(n)$, $k \leq k(n, \varepsilon)$. However, this turns out to be false; see Examples 8.11, 8.12.

Now let M be a smooth manifold and let $\mathfrak{g}$ be a sheaf of connected Lie groups on M. Let $\underline{\mathfrak{g}}$ be the associated sheaf of Lie algebras.

Definition 1.4. An *action* of $\mathfrak{g}$ is a (Lie algebra) homomorphism, h, of $\underline{\mathfrak{g}}$ into the sheaf of smooth vector fields on M.

A metric, g, is called *invariant* for $\mathfrak{g}$ if $h(\underline{\mathfrak{g}})$ is a sheaf of local Killing fields for g.

Note that if $\pi\colon \widetilde{M} \to M$ is a local homeomorphism, then there is an induced action, $\pi^*(h)$, of the pullback sheaf, $\pi^*(\mathfrak{g})$.

A curve $c\colon (a, b) \to M$ is called an *integral curve* if $c \subset V$ for some open set, V, and c is everywhere tangent to the image, $h(X)$, of some section $X \in \mathfrak{g}(V)$ (i.e,. $c'(s) = h(X)(c(s))$). A set $Z \subset M$ is called *invariant* if $c \subset Z$, for all such c with $c \cap Z \neq \varnothing$. The unique minimal invariant set containing p is called the *orbit*, $\mathscr{O}_p$, of p. Clearly, M is the union of its orbits.

Let h be an action of a sheaf, $\mathfrak{n}$, of simply connected nilpotent Lie groups. Let g be a (ρ, k)-round metric and let N_0, V, etc., be as in (1.1.1)–(1.1.6).

Definition 1.5. $(\mathfrak{n}, h)$ defines a *nilpotent Killing structure*, for g, if for all p, we can choose H, V, $\widetilde{V}$ as follows. There is an invariant neighborhood U and normal covering, $\widetilde{U} \subset \widetilde{V}$, such that:

(1.5.1) $\pi^*(h)$ is the infinitesimal generator of a (necessarily unique) action of the group, $\pi^*(\mathfrak{n})(\widetilde{U})$, whose kernel, K, is discrete. $N_0 = \pi^*(\mathfrak{n})(\widetilde{U})/K$ and the action of $N_0|\widetilde{U}$ is the quotient action.

(1.5.2) For all $W \subset \widetilde{U}$ such that $W \cap \pi^{-1}(p) \neq \varnothing$, the structure homomorphism, $\pi^*(\mathfrak{n})(\widetilde{U}) \to \pi^*(\mathfrak{n})(W)$ is an isomorphism.

(1.5.3) The neighborhood U and covering $\widetilde{U}$ can be chosen independent of p, for all $p \in \mathscr{O}_p$.

Clearly, the metric g in Definition 1.5 is an *invariant metric* for $(\mathfrak{n}, h)$.

A structure is called *pure* if the dimension of the stalk is locally constant.

Before going to the next example, we will recall some elementary (but confusing) facts.

Let H be a Lie group. The diffeomorphisms of H obtained by integrating *right* invariant vector fields are *left* translations. Conversely, integrating *left* invariant vector fields yields *right* translations.

In particular, given a *left* invariant metric on H, the *right* invariant fields are Killing fields but *left* invariant fields need not be.

Example 1.6. Let N be a simply connected Lie group and $\Lambda \subset N$ a discrete subgroup. The quotient sheaf, $\mathfrak{n}$, of the constant sheaf, $N \times N \to N$, by the action, $\lambda\colon (n_0, n) \to (\lambda n_0 \lambda^{-1}, \lambda n)$ has an action on $\Lambda \setminus N$ induced by *left* multiplication on N. The image sheaf, $h(\mathfrak{n})$, is the sheaf of *locally defined right invariant vector fields.* Any *left* invariant metric on N induces an invariant metric on $\Lambda \setminus N$ for the action of this sheaf. It follows that $(\mathfrak{n}, h)$ defines a nilpotent Killing structure.

Note, however, that the standard collapsing construction for $\Lambda\setminus N$ involves inhomogeneous scaling of the left invariant metric and hence of the lengths of the *left* invariant vector fields (see [BK]). The right action of N generates the left invariant fields and gives rise to the *nilpotent collapsing structure* in this case. As indicated above, typically, the *right* action of N on $\Lambda \setminus N$ does *not* give rise to a nilpotent Killing structure, because there is no metric that it leaves invariant.

Let M, g, g_ε be as in Theorem 1.3. Our second main result is

Theorem 1.7. *The (ρ, k)-round metric, g_ε, can be chosen such that there is a nilpotent Killing structure, $\mathfrak{N}$, for g_ε whose orbits are all compact with diameter $< \varepsilon$.*

Remark 1.8. The structure described in Theorem 1.7 can be viewed as generalizing the system of fibrations with nilpotent fibre and locally symmetric base that is known to exist near infinity on a noncompact locally symmetric space of finite volume.

Remark 1.9. Theorem 1.7 also provides an alternative means of obtaining an F-structure of positive rank on the collapsed part of M. In fact, replacing each

Lie algebra of local sections, $\mathfrak{n}(U)$, by its center leads to the existence of an F-structure.

Open Problem 1.10. Suppose that the original metric, g, in Theorems 1.3 and 1.7 is Kähler, Einstein, etc. Can one take g_ε within the same category?

In spirit, our construction of the nilpotent Killing structure is similar to the construction of the F-structure in [CG4]. But in carrying out the details, we use the framework of [F1–F3].

As in [CG4], we will fit together a collection of locally defined pure structures. Initially, the collection is organized in such a way that on nonempty intersections of their domains, the structures fit approximately, one inside another. Then using a suitable stability property they are perturbed so as to fit together exactly.

In [CG4] the locally defined pure structures are constructed on the scale of the injectivity radius, with the help of a result on local approximation by complete flat manifolds; see [CG4, §3]. The stability property is a consequence of the stability of compact group actions (in particular of torus actions); see [CG4, §1]. Here, we work on length scales that, though small, may be arbitrarily large compared to the injectivity radius. We also work with nilpotent groups, which typically have no compact quotient groups. As a consequence, neither of the above-mentioned basic tools is available.

Following the approach of [F1–F3], we will construct an $O(n)$-equivariant nilpotent Killing structure on the frame bundle. This structure induces the desired nilpotent structure on the base. The requirement of maintaining $O(n)$-equivariance at all stages of the construction introduces some technical problems; see in particular §3. They are handled by averaging arguments, some of which are very similar to those used to prove the stability of compact group actions. In addition, we use Malcev's rigidity theorem for discrete cocompact subgroups of nilpotent groups, which serves as a partial replacement for the stability of torus actions.

The preliminaries on which our construction is based are given in §§2–4. The construction is carried out in §§5–8 and is organized as follows.

In §§5–6 we manufacture an $O(n)$-equivariant collection of local fibrations of the frame bundle such that if a pair of fibres from two of these fibrations intersect, then one fibre contains (or is equal to) the other.

In §7 flat affine structures are introduced on the fibres. Then, the fibrations are readjusted so that for a pair of fibres as above, the smaller is totally geodesic in the larger, with respect to these affine structures. The affine structures give rise to a nilpotent Killing structure on the frame bundle. Those fibres that are not contained in any other are the orbits of this structure. Their images in the base are the orbits of the structures we are seeking.

In §8 we check that the nilpotent structure and metric on the frame bundle do indeed induce the desired objects on the base.

Before giving a more detailed summary of the contents of the paper, we explain the following basic point that was alluded to in the previous section.

A manifold is called *A-regular* if for some nonnegative sequence $A = \{A_i\}$, we have

$$|\nabla^i R| \le A_i \, . \tag{1.11.1}$$

By the following result of Abresch [A] (see also [Ba, BMR, Shi]), we will ultimately (in §8) be able to replace the given metric in Theorems 1.3 and 1.7 by one that is A-regular, where

$$A_i = A_i(n, \varepsilon) \tag{1.11.2}$$

and ε is as in Theorems 1.3 and 1.7. Thus, prior to §8 we will always work with manifolds that are A-regular with $A_0 = 1$.

Theorem 1.12 (Abresch). *On the set of complete Riemannian manifolds, (M^n, g), with $|K| \leq 1$, there exists for all $\varepsilon > 0$, a smoothing operator, $g \to S_\varepsilon(g) = \tilde{g}$, such that*
(1.12.1) $e^{-\varepsilon} g \leq \tilde{g} \leq e^{\varepsilon} g$,
(1.12.2) $|\nabla - \tilde{\nabla}| \leq \varepsilon$,
(1.12.3) $|\tilde{\nabla}^i \tilde{R}| \leq A_i(n, \varepsilon)$.
Moreover, at $p \in M^n$, the value of $\tilde{g}$ depends only on $g|B_p(\frac{1}{4})$. Finally, any isometry of g is also an isometry of $\tilde{g}$.

Note that since $\tilde{g}(p)$ depends only on $g|B_p(\frac{1}{4})$, the completeness assumption can be removed, provided one stays away from $\overline{M} \setminus M$ (where the bar denotes metric space completion).

In §2 we give a new proof of the fibration theorem of [F1], in the local equivariant form that we need. The projection map of the fibration is obtained by regularizing a Hausdorff approximation. Although there is some freedom in choosing the scale on which the regularization is performed, for the application in §§5–8, it is important that the scale is chosen to be that of the injectivity radius of the base space.

In §3 we remove the dependence on the base point in Ruh's theorem by averaging the base point dependent choices of the flat connection that occur in the initial step of the proof. Since this procedure and the remainder of Ruh's arguments are both canonical, we immediately obtain an equivariant and parameterized version of his theorem.

In §4 we observe that the affine structures on the fibres introduced in §3 allow us to define in a canonical way, a pure nilpotent Killing structure on the total space of the fibration (we also construct a canonical metric that is invariant for the structure). Clearly, on a given fibre we can speak of the local right invariant fields. But the issue is to define these fields (locally) on the total space itself. For this, we note that the affine equivalence that identifies neighboring fibres is unique up to elements of the *identity component*, $\mathrm{Aff}^0(\Lambda \setminus N)$, of $\mathrm{Aff}(\Lambda \setminus N)$. One sees easily that $\mathrm{Aff}^0(\Lambda \setminus N) \subset N_R$, the subgroup of $\mathrm{Aff}(N)$ consisting of right translations. Since N_R acts trivially on right invariant fields, it follows that there is a *canonical* 1-1 correspondence between local *right* invariant fields (at nearby points) of neighboring fibres.

In §5 (using the results of §2) we select a system of $O(n)$-equivariant, local fibrations of the frame bundle with almost flat fibres. On the intersections of their domains, these fit approximately, one inside another. To achieve this requires a suitable mechanism for picking out which directions are to be considered collapsed in cases which might otherwise appear to be ambiguous. Lacking such a mechanism, we could wind up with fibrations whose domains intersect, but whose fibres do not satisfy the above relation of approximate containment. Essentially the same point had to be dealt with in [CG4, see §5-b]. Here we employ what amounts to a standard device from stratification theory.

In §6, by employing an inductive argument that depends on the result of Appendix 2, the local fibrations are modified $O(n)$-equivariantly, so that they fit, one inside another, on the intersections of their domains.

In §7 we complete the construction of the nilpotent Killing structure and invariant metric on the frame bundle, using an inductive argument like that of §6. For the construction of the Killing structure, the main part of the induction step can be described as follows. Note that as a consequence of §3, each fibre of a local fibration in §6 is endowed with a flat affine structure, affinely diffeomorphic to some $\Lambda \backslash N$. Consider a pair of fibrations, $\mathscr{F}_t \subset \mathscr{F}_s$ as in §6 (i.e., the fibres of $\mathscr{F}_t$ are contained in those of $\mathscr{F}_s$). By §4, the fibres carry affine structures that determine a local left action. However, the inclusion $\mathscr{F}_t \subset \mathscr{F}_s$ need not be compatible with the affine structures. Using Malcev's rigidity theorem, we find a unique subfibration, $\mathscr{F}_t' \subset \mathscr{F}_s$, whose fibres are totally geodesic for the affine structure on the fibres of $\mathscr{F}_s$, and such that for each fibre of $\mathscr{F}_t$, there is a small motion carrying it onto some fibre of $\mathscr{F}_t'$. Then, as in §6, we find a small $O(n)$-equivariant diffeomorphism that matches $\mathscr{F}_t$ with $\mathscr{F}_t'$ such that the affine structure on $\mathscr{F}_t$ is carried into that of $\mathscr{F}_t'$.

The remaining sections, §8 and the Appendices, require no further description at this point.

With minor variations, we will employ the same notation as in [F3, see §1] and [F4, see §7]. In particular, we use:

(1.13.1) $d(\cdot\,, \cdot)$: the distance function.

(1.13.2) $B_p(D) = \{x \in M \mid d(p\,, x) \leq D\}$.

(1.13.3) $TB_p(D) = \{v \in T_pM \mid |v| < D\}$.

(1.13.4) $\pi_1(M\,, p\,; \varepsilon) = \{\gamma : TB_p(\varepsilon) \to TB_p(2\varepsilon) \mid \exp_p \circ \gamma = \exp_p\}$: the pseudo-fundamental group. $\pi_1(M\,, p\,; \varepsilon)$ has a pseudogroup structure and it acts on $TB_p(\varepsilon)$ with $TB_p(\varepsilon)/\pi_1(M\,, p\,; \varepsilon) = B_p(\varepsilon)$.

(1.13.5) $d_H(X\,, Y)$: the Hausdorff distance between X and Y. When X and Y have isometric G-action, the G Hausdorff distance is also denoted by $d_H(\cdot\,, \cdot)$.

(1.13.6) $\tau(\varepsilon \mid a\,, b\,, \dots)$ denotes a positive number depending on the numbers in the parentheses and satisfying $\lim_{\varepsilon \to 0} \tau(\varepsilon \mid a\,, b\,, c\,, \dots) = 0$, for each fixed $a\,, b\,, c\,, \dots$.

(1.13.7) If $\{A_i\}$ is a positive sequence $c(\cdot\,, A\,, \cdot)$ will denote a generic constant depending on *finitely many* of the A_i (and possibly on some other parameters).

II. Preliminaries

2. Smoothing Hausdorff approximations

In this section we give a new proof of the fibration theorem of [F1, F3] (see Theorem 2.6).

A map $h\colon X \to Y$ of metric spaces will be called an *δ-Hausdorff approximation* if for all x_1, x_2

(2.1.1) $|d(x_1, x_2) - d(h(x_1), h(x_2))| \le \delta$;
(2.1.2) the range of h is δ-dense.

If G is a group acting by isometries on X and Y, then h is called a *G-δ-Hausdorff approximation* if in addition, for all $g \in G$, $x \in X$,

(2.1.3) $d(h(gx), gh(x)) < \delta$.

Let $\overline{V}$ denote the completion of the metric space V, and put

(2.2) $\partial V = \overline{V} \setminus V$,
(2.3) $V_\eta = \{v \in V \mid d(v, \partial V) > \eta\}$.

Now let X^n, Y^j $(j \le n)$ be Riemannian manifolds such that for some sequence, $\{A_i\}$, with $A_0 = 1$,

(2.4.1) X^n, Y^j are $\{A_i\}$-regular.

Assume in addition, that for all $y \in Y$ and some $\iota \le 1$,

(2.4.2) $\operatorname{inj\,rad}_y \ge \min(\iota^{-1}, d(y, \partial Y))$.

Let G act on X^n, Y^j by isometries. Let distances in X^n_δ, Y^j_δ be measured in X^n, Y^j respectively. If the G-Hausdorff distance, $d_H(X^n, Y^j)$, satisfies

(2.4.3) $d_H(X^n, Y^j) \le \delta/10$,

with $\delta\iota^{-1} \le \frac{1}{2}$, then there is a *continuous* G-δ-Hausdorff approximation,

(2.4.4) $h\colon X^n_\delta \to Y^j$,

with $h(X^n_\delta) \supset Y^j_{3\delta}$ (see [F4, GLP, GrK]). In what follows it will be convenient simply to assume the existence of a continuous, G-δ-Hausdorff approximation, $h\colon X^n \to Y^j$.

A fibration, $f\colon X \to Y$, of Riemannian manifolds is called a *θ-almost Riemannian submersion* if for all $y \in Y$, $x \in f^{-1}(y)$, and $V \in TX_x$, normal to $f^{-1}(y)$,

(2.5.1) $$e^{-\theta}|f_*(V)| \le |V| \le e^{\theta}|f_*(V)|.$$

Let $B = \{B_i\}$, $i = 1, 2, \ldots$. A map, $f\colon X \to Y$, of Riemannian manifolds is called *B-regular* if

(2.5.2) $$|\nabla^i f| \le B_i.$$

Let II_Z denote the second fundamental form of $Z \subset X$.

Fix $\lambda \le \lambda(n)$ sufficiently small.

Theorem 2.6. *Let X^n, Y^j satisfy* (2.4.1), (2.4.2) *and let $h\colon X^n \to Y^j$ be a continuous, G-equivariant, δ-Hausdorff approximation, with $(\delta\iota^{-1})^{1/2} \le \lambda$. Then there exists G-equivariant $W^n \supset X^n_\iota$ and a fibration $f\colon W \to Y^j$ such that*

(2.6.1) $\operatorname{dia}(f^{-1}(y)) \le c(n, A)\delta$, *for all $y \in f(W^n)$. In particular, $f^{-1}(y)$ is a connected manifold.*

(2.6.2) *f is a $c(n, A)\lambda$-almost Riemannian submersion.*
(2.6.3) $|II_{f^{-1}(y)}| \le c(n, A)\iota^{-1}$, *for all* $y \in f(W^n)$.
(2.6.4) *f is $\{C_i(n, A)(1 + \lambda^{2-n})\iota^{1-i}\}$-regular.*
(2.6.5) *f is G-equivariant.*
(2.6.6) *For $c^2(n, A)\delta\iota^{-1} \le \xi(n)$, sufficiently small, $f^{-1}(y)$ is an almost flat manifold, for all $y \in f(W^n)$.*
(2.6.7) $d(h, f) \le c(n)\lambda\iota$.

The proof of Theorem 2.6 will occupy the remainder of this section and will require a number of lemmas.

Before going through the proof of Theorem 2.6, the reader may wish to glance at Example 2.29 below.

Proof of Theorem 2.6. Note that (2.6.6) is a direct consequence of (2.6.1), (2.6.3). Also, given the bound on $\nabla^2 f$ and (2.6.2), the connectedness of $f^{-1}(y)$ can be proved by an argument like the one used to prove (A.2.3.2) of Appendix 2.

As for the remaining statements, we begin by noting that by an obvious scaling argument, we can assume $\iota = 1$.

We will construct f by regularizing the map h.

Given $x \in X^n$ and $\beta\colon X \to Y$, we define

(2.7) $\tilde\beta := \beta \exp_x$,

on the ball $TB_x(1)$, of radius 1 in the tangent space, T_xX.

Let $\zeta\colon [0, 1] \to [0, 1]$ be a smooth function such that $\zeta|[0, \frac{1}{3}] \equiv 1$, $\zeta\,|\,[\frac{2}{3}, 1] \equiv 0$, $|\zeta'| \le 4$, $|\zeta''| \le 12$. For $\tilde x \in TB_x(1)$, put

(2.8.1) $\chi_\varepsilon(\tilde x) = \zeta(\varepsilon^{-1} d(0, \tilde x))$.

Then

(2.8.2) $|\nabla \chi_\varepsilon| \le c(n)\varepsilon^{-1}$,
(2.8.3) $|\nabla^2 \chi_\varepsilon| \le c(n)\varepsilon^{-2}$.

Let $d\tilde x$ denote the volume form for the pullback metric on $TB_x(1)$. Consider (for fixed x and small ε) the function,

(2.9.1) $y \to \int d^2(\tilde\beta(\tilde x), y)\, \chi_\varepsilon(\tilde x)\, d\tilde x$.

If $\beta(B_x(\varepsilon)) \subset B_{\beta(x)}(a)$, for some $a < \frac{\pi}{3}$, then the function in (2.9.1) is a weighted average of *convex* functions on the ball, $B_{\beta(x)}(3a)$, and so, is itself convex on this ball. Clearly, it takes a unique minimum at some $y_1 \in \overline{B_{\beta(x)}(2a)}$. By definition, y_1 is the *center of mass of* $\tilde\beta$, *weighted by the function* χ_ε.

Define the *ε-regularization* of β, by

(2.9.2) $\beta_\varepsilon(x) := y_1$.

We note that the ε-regularization can also be defined for a continuous function, ξ, on $TB_x(1)$, satisfying $(TB_x(\varepsilon)) \subset B_{\xi(0)}(a)$, *by using the pullback metric.* We continue to denote the ε-regularization of such a function by ξ_ε. In particular, for those functions, $\tilde\beta$, as in (2.7), that are pullbacks we see by inspection that the following crucial relation holds (compare [CG3, Lemma 5.3]). On $TB_x(1)$,

(2.9.3) $(\tilde{\beta})_\varepsilon = \tilde{\beta}_\varepsilon$.

Now define

(2.10) $f := h_\lambda$.

Since h is G-equivariant, it follows that f satisfies (2.6.5) and by a standard argument, (2.6.4), (2.6.7) hold as well. However, (2.6.1)–(2.6.3) are not yet apparent.

In order to prove (2.6.1)–(2.6.3), we will compare $\tilde{h}$, with an auxiliary function, k_x, on $TB_x(2\lambda)$. (From now on we just write k for k_x.) The function k has regularity properties like those that we are trying to establish for f. However, k can be constructed directly since it is locally defined, allowed to depend on x, and *not* required to be the pullback of a function on $B_x(2\lambda)$.

The crucial point will be to show that k and $\tilde{h}$ are $c(n)\lambda^2$-close (see (2.11.1)). This degree of closeness will imply that the regularity properties of $\tilde{h}_\lambda = \tilde{f}$ (by (2.9.3)) are like those of k_λ. These, in turn, are like those of k, since k is already regular. Finally (of course) the regularity properties of f and $\tilde{f}$ are the same.

A priori, it is only clear that k and $\tilde{h}$ are $c(n)\lambda$-close. This does not suffice for our purposes since it leads only to a bound on ∇f and not to the assertion that f is an almost Riemannian submersion. It is in establishing the required closeness of k and $\tilde{h}$ that the geometry of our setup enters (in essentially the same way as in [F1, §3]); see Lemmas 2.16 and 2.19.

In the lemmas that follow, $\tilde{\nabla}$ denotes the Levi Civita connection of the pullback metric on $TB_x(1)$.

Lemma 2.11. *For all $x \in X^n$, there is a function, $k\colon TB_x(2\lambda) \to Y^j$, such that*

(2.11.1) $d(k, \tilde{h}) \le c(n, A)\lambda^2$,

(2.11.2) *k is a $c(n)\delta^{1/2}$-almost Riemannian submersion,*

(2.11.3) $|\tilde{\nabla}^2 k| \le c(n, A)$.

Lemma 2.12.

(2.12.1) *k_λ is a $c(n)\lambda$-almost Riemannian submersion.*

(2.12.2) $|\tilde{\nabla}^2 k_\lambda| \le c(n, A)$.

Lemma 2.13.

(2.13.1) $|\tilde{\nabla}\tilde{f} - \tilde{\nabla}k_\lambda| \le c(n, A)\lambda$.

(2.13.2) $|\tilde{\nabla}^2\tilde{f} - \tilde{\nabla}^2 k_\lambda| \le c(n, A)$.

Essentially, to get Lemma 2.13, we can estimate the ith derivative of the regularization of $(\tilde{h} - k)$ by λ^{-i} times the quantity in (2.11.1) (see (2.8)). Similarly, the properties (2.12.1), (2.12.2) are consequences of the corresponding properties (2.11.2), (2.11.3).

Indeed, Lemmas 2.12 and 2.13 would be standard in the familiar case $X^n = R^n$, $Y^j = R^j$. In the present context, their proofs are straightforward, if slightly tedious, exercises in advanced calculus. Hence, the proofs of these lemmas will be omitted.

The proof of Lemma 2.11 will be given at the end of this section.

By combining Lemmas 2.12 and 2.13 we see that

(2.14) f is a $c(n, A)\lambda$-almost Riemannian submersion if (2.6.2) and (2.6.3) holds.

Clearly, we can choose $W^n \supset X_1^n$ such that $f^{-1}(y) \subset W$ is compact for all $y \in f(W^n)$.

We now prove (2.6.1).

Note that by (2.6.3), if some fibre, $f^{-1}(y_0)$, has diameter, $\mu\delta$, then for all $y \in B_{y_0}(\frac{1}{2})$,

(2.15.1) $\operatorname{dia}(f^{-1}(y)) \geq c^{-1}(n, A)\mu\delta$.

In view of (2.14), it follows that for $\delta < c^{-1}(n)$, at least

(2.15.2) $c^{-1}(n, A)\mu\delta^{-j}$

balls of radius δ are required to cover $f^{-1}(B_{y_0}(\frac{1}{2}))$.

But from the existence of the δ-Hausdorff approximation h, it follows that at most

(2.15.3) $c(n)\delta^{-j}$

such balls are required. Therefore we get

(2.15.4) $c(n)c(n, A) \geq \mu$,

which gives (2.6.1).

In order to prove Lemma 2.11 we will need two auxiliary lemmas (compare [F1, F3]).

Lemma 2.16. *Let X, Y be as in Theorem 2.6 and let σ be a geodesic loop of length l on $p \in X_\iota$. Let γ be a minimal geodesic segment with $\frac{1}{3}\iota \leq L[\gamma] \leq \iota$ and $\gamma(0) = p$. Then*

$$(2.16.1) \qquad \left|\measuredangle(\gamma'(0), \sigma'(0)) - \frac{\pi}{2}\right| \leq c\max(l\iota^{-1}, (\delta\iota^{-1})^{1/2}).$$

Proof. By scaling, it suffices to consider the case $\iota = 1$.

The inequality

$$(2.17) \qquad \measuredangle(\gamma'(0), \sigma'(0)) \geq \frac{\pi}{2} - cl$$

is a direct consequence of Toponogov's theorem applied to the degenerate isoceles triangle with sides γ, γ, σ.

On the other hand, put

(2.18.1) $h(\gamma(\frac{1}{3})) = \exp_{h(p)}(u)$,

and choose p' such that

(2.18.2) $d(h(p')\ , \exp_{h(p)}(-u)) \leq \delta$.

By Toponogov's theorem, if ζ is minimal from p to p', then

(2.18.3) $|\measuredangle(\zeta'(0), \gamma'(0)) - \pi| \leq c\delta^{1/2}$.

By using (2.17) with ζ in place of γ, we get (2.16.1) (compare [F1, §3]).

Lemma 2.19. *Let X^n, Y^j be as in Theorem 2.6. Let γ_1, γ_2 be minimal geodesics in X^n joining the point z to points $q_1, q_2 \in X^n_\iota$ respectively. Assume that $\frac{1}{3}\iota \leq L[\gamma_1] \leq \frac{5}{12}\iota$ and that $d(q_1, q_2) = l << \frac{1}{3}\iota$. Then*

(2.19.1) $$\measuredangle(\gamma_1'(0), \gamma_2'(0)) \leq c \max(l\iota^{-1}, (\delta\iota^{-1})^{1/2}) .$$

Proof. By scaling it suffices to consider the case $\iota = 1$.

Write $h(q_1) = \exp_{h(z)} v$ and put $y = \exp_{h(z)}(-v)$. Using the facts that h is a δ-Hausdorff approximation and that Y^j has bounded geometry, we find by a standard comparison argument that

(2.20.1) $d(h(q_2), h(z)) + d(h(z), y) - d(h(q_2), y) \leq c_1(l+\delta)^2$.

(This quantity is the excess of the triangle with vertices $h(q_2), h(z), y$.) Let $w \in X$ be such that $d(h(w), y) \leq \delta$. Then we have

(2.20.2) $d(q_2, z) + d(z, w) - d(q_2, w) \leq c_2[(l+\delta)^2 + \delta]$.

Thus, if τ is minimal from z to w, Toponogov's theorem implies

(2.20.3) $\measuredangle(\gamma_2'(0), \tau'(0)) \geq \pi - c_3 \max(l, \delta^{1/2})$.

Similarly,

(2.20.4) $d(q_1, z) + d(z, w) - d(q_1, w) \leq 3\delta$,

and by Toponogov's theorem,

(2.20.5) $\measuredangle(\gamma_1'(0), \tau'(0)) \geq \pi - c\delta^{1/2}$.

Our claim follows from (2.20.3) and (2.20.5).

Remark 2.21. In the proof of Lemma 2.16 we used only the lower bound on K_X; compare [Y]. But in Lemma 2.19 we also use the two-sided bound on K_Y.

Proof of Lemma 2.11. Let $e_1, \dots, e_j$ be an orthonormal frame at $h(x)$. Pick $x_1 \cdots x_j \in X^n$ such that

(2.22) $$d(h(x_i), \exp_{h(x)}(\frac{1}{3}e_i)) \leq \delta .$$

Let $x_i' \in TX^n_x$, with $\exp_x x_i' = x_i$ and $t \to \exp_x tx_i'$, $t \in [0, 1]$, a minimal segment from x to x_i. For $p \in B_x(2\lambda)$, $p' \in TB_x(2\lambda) \subset TX_x$, put

(2.23.1) $\rho_{x_i}(p) := d(p, x_i)$,

(2.23.2) $\rho_{x_i'}(p') := d(p', x_i')$.

We claim that Lemmas 2.16 and 2.19 imply

(2.24) $$|\tilde{\rho}_{x_i} - \rho_{x_i'}| \leq c\lambda^2 .$$

For the moment, let us grant this. Then if we define k by

(2.25) $$d(\exp_{h(x)}(\frac{1}{3}e_i), k(p')) = \rho_{x_i'}(p') , \quad i = 1, \dots, j,$$

we get (2.11.1). Moreover, (2.11.2), (2.11.3) are direct consequences of the definition of k.

To verify (2.24), fix x_i as above and let γ_1 be minimal from x_i to x. Take $p' \in TB_x(2\lambda) \subset TX_x$ and put $p = \exp_x p'$. Let γ_2 be minimal from x_i to p. Let $\tilde{\gamma}_1$ be the lift of γ_1 running from x_i' to $0 \subset TX_x$. Finally, let $\tilde{\gamma}_2$ be the lift of γ_2 with initial point x_i'.

Since by Lemma 2.19,

(2.26.1) $\measuredangle(\gamma_1'(0), \gamma_2'(0)) \le c\lambda$,

it follows that the end point, p'', of $\tilde{\gamma}_2$ lies in $TB_x(c\lambda)$. Let $\tilde{\sigma}$ be minimal from p'' to p'. Then we have

(2.26.2) $L[\tilde{\sigma}] \le c\lambda$.

The projection of $\tilde{\sigma}$ is a geodesic loop, σ, on p, of length $L[\sigma] = L[\tilde{\sigma}]$. It follows from Lemma 2.16 that

$$(2.27) \qquad \begin{aligned} \measuredangle(\sigma'(0), -\gamma_2'(l)) &= \measuredangle(\tilde{\sigma}'(0), -\tilde{\gamma}_2'(l)) \\ &\le \tfrac{\pi}{2} + c\lambda . \end{aligned}$$

Using (2.26.2), (2.27), and a standard comparison argument, we get

$$(2.28.1) \quad \begin{aligned} d(p', x_i') &\le d(p'', x_i') + (c\lambda)(c\lambda), \\ &= d(p'', x_i') + c^2\lambda^2 . \end{aligned}$$

Since

(2.28.2) $d(p', x_j') = \rho_{x_j'}(p')$,

(2.28.3) $d(p'', x_j') = \tilde{\rho}_{x_j}(p')$,

this suffices to complete the proof.

Example 2.29. For $a > 0$, consider the annulus, $a < r < 2$, in R^2. Let $X^2_{a,\delta}$ (where $\delta = 2\pi/N$) denote its quotient by the action of $\mathbb{Z}_N$; $(r, \theta) \to (r, \theta + 2\pi/N)$. Let Y' be the open interval, $(a, 2)$. Then the map $h((r, \theta)) := r$ is a Riemannian submersion and a δ-Hausdorff approximation. But no matter how small we take δ, the second fundamental form of the fibres has norm $1/r$, which blows up as $a \to 0$. Clearly, no smoothing procedure will improve this situation. This confirms the necessity of restricting f in Theorem 2.6 to points that are far from ∂X^n (e.g., to X_1^n) independent of how small we take λ.

The reader may also wish to verify Lemma 2.11 directly in the context of this example.

For the application in §5, we will need the following sharpening of (2.6.2).

Let f be as in Theorem 2.6 and let v be a tangent vector at $x \in f^{-1}(y)$, with v orthogonal to $f^{-1}(y)$.

Proposition 2.30. *There exists a geodesic γ, with $\gamma(0) = x$, $\gamma | [0, \frac{1}{3}\iota]$ minimal, and*

$$(2.30.1) \qquad \measuredangle\,(\gamma'(0), v) \le c(n, A)(\delta\iota^{-1})^{1/2} .$$

Let τ be the minimal geodesic from $f(\gamma(0))$ to $f(\gamma(\frac{1}{3}\iota))$. Then

$$(2.31.2) \qquad \measuredangle\,(df(v), \tau'(0)) \le c(n, A)(\delta\iota^{-1})^{1/2} .$$

Proof. By Lemma 2.13, it suffices to verify the corresponding assertions for the map k, for which they are clear by inspection.

3. Equivariant and parametrized version of the theorem on almost flat manifolds

The main result of this section is concerned with fibrations such as those obtained in Theorem 2.6. To prove it, we show that one can canonically remove the dependence on the base point in the initial step of the proof of Ruh's theorem [R] (see also [Gh1]). Thus, initially we will be concerned with a single almost flat manifold.

Let N be a nilpotent Lie group (which need not be simply connected). The canonical connection, ∇^{can}, on the tangent bundle of N, is, by definition, the unique connection that makes all the left invariant vector fields parallel. Let $N_L \propto \operatorname{Aut} N$ be the skew product of N_L and $\operatorname{Aut} N$ (N_L denotes an isomorphic copy of N acting on N by left multiplication). It is easy to see that this group coincides with the group, $\operatorname{Aff}(N, \nabla^{\mathrm{can}})$, the group of all affine transformations of $(N, \nabla^{\mathrm{can}})$. If $\Lambda \subset \operatorname{Aff}(N, \nabla^{\mathrm{can}})$ is a subgroup whose action on N is properly discontinuous, we can define the induced connection, ∇^{can}, on the quotient space $\Lambda\backslash N$.

Remark 3.1. The subgroups, N_L and N_R, can be defined intrinsically, just using the affine structure of $(N, \nabla^{\mathrm{can}})$. The group N_L is the kernel of the holonomy homomorphism, i.e., the subgroup that acts trivially on all globally parallel fields. The group N_R is obtained by integrating these fields. On the other hand, the subgroup, $\operatorname{Aut} N$, can be described as the isotropy group of the identity element, $e \in N$. Equivalently, it depends on a choice of base point in the affine homogeneous space $(N, \nabla^{\mathrm{can}})$. Thus, the specific isomorphism, $\operatorname{Aff}(N, \nabla^{\mathrm{can}}) \simeq N_L \propto \operatorname{Aut}(N)$ depends on a choice of base points as well.

Let Z^m be an A-regular Riemannian manifold with $A_0 = 1$ and diameter, $\delta \le \xi(m)$. In [R] Ruh observed that the results of [G] allow one to associate to each point, $z \in Z^m$, a flat orthogonal connection, ∇^z, such that:

(3.2.1) For $p, q \in Z^m$, there is a gauge transformation, $g^{p,q}$, carrying ∇^q to ∇^p.

(3.2.2) $g^{p,q}$ can be chosen such that $|g^{p,q} - \mathrm{Ident}| \le c(m)\delta$, $|\nabla^i g^{p,q}| \le c(A, i)\delta$, where Ident denotes the identity element of the gauge group.

(3.2.3) For all z, $|\nabla^i(\nabla^{\mathrm{LC}} - \nabla^z)| \le c(m, A, i)\delta$, where ∇^{LC} denotes the Levi Civita connection of the underlying metric.

(3.2.4) The connection, ∇^z, depends smoothly on z, (with estimates like those above on derivatives with respect to z).

(3.2.5) The holonomy group of ∇^z has order $< \omega_m$ (see, however, Remark 3.9).

Ruh went on to show that for some simply connected nilpotent Lie group, N, and discrete subgroup, $\Lambda \subset \operatorname{Aff}(N, \nabla^{\mathrm{can}})$, with $\#(\Lambda \cap N_L \setminus \Lambda)$ equal to the order of the holonomy group, one can associate to each connection, ∇^z, a gauge transformation conjugating ∇^z into a connection isomorphic to the connection ∇^{can} on $\Lambda\backslash N$. The fact that $\Lambda\backslash N$ is actually the same for all z, follows from

Malcev's rigidity theorem, (see [Rag, BK] and Theorem 3.7).

We now show that by suitably averaging the family of connections, ∇^z, we can obtain a canonical gauge equivalent flat connection associated to the Riemannian structure (and not depending on a choice of base point). The connection isomorphic to ∇^{can}, associated to this one by Ruh's construction, depends smoothly on the underlying metric and is automatically invariant under all of its isometries. From this, the main result of this section follows immediately.

We now explain the averaging procedure. Let $\mathscr{P}$ denote the bundle associated to the frame bundle, FZ^m, via the adjoint representation. Each fibre of $\mathscr{P}$ has a natural group structure isomorphic to $O(m)$. The *gauge group* is the space of sections of $\mathscr{P}$, equipped with the group structure induced by pointwise multiplication. It has a natural action on FZ^m, which commutes with the action of $O(m)$. Hence it also acts on the space of connections.

A connection, ∇, on FZ^m induces a connection on $\mathscr{P}$. The group, $K(\nabla)$, of gauge transformations fixing ∇ is easily seen to be the group of *parallel sections* of $\mathscr{P}$ with respect to the induced connection. Let $\mathscr{K}(\nabla)\subset\mathscr{P}$ be the bundle whose fibre at $z\in Z^m$ is gotten by evaluating at z, the sections of $K^0(\nabla)$, the identity component of $K(\nabla)$. Then $\mathscr{K}(\nabla)$ is *canonically trivial.*

Let $g^{p,q}$ be as in (3.2.1). Put

$$(3.3.1)\quad \begin{aligned} g^{p,q}(z) &= h^{p,q}(z)k^{p,q}(z)\\ &= e^{U(z)}e^{V(z)}, \end{aligned}$$

where $V(z)$ is in the Lie algebra of $\mathscr{K}(\nabla)_z$ and for all z,

$$(3.3.2)\quad \langle U(z), V(z)\rangle = 0.$$

The inner product in (3.3.2) comes from the negative of the Killing form. Since $g^{p,q}$ is close to the identity, it is uniquely defined up to right multiplication by an element of $K^0(\nabla^q)$. It follows that $h^{p,q}$ is independent of the particular choice of $g^{p,q}$. Also, there is a *unique* choice of $g^{p,q}$ that satisfies

$$(3.3.3)\quad \textstyle\int_{Z^m} V(z)\,dz = 0,$$

where dz is the *normalized* Riemannian volume element, for which

$$(3.3.4)\quad \textstyle\int_{Z^m} dz = 1.$$

This is an immediate consequence of center of mass construction for the compact Lie group $K^0(\nabla)$; see [BK, §8].

Suppose $g^{p,q}$, $g^{q,w}$, $g^{p,w}$ are normalized as in (3.3.2), (3.3.3). Put

$$(3.4.1)\quad g^{p,q} = e^{U_1}e^{V_1},\qquad g^{q,w} = e^{U_2}e^{V_2},\qquad g^{p,w} = e^{U_3}e^{V_3},$$

where $V_1\in\mathscr{K}(\nabla^q)$, $V_2, V_3\in\mathscr{K}(\nabla^w)$. Assume

$$(3.4.2)\quad \|U_j\|, \|V_j\| < \eta << 1.$$

Then since the product of elements in a Lie group that are close to the identity is commutative modulo higher order terms,

$$(3.4.3)\quad \begin{aligned} g^{p,q}g^{q,w} &= e^{U_1}g^{q,w}(g^{q,w})^{-1}e^{V_1}g^{q,w}\\ &= e^{U_1+U_2}e^{V_2+V_1'} + O(\eta^2). \end{aligned}$$

Here,

$$(3.4.4)\quad (g^{q,w})^{-1}e^{V_1}g^{q,w} := e^{V_1'}.$$

Thus, $V_1' \in \mathscr{K}(\nabla^w)$. It follows that

(3.4.5) $\left\langle U_1 + U_2, V_2 + V_1' \right\rangle = O(\eta^2)$,

(3.4.6) $\int_{Z^m}(V_2 + V_1')\,dz = 0$,

which easily implies that

(3.4.7) $g^{p,q}g^{q,w} = g^{p,w} + O(\eta^2)$.

For fixed w and variable p in (3.4.1), we write $U_3 = U_3(p)$, $V_3 = V_3(p)$. Set

(3.5.1) $g^w = e^{\int_Z U_3(p)\,dp} \cdot e^{\int_Z V_3(p)\,dp}$.

By using (3.4.7), we obtain

(3.5.2) $g^q g^{q,w} = g^w + O(\eta^2)$.

Hence, if we put

(3.5.3) $\nabla_1^w = g^w(\nabla^w)$,

we get

(3.5.4) $\nabla_1^q = \nabla_1^w + O(\eta^2)$.

By iterating the above construction, we obtain convergent sequences, ∇_1^q, $\nabla_2^q, \ldots$ such that for all q, w

(3.5.6) $\nabla^\infty := \lim_{j\to\infty} \nabla_j^q = \lim_{j\to\infty} \nabla_j^w$

is independent of the base point and, in particular, invariant under the isometry group of Z^m.

We now turn to our main result, Proposition 3.6.

Let X^n, Y^j be A-regular Riemannian manifolds, with $A_0 = 1$, and let $f\colon X^n \to Y^j$ be a $\{C_i \iota^{1-i}\}$-regular, 1-almost Riemannian submersion, where $C_i = C_i(n, A)$. (This normalization corresponds to that of Theorem 2.6 but no assumption on inj rad_y, $y \in Y^j$, is required here.) Assume that G acts on X^n, Y^j by isometries and that f is G-equivariant.

Let $\nabla^{y,\mathrm{LC}}$ denote the Levi Civita connection for the induced metric on $f^{-1}(y)$. Suppose that $\mathrm{dia}(f^{-1}(y)) \le \delta$ and $|II_{f^{-1}(y)}| \le c\iota^{-1}$, with $c\delta\iota^{-1} \le \xi(n)$, where $c = c(n, A)$ and $\xi(n)$ is so small that $f^{-1}(y)$ is almost flat. Let $\nabla^{y,*}$ denote the affine flat connection on $f^{-1}(y)$ obtained by applying the construction of [R] (or [Gh1]) to the connection, $\nabla^{y,\infty}$, associated to $\nabla^{y,\mathrm{LC}}$ via (3.5.6). Thus, $(f^{-1}(y), \nabla^{y,*})$ is affinely diffeomorphic to some $(\Lambda \setminus N, \nabla^{\mathrm{can}})$ with $\#(\Lambda \cap N_L \setminus \Lambda) \le \omega_n$ (see, however, Remark 3.9).

Let y vary and regard, $\nabla^{y,\mathrm{LC}} - \nabla^{y,\infty}$, $\nabla^{y,\mathrm{LC}} - \nabla^{y,*}$ as tensor fields on X^n, by putting $\nabla_V^{y,\mathrm{LC}} - \nabla_V^{y,\infty} = 0$, $\nabla_V^{y,\mathrm{LC}} - \nabla_V^{y,*} = 0$, for V orthogonal to $f^{-1}(y)$.

Proposition 3.6.

(3.6.1) $|\nabla^i(\nabla^{y,\mathrm{LC}} - \nabla^{y,\infty})| \le C_i(n, A)\delta\iota^{-(2+i)}$.

(3.6.2) $|\nabla^i(\nabla^{y,\infty} - \nabla^{y,*})| \le C_i(n, A)\delta\iota^{-(2+i)}$.

(3.6.3) *If* $h \in G$, *then* $h\colon (f^{-1}(y), \nabla^{y,*}) \to (h(f^{-1}(y)), \nabla^{h(y),*})$ *is an affine diffeomorphism.*

Proof. By our previous discussion, Ruh's method yields a family of affine flat structures on the fibres and it is straightforward to check that the conditions of the proposition hold. Each Z_y is affine equivalent to some $(\Lambda_y \setminus N_y, \nabla^{\mathrm{can}})$. Moreover, since affine structures of this type cannot occur in nontrivial families, $\Lambda_y \setminus N_y = \Lambda \backslash N$ is actually independent of y. This is a weak generalization of the second Bieberbach theorem (the uniqueness of the affine structure on a compact flat Riemannian manifold; see [Char]). For completeness, we give the argument (see, however, Remark 3.9).

A local trivialization of our fibration over an open neighborhood, U, of $y \in Y$, induces isomorphisms, $\Lambda_{y'} \simeq \pi_1(Z_{y'}) \simeq \pi_1(Z_y) \simeq \Lambda_y$. The holonomy homomorphisms vary continuously and, by (3.6.1), have *finite* image. It follows that the identifications, $\Lambda_{y'} \simeq \Lambda_y$, respects the kernels, $\Lambda_{y'} \cap N_{y'}$, $\Lambda_y \cap N_y$ of the holonomy homomorphisms. These are cocompact subgroups of the groups $N_{y'}$, N_y. By a theorem of Malcev the isomorphisms $\Lambda_{y'} \cap N_{y'} \simeq \Lambda_y \cap N_y$ extend uniquely to isomorphisms, $N_{y'} \simeq N_y$.

Theorem 3.7 (Malcev). *Let* N_1, N_2 *be simply connected nilpotent Lie groups and* $\Lambda \subset N_1$ *a cocompact subgroup. Then a homomorphism from* Λ_1 *to* N_2 *extends uniquely to* N_1.

Now the following consequence of the affine center of mass construction for Lie groups [BK, §8] implies the asserted rigidity of $(\Lambda_y \setminus N_y, \nabla^{\mathrm{can}})$.

Proposition 3.8. *Let* $h_t\colon G_1 \to G_2$ *be a continuous family of homomorphisms of Lie groups such that for some subgroup* $H \subset G_1$, *of finite index,* $h_t|H$ *is independent of* t. *Then there is a continuous map,* $t \to k_t \in G_2$ *such that* $h_t = k_t h_0 k_t^{-1}$.

Proof. It suffices to consider t so small that the affine center of mass, k_t, of the finite set $\{h_t^{-1} h_0(g) | g \in G_1\}$ is defined. As in Proposition 8.1.7 of [BK], this choice of k_t has the required property.

Remark 3.9. The results of this section and the next will be applied in §7 to the local fibrations of the frame bundle constructed in §§5, 6. In that case, one actually has $\Lambda \subset N_L$; equivalently, the connection ∇^{can} on $\Lambda\backslash N$ is globally flat (see §7 and Appendix 1). Thus, Proposition 3.8 and the argument given in the proof of Proposition 3.6 are not needed for the construction of the nilpotent Killing structure.

4. Nilpotent Killing structures on fibrations

Let $Z \to X \xrightarrow{f} Y$ be a fibration acted on isometrically by a compact group G, such that the assumptions of Proposition 3.6 hold. By Proposition 3.6, each fibre carries a flat affine structure isomorphic to some $\Lambda\backslash N$.

Let $N_{\mathrm{Aff}(N)}\Lambda$ and $C_{\mathrm{Aff}(N)}\Lambda$ denote respectively the *normalizer* and *centralizer* of Λ in $\mathrm{Aff}(N)$. Then

(4.1.1) $\mathrm{Aff}(\Lambda\backslash N) \simeq (N_{\mathrm{Aff}(N)}\Lambda)/\Lambda$

and since Λ is discrete,

(4.1.2) $\mathrm{Aff}^0(\Lambda\backslash N) \simeq (C_{\mathrm{Aff}(N)}\Lambda)/\Lambda$,

where, $\mathrm{Aff}^0(\Lambda\backslash N) \subset \mathrm{Aff}(\Lambda\backslash N)$ denotes the identity component. Also, $\Lambda \cap N_L \subset N_L$ is cocompact. Thus, by Malcev's theorem (3.7),

(4.1.3) $C_{\mathrm{Aff}(N)}\Lambda \subset C_{\mathrm{Aff}(N)}N_L$.

Moreover,

(4.1.4) $N_R = C_{\mathrm{Aff}(N)}N_L = \{(t, Ad_{t^{-1}})\}$

is just an isomorphic copy of N acting by *right* translations.

The identification $N_R = C_{\mathrm{Aff}(N)}N_L$ depends only on the affine structure of the *affine homogeneous space*, $(N, \nabla^{\mathrm{can}})$; compare Remark 3.1. However, an explicit isomorphism, $N_L \simeq N_R$, or equivalently the representation $N_R = (t, Ad_{t^{-1}})$, does depend on a choice of base point (which can then be viewed as the identity element, $e \in N$). More generally, corresponding to each *normal* subgroup, $B_L \subset N_L$, there is a well-defined isomorphic subgroup, $B_R \subset N_R$. Again, a specific isomorphism, $B_L \simeq B_R$, depends on a choice of base point.

Now let V be a locally defined *right* invariant vector field on a neighborhood $W \subset Z_y$. As in §3, we can find a local trivialization $\phi : U \times Z \to X$ (over a small neighborhood U of y), with respect to which the affine structure on the fibres is constant. Such a trivialization is unique up to a map $U \to \mathrm{Aff}^0(\Lambda\backslash N)$. Since the group $\mathrm{Aff}^0(\Lambda\backslash N)$ is contained in N_R, it follows that this group acts trivially on local *right* invariant fields. Thus, V has a canonical extension to $\phi(U \times W)$. In this way, we obtain a sheaf $\underline{\mathfrak{n}}$, of nilpotent Lie algebras of vector fields on X and an action of the associated sheaf $\mathfrak{n}$, of simply connected nilpotent Lie groups.

The action of G extends in an obvious way to an action on $\mathfrak{n}$ and the actions of $\mathfrak{n}$ and G on X commute in the obvious sense. In general, the action of a group on a sheaf is called *locally trivial* if for each open set U, there is a neighborhood, W, of the identity in G such that for all $g \in W$,

(4.2)
$$\rho_{g(U)\cap U,\, g(U)}\, g = \rho_{g(U)\cap U,\, U}\,.$$

Here, $\rho_{B,A}$ denotes the restriction map from A to B.

Now the same sort of argument as was given above yields

Proposition 4.3. *The action of G on $\tilde{\mathfrak{n}}$ is locally trivial.*

In case G acts freely, it follows directly from Proposition 4.3 that there is an induced sheaf, $\tilde{\mathfrak{n}}$, on X/G (see §8 for the detailed discussion).

We now discuss the quantitative behavior of the local right invariant fields constructed above. This requires a more explicit description of a local trivialization in which the affine structure on the fibres is constant.

Let V be a tubular neighborhood of a fixed fibre, Z_y, such that the normal exponential map of Z_y provides a local trivialization $\phi : U \times Z_y \to V$ (where $\phi(U \times Z_y)$ is the union of all fibres contained in V). By the proof of Proposition A2.2, the normal injectivity radius of Z_y is bounded below by $c(n, A)\min(\iota, d(y, \partial Y))$.

Let $p_{y'} = \phi((y', z)) \in Z_{y'}$. The universal covering space, $(\widehat{V}, \hat{p}_y)$, is fibred by universal covering spaces $(\widehat{Z}_{y'}, \widehat{p}_{y'})$. The covering groups of all of these spaces are canonically isomorphic to Λ. Let $\mathrm{Aff}(\widehat{Z}_{y'}, \widehat{\nabla})$ denote a group of affine automorphisms of $\widehat{Z}_{y'}$, with respect to its canonical flat affine connection, $\widehat{\nabla}$. Let $N_L(\widehat{Z}_{y'}) \subset \mathrm{Aff}(\widehat{Z}_{y'}, \widehat{\nabla})$ be the corresponding canonically defined subgroup. Then (up to natural isomorphism), for all y', we can regard $\Lambda \subset \mathrm{Aff}(\widehat{Z}_{y'}, \widehat{\nabla})$.

By Malcev's theorem (3.7), there is a unique affine equivalence $\psi_{y'}: \widehat{Z}_y \to \widehat{Z}_{y'}$, such that

(4.4.1) $\psi_{y'}(\widehat{p}_y) = \widehat{p}_{y'}$,

and for all $\lambda \in N_L(\widehat{Z}_y)$ (or equivalently for all $\lambda \in \Lambda \cap N_L(\widehat{Z}_{y'})$),

(4.4.2) $\psi_{y'}\lambda = \lambda\psi_{y'}$.

Given that such an affine equivalence exists, it is explicitly determined as follows.

By integrating the left invariant fields (i.e., the parallel fields for ∇^*) we obtain the group $N_R(\widehat{Z}_{y'})$, and hence, the right invariant vector fields. By integrating these, we obtain the group $N_L(\widehat{Z}_{y'})$.

Fix $\lambda \in \Lambda \cap N_L(\widehat{Z}_y) \simeq \Lambda \cap N_L(\widehat{Z}_{y'})$. For each y', there is a unique integral curve, $c_{y',\lambda}$, of a certain right invariant vector field on $\widehat{Z}_{y'}$, such that $c_{y',\lambda}(0) = \widehat{p}_{y'}$, $c_{y',\lambda}(t_\lambda) = c_{y',\lambda}(t_\lambda) = \lambda(\widehat{p}_{y'})$. Here t_λ is independent of y'. Since $\psi_{y'}$ is an affine equivalence satisfying (4.4.1), (4.4.2), we get

$$d\psi_{y'}(c'_{y,\lambda}(0)) = c'_{y',\lambda}(0) . \tag{4.5}$$

Clearly, the collection of vectors, $\{c'_{y,\lambda}(0)\}$, spans the tangent space at p_y. Thus, (4.5) determines the linear map, $d\psi_{y'}$. Then $\psi_{y'}$ itself is determined by the condition that it map a given right invariant field on $\widehat{Z}_y$ to the right invariant field on $\widehat{Z}_{y'}$ to which it corresponds under $d\psi_{y'}$.

Now for all y we have $\mathrm{inj\,rad}_{\hat{p}_y} \geq c(n) > 0$; see [BK, Proposition 4.6.3]. On the other hand, the points $\lambda(y)$ are $c(n, A)\delta$-dense in $\widehat{Z}_y^m$. Thus, we can find $\lambda_1, \dots, \lambda_m$ such that

$$\left| \measuredangle(c'_{y,\lambda_{i_1}}(0), c'_{y,\lambda_{i_2}}(0)) - \frac{\pi}{2} \right| \leq c(n, A)\delta . \tag{4.6}$$

From the preceding explicit description of $\psi_{y'}$, together with Proposition 3.6 and standard bounds on the local trivialization, ϕ, we readily obtain

Proposition 4.7. *Let w be a right invariant field on $\widehat{V} \cap B_{\hat{p}}(2\delta)$, with $|w(\hat{p})| = 1$. Then*

(4.7.1) $|\nabla^i w| \leq c(n, A, i)\iota^{1-i}$.

We now construct a canonical (and hence G-invariant) invariant metric for the action of N_L on U, or equivalently for the action of $\mathfrak{n}$ on X/G (in case G acts freely). Given such a metric, it is obvious that the action of $\mathfrak{n}$ determines a nilpotent Killing structure (see Definition 1.5).

We have $\#(\Lambda \cap N_L \setminus \Lambda) \leq \omega_n$. Thus, it is clear that we can reduce to the case $\Lambda \subset N_L$.

Let v be a tangent vector at $\hat{p} \in \widehat{V}$ and let $\langle \ , \ \rangle$ denote the pullback to $\widehat{V}$, of the original metric on V. Let $h \in N_L$ and let hv denote the image of v under the differential of h. Then, the function, $h \to \langle hv , hv \rangle$, is constant on the left cosets of Λ. Since the group N_L is nilpotent, it is unimodular. Therefore, the space $\Lambda \backslash N$ inherits a canonical invariant measure $d\mu$, of total volume 1. The metric,

$$(v , v) = \int_{N_L/\Lambda} \langle hv , hv \rangle \, d\mu , \tag{4.8}$$

is invariant under N_L and pushes down to the required metric on V.

Clearly, our construction is independent of the choice of U and of the choice of base point used to define $\widehat{V}$. Thus it gives a canonical (and hence G-invariant) metric on X, which is invariant for the nilpotent Killing structure.

Proposition 4.9. *The original metric,* $\langle \ , \ \rangle$, *and invariant metric,* $(\ , \ )$, *satisfy*

(4.9.1) $|\nabla^i(\langle \ , \ \rangle - (\ , \ ))| \leq c(n , A , i)\delta\iota^{-(1+i)}$.

Proof. The estimates on left multiplication that follow immediately from (4.7.1) yield (4.9.1).

Remark 4.10. Note that the right-hand side of (4.9.1) is small provided δ is small relative to ι^{i+1}.

Remark 4.11. One can also construct an equivariant *right* action on X; it gives rise to the nilpotent collapsing structure. The construction, which will be carried out elsewhere, *does* make use of the invariant metrics on fibres.

III. The nilpotent Killing structure and invariant round metric

5. Local fibration of the frame bundle

In this section we begin the construction of the nilpotent Killing structure by constructing local fibrations of the frame bundle.

Let M^n be a complete A-regular Riemannian manifold. A standard computation shows that the frame bundle, FM^n, with its natural metric, is B-regular (for $B = B(A)$).

For fixed n and A, put

(5.5.1) $\mathfrak{F} = \{FB_p(2) \mid M^n \text{ is } A\text{-regular}\}$.

Note, on the right-hand side of (5.1.1), M^n is *not* fixed. (Also, we could replace 2 by any fixed $R > 0$ in (5.1.1)). Let $\mathfrak{C}\mathfrak{F}$ denote the closure of $\mathfrak{F}$ with respect to the $O(n)$-Hausdorff distance, d_H. Then by [F3], $\mathfrak{C}\mathfrak{F}$ consists of $B(n , A)$-regular *manifolds*, Y^i. It will be convenient to assume that A is normalized such that A_0, $B_0(n, A) \leq 1$. The induced action of $O(n)$ on Y^i is $D(n, A)$-regular *but need not be free.*

Put

(5.1.2) $\mathfrak{C}\mathfrak{F}_j := \{Y^j \in \mathfrak{C}\mathfrak{F} \mid \dim Y^j = j\}$.

Then $\mathfrak{C}\mathfrak{F} = \bigcup_j \mathfrak{C}\mathfrak{F}_j$ determines a stratification of $\mathfrak{C}\mathfrak{F}$. This fact, although we do not use it explicitly, puts our constructions in a natural context.

Clearly, $\mathfrak{C}\mathfrak{F}_j$ is empty for $j > n + \frac{n(n-1)}{2}$. One can also show that $\mathfrak{C}\mathfrak{F}_j$ is empty for $j < \frac{n(n-1)}{2} = \dim O(n)$; see Appendix 1. Again, we do not use this.

It follows from [GLP, §8] (together with [CGT, Theorem 4.3]) that there is a positive function, $\phi(\delta, n)$, with $\phi(\delta, n)/\delta$ increasing, such that if

(5.2.1) $Y_1^{j_1} \in \mathfrak{C}F_{j_1}$,

(5.2.2) $\operatorname{inj\,rad}_y \leq \min(\phi(\delta, n), d(y, \partial Y_1^{j_1}))$, for *some* $y \in Y_1^{j_1}$,

then there exists $Y_2^{j_2} \in \mathfrak{C}\mathfrak{F}_{j_2}$ with

(5.2.3) $j_2 < j_1$,

(5.2.4) $d_H(Y_1^{j_1}, Y_2^{j_2}) < \delta$.

From now on we suppress the dependence of ϕ on n.

Put $\iota_0 = 1$. Let $\iota_0 > \iota_1 > \cdots > \iota_{n+[n(n+1)]/2}$ and $1 > \delta_0 > \delta_1 > \cdots > \delta_{n+[n(n+1)]/2}$, be positive sequences, such that for $j \geq 1$,

(5.3.1) $\phi^{-1}(\iota_j) + \delta_j \leq \delta_{j-1}$.

Relation (5.3.1) can be satisfied by taking

(5.3.2) $\iota_j = \phi(\frac{1}{2}\delta_{j-1})$,

(5.3.3) $\delta_j \leq \frac{1}{2}\delta_{j-1}$.

In this and subsequent sections it will be necessary to assume that δ_j is small enough relative to ι_j such that certain additional conditions are satisfied.

Proposition 5.4. *Let $FB_p(1) \in \mathfrak{F}$ and let j be the smallest number such that there exists $Y^j \in \mathfrak{C}\mathfrak{F}_j$ with*

(5.4.1) $d_H(FB_p(1), Y^j) \leq \delta_j$.

Then for any such Y^j and $y \in Y^j$

(5.4.2) $\operatorname{inj\,rad}_y \geq \min(\iota_j, d(y, \partial Y^j))$.

Proof. Note that since $d_H(FB_p(1), FB_p(1)) = 0$, the set of Y^j satisfying (5.4.1) is nonempty. If (5.4.2) failed to hold, then by the definition of the function ϕ, we would have

(5.5.1) $d_H(Y^j, Y_1^{j_1}) \leq \frac{1}{2}\delta_{j-1}$,

for some $Y_1^{j_1}$ with $j_1 \leq j - 1$. Then by (5.3.1) and (5.4.1), it follows that

$$(5.5.2)\quad \begin{aligned} d_H(FB_p(1), Y_1^{j_1}) &\leq d_H(FB_p(1), Y^j) + d_H(Y^j, Y_1^{j_1}) \\ &\leq \delta_j + \phi^{-1}(\iota_j) \leq \delta_{j-1} \leq \delta_j, \end{aligned}$$

which is a contradiction.

Fix $\lambda = \lambda(n) < 1$ to be determined in §§6, 7.

Proposition 5.6. *Let $p_s \in M^n$, $s = 1, 2$, with $j_1 \le j_2$. Let $Y_s^{j_s}$ satisfy* (5.4.1) *with j_s minimal for p_s. Then there exist*

(5.6.1) $f_s \colon FB_{p_s}(1) \to Y_s^{j_s}$,

(5.6.2) $f_{1,2} \colon f_2(FB_{p_1}(1) \cap FB_{p_2}(1)) \to Y_1^{j_1}$,

such that

(5.6.3) f_s *satisfies* (2.6.1)–(2.6.6) *with* $\iota = \iota_{j_s}$, $\delta = \delta_{j_1}$, $\lambda = \lambda(n)$;

(5.6.4) $f_{1,2}$ *satisfies* (2.6.1)–(2.6.7) *with* $\lambda = \iota_{j_2}^{-1}$, $\delta = c(n, A)\delta_{j_1}^{1/2}$;

(5.6.5) $d(f_{1,2}f_2, f_1) \le c(n, A)\lambda\iota_{j_1}$;

(5.6.6) $|\nabla(f_{1,2}f_2) - \nabla f_1| \le c(n, A)\lambda$.

Proof. The existence of the fibrations, f_s, satisfying (5.6.1), (5.6.3) follows from (5.4.1), (5.4.2).

Using (2.4.3), (2.4.4), we construct an $O(n)$-equivariant Hausdorff approximation, h, with domain $f_2(FB_{p_1}(1) \cap FB_{p_2}(1))$ and range in $Y_1^{j_1}$. By regularizing h, we obtain $f_{1,2}$ satisfying (5.6.2), (5.6.4), and (5.6.5). Finally, (5.6.6) follows with the help of Proposition 2.30.

Remark 5.7. The sets $FB_{p_1}(1) \cap FB_{p_2}(1)$ are *not* necessarily of the form $f_1^{-1}(U_1)$ or $(f_{12}f_2)^{-1}(U_2)$ and hence, are not unions of compact fibres. To obtain actual fibrations we must restrict the domain of a map, f_s, to the set consisting of all *compact* fibres whose intersection with $FB_{p_s}(\frac{1}{2})$ is nonempty. This is a slightly smaller set. We will deal with this (minor) point when it arises in the proof of Proposition 6.1. But in the meantime, to simplify notation, we will continue to refer to "the fibration f_s." More importantly, the maps, f_1, f_2 are *not* necessarily compatible in the sense that the fibres of f_2 need not be unions of the fibres of f_1. Equivalently, $f_{1,2}f_2 \ne f_1$ in general. However, by (5.6.5) and (5.6.6), f_1, f_2 are *almost* compatible. This together with the results of Appendix 2, will be used in §6 to construct a collection of local fibrations of FM^n that are compatible in the above sense.

Remark 5.8. The smaller the numbers, δ_j, the more difficult the condition

(5.8.1) $d_H(FB_p(2), Y^j) \le \delta_j$

is to satisfy. In particular, the subsets of elements of $\mathfrak{F}$, for which there exists a *nontrivial* fibration also gets smaller.

Remark 5.9. If we fix $\varepsilon = 1$ in Theorems 1.3 and 1.7 then we can work with a fixed sequence that is small enough for the arguments of subsequent sections to go through. But if we let $\varepsilon \to 0$, then necessarily $\delta_j \to 0$ as well. As a consequence, for ε very small, our structure will be non*trivial* only on the part of M^n that is very collapsed.

6. Making the local fibrations compatible

Let M^n be as in §5.

The fibrations constructed in this section will be obtained by slightly modifying those constructed in §5 and restricting their domains. After this has been done, to simplify notation, we will continue to denote the modified fibrations by f_s, $f_{s,t}$ and their base spaces by $Y_s^{j_s}$.

Let $\lambda = \lambda(n) < 1$ be a sufficiently small constant. The constraints on $\lambda(n)$ will be determined in the course of the proof of Proposition 6.1. These and the constraints entailed in the analogous constructions of §7 allow us to fix the values of $\lambda(n)$. We will assume without further mention at the end of §7 that this has been done.

Let $b\colon [0, 1] \to [0, 1]$ be an increasing function, with $b(u) \le u$.

Proposition 6.1. *Given b, there exists a decreasing sequence, $\iota_j = \iota_j(b, n, A)$, such that the following holds. There is a covering, $M^n = \bigcup_s B_{p_s}(\frac{1}{16})$, and $O(n)$-equivariant fibrations,*

(6.1.1) $f_s\colon FB_{p_s}(\frac{1}{2}) \to Y_s^{j_s}$,

such that for $y \in Y_s^{j_s}$,

(6.1.2) $\operatorname{inj\,rad}_y \ge \min(\iota_{j_s}, d(y, \partial Y_s^{j_s}))$.

Moreover, if $B_{p_s}(\frac{1}{2}) \cap B_{p_t}(\frac{1}{2}) \neq \varnothing$, $j_s \le j_t$, then there is an $O(n)$-equivariant fibration,

(6.1.3) $f_{s,t}\colon f_t(FB_{p_s}(\frac{1}{2}) \cap FB_{p_t}(\frac{1}{2})) \to f_s(FB_{p_s}(\frac{1}{2}) \cap FB_{p_t}(\frac{1}{2}))$,

such that

(6.1.4) $f_{s,t} f_t = f_s$.

The fibrations, f_s, satisfy:

(6.1.5) $\operatorname{dia}(f_s^{-1}(y)) \le b(\iota_{j_s})$.

(6.1.6) *f_s is a $c(n, A)\lambda$-almost Riemannian submersion.*

(6.1.7) *f_s is $\{C_i(n, A)\iota_{j_s}^{1-i}\}$-regular.*

(6.1.8) $|II_{f_s^{-1}(y)}| \le c(n, A)\iota_{j_s}^{-1}$.

(6.1.9) *The maps $f_{s,t}$ satisfy* (6.1.5)–(6.1.8).

(6.1.10) *The (compact) fibres, $f_s^{-1}(y)$, $f_{s,t}^{-1}(y)$ are diffeomorphic to nilmanifolds.*

Proof. The fact that the fibres, $f_s^{-1}(y)$, $f_{s,t}^{-1}(y)$ are diffeomorphic to nilmanifolds (and not just infranilmanifolds) was mentioned in the introduction and is explained further in §7 and Appendix 1.

Pick a maximal collection of points, p_s, such that for all s, t,

(6.2.1) $d(p_s, p_t) \ge \frac{1}{64}$.

In particular,

(6.2.2) $M^n = \bigcup_s B_{p_s}(\frac{1}{16})$.

Fix a decreasing sequence,

(6.3.1) $\delta_0 > \delta_1 > \cdots > \delta_{n+[n(n+1)]/2}$,

to be determined later. As in (5.3.2), define

(6.3.2) $\iota_j = \phi(\frac{1}{2}\delta_{j-1})$.

Relative to the sequence $\{\delta_j\}$, choose for each p_s, a fibration,

(6.3.3) $f_s\colon FB_{p_s}(1) \to Y_s^{j_s}$,

satisfying (5.4.1) with j_s minimal. Let the corresponding fibrations, $f_{s,t}$ be as in Proposition 5.6. We can assume that $\{\delta_j\}$ is such that (6.1.9) holds.

In order to make it clear that when we repeatedly modify our fibrations, approximately compatible fibrations do not eventually become too far apart, we use a technical device.

As in Lemma 2.2 of [CG1], we partition the set $\{p_s\}$ into disjoint subsets $S_1, \ldots, S_{N(n)}$, such that if $p_s, p_u \in S_k$, then

(6.4.1) $d(p_s, p_u) > 4$.

In particular, those balls, $B_{p_t}(1)$, whose intersection with a fixed ball, $B_{p_s}(1)$, is nonempty, all belong to different subsets, S_k. Thus, there are at most $N(n)$ such balls.

Put

(6.4.2) $S_{k,j} = \{p_s \in S_k \mid j_s = j\}$.

There are $T(n) = N(n)(n+1+\frac{n(n+1)}{2})$ of the $S_{k,j}$, some of which might be empty. Put

(6.4.3) $S_{k,j} = S^{k+N(n)j}$.

Note that if $S^\alpha = S_{k(\alpha),j(\alpha)}$, then

(6.4.4) $\alpha < \alpha'$ implies $j(\alpha) \le j(\alpha')$.

Also, $p_s \in S^\alpha$, $f_s\colon B_{p_s}(1) \to Y_s^{j_s}$, implies

(6.4.5) $j_s = j(\alpha)$.

In order to make our fibrations compatible, we now modify them in a total of $T(n)$-stages, one for each S^α. Each stage, say α_0, is divided into $(2^{T(n)-\alpha_0-1})$ steps, one for each nonempty subset, $(\alpha_1, \ldots, \alpha_m)$, with $\alpha_0 < \alpha_1 < \cdots < \alpha_m \le T(n)$. Thus, there are $N'(n) = 2^{T(n)-1} - (T(n)-1)$ steps in all. (The order in which the steps are performed is specified below.)

At a given step we must also decrease the radii of the balls involved by a definite amount. The notation is simplest if at the end of each step α_0, we actually decrease the radii of *all* balls (i.e., with centers $p_1, p_2, p_3, \ldots$) by an amount $\frac{1}{2N'(n)}$. Thus, at the beginning of a given step, every ball has radius, $r = 1 - \frac{1}{2N'(n)} \times$ the number of steps already peformed.

Note that since at each stage we decrease the radii of our balls by *exactly* $\frac{1}{2N'(n)}$ we certainly want $\lambda = \lambda(n) < \frac{1}{2N'(n)}$.

At step $(\alpha_1, \dots, \alpha_m)$ of stage α_0, we modify only the fibrations f_{i_l}, $1 \le l \le m$, and $f_{i_{l_1}, i_{l_2}}$, $1 \le l_1 < l_2 \le m$, over sets of the form $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)$ (respectively $f_{i_{l_2}}(FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r))$) where $p_{i_l} \in S^{\alpha_l}$. Note that for certain $(\alpha_1, \dots, \alpha_m)$ (e.g., unless $S^{\alpha_0}, \dots, S^{\alpha_m}$ all belong to distinct S_u) there will be no such nonempty intersections. However, if at any step there are no nonempty intersections, we simply decrease the radii of all balls by $\frac{1}{2N'(n)}$ and proceed to the next step.

Now we can explain the reason for introducing the sets S_u. If $\hat{p}_{i_l} \in S^{\alpha_l}$ and $(\hat{p}_{i_0}, \dots, \hat{p}_{i_m})$ is distinct from $(p_{i_0}, \dots, p_{i_m})$ then (by construction)

$$(6.5) \quad (FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)) \cap (FB_{\hat{p}_{i_0}}(r) \cap \cdots \cap FB_{\hat{p}_{i_m}}(r)) = \varnothing.$$

This guarantees that the various modifications performed at step $(\alpha_1, \dots, \alpha_m)$ of stage α_0 do not interact with one another (and that a given fibration is modified at most $N'(n)$ times). As a consequence, fibrations that are initially almost compatible do not grow uncontrollably further apart as the construction progresses.

It is important that in carrying out the modifications, the stages are arranged in descending order; i.e., we start with stage $T(n)$, then pass to stage $T(n) - 1$, etc. It is also important that the steps of stages α_0 are arranged as follows. First we do step $(\alpha_0 + 1, \alpha_0 + 2, \dots, T(n))$. Then, in some (arbitrary) order, we do the steps corresponding to subsets of $(\alpha_1, \dots, \alpha_m)$ of cardinality, $T(n) - \alpha_0 - 1$; then, in some (arbitrary) order, the steps corresponding to subsets of cardinality $T(n) - \alpha_0 - 2$, etc.

Let r be the common radius of all balls at the beginning of step $(\alpha_1, \dots, \alpha_m)$ of stage α_0. At the beginning of this step, we can assume by induction that the following holds.

Let $\alpha_0' \ge \alpha_0$ and let $(\alpha_1', \dots, \alpha_{m'}')$ be a step of stage α_0' that has already been completed (automatic unless $\alpha_0' = \alpha_0$, $m' \le m$). Let $\underline{p}_{i_{l'}} \in S^{\alpha_{l'}'}$, $0 \le l' \le m'$. Then for $0 \le l_1' < l_2' \le m'$, our (previously redefined) fibrations satisfy

$$(6.6.1) \quad f_{i_{l_1'} i_{l_2'}} f_{i_{l_2'}} = f_{i_{l_1'}} \quad \text{on } FB_{\underline{p}_{i_0}}(r) \cap \cdots \cap FB_{\underline{p}_{i_m}}(r).$$

In addition, we can assume by induction that for all s, t, u with $j_s \le j_t \le j_u$,

(6.6.2) $\operatorname{dia}(f_s^{-1}(y))$, $\operatorname{dia}(f_{s,t}^{-1}(y)) \le c(n, A)\delta_{j_s}$.

(6.6.3) f_s, $f_{s,t}$ are $c(n, A)\lambda$-almost Riemannian submersions.

(6.6.4) f_s, $f_{s,t}$ are $\{C_i(n, A)\iota_{j_s}^{-1}\}$-regular.

(6.6.5) $d(f_{s,t} f_t, f_s) \le c(n, A)\lambda\iota_{j_s}$.

(6.6.6) $|\nabla(f_{s,t} f_t) - \nabla f_s| \le c(n, A)\lambda$.

(6.6.7) $d(f_{s,t} f_{t,u}, f_{s,u}) \le c(n, A)\lambda\iota_{j_s}$.

(6.6.8) $|\nabla(f_{s,t} f_{t,u}) - \nabla f_{s,u}| \le c(n, A)\lambda$.

We now define certain $O(n)$-equivariant self-diffeomorphisms

(6.7.1) $\psi_{i_l}: FB_{p_{i_l}}(r) \to FB_{p_{i_l}}(r)$, $1 \leq l \leq m$

and

(6.7.2) $\xi_{i_l}: f_{i_l}(FB_{p_{i_l}}(r)) \to f_{i_l}(FB_{p_{i_l}}(r))$, $1 \leq l \leq m$.

(We suppress the dependence of these maps on $(\alpha_0, \dots, \alpha_m)$.) Eventually, we will redefine f_{i_l} to be

(6.7.3) $\xi_{i_l}^{-1} f_{i_l} \psi_{i_l}$, $1 \leq l \leq m$,

and redefine $f_{i_{l_1}, i_{l_2}}$ to be

(6.7.4) $\xi_{i_{l_1}}^{-1} f_{i_{l_1}, i_{l_2}} \xi_{i_{l_2}}$, $1 \leq l_1 < l_2 \leq m$.

Until this is done explicitly, f_{i_l}, $f_{i_{l_1}, i_{l_2}}$ retain their previous meanings.

We now use Proposition A2.2 to construct the diffeomorphism ψ_{i_l}. (Remarks similar to those that follow also apply to the construction of the diffeomorphism ξ_{i_l} below). Since Proposition A2.2 holds for fibrations with compact fibres, we first restrict the map f_{i_0} (which is used in defining the various ψ_{i_l}) to the subset of $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)$ consisting of the union of all *compact* fibres of f_{i_0}. The set contains $(FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r))_{c(n,A)\delta_{j(\alpha_0)}}$ (the notation is as in (2.3)).

In view of (6.6.3)–(6.6.6), by Proposition A.2.2 we can find an $O(n)$-equivariant map, ψ $(= \psi_{i_{l_1}, \dots, i_{l_m}})$ such that:

(6.8.1) ψ is a self-diffeomorphism of $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)$.

(6.8.2) ψ is the identity near the boundary.

(6.8.3) $f_{i_0, i_1}(f_{i_1} \psi) = f_{i_0}$ on $FB_{p_{i_0}}(r - \frac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \frac{1}{2N'(n)})$.

(6.8.4) ψ is the identity on the subset of

$$FB_{p_{i_0}}(r - \tfrac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \tfrac{1}{2N'(n)})$$

on which $f_{i_0, i_1} f_{i_1} = f_{i_0}$.

(6.8.5) $d(\psi, \mathrm{Ident}) \leq c(n, A)\lambda \iota_{j(\alpha_0)}$,

(6.8.6) $|\nabla \psi - \mathrm{Ident}| \leq c(n, A)\lambda$.

(6.8.7) ψ is $\{C_i(n, A)\iota_{j(\alpha_1)}^{1-i}\}$-regular.

Define ψ_{i_l} as in (6.7.1) by

$$\psi_{i_l}(x) = \begin{cases} \psi(x) & x \in FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r), \\ x & x \notin FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r). \end{cases}$$

We now define the diffeomorphisms, ξ_{i_l}.

In view of (6.6.3), (6.6.4), (6.6.7), (6.6.8), by Proposition A2.2, we can find an $O(n)$-equivariant map, ξ_{i_l} $(l \geq 2)$ such that:

(6.9.1) ξ_{i_l} is a self-diffeomorphism of $f_{i_l}(FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r))$.

(6.9.2) ξ_{i_l} is the identity near the boundary.

(6.9.3) $f_{i_0,i_1} f_{i_1,i_l} \xi_{i_l} = f_{i_0,i_l}$ on $f_{i_l}(FB_{p_{i_0}}(r - \frac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \frac{1}{2N'(n)}))$.

(6.9.4) ξ_{i_l} is the identity on the subset of

$$f_{i_l}(FB_{p_{i_0}}(r - \tfrac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \tfrac{1}{2N'(n)}))$$

on which $f_{i_0,i_1} f_{i_1,i_l} = f_{i_0,i_l}$.

(6.9.5) $d(\xi_{i_l}, \text{Ident}) \le c(n, A)\lambda \iota_{j(\alpha_0)}$.

(6.9.6) $|\nabla\xi_{i_l} - \text{Ident}| \le c(n, A)\lambda$.

(6.9.7) ξ_{i_l} is $\{C_i(n, A)\iota_{j(\alpha_1)}^{1-i}\}$-regular.

Extend ξ_{i_l} to all of $f_{i_l}(FB_{p_{i_l}}(r))$ by defining it to be the identity map off $f_{i_l}(FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r))$. Also define ξ_{i_1} to be the identity map.

We now examine the effect of modifying f_{i_l}, $f_{i_{l_1}, i_{l_2}}$ as in (6.7.3), (6.7.4) on our induction hypotheses.

First of all, it follows from (6.6.1), (6.8.4), (6.9.4) that ψ_{i_l}, ξ_{i_l} is equal to the identity over the subset of $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_l}}(r)$ that intersects any $FB_{p_i}(r)$, where $p_i \in S^\beta$, $\beta > \alpha_0$, $\beta \ne \alpha_1, \ldots, \alpha_m$. Moreover, the corresponding statement holds for ξ_{i_l}. As a consequence, in examining the effect of the proposed modifications on (6.6.1), we can assume that $\alpha_{l'_u} = \alpha_{l_u}$, for some $\alpha_{l_u} \in S^{\alpha_{l_u}}$, $u = 1, 2$ (since otherwise, nothing changes).

Next observe that for $1 \le l_1 < l_2 \le m$, on $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)$, by (6.6.1) we have

$$\begin{aligned}(6.10.1)\qquad (\xi_{i_{l_1}}^{-1} f_{i_{l_1}, i_{l_2}} \xi_{i_{l_2}})(\xi_{i_{l_2}}^{-1} f_{i_{l_2}} \psi_{i_{l_2}}) &= \xi_{i_{l_1}}^{-1} f_{i_{l_1}, i_{l_2}} f_{i_{l_2}} \psi \\ &= \xi_{i_{l_1}}^{-1} f_{i_{l_1}} \psi \\ &= (\xi_{i_{l_1}}^{-1} f_{i_{l_1}} \psi_{i_{l_1}})\end{aligned}$$

(while outside $FB_{p_{i_0}}(r) \cap \cdots \cap FB_{p_{i_m}}(r)$, the maps ψ_{l_u}, ξ_{l_u} are the identity).

Now by construction,

(6.10.2) $f_{i_0,i_1}(\xi_{i_1}^{-1} f_{i_1} \psi_{i_1}) = f_{i_0}$ on $FB_{p_{i_0}}(r - \frac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \frac{1}{2N'(n)})$

(recall ξ_{i_1} is the identity map).

Finally, for $l \ge 2$, by (6.6.1), on $FB_{p_{i_0}}(r - \frac{1}{2N'(n)}) \cap \cdots \cap FB_{p_{i_m}}(r - \frac{1}{2N'(n)})$, we have

$$\begin{aligned}(6.10.3)\qquad f_{i_0,i_l}(\xi_{i_l}^{-1} f_{i_l} \psi_{i_l}) &= f_{i_0,i_1} f_{i_1,i_l} \xi_{i_l} \xi_{i_l}^{-1} f_{i_l} \psi_{i_l} \\ &= f_{i_0,i_1} f_{i_1,i_l} f_{i_l} \psi_{i_l} \\ &= f_{i_0,i_1} f_{i_1} \psi_{i_1} \\ &= f_{i_0}.\end{aligned}$$

Thus, if we redefine the maps f_{i_l}, $1 \le l \le m$ and $f_{i_{l_1}}$, $f_{i_{l_2}}$, $1 \le l_1 < l_2 \le m$ as in (6.7.3), (6.7.4), then the part of the induction hypothesis corresponding to (6.6.1) holds (here, of course, r is replaced by $r - \frac{1}{2N'(m)}$). Moreover, it is straightforward to verify that the redefined maps satisfy the induction hypotheses corresponding to (6.6.2)–(6.6.7). This completes the step $(\alpha_1, \dots, \alpha_m)$.

By taking $\lambda = \lambda(n)$ sufficiently small and each δ_j of our sequence sufficiently small relative to ι_j, relations (6.6) guarantee that when the whole modification process has been completed, the resulting maps will satisfy the conditions of Proposition 6.1. This completes the proof.

Remark 6.11. Examination of the proof of Proposition 6.1 shows why we stated Proposition A2.2 in such a way that a bound on the Hessian of *only one* of the maps in the proposition is required. For example, the Hessian of the map, f_{i_1}, and hence of the map, $f_{i_0, i_1} f_{i_1}$, used in defining ψ, is bounded by a constant times $\iota^{-1}_{j(\alpha_1)}$, rather than by $\iota^{-1}_{j(\alpha_0)}$, as is the case for the map f_{i_0}. Note that $\iota^{-1}_{j(\alpha_1)} >> \iota^{-1}_{j(\alpha_0)}$, if $j(\alpha_0) < j(\alpha_1)$. Also, care had to be taken in choosing the method of redefining the maps f_{i_l}, $f_{i_{l_1}, i_{l_2}}$, in order to ensure that control over Hessians of relevant maps was not lost in the induction process.

Remark 6.12. By §§3, 4, the fibres of our maps, f_s, carry canonical affine structures. However, the inclusions of fibres implied by (6.1.4) (namely, $f_t^{-1}(y_t) \subset f_s^{-1}(y_s)$, where $y_t \in f_{s,t}^{-1}(y_s)$) need not be compatible with these affine structures. Arranging this is the subject of §7. However, if we pretend that it is already the case, then Proposition 6.1 summarizes much of what we aim to accomplish in this paper.

7. Making the local group actions compatible

In this section we complete the construction of the nilpotent Killing structure on the frame bundle.

By §6 we have a mutually compatible system of maps,

$$Z_s \to FB_{p_s}(\tfrac{1}{2}) \xrightarrow{f_s} Y_s^{j_s}, \tag{7.1}$$

such that $M^n = \bigcup_s B_{p_s}(\frac{1}{16})$. As pointed out in the proof of Proposition 6.1, to obtain actual fibrations with compact fibres, we must replace the sets $FB_{p_s}(\frac{1}{2})$ in (7.1) by slightly smaller sets, i.e., the union of all compact fibres intersecting $FB_{p_s}(\frac{1}{2})$. This is to be understood (sometimes without explicit mention) in what follows. We denote by $\mathscr{F}_s$ the fibration corresponding to (7.1).

By §3, each fibre, Z_s, carries a canonical flat affine structure, affine isomorphic to some $(\Lambda_s \backslash N_s, \nabla^{\mathrm{can}})$, and a canonical metric, whose image under such an isomorphism lifts to a left invariant metric on N_s.

In our case, we actually have

$$\Lambda_s \subset (N_s)_L \subset \mathrm{Aff}(N_s, \nabla^{\mathrm{can}}) \tag{7.2}$$

(and not just $\#(\Lambda_s \cap (N_s)_L \setminus \Lambda_s) \leq \omega_n$). This follows from the fact that short closed loops on the frame bundle of an A-regular Riemannian manifold (in this case Z_s) automatically have small holonomy (compare [G, R] and Appendix 1).

If $B_{p_s}(\frac{1}{2}) \cap B_{p_t}(\frac{1}{2})$ is nonempty, then (say) each fibre Z_t of $\mathscr{F}_t$ is contained in some fibre, Z_s of $\mathscr{F}_s$. However, this inclusion need *not* be compatible with the affine structures. We now show that $\mathscr{F}_t$ lies *close to* a unique $O(n)$-equivariant subfibration, $\mathscr{F}_t'$ of $\mathscr{F}_s$, such that the tangent bundle to the fibres, TZ_t', is a totally geodesic sub-bundle of $(TZ_s, \nabla^{\mathrm{can}})$. Given this and an argument that replaces Proposition A2.2, the construction of the nilpotent Killing structure can be completed by arguments like those in §6. Specifically, we will obtain modified fibrations such that on nonempty intersections of their domains, the inclusions of fibres are compatible with affine structures. Then we construct nilpotent Killing structures and invartiant metrics as in §4.

For each fibre, Z_s of $\mathscr{F}_s$, there is a fibration

$$Z_t \to Z_s \longrightarrow W. \tag{7.3}$$

Using the fibration in (7.3) and the affine structure on Z_s, we will construct for each Z_s, a fibration,

$$Z_t' \to Z_s \to W', \tag{7.4}$$

which has totally geodesic fibres and that lies close to the one in (7.3). Then we define the fibration $\mathscr{F}_t'$ to be the one whose fibres are all Z_t' in (7.4) (as Z_s in (7.4) varies).

Let $\hat{Z}_s$ be the universal covering space of Z_s. Although a specific choice of $\hat{Z}_s$ is gotten by choosing a base point, the construction of $\mathscr{F}_t'$ that follows will not depend on the particular choice of base point. Thus, our construction will automatically be $O(n)$-equivariant.

Write $Z_s = \Lambda_s \backslash \hat{Z}_s$ and let $\pi\colon \hat{Z}_s \to Z_s$. The group $\pi_1(Z_t)$ does depend on a choice of base point. However, the existence of the fibration in (7.3) implies that

$$i(\pi_1(Z_t)) := \Lambda_t \subset \Lambda_s, \tag{7.5}$$

the image of $\pi_1(Z_t)$ under the map induced by the inclusion, $Z_t \to Z_s$, is a well-defined *normal* subgroup.

Lemma 7.6. *The map* $\pi_1(Z_t) \xrightarrow{i} \pi_1(Z_s)$ *is an injection.*

Proof. Let Z_s, Z_t be the fibres of f_s, f_t, respectively. Then W is a fibre of $f_{s,t}$. Hence, by (6.1.10), W is almost flat, and, in particular, aspherical. Then, by applying the homotopy sequence for fibrations to the fibration in (7.3), our claim follows.

Let $\hat{\nabla}$ denote the pullback to $\hat{Z}_s$, of the flat affine connection on Z_s. Then

$$(\hat{Z}_s, \hat{\nabla}) \simeq (N_s, \nabla^{\mathrm{can}}), \tag{7.7}$$

where we view N_s as an affine homogeneous space; i.e., we do *not* distinguish a base point. We regard the invariantly defined subgroup $(N_s)_L$ as contained in $\mathrm{Aff}(\hat{Z}_s, \widehat{\nabla})$; compare Remark 4.2. Then

(7.8) $$\Lambda_s \subset (N_s)_L.$$

By Malcev's theorem (3.7), there is a unique simply connected subgroup,

(7.9.1) $(N_t')_L \subset (N_s)_L$,

which contains Λ_t as a cocompact subgroup. Since $\Lambda_t \subset \Lambda_s$ is normal, $(N_t')_L \subset (N_s)_L$ is normal as well. Define $\Lambda_t' \supseteq \Lambda_t$ by

(7.9.2) $\Lambda_t' = \Lambda_s \cap (N_t')_L$.

We will show in Lemma 7.13 that, in fact, $\Lambda_t' = \Lambda_t$.

Let the fibration, $Z_t' \to Z_s \to W'$, in (7.4) be the one whose fibres are the orbits,

(7.9.3) $Z_t' = \Lambda_t' \backslash (N_t')_L(z)$,

where $z \in \widehat{Z}_s$ (compare Remark 3.9). Then the fibration, $\mathscr{F}_t'$, is as specified after (7.4). We now show that this $\mathscr{F}_t'$ is close to $\mathscr{F}_t$ over $B_{p_s}(\frac{1}{2}) \cap B_{p_t}(\frac{1}{2})$.

Note that Lemma 7.6 already implies

(7.10) $$\dim Z_t = \dim Z_t'.$$

By (6.1.8),

(7.11.1) $|II_{Z_t}| \le c(n, A)\iota_{j_t}^{-1}$,

while from (3.6.1), (6.1.5), and (6.1.8) we get

(7.12.2) $|II_{Z_t'}| \le c(n, A)(b(\iota_{j_s})\iota_{j_s}^{-2} + \iota_{j_s}^{-1}) \le 2c(n, A)\iota_{j_s}^{-1}$,

(where $\iota_{j_s}^{-1} < \iota_{j_t}^{-1}$ and j_s, j_t are as in §6).

Now suppose

(7.12.1) $b(u) \le \theta u$,

so that

(7.12.2) $b(\iota_{j_t}) \le \theta \iota_{j_t}$,

where $\theta = \theta(n, A)$ is a small constant. Then we obtain

Lemma 7.13. (1) *There exists* $c^*(n, A) > 0$, *such that the normal injectivity radius of a fibre* Z_t *is bounded below by* $\min(c^*(n, A)\iota_{j_t}, d(Z_t, \partial(\mathrm{dom}\mathscr{F}_t))$.

(2) *If* Z_t, Z_t' *are fibres of* $\mathscr{F}_t$, $\mathscr{F}_t'$ *passing through* z, *then*

(7.13.1) $d(Z_t, Z_t') \le \frac{1}{2}c(n, A)\theta^2(n, A)\iota_{j_t}$.

(3) *For* $\theta(n, A)$ *sufficiently small, if* $d(z, \partial(\mathrm{dom}(\mathscr{F}_t))) > c^*(n, A)\iota_{j_t}$, *then normal projection onto* Z_t *defines a diffeomorphism from* Z_t' *to* Z_t. *In particular,* $\Lambda_t' = \Lambda_t$.

Proof. (1) The estimate on the normal injectivity radius is contained in the proof of Proposition A2.2.

(2) By Proposition 4.6.3 of [BK] there exists $c^*(n, A) > 0$, such that

(7.14.1) $\mathrm{inj\,rad}\, Z_t \ge c^*(n, A)\iota_{j_t}$

induced by the invariant metric constructed in Proposition 4.9. By (4.9.1) the same holds for the metric induced by the given one.

Let $\hat{z} \in \pi^{-1}(z) \subset \widehat{Z}_t$ and let $\widehat{Z}_t$, $\widehat{Z}'_t$ be the components of $\pi^{-1}(Z_t)$, $\pi^{-1}(Z'_t)$ through $\hat{z}_t$. These have in common the $b(i_j)$-dense set, $\Lambda_t(z)$ with $b(u)$ as in (7.12). Additionally, they satisfy the bounds of (7.10), (7.11). It follows easily that the tangent spaces, $(\widehat{Z}_t)_{\hat{z}}$, $(\widehat{Z}'_t)_{\hat{z}}$ satisfy

(7.14.2) $\measuredangle((\widehat{Z}_t)_{\hat{z}}, (\widehat{Z}'_t)_{\hat{z}}) \le c(n, A)\theta$.

Similarly, for each $\hat{q}' \in \widehat{Z}'_t$ there is a unique closest $\hat{q} \in \widehat{Z}_t$, such that

(7.14.3) $d(\hat{q}', \hat{q}) \le c(n, A)\theta^2 \iota_{j_t}$.

For $\theta(n, A)$ sufficiently small, this yields (2).

(3) Finally, the angle between $(\widehat{Z}_t)_{\hat{q}}$ and the parallel translate of $(\widehat{Z}_t)_{\hat{q}'}$ (along, unique minimal geodesic from $\hat{q}'$ to $\hat{q}$) is at most $c(n, A)\theta$. It is now clear that a normal projection to Z_t defines a covering map from Z'_t to Z_t. Since $\Lambda_t \subset \Lambda'_t$, this must be a diffeomorphism.

Remark 7.15. The principle behind (2) above is the following. If two functions agree on an ε-dense set and have derivatives up to order N bounded, then they are close to order ε^{N-1}, their derivatives are close to order ε^{N-2}, etc.

Now suppose that for $\theta = \theta(n, A, i_0)$, $i_0 \ge 1$, in fact,

(7.16.1) $b(u) \le \theta\, u^{1+i_0}$.

Then by Proposition 4.9, we obtain an ($O(n)$-invariant) invariant metric, $(\ ,\ )$ on dom $\mathscr{F}_s$ with the following property. If $\langle\ ,\ \rangle$ denotes the restriction to dom $\mathscr{F}_s$ of our original metric, g, and $\theta(n, A, i_0)$ is sufficiently small, then for $i \le i_0$,

(7.16.2) $|\nabla^i(\langle\ ,\ \rangle - (\ ,\ ))| \le \iota_{j_s}^{i_0 - i}$.

At this point, we can match the affine structures on $\mathscr{F}_t$, $\mathscr{F}'_t$, by adapting to our situation, the center of mass argument of [GrK], used there to prove the stability of compact group actions.

Let $y \in Y^{j_t}$. Put $V = f_t^{-1}(B_y(\frac{1}{2}\iota_{j_t}))$. By shrinking V slightly, we obtain a subdomain $\underline{V} \subset V$ such that if $q \in \underline{V}$, then the fibres of $\mathscr{F}_t$ and $\mathscr{F}'_t$ through q are both contained in V. Let $(\widehat{V}, \hat{q})$ denote the universal covering space of V. Then we have the following preliminary result.

Lemma 7.17. *There exists $c^*(n, A) > 0$ such that for $\hat{q}_1 \in \widehat{V}$,*

(7.17.1) $\operatorname{inj\,rad}_{\hat{q}_1} \ge \min(c^*(n, A)\iota_{j_s}, d(\hat{q}_1, \partial\widehat{V}))$.

Proof. This follows easily from (7.11.1), Lemma 7.13 (1) and (7.14).

The isomorphism, $\Lambda_t \simeq \Lambda'_t$, extends to a canonical isomorphism $(N_t)_L \simeq (N'_t)_L$. Let μ_t, μ'_t denote the actions of $(N_t)_L$, $(N'_t)_L$. Then for all $\lambda \in \Lambda_t \simeq \Lambda'_t$, we have

(7.18.1) $\mu_t(\lambda) = \mu'_t(\lambda)$.

Thus, the map $\mu_t(h)(\mu_t')^{-1}(h)$ is well defined on $(N_t)_L/\Lambda_t \simeq (N_t')_L/\Lambda_t'$. A slight variant of the argument leading to Proposition 4.7 shows that for all h,

(7.18.2) $|\nabla^i(\mu_t(h)(\mu_t')^{-1}(h)) - \text{Ident}| \leq c(n, A, i_0)\iota_{j_s}^{i_0-i}$.

Similarly, by integrating over $(N_t)_L/\Lambda_t \simeq (N_t')_L/\Lambda_2'$ (rather than all of N_t) we can define the center of mass of the map $\mu_2(\mu_2')^{-1}$; compare [GrK] and (4.8). As in [GrK], this yields an $O(n)$-equivariant diffeomorphism, $\hat{\psi}$, such that

(7.19.1) $\hat{\psi}\mu_t = \mu_t'\hat{\psi}$,

(7.19.2) $|\nabla^i\hat{\psi} - \text{Ident}| \leq c(n, A, i_0)\iota_{j_s}^{i_0-i}$.

It follows from (7.17) and (7.19.1) that $\hat{\psi}$ commutes with the action of Λ_t. Thus, the collection of maps, $\hat{\psi}$, obtained by varying $y \in Y^{j_t}$, induces a well-defined embedding,

(7.19.3) $\psi: FB_{p_s}(\frac{1}{2} - \frac{1}{8N'(n)}) \cap FB_{p_t}(\frac{1}{2} - \frac{1}{8N'(n)}) \to FB_{p_s}(\frac{1}{2}) \cap FB_{p_t}(\frac{1}{2})$.

The map ψ sends fibres of $\mathscr{F}_t$ to fibres of $\mathscr{F}_t'$, preserving affine structures and corresponding local actions. Here, the number $N'(n)$ is as in §6. By modifying ψ with the aid of a cutoff function, we obtain an $O(n)$-equivariant map (also denoted ψ) such that

(7.20.1) ψ is a self-diffeomorphism of $FB_{p_s}(\frac{1}{2}) \cap FB_{p_t}(\frac{1}{2})$.

(7.20.2) ψ is the identity near the boundary.

(7.20.3) $d(\psi, \text{Ident}) \leq c(n, A, i_0)\iota_{j_s}^{i_0}$.

(7.20.4) $|\nabla^i\psi - \text{Ident}| = c(n, A, i_0)\iota_{j_s}^{i_0-i}$.

(7.20.5) If $\psi(Z_t) \cap (FB_{p_s}(\frac{1}{2} - \frac{1}{4N'(n)}) \cap FB_{p_t}(\frac{1}{2} - \frac{1}{4N'(n)})) \neq \varnothing$, then for some Z_t', $\psi: Z_t \to Z_t'$, preserving the affine structure.

Let $\underline{\tilde{\mathfrak{n}}}_s$, $\underline{\tilde{\mathfrak{n}}}_t$ denote the sheaves of local right invariant vector fields associated to the affine structures on the fibres of $\psi(\mathscr{F}_t)$, $\mathscr{F}_s$, as in §4. Let $\tilde{\mathfrak{n}}_s$, $\tilde{\mathfrak{n}}_t$ denote the associated sheaves of simply connected nilpotent Lie groups. By arguing as in §4, it is easy to check that on their common domain $\tilde{\mathfrak{n}}_t$ is a subsheaf of $\tilde{\mathfrak{n}}_s$. Then there is an obvious sheaf, $\tilde{\mathfrak{n}}_s \cup \tilde{\mathfrak{n}}_t$, over $FB_{p_s}(\frac{1}{2} - \frac{1}{4N'(n)}) \cap FB_{p_t}(\frac{1}{2} - \frac{1}{4N'(n)})$, whose stalk at points of $FB_{p_s}(\frac{1}{2} - \frac{1}{4N'(n)})$ coincides with $\mathfrak{n}_s$, and elsewhere, coincides with that of $\tilde{\mathfrak{n}}_t$.

Now (assuming that $\theta(n, A, i_0)$ is chosen sufficiently small) as in the proof of Proposition 6.1, we obtain a sheaf, $\tilde{\mathfrak{n}}^*$, defined all of FM^n, on which the natural action of $O(n)$ is trivial (in the sense of Proposition 4.3). This sheaf is associated to a system of (modified) maps, f_s, $f_{s,t}$, where $f_s: FB_{p_s}(\frac{1}{4}) \to Y_s^{j_s}$. These maps satisfy (6.1.4)–(6.1.10) (with the function b as in (7.16.1) and $\lambda = \lambda(n, A)$). Moreover the affine structures on the fibres of the associated fibrations are mutually compatible.

Let $f_s: FB_{p_s}(\frac{1}{16}) \to Y_s^{j_s}$ be obtained by restriction. Denote by $\tilde{\mathfrak{n}}$ the sheaf of FM gotten by applying the above construction to these maps. The sheaf $\tilde{\mathfrak{n}}$ will be shown in §8 to induce the desired structure on M^n. The fact that the domains of its defining system of maps can be enlarged to the sets $FB_{p_s}(\frac{1}{4})$, is required in order to verify Theorems 1.3 and 1.7.

Finally, we observe that an invariant metric close to the original one, can be constructed for our structure. Let S^{α} be as in (6.4.3). Recall that there are $T(n)$ of these sets. Start with those balls with centers in $S^{T(n)}$. Over each such ball, $B_{p_s}(\frac{1}{4})$, construct an ($O(n)$-invariant) invariant metric on $FB_{p_s}(\frac{1}{4})$ as in §4. Using a suitable cutoff function, modify the original metric over each $FB_{p_s}(\frac{1}{4})\backslash FB_{p_s}(\frac{1}{4}-\frac{1}{8T(n)})$, so as to obtain a new metric agreeing with the original one outside the union of the $FB_{p_s}(\frac{1}{4})$ and with the invariant one on the union of the $FB_{p_s}(\frac{1}{4}-\frac{1}{8T(n)})$. By applying this construction successively to $S^{T(n)-1}$, $S^{T(n)-2}$, ... we obtain an $O(n)$-invariant metric that is invariant, for the sheaf associated to the covering $\bigcup_s FB_{p_s}(\frac{1}{4})$ and thus, for the sheaf $\mathfrak{n}$ as well. More precisely we get the following:

Let $\tilde{\mathfrak{n}}$ denote the metric on FM^n induced by our original metric g on M^n.

Proposition 7.21. *Given ε, i_0, there exists $\theta(n, A, i_0, \varepsilon)$ such that by choosing $\theta = \theta(n, A, i_0, \varepsilon)$ in (7.16.1), we obtain a metric $\tilde{g}_E$ that is $O(n)$-invariant and invariant for $\mathfrak{n}^*$, such that for $i \leq i_0$,*

(7.21.1) $|\nabla^i(\tilde{g} - \tilde{g}_{\varepsilon})| \leq \varepsilon$.

It is now routine to verify that the action of the sheaf $\mathfrak{n}$ defines a nilpotent Killing structure for the $(\rho(n, \varepsilon), 1)$-round metric, $\tilde{g}_{\varepsilon}$.

8. The induced structure and metric on the base

In §7 we constructed an $O(n)$-invariant Riemannian metric, $\tilde{g}_{\varepsilon}$, and an associated nilpotent Killing structure, $\widetilde{\mathfrak{N}}$, on the total space, FM^n, of the frame bundle. Here, we construct the corresponding objects, g_{ε}, $\mathfrak{N}$, on the base, M^n, and show that the assertions of Theorem 1.3 and 1.7 hold. The statements of Theorems 1.3 and 1.7 are closely related and it will be convenient to prove them simultaneously. Briefly, we must

(1.3.A) construct the metric, g_{ε}, satisfying (1.3.1)–(1.3.3),
(1.3.B) show that g_{ε} is (ρ, k)-round for suitable ρ, k depending on n, ε ((1.1.1)–(1.1.6)),
(1.7.A) construct the nilpotent Killing structure $\mathfrak{N}$, compatible with g_{ε}, and
(1.7.B) show that $\mathfrak{N}$ has compact orbits of diameter $< \varepsilon$.

Proof of Theorems 1.3 *and* 1.7. (1.3.A) The results of §§5–7 were stated for A-regular Riemannian manifolds, with the sequence, A, normalized in §5. However, the metric, g, in Theorem 1.3 is assumed only to satisfy $|K| \leq 1$. Therefore, given ε as in Theorem 1.3, we begin by replacing g by the metric, $S_{\varepsilon/2}(g)$ of Theorem 1.12. This metric is $A(n, \varepsilon/2)$-regular. Although the sequence, $A(n, \varepsilon/2)$, is not normalized as in §5, by an obvious scaling argument the results of §§5–7 can still be applied to $S_{\varepsilon/2}(g)$.

Fix i_0 (large) and ε'. Starting with the metric $S_{\varepsilon/2}(g)$, construct a metric, $\tilde{g}_{\varepsilon'}$, on FM^n as in Proposition 7.21. Since $\tilde{g}_{\varepsilon'}$ is $O(n)$-invariant, there is a unique metric, $g_{\tau(\varepsilon'|n)}$ on M^n, such that $\pi: FM^n \to M^n$ is a Riemannian submersion (see (1.13.6) for the $\tau(\ )$ notation). Here the notation is understood to be such that if $\tau(\varepsilon'|n) = \varepsilon$, then g_ε satisfies (1.3.1)–(1.3.3).

(1.7.B) The structure, $\widetilde{\mathfrak{N}}$, on FM^n is $O(n)$-equivariant. In particular, the action of $O(n)$ maps orbits to orbits, and hence, induces a partition of M^n into compact submanifolds, $\{\mathcal{O}\}$. These will be seen to be the orbits of the nilpotent Killing structure on M^n. Since $\pi: FM^n \to M^n$, is distance nonincreasing, it follows that for all $\mathcal{O}$, $\mathrm{dia}(\mathcal{O}) < \varepsilon$, provided that the same is true for the orbits of $\widetilde{\mathfrak{N}}$. Clearly, if the constant $\theta = \theta(n, \varepsilon)$ in §7 is chosen sufficiently small, this will be the case.

(1.7.A) For the remainder of this section we will use a tilde to indicate that a point lies in the frame bundle. Let $p \in M^n$, $\tilde{p} \in \pi^{-1}(p)$ and let $Z_{\tilde{p}}$ denote the orbit of $\widetilde{\mathfrak{N}}$ through $\tilde{p}$. Choose $\eta > 0$ so small that

(8.1.1) $B_{\tilde{p}}(\eta)$ is simply connected,
(8.1.2) $B_{\tilde{p}}(\eta) \cap O(n)(B_{\tilde{p}}(\eta))$ is connected,
(8.1.3) the restriction of $\tilde{\mathfrak{n}}$ to the η-tubular neighborhood, $T_\eta(Z_{\tilde{p}})$, of the fibre through $\tilde{p}$ is pure,
(8.1.4) the normal injectivity radius to the orbit $\mathcal{O}_p$ is $\geq \eta$.

The space of local sections, $\underline{\tilde{\mathfrak{n}}}(B_{\tilde{p}}(\eta))$, is the nilpotent Lie algebra of local right invariant fields. In view of (8.1.1), (8.1.2), it follows from Proposition 4.3 that each local vector field in $\underline{\tilde{\mathfrak{n}}}(B_{\tilde{p}}(\eta))$ is π-related to some vector field on $\pi(B_{\tilde{p}}(\eta)) = B_p(\eta)$. Thus we get a nilpotent Lie algebra of local Killing fields on $B_p(\eta)$, which, by Proposition 4.3, is independent of the choice of $\tilde{p} \in \pi^{-1}(p)$.

In a standard way, the collection of Lie algebras on the various $B_p(\eta)$ determines a sheaf of nilpotent Lie algebras of Killing fields on M^n. Let $\mathfrak{n}$ be the associated sheaf of simply connected Lie groups and h its natural action. Obviously the orbits of this action are those considered in (1.7.B).

We now show that $\mathfrak{n}$ defines a nilpotent Killing structure.

For $\tilde{p}$, η as above, we put $U = T_\eta(\mathcal{O}_p)$, where U is as in Definition 1.5. Since it is clear that we can choose the same η for all points on $\mathcal{O}_p$, it follows that (1.5.3) holds.

Recall that the local fibration with fibre, $Z_{\tilde{p}}$, is the restriction of a fibration defined on an open set containing $T_{1/8}(Z_{\tilde{p}})$. (This fibration comes from $\widetilde{\mathfrak{N}}^*$; see the end of §7). Let $\mathfrak{N}'$ denote the corresponding pure nilpotent Killing structure on $T_{1/8}(\mathcal{O}_p)$. By (8.1.3), $\mathfrak{N}'$ extends $\mathfrak{N}|U$. An orbit of $\mathfrak{N}'$ will be denoted by $\mathcal{O}'$.

For the construction of the neighborhood, V, appearing in (1.5.1), (1.5.2) and in (1.3.A), we need ($\mathfrak{N}'$ and) Lemma 8.5 below, the statement of which requires some terminology.

Let W be a length space (see [GLP]).

Definition 8.2. $B_{\underline{q}}(r) \subset W$ is *star shaped* if for all $w \in B_{\underline{q}}(r)$, there is a unique geodesic in $B_{\underline{q}}(r)$ joining w to $\underline{q}$.

Let $\underline{q} \in W^l = Y^m/G$, where Y^m is a Riemannian manifold and G is a compact group of isometries of W. Put

$$Ang_{\underline{q}} W = \frac{\mathrm{Vol}(C_{\underline{q}} W)}{\mathrm{Vol}(S^{l-1})}, \tag{8.3}$$

where $C_{\underline{q}} W$ is the set of unit vectors in the tangent cone at $\underline{q} \in W$.

Definition 8.4. $W = Y/G$ has (ρ, k)*-bounded geometry* at $w \in W$ if there exists $\underline{q} \in W$ such that

(8.4.1) $B_w(\rho) \subset B_{\underline{q}}(R)$ for some star shaped $B_{\underline{q}}(R)$.
(8.4.2) $Ang_{\underline{q}} W \geq k$.

Let $\mathfrak{CF}(r, A)$ be the Hausdorff closure of the collection of $FB_p(r)$, such that the ball, $B_p(r)$, is contained in an A-regular manifold; compare §5. Here we do not assume that A is normalized. Let $\mathfrak{CF}_m(r, A)$ be defined as in §5.

Lemma 8.5. *For all $\delta > 0$ there exists $\rho = \rho(r, A, \delta)$ and $k = k(r, A, \delta)$ such that if $Y \in \mathfrak{CF}_{m+1}(r, A)$, satisfies*

(8.5.1) $d_H(Y, \mathfrak{CF}_m(r, A)) \geq \delta$,

then $Y/O(n)$ has (ρ, k)-bounded geometry.

Proof. This is essentially a restatement of Theorem 0.14 of [F3] (compare also (4.6.2)). It follows from Theorem 10.1 of [F3].

Let FM^n carry the metric $\tilde{g}_{\tau(\varepsilon'|n)}$ $(\varepsilon' = \varepsilon'(\varepsilon, n))$ inducing the metric g_ε on M^n.

Fix $p \in M^n$. For some $Y_s^{j_s}$ as in §7, we can identify $Y_s^{j_s}/O(n)$ with the orbit space of the action of $\mathfrak{n}'$. Let $w \in Y_s/O(n)$ be the projection of p. Let q project to $\underline{q} \in Y_s/O(n)$ as in (8.4.1).

Remark 8.6. By way of explanation, we mention that the orbit, $\mathscr{O}'_q$, should be thought of as one that has minimal dimension among all orbits in $T_R(\mathscr{O}'_q)$.

It is an easy consequence of Lemma 8.5 and (8.4.1) that $\mathscr{O}'_q$ has normal injectivity radius $\geq R = R(n, \varepsilon)$. For future reference, we note that since M^n is $A(n, \varepsilon)$-regular, it follows that the second fundamental form, $II_{\mathscr{O}'_q}$, satisfies

$$|II_{\mathscr{O}'_q}| \leq c(n, \varepsilon). \tag{8.7}$$

Put $V = T_R(\mathscr{O}'_q)$, where V is the neighborhood occurring in (1.5.1), (1.5.2), and (1.1.1)–(1.1.6).

Since V is a union of orbits and $\mathfrak{N}'$ is pure, the action of $\mathfrak{n}'$ on V lifts to the action of the simply connected nilpotent Lie group, $\beta^*(\mathfrak{n}')(\widehat{V})$, on the universal covering space, $\widehat{V} \xrightarrow{\beta} V$.

Let K be the kernel of the action of $\beta^*(\mathfrak{n}')(\widehat{V})$ and put

$$N_0 = \beta^*(\mathfrak{n}')(V)/K. \tag{8.8}$$

Apart from the assertion that K is discrete (the proof of which is given in Appendix 1) it is now clear that (1.5.1), (1.5.2) hold. Modulo the proof of (1.3.B), this completes the proof of (1.7.A).

(1.3.B) Let $\tilde{q} \in \pi^{-1}(q)$. The action of $\tilde{\mathfrak{n}}'$ on $T_R(Z'_q)$ lifts to the action of a simply connected Lie group, $\underline{N}_0$, on the universal covering space, $T_R(\widehat{Z}'_{\tilde{q}})$. Moreover, it follows easily from Proposition 4.3 that the natural map, $\tau\colon T_R(\widehat{Z}'_{\tilde{q}}) \to T_R(\mathscr{O}'_q)$, intertwines the actions of $\underline{N}_0$ and N_0. Since $\pi_1(Z'_{\tilde{q}}) = \Lambda \subset \underline{N}_0$, we have $\tau(\Lambda) \subset N_0$. Also, by Proposition 4.3 and Lemma 8.5, $Z_{\tilde{q}}$ fibres (with a torus as fibre) over a finite covering of $\mathscr{O}_q$, of index at most k. Then the homotopy sequence for fibrations, implies that $\tau(\Lambda)$ has index at most k in $\pi_1(\mathscr{O}'_q) = \pi_1(V)$.

Let H of (1.1.1)–(1.1.6) be the Lie group generated by $\pi_1(V)$ and N_0. Now (1.1.1)–(1.1.4) and (1.1.6) are obvious.

To see (1.1.5), first note that the normal projection, $\psi\colon \widetilde{V} \to \widetilde{\mathscr{O}}'_q$, is well defined and by (8.7) increases distances by a factor of at most $c = c(n, \varepsilon)$. If γ is a geodesic loop on $\hat{q} \in \widehat{\mathscr{O}}'_q$, then γ is homotopic to $\underline{\gamma} \subset \widehat{\mathscr{O}}'_q$ over curves of length at most $c \cdot L[\gamma]$. But since $\widehat{\mathscr{O}}'_q$ is isometric to a simply connected nilpotent Lie group with left invariant metric, any closed curve in $\widehat{\mathscr{O}}'_q$ is contractible to a point over curves of shorter length; see Proposition 4.6.3 of [BK]. By Klingenberg's lemma on lifting homotpies inside the conjugate locus, we get

$$L[\gamma] \geq c(n, \varepsilon)\pi . \tag{8.9}$$

This suffices to complete the proof of (1.3.B).

Remark 8.10. The orbit space $Y/O(n)$ coincides locally with the orbit space of our structure $\mathfrak{N}$. In this connection, Lemma 8.5 expresses the fact that the orbits of our structure absorb all collapsed directions.

The following examples show that in general, the numers ρ, k of Theorem 1.3 cannot be chosen independent of ε.

Example 8.11. We consider certain metrics on a nilmanifold M^3, viewed as the total space of a fibration, $S^1 \to M^3 \xrightarrow{\pi} T^2$ (with nonzero Euler class). We assume that all fibres have length δ', that π is a Riemannian submersion and that the metric on the base is chosen as follows. Start with a metric, g, which is the product of two short circles of length δ. Deform g slightly by introducing a small bump centered at p such that the new metric, g', satisfies, $|K_{g'}| \leq 1$, $K_{g'}(p) = 1$. Then the isometry group of R^2 equipped with the pullback metric, $\hat{g}'$, is discrete. Moreover, for some $\eta > 0$, any metric that is η-quasi-isometric to g' also has this property. Thus, for the corresponding metric on M^3, the isometry group of $\widetilde{M}^3$, the universal covering of M^3, contains the skew product, $R \times \mathbb{Z}^2$ as a subgroup of finite index. Here, the center, R, acts by translation in the direction of the universal coverings of the S^1 fibres.

Now consider a sequence of such manifolds where $\delta' \to 0$, while δ stays fixed, $\varepsilon = \varepsilon(\delta)$ is as in Theorem 1.3 and $\varepsilon << \eta$. Suppose we assume that for this sequence and $p \in M^3$, we can find ρ, k as in Theorem 1.3 with $\rho > \operatorname{dia}(T^2, g')$. Then neighborhood V of (1.1) must be $\pi^{-1}(T^2) = M^3$, and it is easy to check that for any group H satisfying (1.1.1)–(1.1.6) (the definiton of (ρ, k)-round)) we must have $k \to \infty$ as $\delta' \to 0$. Thus for (ρ, k) as in Theorem 1.3, we must have $\rho \le \operatorname{dia}(T^2, g') < 3\delta$, as soon as δ' is sufficiently small.

Finally, let $\delta \to 0$, $\delta'/\delta \to 0$ sufficiently fast, and choose $\varepsilon = \varepsilon(\delta) << \eta(\delta)$ as above. Then for such a sequence if ρ, k are as in Theorem 1.3, it follows that $\rho \to 0$.

On the other hand, if for each such manifold, we take $\rho = \operatorname{inj\,rad}_p$, $V = \pi^{-1}(B_p(\rho))$, and $\widetilde{V}$ the universal covering of V, we find that we can choose k as in Theorem 1.3 equal to 1 (and $H = N = R$).

Example 8.12. Start with flat R^2 and introduce k mutually isometric tiny bumps as above centered at points with polar coordinates $(r, \frac{2\pi j}{k})$, where $j = 0, \dots, k-1$ and $r > 0$ is a small fixed number. Then $\mathbb{Z}_k$ acts isometrically on R^2 by rotation about the origin. Moreover, for some $\eta > 0$, any metric on R^2 that is η-quasi-isometric to the given one has at most k orientation preserving isometries.

Form $R \times R^2$, with the product metric, where the metric on R^2 is the one just described. Let $(t, \Theta_{2\pi/k})$ denote the isometry of R^3 that acts by translation by t units in the R factor and by rotation through an angle, $2\pi/k$, about the origin in R^2. Let Λ be the group of isometries generated by this transformation, and put $M^3 = \Lambda \setminus R^3$.

The image of the axis $(x, 0, 0)$ is a circle, $S_t^1 \subset M^3$, of length t. Fix $\varepsilon < \eta$ and take sufficiently small such that in particular, $t \cdot k < \varepsilon$. Then the group N_0 will be the 1-parameter group $s \to (st, \text{Ident})$. (Note that apart from S_t^1, the orbits in M^3 have length kt.) In this case, $\#(\Lambda/\Lambda \cap N_0) = k$ and k can be taken arbitrarily large.

By taking products of the above manifolds with the ones in Example 8.11, we get examples for which necessarily, $\rho \to 0$, and $k \to \infty$ as $\varepsilon \to 0$.

Appendix 1. Local structure of manifolds of bounded curvature

The proof of Theorems 1.3 and 1.7 given in §8 has as a consequence that every point $p \in M^n$ is contained in a neighborhood of the form $T_R(\mathscr{O}_q')$, for the metric g_ε. We now give a more explicit desription of the metric structure of $T_R(\mathscr{O}_q')$.

First of all, examination of the proof of Theorem 10.1 of [F3] shows that given R_2 there exists $R_1(n, \varepsilon, R_2)$ such that we can choose $R_1(n, \varepsilon, R_2) \le R \le R_2$, provided we take $\rho = \rho(n, \varepsilon, R_2)$ sufficiently small and $k = k(n, \varepsilon, R_2)$ sufficiently large.

For $R_2 = R_2(n, \varepsilon)$ sufficiently small, we can replace $g_\varepsilon | T_R(\mathscr{O}_q')$ by the natural metric on the tube of radius R in the normal bundle, $\nu(\mathscr{O}_q')$, and obtain a metric that satisfies (1.3.1)–(1.3.3) (with ε replaced by 2ε). Here, we identify

$T_R(\mathscr{O}'_q)$ and this tube via the normal exponential map. Relations (1.3.1)-(1.3.3) continue to hold for the new metric (and 2ε), provided we take

(A1.1.1) $\rho = \rho(n, \varepsilon, R_2(n, \varepsilon))$

(A1.1.2) $k = k(n, \varepsilon, R_2(n, \varepsilon))$,

(which still depend only on n, ε). In this way, we obtain a more canonical local model for the geometry.

The bundle $\nu(\mathscr{O}'_q)$ can be described up to isometry as follows.

It follows easily from (4.1) that there is a principal bundle,

(A1.2) $$C' \to Z_{\tilde{q}} \to \mathscr{O}'_q .$$

Here $Z_{\tilde{q}}$ is isometric to the almost flat manifold $\Lambda \backslash N$, with some left invariant metric, and $\Lambda \subset N_L$. The group $C' \subset O(n)$, is the isotropy group of $Z_{\tilde{q}}$. It satisfies

(A1.3) $$e \to (\Lambda \cap C) \backslash C \to C' \to B \to e ,$$

where C is contained in the center of $\mathfrak{n}$, $(\Lambda \cap C) \backslash C$ is a torus and B is a finite group of order $\leq k$.

Since the local fibration, $Z \to FM \xrightarrow{f_s} Y_s$ is a Riemannian submersion for the metric, $\tilde{g}_\varepsilon$, the normal bundle to any fibre has a natural isometric trivialization, $\nu(Z_{\tilde{q}}) = Z_q^{n-j} \times R^j$. The action of $O(n)$ preserves the fibration f_s. Thus, the action of C on $\nu(Z_{\tilde{q}})$ is of the form

(A1.4) $$g(x, y) = (gx, \phi(y)) ,$$

where gx denotes the natural action of C on $Z_{\tilde{q}}$ and $\phi: C \to O(j)$ is an orthogonal representation of C. The representation ϕ preserves the splitting, $R^j = R^m \oplus R^{j-m}$, where R^m is the subspace orthogonal to *both* the fibre, $Z_{\tilde{q}}$, *and* the fibres of the princpal bundle (i.e., $O(n)$ orbits). The following proposition is then obvious.

Proposition A1.5. *The bundle $\nu(\mathscr{O}'_q)$, with its natural metric, is isometric to the vector bundle (with its natural metric) associated to the principal fibration,* (A1.2), *via the representation $\phi: C' \to O(m)$.*

We now give a relation between the algebraic structure of the stalk, $\mathfrak{n}_p$, of our N-structure and the volume of the ball, $B_p(\rho)$.

Given a Lie algebra $\mathfrak{n}$, let $\mathfrak{n} = \mathfrak{n}_0 \supset [\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}_1 \supset \cdots$ be its lower central series. Put

(A1.6) $$d(\mathfrak{n}) = \sum \dim \mathfrak{n}_k .$$

Theorem A1.7. *There exists a constant $C = C(n, \rho, k)$ such that, if $\mathfrak{n}$ is an N-structure on M and g is an $\mathfrak{n}$-invariant (ρ, k)-round metric as in Theorem* 1.7, *then*

(A1.7.1) $\operatorname{Vol}(B_p(\rho)) \leq C\,\varepsilon^{d(\mathfrak{n}_p)}$.

Corollary A1.8. *For each C there exists ε_0 such that if in the situation of Theorem* A1.7,

(A1.8.1) $\operatorname{Vol}(B_p(\rho)) \geq C\,\varepsilon^{\dim \mathfrak{n}_p}$,

for $\varepsilon < \varepsilon_0$, then $\mathfrak{n}_p$ is Abelian.

Remark A1.9.

(A1.9.1) $\quad \mathrm{Vol}(B_p(\rho)) \geq C\,\varepsilon^{d(\mathfrak{n}_p)}$,

does not hold in general. The flat metric $\varepsilon^2\,dx^2 + \varepsilon^4\,dy^2$ on T^2 is a counterexample.

Remark A1.10. We say that $\mathfrak{n}$ is *filtered* if there exists $\mathfrak{n}_{(i)} \subset \mathfrak{n}$ such that $[\mathfrak{n}_{(i)}, \mathfrak{n}_{(j)}] \subset \mathfrak{n}_{(i+j)}$. Put $d'(\mathfrak{n}) = \sum \dim \mathfrak{n}_{(i)}$. Then there exists a locally homogeneous metric on $M = \Lambda \setminus N$ such that

(A1.10.1) $\quad \mathrm{Vol}(B_p(\rho)) \sim C\,\varepsilon^{d'(\mathfrak{n}_p)}$.

Proof of Theorem A1.7. We replace $B_p(\rho)$ by a normal covering space $\widetilde{FB}_p(\rho)$ of $FB_p(\rho)$, of order $\leq k$. Then, in view of the local description of the metric given at the beginning of this appendix, it suffices to show the following

Lemma A1.11. *There exist ε_n and C_n such that if $\varepsilon < \varepsilon_n$ then the following holds: Let $(\Lambda \setminus N, \overline{g})$ be a compact n-dimensional nilmanifold equipped with a locally homogeneous metric. Suppose*

(A1.11.1) $\quad \mathrm{dia}(\Lambda \setminus N, \overline{g}) < \varepsilon$,
(A1.11.2) $\quad |K_{(\Lambda\setminus N, \overline{g})}| < 1$;

then we have

(A1.11.3) $\quad \mathrm{Vol}(\Lambda \setminus N) < C_n\,\varepsilon^{d(n)}$,

where $\mathfrak{n}$ is the Lie algebra of N.

Proof. Let $\Lambda_1 = [\Lambda, \Lambda] \supset \cdots \supset \Lambda_{k+1} = [\Lambda, \Lambda_k] \cdots$. By [BK, 2.4.2], there exists C such that Λ_k is generated by homotopy classes of loops whose length is smaller than ε^k.

Let $N_0 \supset \cdots \supset N_k \cdots$ be the lower central series of N. Suppose $N_k \neq 1$, $N_{k+1} = 1$. Since $\Lambda_k \setminus N_k$ is flat and since its fundamental group is generated by loops whose length is smaller than ε^k, it follows that

(A1.12.1) $\quad \mathrm{Vol}(\Lambda_k \setminus N_k) < C\,\varepsilon^{k \dim N_k}$.

Similarly we have

(A1.12.2) $\quad \mathrm{Vol}(\Lambda_{k-1} \setminus N_{k-1}/N_k) < C\,\varepsilon^{(k-1)(\dim N_{k-1} - \dim N_k)}$.

Therefore

(A1.12.3) $\quad \mathrm{Vol}(\Lambda_{k-1} \setminus N_{k-1}) < C\,\varepsilon^{\dim N_k + (k-1)\dim N_{k-1}}$.

Inductively we have

(A1.12.4) $\quad \mathrm{Vol}(\Lambda_{k-i} \setminus N_{k-i}) < C\,\varepsilon^{\Sigma_{j=0}^{i-1} \dim N_{k-j} + (k-i)\dim N_{k-i}}$.

The lemma follows immediately.

Remark A1.13. The construction in [FY, §7] is closely related to the lemma.

Lastly, we return to the point in the proof of Theorem 1.7 whose proof was deferred. Namely, we show that the group K in (8.8) is discrete. For this it suffices to show the following. Let M^n, ε be as in Theorem 1.7. Let $p \in M^n$, $\tilde{p} \in \pi^{-1}(p)$ and let $Z_{\tilde{p}}$ be the corresponding orbit of $\tilde{\mathfrak{n}}$.

Proposition A1.14. *If $\varepsilon < \varepsilon(n)$ is sufficiently small, then for all $p \in M^n$ and $\delta > 0$, there exists $p_1 \in M^n$ such that $d(p, p_1) < \delta$ and* $\dim \mathscr{O}_{p_1} = \dim Z_{\tilde{p}}$.

Proof. The identity component, C, of $C' \subset O(n)$, the isotropy group of $Z_{\tilde{p}}$, is a torus whose Lie algebra lies in the center of $\tilde{\mathfrak{n}}$. If $\dim C = m$, then

$$\dim Z_{\tilde{p}} - m = \dim \mathscr{O}_p \,. \tag{A1.15}$$

The torus C acts on a covering space $\widehat{T}_\eta(\mathscr{O}_p)$, of order at most k, by rotation in the fibres normal to the lifted orbit, $\widehat{\mathscr{O}}_p$. If this action is effective, its principal orbits have dimension m. It follows easily that if p_1 lies on a principal orbit sufficiently close to p, then

$$\dim(\mathscr{O}_{p_1}) = \dim Z_{\tilde{p}} \,. \tag{A1.16}$$

To see that the action of C (or equivalently, that of $\mathfrak{n}$) is effective, recall the following standard fact.

Lemma A1.17. *There exists $c_1(n, \varepsilon)$, $c_2(n, \varepsilon)$ such that the map of pseudo-groups, $\pi_* : \pi_1(FM, \tilde{p}, c_1) \to \pi_1(M, p, c_2)$, is an injection.*

As a consequence of Lemma A1.17, the *unique* inverse to the map π_* is gotten by lifting elements of $\pi_1(M, p, c_2)$ via their differentials, to obtain elements of $\pi_1(FM, \tilde{p}, c_1)$.

Put $\pi_*^{-1}(h) = \tilde{h}$. Let $d(\ ,\ )$ denote the uniform norm. Clearly, there exists $c(n) > 0$, such that for all $\tilde{h} \in \pi_1(FM, \tilde{p}, c_1)$,

$$d(\tilde{h}, \text{Ident}) \le c(n)\, d(h, \text{Ident}). \tag{A1.18}$$

(Recall that a local isometry that fixes a point and a frame at that point is the identity map.)

Now identify $\tilde{\mathfrak{n}}_{\tilde{p}}$ with the universal covering, $\widehat{Z}_{\tilde{p}}$, of $Z_{\tilde{p}}$. Then we can regard $\pi_1(FM, \tilde{p}, c_1) \subset \Lambda$. Since Λ is $b(i_{j_s})$-dense in $\widehat{Z}_{\tilde{p}}$, it follows with the help of (4.7) that an estimate like (A1.18) holds for all elements of $\tilde{\mathfrak{n}}_p$ (possibly with a different constant $c(n)$). This suffices to complete the proof.

Remark A1.19. In case M^n itself is almost flat, the statement of Proposition A1.14 yields a first main step in the proof of the theorem on almost flat manifolds; i.e., short loops with not too big holonomy have holonomy at most comparable to their lengths; compare [Gh2]. This argument might appear to be circular since it depends on constructing a fibration of the frame bundle with nilmanifold fibres. However, in this case the first step of the almost flat theorem is trivial since a short loop in the frame bundle automatically has holonomy at most of size comparable to its length.

Appendix 2. Fibration isotopy

In this appendix we give a version of a well-known result to the effect that if two fibrations are sufficiently C^1-close, then one of them can be deformed onto the other. For the applications in §6 and §7, it is important that the required degree of closeness is independent of the injectivity radius of the total space and of the Hessian of one of the projection maps.

Let X, Y be A-regular Riemannian manifolds with $A_0 = 1$, on which a compact group, G, acts by isometries. Assume that for all $y \in Y$,

(A2.1) $\quad \operatorname{inj}\operatorname{rad}_y \geq \min(\iota, d(y, \partial Y))$.

(Here and below we use the notation of (2.2), (2.3).)

Proposition A2.2. *Let $f, g : X \to Y$ be G-equivariant maps for which the following hold*:

(A2.2.1) *f is a 1-almost Riemannian submersion.*
(A2.2.2) *f is $\{B_i \iota^{1-i}\}$-regular.*
(A2.2.3) *g is C-regular.*
(A2.2.4) *For some $\eta > 0$, f, g are ε-close in the C^1-topology, with $\varepsilon(\iota\eta)^{-1} < \beta_0(B_1, B_2)$, sufficiently small.*

Then there exists a G-equivariant self-diffeomorphism, ψ, of X, such that

(A2.2.5) *$f = g\psi$ on X_η.*
(A2.2.6) *ψ is the identity near ∂X.*
(A2.2.7) *ψ is $c(n, B_0, B_1)\varepsilon$ close to the identity map in the C^1-topology.*
(A2.2.8) *ψ is $\{D_i(A, B, C)(\eta\iota)^{1-i}\}$-regular. Moreover, D_i depends only on B_k, C_k, $k \leq i$ (and finitely many A_i).*

Proof. By scaling the metrics on X and Y we can assume $\iota, \eta \geq 1$.
By a standard computation, for all $y \in Y$,

(A2.3.1) $\quad |II_{f^{-1}(y)}| \leq c_1(B_2)$.

Moreover, for $y \in Y_1$, the normal injectivity radius, l, of $f^{-1}(y)$, satisfies

(A2.3.2) $\quad l \geq c_2(B_2)$.

Indeed, the normal exponential map is nonsingular on a tube of radius $c_3(B_2)$. Thus, by a standard argument, if $l < c_2(B_2)$ there exists a geodesic segment, γ, of length $2l$, with $\gamma(0)$, $\gamma(2l) \in f^{-1}(y)$ and $\gamma'(0)$, $\gamma'(2l)$ normal to $f^{-1}(y)$. For l sufficiently small relative to c_1, $\gamma'(t)$ is almost normal to the fibre through $\gamma(t)$, for all $0 \leq t \leq 2l$. It follows that $f(\gamma)$ is a short loop on y of small geodesic curvature. For l sufficiently small this contradicts $\operatorname{inj}\operatorname{rad}_y \geq 1$.

Note that all fibres $f^{-1}(y)$, $g^{-1}(y)$ are $c\varepsilon$-close, together with their tangent planes. For ε-sufficiently small, normal projection from $g^{-1}(y)$ to $f^{-1}(y)$ is a covering map. Moreover an argument like that of the previous paragraph shows that this map is actually a diffeomorphism. The collection of all such maps defines a G-equivariant map $\hat{\psi}$ with domain say, $X_{1/2}$, satisfying (A2.2.5). Let χ be a function such that

(A2.4.1) $\chi \mid X_1 \equiv 1$,
(A2.4.2) $\chi \mid X \setminus X_{1/2} \equiv 0$,

constructed from a G-equivariant smoothing of the distance function as in §2. For ε, β_0 as in (A2.2.4), sufficiently small, put

(A2.4.3) $\psi = \chi \hat{\psi} + (1 - \chi)\,\mathrm{Ident}.$

Then, ψ is easily seen to satisfy (A2.2.6)–(A2.2.8).

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RICCI CURVATURE BOUNDS AND EINSTEIN METRICS ON COMPACT MANIFOLDS

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1. INTRODUCTION

Let M_0 and M_1 be compact Riemannian manifolds. In [20], Gromov introduced the notion of Lipschitz distance between M_0 and M_1, defined by

$$d_L(M_0, M_1) = \inf_f [|\log \operatorname{dil} f| + |\log \operatorname{dil} f^{-1}|],$$

where $f\colon M_0 \to M_1$ is a homeomorphism and $\operatorname{dil} f$ is the dilatation of f given by $\operatorname{dil} f = \sup_{x_1 \neq x_2} \operatorname{dist}(f(x_1), f(x_2))/\operatorname{dist}(x_1, x_2)$. If M_0 and M_1 are not homeomorphic, define $d_L(M_0, M_1) = +\infty$. Gromov [20] proves the remarkable result that the space of compact Riemannian manifolds $\mathscr{M}(\Lambda, \delta, D)$ of sectional curvature $|K| \leq \Lambda$, injectivity radius $i_M \geq \delta > 0$, and diameter $d_M \leq D$, is $C^{1,1}$ compact with respect to the Lipschitz topology. By $C^{1,1}$ compact we mean that any sequence in $\mathscr{M}(\Lambda, \delta, D)$ has a subsequence which converges, in the Lipschitz topology, to a C^∞ manifold M with C^0 Riemannian metric and $C^{1,1}$ distance function $\rho\colon M \times M \to \mathbf{R}$. Related but different proofs of this result obtaining a limit $C^{1,\alpha}$, $\alpha < 1$, Riemannian metric on M appear in [19, 25]. A number of applications of the Gromov compactness theorem have now been obtained, for example in [4, 25]. For an interesting discussion of this result in the context of more general studies, we refer to [30].

An important antecedent of Gromov's compactness theorem is Cheeger's finiteness theorem [8] that the set $\mathscr{M}(\Lambda, v, D)$ of compact Riemannian manifolds of curvature $|K| \leq \Lambda$, volume $V_M \geq v$, and diameter $d_M \leq D$, has only finitely many diffeomorphism types (cf. also [31]). A basic step in this theorem is a lower bound estimate for the injectivity radius $i_M \geq c(|K|, d_M, V_M^{-1})$. In particular, Gromov's compactness theorem may be strengthened to the statement that $\mathscr{M}(\Lambda, v, D)$ is $C^{1,1}$ compact in the Lipschitz topology.

In this paper, we study the question of Lipschitz convergence of compact Riemannian manifolds with bounds imposed on the Ricci curvature Ric in

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place of bounds on the sectional curvature. Of course, this is interesting only if $\dim M \geq 4$.

Besides its intrinsic interest, one motivation for this is the study of the existence and the moduli space of Einstein metrics on a compact manifold M. Recall that Einstein metrics are exactly critical points of the total scalar curvature functional $\mathscr{R}: \mathscr{M}_1/\mathscr{D} \to \mathbf{R}$,

$$\mathscr{R} = \int_M \tau_g \, dv_g,$$

where $\mathscr{M}_1$ is the space of Riemannian metrics of volume 1 on M, $\mathscr{D}$ is the diffeomorphism group, acting on $\mathscr{M}_1$ by pullback, and τ_g is the scalar curvature. $\mathscr{R}$ may be viewed as a smooth function on $\mathscr{M}_1$, and one has [5]

$$d\mathscr{R}_g = \mathrm{Ric}_g - (\tau_g/n)g.$$

The existence of critical points of a functional can often (but certainly not always) be deduced if the functional satisfies a Morse or Lusternik-Schnirrelman theory. Crucial for this is a condition such as Condition C of Palais-Smale; namely, in this case, if g_i is a sequence of metrics such that $|\mathscr{R}(g_i)| \leq K$ and $\|d\mathscr{R}(g_i)\| \to 0$ as $i \to \infty$, then a subsequence converges to a critical point of $\mathscr{R}$. Here the norm and topology may be given by a suitable Sobolev or sup norm on $\mathscr{M}_1$. This condition then involves bounds on the Ricci tensor, but not on the sectional curvature. Similarly, on the moduli space of Einstein metrics on a given manifold, one has no a priori bounds on the sectional curvature.

Another motivation is the recent result of L. Gao [14] that the space of $+1$ Einstein metrics on a 4-manifold M is compact in the C^∞ topology. This result, the first of its kind on Einstein metrics (known to the author), shows that one can control the geometry of M in terms of the injectivity radius.

Our first result in dimension 4 is as follows. Let l_M be the length of the shortest inessential (i.e., null-homotopic) nontrivial geodesic loop on the Riemannian manifold M. (If no such loop exists, define l_M to be $+\infty$.)

Theorem A. *The space of compact* 4-*dimensional Riemannian manifolds* M *such that*

(i) $|\mathrm{Ric}| \leq c_1$,
(ii) $\sup_{r \leq \varepsilon} r^{2(1-4/q)}[\int_{B(r)} |D^2 \mathrm{Ric}|^{q/2}]^{2/q} \leq c_2(\varepsilon, q)$, *for some* $q > 4$, $\varepsilon > 0$,
(iii) $l_M \geq c_3$,
(iv) $\mathrm{vol}_M \geq c_4$,
(v) $\mathrm{diam}_M \leq c_5$,
(vi) $\dim H_2(M, \mathbf{R}) \leq c_6$

is $C^{1,\alpha}$ *compact in the Lipschitz topology.*

Here in (ii), $D^2 \mathrm{Ric}$ is the second covariant derivative of the Ricci tensor. The bounds (i) and (ii) are implied by a bound on the Holder space norm $\|\nabla \mathscr{R}\|_{3,\alpha}$ of the gradient of the total scalar curvature. We note that the bound

(iii) may be replaced by other bounds. For instance, it may be replaced by $\mathscr{A}_M = \inf\{\text{area } \Sigma \colon [\Sigma] \neq 0 \in H_2(M, \mathbf{Z})/\text{Torsion}\} \geq a > 0$, cf. §6. If $H_2(M, \mathbf{Z})$ is torsion, this condition becomes vacuous. There is an analogous result in dimensions > 4 when the bound on the second Betti number $b_2(M)$ is replaced by a bound on the curvature integral $\int_M |R|^{n/2}$, where R is the curvature tensor of M.

Theorem A′. (I). *The space of compact n-dimensional Riemannian manifolds M such that*

(i) $|\mathrm{Ric}| \leq c_1$,
(ii) $\sup_{r \leq \varepsilon} r^{2(1-n/q)} [\int_{B(r)} |D^2 \mathrm{Ric}|^{q/2}]^{2/q} \leq c_2(\varepsilon, q)$, *for some* $q > n$, $\varepsilon > 0$,
(iii) $l_M \geq c_3$,
(iv) $\mathrm{vol}_M \geq c_4$,
(v) $\mathrm{diam}_M \leq c_5$,
(vi) $\int_M |R|^{n/2} \leq c_6$

is $C^{1,\alpha}$ compact in the Lipschitz topology.

(II) *If n is odd, condition* (iii) *on l_M may be dropped.*

Notice that in Theorem A no assumption is made regarding the sectional curvature, while in Theorem A′ a comparatively weak assumption, namely a bound on the scale-invariant integral $\int_M |R|^{n/2} dV$, is made. The quantity in (ii) scales in the same way as the sectional curvature (or the norm of any (3, 1) tensor), and again may be replaced by a bound on $\|\nabla \mathscr{R}\|_{3,\alpha}$. Theorems A and A′ apply naturally to Einstein manifolds, or their products, since then $D\,\mathrm{Ric} \equiv 0$, and lead to the following consequences, proved in §4.

Corollary B. (1)(a) *The space of Einstein metrics of Ricci curvature $+1$ on compact 4-manifolds M such that $l_M \geq c_1$, $\mathrm{vol}_M \geq c_2$, and $b_2(M) \leq c_3$ is compact in the C^∞ topology. In particular, there are only finitely many diffeomorphism types.*

(b) *On each component D of the moduli space of $+1$ Einstein metrics on a given 4-manifold M, the function $l_M \colon D \to \mathbf{R}^+$ is a proper exhaustion function.*

(2) *The space of Einstein metrics of Ricci curvature 0 or -1 on compact 4-manifolds M such that $l_M \geq c_1$, $\mathrm{vol}_M \geq c_2$, $\mathrm{diam}_M \leq c_3$, and $b_2(M) \leq c_4$ is compact in the C^∞ topology. In this case, the function $l_M^{-1} + \mathrm{diam}_M$ is a proper exhaustion function on each component D of the moduli space.*

These results hold with $\mathscr{A}_M$ in place of l_M.

(3) *Results* (1) *and* (2) *above hold for the space of Einstein metrics on compact n-dimensional manifolds M^n, provided the bound on $b_2(M)$ is replaced by a bound on $\int_M |R|^{n/2}$. If n is odd, the lower bound on l_M may be dropped. Thus, for example, the function $\int_M |R|^{n/2}$ is a proper exhaustion function on the moduli space of $+1$ Einstein metrics on a given manifold M in this case.*

(4) (*Almost Einstein metrics*) *There is a constant*

$$\delta = \delta(|\operatorname{Ric}|, \sup_{r \leq \varepsilon} r^{2(1-4/q)} \|D^2 \operatorname{Ric}\|_{B(r),q/2}, l_M, \operatorname{vol}_M, \operatorname{diam}_M, b_2(M))$$

such that if M *is a compact* 4*-manifold with* $\int_M |\operatorname{Ric} - \lambda g| \leq \delta$, *then* M *admits an Einstein metric with* $\operatorname{Ric} = \lambda g$. *The same result holds if* M *is a compact* n*-manifold with the bound on* $b_2(M)$ *replaced by a bound on* $\int_M |R|^{n/2}$ (*and* 4 *replaced by* n). *Again, if* n *is odd, the bound on* l_M *may be dropped.*

Before proceeding, we make some remarks on the hypotheses of Theorems A and A$'$, namely, whether any bound may be dropped in the presence of the others.

Remarks. (i) It is very possible that the two-sided condition $|\operatorname{Ric}| \leq c_1$ can be weakened to a lower bound $\operatorname{Ric} \geq -c_1$ especially in the presence of the bound (ii).

(ii) It is not known if the bound (ii) is necessary in Theorems A and A$'$, and it would be very interesting to know if, or to what extent, it can be removed. In spirit, it is similar to an α-Hölder bound on the Ricci tensor.

(iii) A condition of the type (iii) is necessary in even dimensions. In fact, Tian and Yau [35] construct a noncompact connected family of Kähler-Einstein metrics with $c_1 > 0$ (and so Ricci curvature $+1$) on a simply connected 4-manifold $(\mathbf{CP}^2 \# 8\overline{\mathbf{CP}^2})$. The first Chern class degenerates at the boundary so there are sequences with no smoothly convergent subsequences. In particular, $l_M \to 0$ (and $\mathscr{A}_M \to 0$) for elements in this family; cf. also [27] and the discussion in §5.

(iv) The lower bound on the volume is necessary, since for instance any manifold $M = N \times S^1$ collapses with bounded curvature, cf. [30].

(v) Similarly, a diameter bound is also necessary. The space of flat metrics on an n-torus with $v_M \geq c > 0$ is still noncompact.

(vi) In dimension 4, it is an interesting open question whether the assumption (vi) is necessary. It is possible that it is a consequence of the bounds (i), (iv), (v). In higher dimensions, the bound on the curvature integral is also not known to be necessary and it would again be interesting to replace this by a weaker curvature invariant, cf. §4.

In §5, we turn to the question of the compactification of the space of Einstein manifolds. Consider for instance Corollary B(1)(b). One is led to study the behavior of metrics in the moduli space $\mathscr{E}^{+1}$ of $+1$ Einstein metrics on a 4-manifold M as $l_M \to 0$, say at a point $p \in M$. The methods developed in the proof of Theorems A and A$'$ (cf., in particular, Theorem 3.5) tend to indicate that M develops a singularity near p; namely, a neighborhood of p is replaced by a cone on a spherical space form. In fact, the following result holds (conjectured independently by H. Nakajima).

Theorem C. *Let* $\{(M_i, g_i)\}$ *be a sequence of compact connected* n*-dimensional Einstein manifolds, normalized so that* $\operatorname{tr}\operatorname{Ric} \in \{-1, 0, +1\}$ *such that*

(i) $\operatorname{vol}_{M_i} \geq c_1$,
(ii) $\operatorname{diam}_{M_i} \leq c_2$ *if* $\operatorname{tr}\operatorname{Ric} = -1$ *or* 0,
(iii) $b_2(M) \leq c_3$ *if* $n = 4$, $\int_{M_i} |R|^{n/2} \leq c_3$ *if* $n > 4$.

Then a subsequence converges, in the Hausdorff topology, to a connected Einstein orbifold M_∞ *with a finite number of singular points* $\{p_i\} \in M_\infty$. *If* $G_\infty \equiv M_\infty - \bigcup\{p_i\}$, *then* G_∞ *has a* C^∞ *Einstein metric* g_∞ *and there are* C^∞ *embeddings* $F_i \colon G_\infty \to M_i$, *for* i *sufficiently large, such that* $(F_i)^* g_i$ *converges, uniformly on compact subsets in the* $C^{k,\alpha}$ *topology on* G_∞, *to* g_∞. *Each singular point* p_i *has a neighborhood which is homeomorphic to a cone on a spherical space form* $C(S^{n-1}/\Gamma)$. *If the metric* g_∞ *is lifted to* $B^n - \{0\}$ *via* Γ, *then there is a* Γ*-equivariant diffeomorphism* $\phi \colon B^n - \{0\} \to B^n - \{0\}$ *such that* $\phi^* g_\infty$ *extends smoothly over* $\{0\}$ *to a smooth Einstein metric on* B^n.

Further, if n *is odd, there are no singular points and* M_∞ *is an Einstein manifold diffeomorphic to* M_i, *for* i *sufficiently large. In this case,* (M_i, g_i) *(sub)converges smoothly to* (M_∞, g_∞).

Remarks. (1) Condition (i) is automatically satisfied if $M_i = M$ and g_i are on a connected component of the moduli space of Einstein metrics on M.

(2) The number of singular points $\{p_i\}$ and the orders of the local fundamental groups $|\Gamma_i|$ may be bounded above in terms of the bounds (i), (ii), (iii).

(3) Parts of Theorem C have been proved independently by Nakajima [29], cf. also [2] for a sketch of a proof of Theorem C obtained later, but essentially similar to the proof here.

(4) Recent work of Kobayashi-Todorov [27] indicates that Einstein orbifolds do actually arise as Hausdorff limits of Einstein metrics on K3 surfaces; cf. §5 for further discussion.

In §6, we study the moduli space of positive Einstein metrics on compact 4-manifolds of low Euler characteristic.

Theorem D. *Let* M *be a compact* 4*-manifold with Euler characteristic* $0 < \chi(M) < 4$. *Then each component of the moduli space* $\mathscr{E}^+$ *of positive Einstein metrics on* M *is compact in the* C^∞ *topology. Further, there are only finitely many components of the space* $\mathscr{E}^{+1}$ *of* $+1$ *Einstein metrics on* M *with* $\operatorname{vol}_M \geq c > 0$.

This holds for instance for $M = S^4$ or $\mathbf{CP}^2$. These results bear some resemblance with the compactness theorems of Uhlenbeck for the space of connections [37] and the space of Yang-Mills fields [36] on principle bundles over a compact manifold. In the latter case, this is, of course, not surprising, since Einstein metrics yield Yang-Mills connections on the tangent bundles. However, spaces of Einstein metrics are likely to be more complicated than spaces of Yang-Mills fields, since the (base) metrics may degenerate both locally and globally. The results above indicate what happens when the appropriate bounds are imposed, on the space of Einstein manifolds for instance. It remains open, for example, what happens in Theorem C when the volume of $(M_i, g_i) \to 0$, or $\operatorname{diam}(M_i, g_i) \to \infty$. This behavior may actually occur on a given manifold M, as shown by the examples of Wang and Ziller [38].

The origin of this paper owes much to questions and discussions with L. Gao. His result [14] led the author to consider the more general questions here.

The author would also like to thank S. Bando for enlightening him on the existence of the metrics in [7, 13]; this set the framework for the current work. Finally, I also thank H. Nakajima for his correspondence on these topics as well as the Taniguchi Foundation for making these latter contacts possible.

2. Preliminary results

In this section, we will discuss several preliminary results that will be used for the proof of the main theorems. This section may be skipped and referred back to, when necessary.

2.1. First, we unify the discussion of dimensions. Recall the formula of Avez [5] expressing the Euler characteristic $\chi(M)$ of a compact 4-manifold in terms of a curvature integral;

$$\chi(M) = \frac{1}{8\pi^2} \int_M |R|^2 - 4|\operatorname{Ric}|^2 + \tau^2, \tag{2.1}$$

where τ is the scalar curvature. Clearly, $\chi(M) \leq 2 + b_2(M)$ and $\int_M \tau^2 \leq 4\int_M |\operatorname{Ric}|^2$. Thus, a bound on $\int_M |\operatorname{Ric}|^2$ and $b_2(M)$ implies a bound on $\int_M |R|^2$. The Bishop comparison theorem (cf. §2.5) implies there is a bound $\operatorname{vol}_M \leq c(\inf \operatorname{Ric}_M, \operatorname{diam}_M)$, so that the bounds (i), (v), (vi) imply a bound on $\int_M |R|^2$.

2.2. The hypotheses (i), (iv), (v) in Theorems A and A$'$ lead to a lower bound on the isoperimetric and Sobolev constants of M. In fact, let h_M be the isoperimetric constant given by

$$h_M = \inf_S \frac{[\operatorname{vol} S]^n}{[\min(\operatorname{vol} M_1, \operatorname{vol} M_2)]^{n-1}}, \tag{2.2}$$

where S varies over closed hypersurfaces of M such that $M - S = M_1 \cup M_2$. Croke [10, Theorem 13] shows that h_M is bounded below by a constant depending only on a lower bound for the Ricci curvature and volume, and an upper bound on the diameter. In particular, if $B_x(r)$ is a geodesic ball of radius r about $x \in M$ and $S_x(r) = \partial B_x(r)$, $v(r) = \operatorname{vol} B_x(r)$, then it follows that $(v'(r))^n / v(r)^{n-1} \geq h_M$ for $v(r) < \frac{1}{2} \operatorname{vol} M$. Integrating this inequality, one obtains

$$v(r)/r^n \geq c_M, \tag{2.3}$$

if $v(r) < \frac{1}{2}\operatorname{vol} M$. Also, it is well known that a lower bound for h_M gives a lower bound for the Sobolev constant c_S of M. In fact, cf. [40],

$$\|f\|_{2n/(n-2)} \leq \frac{1}{c_S}\|df\|_2 + \operatorname{vol}_M^{-2/n} \|f\|_2, \tag{2.4}$$

for any Lipschitz function for M. Note that the bounds in (2.3) and (2.4) are scale invariant.

2.3. A basic tool in the arguments to follow will be the equation for the (rough) Laplacian of the curvature tensor R of M. It is shown in [21, Lemma 7.2] that

$$\Delta R = R * R + R * \operatorname{Ric} + P^2(\operatorname{Ric}),$$

where $A * B$ denotes a linear combination of tensors A, B obtained by contracting A, B with the metric g and $P^2(\operatorname{Ric})$ is a linear combination of second covariant derivatives of the Ricci tensor. In particular, one obtains

$$|\Delta R| \leq c_1|D^2 \operatorname{Ric}| + c_2|R|^2, \tag{2.5}$$

where c_1 and c_2 are constants depending only on dimension. One has $\langle \Delta R, R\rangle + |DR|^2 = \frac{1}{2}\Delta|R|^2 = |R|\Delta|R| + |d|R||^2$. An application of the Schwartz inequality shows $|d|R||^2 \leq |DR|^2$ so that from (2.5) one has

$$\Delta|R| + c_1|D^2 \operatorname{Ric}| + c_2|R|^2 \geq 0. \tag{2.6}$$

Elliptic inequalities of this type have now been used in many geometric contexts to derive pointwise bounds, the basic idea going back to Uhlenbeck [32]. Since such a pointwise bound is crucial in our arguments, we will include a full proof, following the lines of [16, Theorem 8.17].

Lemma 2.1. *There is a constant $C = C(n, c_S)$ and $\varepsilon_0 = \varepsilon_0(n, c_S)$ such that if $B(t)$ is a geodesic ball of radius t in M and*

$$\int_{B(t)} |R|^{n/2}\, dV < \varepsilon_0, \tag{2.7}$$

then

$$\sup_{B(t/2)} |R|^2 \leq C \cdot \left[\frac{1}{t^2}\int_{B(t)} |R|^{n/2} + t^{2\delta}\left[\int_{B(t)} |D^2 \operatorname{Ric}|^{q/2}\right]^{q/2}\right] \tag{2.8}$$

for a fixed $q > n$ and $\delta = 1 - \frac{n}{q}$.

Proof. It simplifies matters if we assume $t = 1$. Note that inequality (2.6), condition (2.7) and claim (2.8) are all scale invariant, so that by rescaling the metric on M, we may assume that $t = 1$.

Let $k = (\int_{B(1)} |D^2 \operatorname{Ric}|^{q/2})^{2/q}$ and let $u = |R| + k$. First multiply (2.6) by $\zeta^2 u^\alpha$, $\alpha \geq 1$, where ζ is a cutoff function of compact support in $B(1)$, to be determined below. Integrating by parts, one obtains

$$\frac{4\alpha}{(\alpha+1)^2}\int \zeta^2|du^{(\alpha+1)/2}|^2 - 2\zeta u^\alpha|d\zeta||du| \leq c_3\int \zeta^2 u^\alpha[|D^2 \operatorname{Ric}| + |R|^2].$$

By the Young inequality

$$2\zeta u^{\alpha}|d\zeta|\,|du| \le 2(u^{(\alpha+1)/2}|d\zeta|)^2 + \frac{2\zeta^2}{(\alpha+1)^2}|du^{(\alpha+1)/2}|^2$$

so that one obtains

$$\frac{4(\alpha-\frac{1}{2})}{(\alpha+1)^2}\int \zeta^2|du^{(\alpha+1)/2}|^2 \le c_3\int \zeta^2 u^{\alpha}[|D^2\operatorname{Ric}| + |R|^2] + 2u^{\alpha+1}|d\zeta|^2.$$

This gives the estimate

$$\int |d\zeta u^{(\alpha+1)/2}|^2 \le c_4\cdot\alpha\int \zeta^2 u^{\alpha}[|D^2\operatorname{Ric}| + |R|^2] + u^{\alpha+1}|d\zeta|^2,$$

so that by the Sobolev inequality (2.4)

$$\text{(2.9)}\qquad \left[\int (\zeta u^{(\alpha+1)/2})^{2n/(n-2)}\right]^{(n-2)/n} \le c_5\alpha\int \zeta^2 u^{\alpha}[|D^2\operatorname{Ric}| + |R|^2] + u^{\alpha+1}[|d\zeta|^2 + \zeta^2].$$

First we set $\alpha+1 = n/2$. Then

$$\int \zeta^2|R|^2u^{\alpha} \le \int \xi|R|\zeta^2u^{\alpha+1} \le \left(\int (\xi|R|^{n/2})^{2/n}\right)\cdot\left(\int (\zeta^2u^{\alpha+1})^{n/(n-2)}\right)^{(n-2)/n},$$

where ξ is an auxiliary cutoff function with $\xi\equiv 1$ on $\operatorname{supp}\zeta$. Since $\int(\xi|R|)^{n/2} \le \varepsilon_0$, if $\varepsilon_0 < \frac{1}{2}c_5(\frac{n}{2}-1)$, this term may be absorbed into the left side of (2.9). Next,

$$\int (\zeta^2|D^2\operatorname{Ric}|)u^{(n-2)/2} \le \left(\int |D^2\operatorname{Ric}|^{n/2}\right)^{2/n}\cdot\left(\int \zeta^2u^{n/2}\right)^{(n-2)/n} \le \operatorname{vol}B(1)^{1-n/q}k[\|\zeta^2|R|\|_{n/2} + k]^{(n-2)/n}.$$

Further, $\int u^{\alpha+1}|d\zeta|^2 \le c_6[\varepsilon_0 + k]$, where c_6 depends on $|d\zeta|$. If we set $\rho_0 = (n/2)(n/(n-2))$, the above estimates combine to give the bound

$$\text{(2.10)}\qquad \|\zeta u\|_{\rho_0} \le c_7[\varepsilon_0 + k].$$

Now we return to (2.9) with $\alpha + 1 > \frac{n}{2}$. Note that

$$u^{\alpha}[|R|^2 + |D^2\operatorname{Ric}|] \le u^{\alpha+1}\left[u + \frac{|D^2\operatorname{Ric}|}{k}\right],$$

so that

$$\text{(2.11)}\qquad \int \zeta^2u^{\alpha}[|R|^2 + |D^2\operatorname{Ric}|] \le \left\|\zeta\left(u + \frac{|D^2\operatorname{Ric}|}{k}\right)\right\|_{q/2}\cdot\|\zeta u^{(\alpha+1)/2}\|^2_{2q/(q-2)}.$$

Then

$$\left\| \zeta \left(u + \frac{|D^2 \operatorname{Ric}|}{k} \right) \right\|_{q/2} \leq \|\zeta u\|_{q/2} + \|\zeta |D^2 \operatorname{Ric}|\|_{q/2} \cdot \frac{1}{k} \leq c_8[\varepsilon_0 + k],$$

provided $\frac{q}{2} \leq \rho_0$. Further, using the interpolation inequality $\|f\|_s \leq \varepsilon \|f\|_r + \varepsilon^{-\mu} \|f\|_t$, where $r > s > t$, $\mu = (\frac{1}{r} - \frac{1}{s})/(\frac{1}{s} - \frac{1}{t})$ with $r = 2n/(n-2)$, $s = 2q/(q-2)$, $t = 2$, we obtain from (2.9)–(2.11) that

(2.12)

$$\left[\int (\zeta u^{(\alpha+1)/2})^{2n/(n-2)} \right]^{(n-2)/n} \leq c_9 \cdot \alpha [(\varepsilon \|\zeta u^{(\alpha+1)/2}\|_{2n/(n-2)} + \varepsilon^{-\mu} \|\zeta u^{(\alpha+1)/2}\|_2]^2 + \int u^{\alpha+1} [|d\zeta|^2 + \zeta^2],$$

where $\mu = n/(q-n)$. If we choose $\varepsilon^2 = \frac{1}{4}(c_9 \alpha)^{-1}$, then the first term on the right side of (2.12) may again be absorbed into the left side. We set $v = u^{(\alpha+1)/2}$ and $\chi = n/(n-2)$ and thus obtain

$$\|\zeta v\|_{2\chi} \leq c_{10} \alpha (1 + \alpha^\mu) [\|(\zeta + |d\zeta|) v\|_2]. \tag{2.13}$$

We may choose the cutoff function $\zeta = \zeta(r)$, where r is the distance to $0 \in B(1)$, and require $\zeta(r) \equiv 1$ for $r \leq r^-$ with $r^- \geq \frac{1}{2}$, $\zeta(r) \equiv 0$ for $r \geq r^+$, where $r^+ \leq 1$ and $|d\zeta| \leq c/(r^+ - r^-)$. Let $T(p, r) = (\int_{B(r)} u^p)^{1/p}$. Then (2.13) gives

$$T(\chi p, r^-) \leq \left(\frac{c_{11} p (1 + p^\mu)}{r^+ - r^-} \right)^{2/p} T(p, r^+). \tag{2.14}$$

This inequality may now be iterated in the standard fashion. Let $p_1 = \frac{n}{2}$, $p_m = \chi^m p$, $r_m^- = \frac{1}{2} + 2^{-(m+3)}$, $r_m^+ = 1 - 2^{-(m+3)}$. We then obtain

$$T(\chi^m \tfrac{n}{2}, \tfrac{1}{2}) \leq (c_{12} \chi)^{2[1 + \sum_1^m i\chi^{-i}]} T(\tfrac{n}{2}, 1) \leq c_{13} \cdot T(\tfrac{n}{2}, 1).$$

Letting $m \to \infty$ gives

$$\sup_{B(1/2)} u \leq c \cdot \left(\int_{B(1)} u^{n/2} \right)^{2/n}. \tag{2.15}$$

Lemma 2.1 then follows by applying the triangle inequality to the integral in (2.15). □

2.4. We will make frequent use of a local lower bound estimate on the injectivity radius of a manifold, due to Cheeger, Gromov, and Taylor [9, Theorem 4.3]. Namely, let N be a complete manifold and $p \in N$. Let l_p denote the length of the shortest geodesic loop in N based at p. If $K_N \leq \Lambda^2$ on $B_p(r)$, and $r \leq l_p/2$, then an estimate of Klingenberg [26, Lemma 1], implies that the injectivity radius i_p of N at p satisfies

$$i_p \geq \min(r, \tfrac{\pi}{\Lambda}). \tag{2.16}$$

Now suppose $|K_N| \le \Lambda^2$ on $B_p(r)$, where $r \le \frac{\pi}{\Lambda}$. Choose any numbers r_0 and s such that $r_0 + 2s \le r$, $r_0 \le \frac{r}{4}$. Then one has the estimate [9]

$$l_p \ge r_0[1 + v^{-\Lambda}(r_0 + s)/v_p(s)]^{-1}, \tag{2.17}$$

where $v_p(s)$ is the volume of $B_p(s)$ and $v^{-\Lambda}(r_0 + s)$ is the volume of the geodesic ball of radius $r_0 + s$ in the space form of constant curvature $-\Lambda$.

In particular, a lower bound on $v_p(s)$ and the bound $|K_N| \le \Lambda^2$ imply a lower bound on the injectivity radius of N of p. Note that from (2.3), one obtains a lower bound for $v_p(s)$ (depending only on s), from bounds $\mathrm{vol}_N \ge c_4$, $\mathrm{diam}_N \le c_5$ (since $|K_N| \le \Lambda^2$).

2.5. We will also make use of the well-known Bishop comparison theorem [6, 9]. If M is a complete manifold with $\mathrm{Ric}_M \ge (n-1)\lambda$, then

$$\frac{v(r)}{v^\lambda(r)} \downarrow, \tag{2.18}$$

where $v(r)$, $v^\lambda(r)$ are as in §2.4. Clearly $\lim_{r\to 0} v(r)/v^\lambda(r) = \omega_n$, the volume of the unit n-ball in $\mathbf{R}^n$. If $v(r)/v^\lambda(r) \equiv \omega_n$ for all r, then M is isometric to the simply connected space form of constant curvature λ. We note, in particular, that if $\mathrm{Ric}_M \ge 0$, then $v(r)/r^n \downarrow$.

2.6. We will need to apply a local form of the Gromov compactness theorem (related to [20, 8.2]) a number of times. First, we give a definition.

Definition. A sequence of Riemannian manifolds V_i converges uniformly on compact sets in the $C^{k,\alpha}$ topology to a Riemannian manifold V if, for any compact domain $D \subset V$ and i sufficiently large, there are compact domains $D_i \subset V_i$ and $C^{k,\alpha}$ diffeomorphisms $F_i\colon D \to D_i$ such that the pull-back $(F_i)^* g_i$ of the metrics on D_i to D converge, in the $C^{k,\alpha}$ topology on D, to the metric on $D \subset M$.

Theorem 2.2 (Gromov compactness). *Let V_i be a sequence of closed C^∞ Riemannian manifolds and Ω_i a sequence of domains in V_i, with smooth boundary $\partial\Omega_i$. Suppose, for all i,*

(i) $|D^l R|(x) \le \Lambda(l)$, $l = 0, \dots, k$,
(ii) $\mathrm{Inj}(x) \ge c_1$,
(iii) $c_2 \le \mathrm{vol}\,\Omega_i \le c_3$

for all $x \in \Omega_i$, where $D^l R$ is the lth covariant derivative of the curvature tensor on V_i. Then given $\varepsilon > 0$, the Riemannian manifolds $\Omega_i(\varepsilon) \equiv \{x \in \Omega_i\colon \mathrm{dist}(x, \partial\Omega_i) > \varepsilon\}$ (assumed nonempty) have a subsequence which converges, uniformly on compact sets in the $C^{k,\alpha}$ topology, to a $C^{k+1,\alpha}$ Riemannian manifold $\Omega_\infty(\varepsilon)$. In particular, $\Omega_i(\varepsilon)$ is diffeomorphic to $\Omega_\infty(\varepsilon)$ for i sufficiently large.

Although this specific form of the Gromov compactness theorem does not seem to appear in the literature, a number of proofs that do appear are easily adapted to give a proof of this situation. For completeness, we give a proof of Theorem 2.2, following closely the lines of [24].

First, we recall the following result from [23].

Fact. There are positive constants δ_0 and c_0, depending on the bounds (i), (ii), (iii), and ε, such that for any $\delta \le \delta_0$, there is a harmonic coordinate chart $H = (h_1, \dots, h_n): B_x(\delta) \to \mathbf{R}^n$ satisfying

(1) $c_0^{-1}\operatorname{dist}(x, y) \le |H(y)| \le c_0 \operatorname{dist}(x, y)$,

(2) $c_0^{-1}|\xi|^2 \le \sum g_{ij}\xi^i\xi^j \le c_0|\xi|^2$, where $g_{ij} = g\langle \nabla h_i, \nabla h_j\rangle$,

(3) $\|g_{ij}\|_{C^{k+1,\alpha}(B_x(\delta))} \le c_0$,

(4) if f is any harmonic function in $B_x(\delta)$, then
$$\|f\|_{C^{k+2,\alpha}(B_x(\delta/2))} \le c_0|f|_{C^0}$$

for any $x \in \Omega_i(\varepsilon)$.

Proof of Theorem 2.2. Let $\delta_1 = \frac{1}{2}c_0^{-5}\delta_0$, where c_0, δ_0 are chosen from the above fact, and we will also assume $\delta_0 < \varepsilon/2$. In each domain $\Omega = \Omega_i(\varepsilon)$, choose a maximal $\delta_1/2$ separated set Γ of points $\{x_j\}$. Thus, $\operatorname{dist}_\Omega(x_j, x_k) > \delta_1/2$ and $\operatorname{dist}_\Omega(x, \Gamma) < \delta_1$. In particular, the balls $B_{x_j}(\delta_1/4)$, $x_j \in \Gamma$, are disjoint and the balls $B_{x_j}(\delta_1)$ cover $\Omega_i(\varepsilon)$. By the Rauch comparison theorem, $\operatorname{vol} B_{x_j}(\delta_1/4) \ge c(\Lambda)\cdot\delta_1^n$ and $\operatorname{vol} B_{x_j}(\delta_1) \le c'(\Lambda)\cdot\delta_1^n$. Thus, by (iii), there is a fixed upper bound Q to the cardinality of Γ and by passing to a subsequence, we may assume $\#\Gamma = \#\Gamma_i(\varepsilon) = Q$ (ε fixed). Let $U = U_i = \bigcup_{j=1}^{Q} B_{x_j}(\delta_1)$, so that U_i covers $\Omega_i(\varepsilon)$, but is contained in $\Omega_i(\varepsilon/2)$.

For each x_j, we have harmonic coordinates $H_j: B_j(x_j) \to \mathbf{R}^n$ satisfying
$$B^n(c_0^{-1}\delta) \subset H_j(B_{x_j}(\delta)) \subset B^n(c_0\delta)$$
for $\delta < \delta_0$ and $B^n(s)$ the ball of radius s about 0 in $\mathbf{R}^n$. Let ζ be a fixed smooth cutoff function satisfying $\zeta(t) \equiv 1$ on $[0, \delta_2]$, $\zeta(t) \equiv 0$ on $[\delta_3, \infty)$, where $\delta_2 = c_0^3\delta_1$, $\delta_3 = c_0^4\delta_1$. Define $\zeta_j: U \to \mathbf{R}$ by $\zeta_j(x) = \zeta(|H_j(x)|)$, so that $\operatorname{supp}\zeta_i \subset B_{x_j}(\delta_5)$ and $\zeta_j \equiv 1$ on $B_{x_j}(\delta_4)$, where $\delta_5 = c_0^5\delta_1$, $\delta_4 = c_0^2\delta_1$.

For each $U = U_i$, we may define a smooth map $E_U: U \to \mathbf{R}^N$ ($N = nQ + Q$), by
$$E_U(x) = (\zeta_1(x)\cdot H_1(x), \dots, \zeta_Q(x)\cdot H_Q(x), \zeta_1(x), \dots, \zeta_Q(x)).$$
It is clear that E_U is an embedding of U into a ball $B^N(r_0)$ of fixed size in $\mathbf{R}^N$. Now observe that for any $x_j \in \Gamma$, $E_U(B_{x_j}(\delta_4))$ is naturally a graph over $H_j(B_{x_j}(\delta_4)) \subset \mathbf{R}^n = (\mathbf{R}^N)_j \subset \mathbf{R}^N$. Namely, for $x \in H_j(B_{x_j}(\delta_4))$,
$$E_U(B_{x_j}(\delta_4)) = \{(f_1(x)F_1(x), \dots, f_{j-1}(x)F_{j-1}(x), x, f_{j+1}(x)F_{j+1}(x), \dots, f_Q(x)F_Q(x), f_1(x), \dots, f_Q(x))\},$$

where $F_l = H_l \circ H_j^{-1}$ and $f_l = \zeta(|F_l|)$. Further, note that there is a fixed lower bound on the size of $H_j(B_{x_j}(\delta_4))$; in fact, $H_j(B_{x_j}(\delta_1)) \subset B^n(c_0\delta_1) \subset H_j(B_{x_j}(\delta_4))$. Clearly, the balls $B_{x_j}(\delta_4)$ cover U. Since the maps H_j are harmonic, the maps F_j and f_j have uniformly bounded $C^{k+2,\alpha}$ norms, by (4) of the fact. In particular, all maps E_{U_i} have uniformly bounded $C^{k+2,\alpha}$ norm.

We will now identify U_i with its image D_i in $\mathbf{R}^N$ under E_{U_i} and the metric g_i on U_i with its image (call it again g_i), under the push forward $(E_{U_i})_* g_i$ on D_i. We then have a covering of D_i by neighborhoods $N_j = N_j(i)$ which are graphs over a fixed domain $B^n(\delta_6) \subset (\mathbf{R}^n)^j \subset \mathbf{R}^N$, with $\delta_6 = c_0\delta_1$, by graphing functions with uniformly bounded $C^{k+2,\alpha}$ norm. We may apply the Arzela-Ascoli theorem to each graphing chart and obtain a subsequence of $\{D_i\}$ which converges, in the $C^{k+2,\alpha'}$ topology, for $\alpha' < \alpha$ to an embedded graph D_∞ of class $C^{k+2,\alpha'}$. For i sufficiently large, any compact subset of D_i may then be graphed over the normal exponential map X_∞ to D_∞, so that, pushing forward the metric g_i to D_∞ by X_∞, we obtain a sequence of $C^{k+1,\alpha'}$ metrics, also called g_i, on any given compact subset $K \subset D_\infty$.

We will now verify that this sequence of metrics has a subsequence, converging uniformly in the $C^{k+1,\alpha}$ topology on K, to a $C^{k+1,\alpha}$ metric on K. To see this, recall $N = Qn + n$ and let P^j be the orthogonal projection of $\mathbf{R}^N$ onto $(\mathbf{R}^n)^j$, $1 \le j \le Q$. Let Π_i^j be the restriction of P^j to the geodesic ball $B_{x_j}(\delta_4) \subset U_i \cong D_i \subset \mathbf{R}^N$. Consider the sequence of metric tensors

$$h_{\alpha\beta}^i = (\Pi_i^j)_* g_i \left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right)$$

defined on $B^n(\delta_6) \subset (\mathbf{R}^n)^j$. To compare these metrics, note that the maps $\phi_i^j = P^j \circ X_\infty \circ (\Pi_i^j)^{-1}$ are $C^{k+2,\alpha}$ diffeomorphisms from $B^n(\delta_6)$ into $\mathbf{R}^n$, with uniformly bounded $C^{k+2,\alpha}$ norm in i, for each fixed j. Thus, they converge to the identity in the $C^{k+2,\alpha'}$ topology. It follows that the $C^{k+1,\alpha}$ norms of $h_{\alpha\beta}^i$ on $B^n(\delta_6)$ are uniformly bounded, so by the Arzela-Ascoli theorem, a subsequence converges to a $C^{k+1,\alpha}$ metric $h_{\alpha,\beta}^\infty$ in the $C^{k+1,\alpha'}$ topology on $B^n(\delta_6)$. Taking then $(\Pi_i^j)^* h_{\alpha\beta}^\infty$ one verifies the claim, and this proves the theorem. □

3. Proof of Theorems A and A$'$

This section will be concerned with the proof of Theorems A and A$'$. Note that by §2.1, we may treat the cases $\dim M = 4$ and $\dim M > 4$ on an equal footing, so we will assume $n = \dim N$ is arbitrary.

We will prove that conditions (i)–(vi) in Theorem A (A$'$) imply a bound on the sectional curvature $|K_M| \le \Lambda^2$ of M. The result then follows from the

Gromov compactness theorem. Suppose $|K_M|$ is not bounded on this set of Riemannian manifolds. Then there is a sequence of Riemannian n-manifolds (M_i, g_i) such that (i)–(vi) hold uniformly for (M_i, g_i) but

$$\sup_{M_i} |K_{M_i}| = R_i \to \infty \quad \text{as } i \to \infty. \tag{3.1}$$

It will be shown that this leads to a contradiction. Suppose the supremum above is achieved at $x_i \in M_i$ and consider the sequence of rescaled metrics (M_i, h_i) with

$$h_i = R_i^{1/2} \cdot g_i. \tag{3.2}$$

This sequence satisfies

$$\begin{aligned}
&(1) \quad |\operatorname{Ric}(h_i)| \le c_1/R_i \to 0,\\
&(2) \quad \sup_{r \le \varepsilon R_i} r^{2(1-n/q)}\Big[\int_{B(r)} |D^2 \operatorname{Ric}(h_i)|\Big]^{2/q} \to 0,\\
&(3) \quad |K(h_i)| \le 1,\\
&(4) \quad |K(h_i)(x_i)| = 1,\\
&(5) \quad \int_{M_i} |R(h_i)|^{n/2} \le c_6,\\
&(6) \quad l_{M_i}(h_i) \ge c_3 R_i \to \infty,\\
&(7) \quad v(r)/r^n = \operatorname{vol}_{h_i} B_x(r)/r^n \ge c_1,\\
&(8) \quad \operatorname{inj}_{M_i}(x) \ge c_0.
\end{aligned} \tag{3.3}$$

Here (1)–(6) follow from the bounds on (M_i, g_i) and scaling properties, (7) follows from (2.3) and scale invariance (we assume $r \le R_i/2 \cdot \operatorname{vol}(M_i, g_i)$), and (8) follows from §2.4 via (3) and (7).

We wish to apply the Gromov compactness theorem (in local form) to the sequence (M_i, g_i). However, a limit metric need only be $C^{1,\alpha}$ and we require some curvature properties, in particular, an analogue of (4), to apply to the limit. Thus, it is useful first to smooth the metrics and then to pass to a limit.

Let $h = h_i$ and consider the evolution equation of Hamilton [21]

$$\partial h/\partial t = -2 \operatorname{Ric}(h) \tag{3.4}$$

on $M = M_i$. It is shown in [3] that there is an $\varepsilon_1 > 0$, depending only on dimension (since $|K(h_i)| \le 1$), such that solutions $h(t)$ to (3.4) exist for $0 \le t \le \varepsilon_1$, and

$$\begin{aligned}
&\|h(t) - h\|_{C^0} \le C(t), \quad \text{with } C(t) \to 0 \text{ as } t \to 0,\\
&\|D^k R(t)\|_{C^0} \le C(\tfrac{1}{t}, k), \qquad k = 1, 2, 3, \dots,\\
&\|R(t)\|_{C^0} \le 1.
\end{aligned} \tag{3.5}$$

One sees that the bounds (3.3)(3), (7) and thus (8) remain valid for $h(t)$, for all i, t with t sufficiently small $t \le \varepsilon_1$. We will show in a sequence of lemmas that the other bounds (except (2) and (6)) are also preserved, with (4) replaced by an analogue. In the following lemmas, we will suppress the dependence on i, but it is important to keep in mind that all estimates are independent of i. The "centerpoint" $x_i \in M_i$ will be denoted by x_0.

Lemma 3.1. *For the metric $h = h(0)$, there are fixed positive constants r_0, δ_0 such that*

$$\int_{B_{x_0}(r_0)} |R|^2 \geq \delta_0 . \tag{3.6}$$

Proof. Since $|R| \leq 1$, we may choose r sufficiently small to apply Lemma 2.1 and obtain

$$\sup_{B_{x_0}(r/2)} |R|^2 \leq C \cdot \left[\frac{1}{r^2} \int_{B_{x_0}(r)} |R|^{n/2} + r^{2\delta} \left(\int_{B_{x_0}(r)} |D^2 \operatorname{Ric}|^{q/2} \right)^{2/q} \right] ,$$

where $\delta = 1 - \frac{n}{q}$. By (3.3)(2), the second integral converges to zero as $i \to \infty$. On the other hand, by (3.3)(4), $\sup_{B_{x_0}(r/2)} |R|^2 = 1$, so that

$$\int_{B_{x_0}(r)} |R|^{n/2} \geq c \cdot r^2 .$$

Again, since $|R| \leq 1$, the result follows. □

Lemma 3.2. *For the metric $h = h(0)$, there is a constant K such that*

$$\int_{B_x(1)} |\nabla R|^2 \leq K \tag{3.7}$$

for any $x \in M$.

Proof. From (2.5), it follows that one has the inequality

$$\Delta |R|^2 \geq -c_1 |D^2 \operatorname{Ric}| \cdot |R| - c_2 |R|^3 + |\nabla R|^2 . \tag{3.8}$$

Let $\zeta = \zeta(r)$ be a fixed cutoff function such that $\zeta(r) \equiv 1$ for $r \leq 1$ and $\zeta \equiv 0$ for $r \geq 2$. Multiply (3.8) by ζ and integrate by parts to obtain

$$\int_{B_x(1)} |\nabla R|^2 \leq \int_M \zeta \Delta |R|^2 + c_1 \int_M \zeta |D^2 \operatorname{Ric}| \cdot |R| + c_2 \int_M \zeta |R|^3 .$$

The latter two integrals on the right side are clearly bounded above. Noting that $\int_M \zeta \Delta |R|^2 = \int_M |R|^2 \Delta \zeta \leq \int_M |\Delta \zeta|$, this term may also be bounded from above by the local geometry of M. □

Remark. Lemmas 3.1 and 3.2 are the only places where condition (ii) in Theorems A and A$'$ is required in the proof.

Lemma 3.3. *There is a constant L such that for $0 \leq t \leq \varepsilon_1$,*

$$\int_{B_x(1)} |\nabla R(t)|^2 \leq L e^{Lt} . \tag{3.9}$$

Proof. It follows from equation (13.3) of Hamilton [21] that one has the inequality

$$\frac{d}{dt}|\nabla R|^2 \leq \Delta|R|^2 + c_3|\nabla R|^2 \cdot |R|. \tag{3.10}$$

Choose a locally finite cover of M by geodesic balls $B_{p_j}(r_1)$ in the $h = h(0)$ metric, where $r_1 < \frac{1}{2}\operatorname{inj}_M(h(0))$ and such that the balls $B_{p_j}(r_1/2)$ also cover M. There is a $p_1 \in \{p_j\}$ such that $\int_{B_{p_1}(r_1)}|\nabla R|^2 \geq \int_{B_{p_j}(r_1)}|\nabla R|^2$, for all j. Further, since for any $D \subset M$, $\int_D |\nabla R(t)|^2\, dV_t$ is a continuous function in t, there is a $t_1 \leq \varepsilon_1$ and p_1 such that, for all j,

$$\int_{B_{p_1}(r_1)} |\nabla R(t)|^2\, dV_t \geq \int_{B_{p_j}(r_1)} |\nabla R(t)|^2\, dV_t, \tag{3.11}$$

for $t \leq t_1$. Let $\zeta = \zeta(r)$ be a fixed cutoff function with $\zeta(r) \equiv 1$ for $r \geq r_1/2$ and $\zeta(r) \equiv 0$ for $r \geq 3r_1/4$. Multiply (3.10) by ζ and integrate by parts to obtain

$$\begin{aligned}\frac{d}{dt}\int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2\, dV_t &= \int_{B_{p_1}(r_1)} \zeta \frac{d}{dt}|\nabla R(t)|^2\, dV_t + \int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2 \tau\, dV_t \\ &\leq \int_{B_{p_1}(r_1)} |\nabla R(t)|^2 |\Delta\zeta|\, dV_t + c_4 \int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2\, dV_t,\end{aligned}$$

where τ is the scalar curvature. One may bound $|\Delta\zeta| \leq c_5$ by a constant depending only on the local geometry, so that

$$\begin{aligned}\frac{d}{dt}\int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2\, dV_t &\leq c_6 \int_{B_{p_1}(r_1)} |\nabla R(t)|^2\, dV_t \\ &\leq c_7 \int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2\, dV_t,\end{aligned}$$

where we have used (3.11) in the last inequality. Setting

$$f(t) = \int_{B_{p_1}(r_1)} \zeta|\nabla R(t)|^2\, dV_t,$$

we obtain $f' \leq c_7 f$ for $t \leq t_1$. Thus, $f(t) \leq f(0)e^{c_7 t} \leq K \cdot e^{c_7 t}$ so that (3.9) holds for $t \leq t_1$. Repeating this argument with t_1 in place of $t_0 = 0$ and continuing by induction gives the result. □

Lemma 3.4. *For all $t \leq \varepsilon_1$ there is a constant C such that*

$$\begin{aligned}&\text{(i)} \quad \int_{B_{x_0}(1)} |R(t)|^2\, dV_t \geq C^{-1}, \\ &\text{(ii)} \quad \int_M |R(t)|^{n/2}\, dV_t \leq C, \\ &\text{(iii)} \quad |\operatorname{Ric}(t)| \leq C \cdot |\operatorname{Ric}(h(0))|.\end{aligned} \tag{3.12}$$

Proof. (i) Using equation (13.3) of Hamilton [21] again, one obtains the inequality

$$\frac{d}{dt}|R|^2 \geq \Delta|R|^2 - 2|\nabla R|^2 - c_3|R|^3 .$$

Choosing ζ as in Lemma 3.2 and integrating by parts gives

$$\frac{d}{dt}\int_{B_{x_0}(1)} \zeta|R|^2\, dV \geq -\int_{B_{x_0}(1)} |\Delta\zeta| - 2\int_{B_{x_0}(1)} |\nabla R|^2 \int_{B_{x_0}(1)} |R|^3$$

The result then follows from Lemmas 3.1 and 3.3.

(ii) From equation (13.3) of [21] one obtains

$$\frac{d}{dt}|R|^2 \leq \Delta|R|^2 + c_3|R|^3 ,$$

from which one deduces

$$\frac{d}{dt}|R|^{n/2} \leq \Delta|R|^{n/2} + c_3|R|^{(n+2)/2} .$$

Integrating this inequality over $(M, h(t))$, and using $|R| \leq 1$ and (3.3)(5) gives the result.

(iii) From Corollary 7.3 of [21], the Ricci curvature satisfies

$$\frac{d}{dt}|\operatorname{Ric}|^2 \leq \Delta|\operatorname{Ric}|^2 + c_3|R| \cdot |\operatorname{Ric}|^3 .$$

Taking into account that $|R| \leq 1$, the result follows by applying the maximum principle on M, cf. [3, 21]. □

Returning now to the sequence of metrics $\{h_i\}$ on M_i, the above results show that there are metrics $h_i(t)$, $t \leq \varepsilon_1$, on M_i, such that

$$\begin{aligned}
&\text{(i)} && \|h_i(t) - h_i\|_{C^0} \leq C(t), \\
&\text{(ii)} && \|\nabla^k R_i(t)\|_{C^0} \leq C(\tfrac{1}{t}, k), \\
&\text{(iii)} && \|R_i(t)\|_{C^0} \leq 1, \\
&\text{(iv)} && \int_{M_i} |R_i(t)|^{n/2} \leq C, \\
&\text{(v)} && \int_{B_{x_i}(1)} |R_i(t)|^2 \geq C^{-1}, \\
&\text{(vi)} && \operatorname{vol}_{h_i(t)} B(r) \cdot r^{-n} \geq C^{-1}, \\
&\text{(vii)} && \operatorname{inj}_{h_i(t)} M_i \geq C^{-1}.
\end{aligned} \tag{3.13}$$

We now apply the Gromov compactness theorem to the pointed manifolds $(M_i, x_i, h_i(t))$. By Theorem 2.2, the sequence $(B_{x_i}(R), h_i(t))$ of Riemannian manifolds has a subsequence which converges, uniformly in the $C^{k,\alpha}$ topology for any k, to a C^∞ Riemannian manifold $(B_{x_\infty}(R), h_\infty(t))$, provided $t > 0$. By choosing a sequence $\{R_j\} \to \infty$ and taking a diagonal subsequence, it follows that a subsequence of $(M_i, x_i, h_i(t))$ converges, smoothly on compact subsets, to a C^∞ Riemannian manifold N, with smooth metric $h_N(t)$, $0 < t < \varepsilon_1$, satisfying

$$(3.14)\qquad \begin{aligned} &\text{(i)} && \mathrm{Ric}_N \equiv 0, \\ &\text{(ii)} && \int_N |R|^{n/2} < \infty, \\ &\text{(iii)} && v(r)/r^n \geq C, \\ &\text{(iv)} && \int_{B_z(1)} |R|^2 > 0 \end{aligned}$$

for some point $z \in N$. We note that (3.14)(i) follows from (3.3)(1) and (3.12)(iii).

Let us indicate briefly how the result (3.14) leads to the required contradiction in the proof of Theorems A and A$'$. We will show below, and this is the crucial result, that a complete Riemannian manifold V satisfying (3.14)(i)–(iii) is asymptotic to a cone on a spherical space form S^{n-1}/Γ. Further, if V is simply connected at infinity, then V is flat. Below, we will use the bound (iii) in Theorems A and A$'$, or equivalently (3.3)(6), to show that N, constructed above, is simply connected at infinity. Thus, N must be flat, contradicting (3.14)(iv).

Theorem 3.5. *Let V be a complete, noncompact Riemannian manifold satisfying (3.14)(i)–(iii). Then given $z \in V$, there is an R_0 such that $V \setminus B_z(R_0)$ is diffeomorphic to $(R_0, \infty) \times S^{n-1}(1)/\Gamma$, where $S^{n-1}(1)/\Gamma$ is a spherical space form. Further, V is asymptotically flat, i.e., the metric on V approaches the flat metric on $(R_0, \infty) \times S^{n-1}(1)/\Gamma$ at a rate $O(1/r^2)$.*

If V is simply connected at infinity, i.e., $\Gamma = \{e\}$, then V is isometric to $\mathbf{R}^n$.

Proof. First note that from Bishop's inequality (§2.5) and (3.14)(iii), there is a constant c_0 such that $c_0^{-1} \leq v(r)/r^n \leq c_0$. It follows from an observation of Yau [40], based on the methods of Croke [10], that there is a global Sobolev constant c_s for functions of compact support on V, i.e.,

$$\|f\|_{2n/(n-2)} \leq c_s \|df\|_2 .$$

Consider now the annulus $A(\frac{1}{2}r, 2r) = \{x \in V : \frac{1}{2}r \leq \operatorname{dist}(x, z) \leq 2r\}$. Since $\int_V |R|^{n/2} < \infty$, we have $\int_{A(r/2, 2r)} |R|^{n/2} \to 0$ as $r \to \infty$. We may apply Lemma 2.1 and find there is an $R_0 > 0$ such that, for all $r \geq R_0$,

$$\sup_{S(r)} |R| \leq \frac{c}{r^2} \int_{A(r/2, 2r)} |R|^{n/2} \equiv \frac{\varepsilon(r)}{r^2},$$

where $\varepsilon(r) \to 0$ as $r \to 0$. In other words, the curvature of V decays faster than quadratically. It follows that the rescaled Riemannian manifolds $(A^r, g^r) = (A(r/2, 2r), g/r^2)$ have sectional curvature satisfying

$$(3.15)\qquad \sup_{A^r} |R^r| \leq 2\varepsilon(r) \to 0 .$$

Clearly, by Bishop's inequality, $\operatorname{vol} A^r \le c_6$. If $B^r(s)$ is a geodesic ball of radius s in A^r, then

$$\operatorname{vol} B^r(s) = \frac{\operatorname{vol} B(rs)}{r^n} = s^n \cdot \frac{\operatorname{vol} B(rs)}{(rs)^n}$$
$$\ge s^n \lim_{s\to\infty} \frac{\operatorname{vol} B(rs)}{(rs)^n} = c_0 s^n.$$

Note also that the diameter of A^r is bounded above.

These results imply by §2.4 that there is a uniform lower bound on the injectivity radius $\operatorname{inj}^r(x)$ in the g^r metric at each $x \in A^r$,

$$\operatorname{inj}^r(x) \ge \delta_0 > 0. \tag{3.16}$$

We may now apply the Gromov compactness theorem of §2.6 to an arbitrary sequence of components of (A^{r_i}, g^{r_i}). It follows that a subsequence converges, in the $C^{1,\alpha}$ topology, to a connected $C^{1,\alpha}$ Riemannian manifold $A^\infty = A^\infty(\frac{1}{2}, 2)$ of the same dimension. It is not difficult to verify that A^∞ is flat. One way to do this is to use the smoothing procedure carried out above to obtain bounds on $|D^k R|$ and pass to a (different) limit $(A^\infty)'$, which is then flat. However, more directly, as we will see in §4, since V (and so A^r) is Einstein, there is a curvature bound of the form $|D^k R_{A^r}| \le c(k, |R_{A^r}|, \operatorname{inj}_{A^r})$, so that in fact (A^{r_i}, g^{r_i}) converges smoothly to its limit A^∞. By (3.15), it follows that A^∞ is a smooth, flat Riemannian manifold.

Clearly, this process may be carried out with the manifolds $(A(r/n, nr), g/r^2)$ for any $n > 1$ and by passing to subsequences, one obtains connected limit manifolds A_n^∞ with $A_m^\infty \subset A_n^\infty$ if $m < n$. Taking a sequence $n_i \to \infty$, we obtain a limit flat connected, Riemannian manifold B_∞ $(= B_\infty(0, \infty))$ such that $A_n^\infty \subset B_\infty$ for any n.

We claim that B_∞ is the cone on a spherical space form $C(S^{n-1}(1)/\Gamma) - \{0\}$. First note that B_∞ has a distinguished distance function ρ, namely the limit of the distance function $r(x)$ to z on $A(r/n, nr)$. The function $\rho\colon B_\infty \to \mathbf{R}^+$ is a Lipschitz function, with Lipschitz constant 1, with $|\nabla\rho| = 1$ a.e. Let $C(s) = \{x \in B_\infty : \rho(x) = s\}$ so that $C(s)$ is the limit of a sequence of geodesic spheres $(S_z(r_i s), g^{r_i})$ in V. Thus, $\{C(s)\}$ is a family of equidistants in B_∞, i.e., $\operatorname{dist}_{B_\infty}(C(s), C(t)) = |s - t|$, with $|\rho(x) - \rho(y)| = |s - t|$ for $x \in C(s)$, $y \in C(t)$. We claim that $\operatorname{diam}_{B_\infty} C(s) \to 0$ as $s \to 0$. To see this, let $D(\frac{s}{2}) = \{x \in B_\infty : \operatorname{dist}(x, C(s)) \le \frac{s}{2}\}$ and note that $\operatorname{vol} D(\frac{s}{2}) \le c_1 \cdot s^n$. We may cover $D(\frac{s}{2})$ by a finite number, say Q, of balls $B_{x_i}(\frac{s}{4})$ in B_∞ with $x_i \in D(\frac{s}{2})$ such that the balls $B_{x_i}(\frac{s}{8})$ are disjoint. Then $\operatorname{vol} B_{x_i}(\frac{s}{8}) \ge c_2 \cdot s^n$, so that

$$Q c_2 s^n \le \operatorname{vol} D(s) \le c_3 s^n.$$

Thus, Q is bounded from above by a constant independent of s. In particular, any two points in $C(s)$ may be joined by a broken geodesic in $D(\frac{s}{2})$ of length $\le c_4 \cdot Q \cdot s$.

It follows that if we attach a point 0 to B_∞ via ρ, then $\overline{B}_\infty = B_\infty \cup \{0\}$ is a complete length space in the sense of Gromov [20, 1.7]. In particular, ρ extends continuously over $\{0\}$ and represents distance to $\{0\}$ in $\overline{B}_\infty$. Any point $x \in C(1)$ may be joined to 0 by a minimizing geodesic γ_x (not necessarily unique), such that $\rho(\gamma_x(t)) = t$.

Let $\widetilde{B}_\infty$ be the universal cover of B_∞. Since $\widetilde{B}_\infty$ is flat and simply connected, the developing map $D: \widetilde{B}_\infty \to \mathbf{R}^n$ is an isometric immersion. In particular, D maps geodesics in $\widetilde{B}_\infty$ to geodesics (straight line segments) in $\mathbf{R}^n$. Choose a geodesic γ_x in B_∞ as above and let $\tilde{\gamma}_x$ be a lift of γ_x to $\widetilde{B}_\infty$. Then $D(\tilde{\gamma}_x)$ is a line segment in $\mathbf{R}^n$ which we may normalize to start at $0 \in \mathbf{R}^n$. Let $y \in C(1)$ and choose points $p_i \in C(s_i)$, with $s_i \to 0$, and curves σ_i, σ joining p_i to $\gamma_x \cap C(s_i)$ and x to y, respectively. Let $\gamma_y(i)$ be minimizing geodesics joining p_i to y in the homotopy class of the curves $\sigma \circ \gamma_x \circ \sigma_i$. Then a subsequence of the curves $\gamma_y(i)$ converges to a minimizing geodesic γ_y joining $0 \in B_\infty$ and $y \in C(1)$. Let $\tilde{\gamma}_y$ be the lift of γ_y to $\widetilde{B}_\infty$ corresponding to $\tilde{\gamma}_x$. Since $\operatorname{dist}(\gamma_x(t), \gamma_y(t)) \to 0$, as $t \to 0$, $\operatorname{dist}(\tilde{\gamma}_x(t), \tilde{\gamma}_y(t)) \to 0$ also, so that $\operatorname{dist}_{\mathbf{R}^n}(D(\tilde{\gamma}_x(t)), D(\tilde{\gamma}_y(t))) \to 0$.

If $\Sigma(s)$ is the inverse image of $C(s)$ under $\Pi: \widetilde{B}_\infty \to B_\infty$, then we see that the developing map D maps $\Sigma(s)$ into $S(s)$, the sphere of radius s about 0 in $\mathbf{R}^n$. It is clear that D is an isometric immersion of $\Sigma(s)$ onto a domain in $S(s)$, and since $\Sigma(s)$ is complete, D maps $\Sigma(s)$ onto $S(s)$. Thus, D is a covering map and since $n > 2$, it follows that D is a diffeomorphism and thus an isometry.

It follows that $\widetilde{B}_\infty$ is isometric to $\mathbf{R}^n - \{0\}$ and $\Gamma = \pi_1(B_\infty)$ acts by isometries on $\mathbf{R}^n - \{0\}$ and thus on $\mathbf{R}^n$ fixing $\{0\}$. So Γ acts freely and isometrically on $S^{n-1}(1)$ so that B_∞ is isometric to $\mathbf{R}^n - \{0\}/\Gamma = C(S^{n-1}/\Gamma) - \{0\}$, as claimed. The manifolds $A_n^\infty \subset B_\infty$ are the domains $A_n^\infty = \{x \in B_\infty : \frac{1}{n} \leq \rho(x) \leq n\}$, so that A_n^∞ is isometric to the truncated cone on S^{n-1}/Γ. In particular, by the Gromov compactness theorem, the original manifolds (A^{r_i}, g) have subsequences which are diffeomorphic to $(\frac{1}{2}, 2) \times S^{n-1}(1)/\Gamma$, for some Γ.

We see there is an $R_0 > 0$ such that, for all $r \geq R_0$, $A(r/2, 2r)$ is diffeomorphic to $(\frac{1}{2}, 2) \times S^{n-1}/\Gamma$, where Γ may possibly depend on r. However, since the convergence $A^r \to A^\infty$ is smooth, we now see that $|\nabla \rho(x)| > 0$, for $\rho(x) \geq R_0$, where ρ is a smooth approximation of the distance function r. Since $(\frac{1}{2}, 2) \times S^{n-1}/\Gamma$ and $(2, 4) \times S^{n-1}/\Gamma'$ are diffeomorphic only if $\Gamma = \Gamma'$, we have $A(R_0, \infty)$ is diffeomorphic to $(R_0, \infty) \times S^{n-1}/\Gamma$. The volume growth assumption (3.14)(iii), together with the Cheeger-Gromoll splitting theorem, implies that $A(R_0, \infty)$ is connected.

This completes the first part of Theorem 3.5. To prove the second part, if V is simply connected at infinity, then $\Gamma = \{e\}$. However, the arguments above imply

$$\lim_{r\to\infty} \frac{\operatorname{vol} B_z(r)}{r^n} \geq \lim_{r\to\infty} \operatorname{vol} A^r\left(\frac{1}{m}r, r\right)$$
$$= \operatorname{vol} B^n(1) - \operatorname{vol} B^n\left(\frac{1}{m}\right),$$

for any $m > 1$, where $B^n(r)$ is the Euclidean ball of radius r. Thus,

$$\lim_{r\to\infty} \frac{\operatorname{vol} B_z(r)}{r^n} \geq \omega_n$$

so that the result follows from Bishop's inequality (§2.5). □

We may now complete the proof of Theorems A and A$'$. Recall that prior to Theorem 3.5, we constructed a complete, noncompact Riemannian manifold N satisfying (3.14)(i)–(iv). By Theorem 3.5, it follows that N is diffeomorphic to the interior of a compact manifold N with boundary $\partial N = S^{n-1}/\Gamma$.

So far, we have not used condition (iii) in Theorems A and A$'$, and we now use it to prove that $\Gamma = \{e\}$. First, from (3.14)(iii) and Proposition 1.2 of [1], it follows that $\pi_1(N)$ is a finite group. If we prove $\Gamma = \{e\}$ for $\widetilde{N}$, so that $\widetilde{N} = \mathbf{R}^n$, then clearly $N = \mathbf{R}^n$ also, since a finite group cannot act freely on $\mathbf{R}^n$. Thus, we may assume $\pi_1(N) = \{e\}$.

Recall that the manifold N is constructed as a limit $(M_i, h_i(t))$ with $0 < t \leq \varepsilon_1$. The manifolds $(M_i, h_i(t))$ are close, in the Lipschitz topology, to the (unsmoothed) manifolds (M_i, h_i), $h_i = h_i(0)$. Note that, by Gromov's compactness theorem, a subsequence of $\{(M_i, h_i)\}$ converges, uniformly in the $C^{1,\alpha}$ topology on compact sets, to a complete Riemannian manifold (M_∞, h_∞), with $C^{1,\alpha}$ metric h_∞. In particular, geodesics in (M_i, h_i) converge, uniformly on compact sets, to geodesics in (M_∞, h_∞). Clearly, (M_∞, h_∞) and $(N, h_\infty(t))$ are close in the Lipschitz topology, so in particular they are diffeomorphic.

It follows that there is a sequence of domains $B_i = B_{x_i}(R) \subset (M_i, h_i)$, for an arbitrary but fixed R, such that B_i is diffeomorphic, and metrically close, to $B_z(R) \subset (N, h_N(t))$, for i sufficiently large. In particular, $\partial B_i \approx S^{n-1}/\Gamma$ and since $B_z(R) \subset (N, h_N(t))$ is a strictly convex domain, for R large, $B_i \subset (M_i, h_i)$ is also (for i sufficiently large). Now there is an $R_0 > 0$ such that if $B_0 = B_{x_i}(R_0) \subset B_i$, then $\pi_1(B_i - B_0) = \pi_1(\partial B_i) = \Gamma$. Suppose $\Gamma \neq \{e\}$. Fix a point $p = p_i \in \partial B_i$ and let $\gamma = \gamma_i$ be a curve of shortest length in $B_i - B_0$ representing a nonzero class in $\pi_1(B_i - B_0)$. We claim that if $|\Gamma| > 3$, then γ does not intersect B_0, and thus γ is a geodesic loop in $B_i - B_0$ (since ∂B_i is convex). To see this, if $\gamma \cap B_0 \neq \varnothing$, then for R_0 sufficiently large, and $R \gg R_0$, $l(\gamma) \approx 2(R - R_0)$, since the metric on $B_i - B_0$ is close to the flat metric on $A(R_0, R) \subset C(S^{n-1}/\Gamma)$. However, since the metric on ∂B_i is close to the canonical metric on $S^{n-1}(R)/\Gamma$, there is a curve σ in ∂B_i of length $l(\sigma) \approx 2\pi R/|\Gamma|$. Since $2\pi/|\Gamma| < 2$ if $|\Gamma| > 3$, this shows $\gamma \cap B_0 = \varnothing$, for R_0, R, and i sufficiently large.

Thus, in this case, B_i contains geodesic loops $\gamma = \gamma_i$ of a bounded length $l(\gamma_i) \approx 2\pi/|\Gamma| \cdot R$. Since B_i is simply connected, these geodesic loops are inessential. However, by (3.3)(6), the length of an inessential geodesic loop in (M_i, h_i) (and thus in a finite cover of M_i), satisfies

$$l_{M_i}(h_i) \geq c_3 \cdot R_i \to \infty, \quad \text{as } i \to \infty. \tag{3.17}$$

This contradiction shows that we must have $|\Gamma| \leq 3$. In this case, we may argue as follows. By (3.17), and Morse theory for the length functional of the h_i metric, it follows that $\pi_k(\Omega^{t_i}) = 0$ for all k, where $t_i = c_3 \cdot R_i$ and Ω^{t_i} is the space of loops of length $< t_i$ in the h_i metric. Since the metrics h_i converge smoothly to the metric h_∞, the length functional converges also and it follows, for instance from the minimax characterization, that $\pi_k(\Omega) = 0$ for all k. (There are no critical points of the length functional of h_∞ associated to any homotopy class $S^k \to \Omega$.) Thus, Ω and so $B_i \approx B$ is contractible. An elementary exact homology sequence argument for the pair $(B, \partial B)$, cf. also Lemma 6.3, shows that $\partial B \approx S^{n-1}/\Gamma$ is a homology sphere. Since $|\Gamma| \leq 3$, this is only possible if $\Gamma = \{e\}$, as required.

Theorem 3.5 now implies that N is flat, which contradicts property (3.14)(iv) of N. This completes the proof of Theorems A and A$'$ (I). To prove A$'$ (II), note that we may take the oriented double cover of the manifold N satisfying (3.14)(i)–(iv) and obtain an oriented manifold, call it N again, satisfying (3.14)(i)–(iv). Now by Theorem 3.5, N has a well-defined boundary S^{n-1}/Γ at infinity. If n is odd, S^{n-1} is an even-dimensional sphere, so the only space forms are $\Gamma = \{e\}$ or $\Gamma = \mathbf{Z}_2$. Since ∂N carries a canonical orientation, and $\mathbf{RP}^{n-1}$ is nonorientable if n is odd, this case is ruled out. Thus, N is simply connected at infinity and the arguments above complete the proof. □

Remarks. (1) The condition $l_M \geq c$ in Theorems A and A$'$ could be dropped if the hypotheses of Theorem 3.5 implied that V is simply connected at infinity, and thus flat. However, this is false in even dimensions. In fact, a number of authors, [7, 13, 22] among others, have produced examples of Ricci-flat metrics on simply connected 4-manifolds, which are asymptotic to lens spaces at infinity, and so in particular are not flat. These are examples of gravitational instantons, or gravitational analogues of the self-dual Yang-Mills fields on $\mathbf{R}^4$. In higher dimensions, Calabi [7] has produced examples of Ricci-flat Kahler metrics on the canonical line bundle over $\mathbf{CP}^{n-1}$. At infinity, these manifolds have the topology of $S^{2n-1}/\mathbf{Z}_n$, and one may verify that their curvature tensor is in fact in L^2 for all n.

(2) For the proof of Theorem A under the assumption $\mathscr{A}_M \geq a > 0$, we refer to the remark following the proof of Lemma 6.3.

(3) It would be interesting to know if the bound on $\int_M |R|^{n/2}$ in Theorem A$'$ can be replaced by a bound on a weaker curvature integral. For instance, is it

sufficient to bound $\sup_{r>0} r^{4-n} \int_{B(r)} |R|^2$? If M is a Kahler-Einstein manifold, then this integral is related to topological invariants of M.

4. Proof of Corollary B

In this section, we will prove the parts of Corollary B. For the most part, the results follow in a straightforward way from Theorems A and A$'$.

If (M, g) is an Einstein manifold, then $\mathrm{Ric}_M = \lambda g$ and we will assume in this section that the metric on M is normalized so that $\lambda = -1, 0$, or $+1$. Since $Dg = 0$, we have $D\,\mathrm{Ric} = 0$ so that condition (ii) in Theorems A and A$'$ is trivially satisfied.

(1)(a) If $\mathrm{Ric}_M = +1g$, then $\mathrm{diam}_M \leq \pi$ by Myer's theorem. Thus, by Theorem A, the space $\mathscr{E}^{+1}(l_M, v_M, b_2(M))$ of $+1$ Einstein metrics on compact 4-manifolds such that $l_M \geq c_1$, $\mathrm{vol}_M \geq c_2$, $b_2(M) \leq c_3$ is $C^{1,1}$ compact. Further, by the proof of Theorem A, the sectional curvature $|K_M|$ is uniformly bounded on $\mathscr{E}^{+1}(l_M, v_M, b_2(M))$. By Cheeger's theorem (§2.4), there is a uniform lower bound on the injectivity radius.

Thus, we have uniform lower bounds for the size of harmonic balls for metrics (M, g) in $\mathscr{E}^{+1}(l_M, v_M, b_2(M))$, and also uniform bounds $\|g\|_{1,\alpha}$ for the metric tensor in these coordinates (cf. Fact of §2.6). Now in harmonic coordinates, the Einstein equation is an elliptic system of partial differential equations

$$(4.1) \qquad -g^{rs}\frac{\partial g_{ij}}{\partial x^r \partial x^s} + g^{rs}\left[\frac{\partial^2 g_{ri}}{\partial x^s \partial x^j} + \frac{\partial^2 g_{ri}}{\partial x^s \partial x^i} - \frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right] + \cdots = 2\lambda g_{ij},$$

where the dots indicate lower order terms involving only one derivative of the metric, cf. [11]. It follows from standard elliptic theory [28] that one has bounds

$$(4.2) \qquad \|g\|_{k,\alpha} \leq C(k, \|g\|_{1,\alpha}),$$

so that all covariant derivatives of the curvature tensor $D^k R$ have uniform bounds on $\mathscr{E}^{+1}(l_M, v_M, b_2(M))$. By the Gromov compactness theorem (Theorem 2.2), we see that $\mathscr{E}^{+1}(l_M, v_M, b_2(M))$ is thus compact in the $C^{k,\alpha}$ topology for any k.

(1)(b) Let D be a connected component of the moduli space of Einstein metrics on a compact manifold M. It is well known that, first, tr Ric is constant on D, and, secondly, that the volume function

$$(4.3) \qquad \mathrm{vol}(g_0) = \mathrm{vol}\colon D \to \mathbf{R}$$

is a constant function on D [5]. In particular, the volume is bounded below on D so the result follows from (1)(a). □

(2), (3) These results follow in exactly the same way as in (1) above. □

(4) We argue by contradiction. If the statement were false, there would exist a sequence (M_i, g_i) satisfying the required bounds such that

$$\int_{M_i} |\operatorname{Ric}(g_i) - \lambda g_i| \to 0,$$

but the M_i have no Einstein metric $\operatorname{Ric} = \lambda g$ for all i. However, the bounds on (M_i, g_i) imply by Theorems A and A$'$ that the sectional curvature on (M_i, g_i), $|K(g_i)|$ is uniformly bounded. Further, the metrics g_i have a $C^{1,\alpha}$ convergent subsequence to a $C^{1,\alpha}$ metric g on a smooth manifold M. Clearly, M is diffeomorphic to $M_{i'}$ for a subsequence $i' \to \infty$. Note, however, that g is a weak $(C^{1,\alpha})$ solution of the Einstein equation (4.1). The regularity theory for elliptic systems [28] then implies that g is smooth and satisfies the Einstein equation $\operatorname{Ric}(g) = \lambda g$. This contradicts the assumption, which proves the first statement. The other statements are proved in the same way. □

5. Convergence of Einstein metrics

This section will be devoted to the proof of Theorem C. The proof uses a number of methods and results from the previous sections. Some of these techniques are also used in [14, 29]; cf. also [33] for a very readable exposition. Let (M_i, g_i) be a sequence of Einstein manifolds satisfying the conditions of Theorem C. It follows from §2.2 that one has uniform lower bounds for the Sobolev constants c_s of (M_i, g_i) and also for the growth of small geodesic balls $B_x(r) \subset M_i$, namely $v(r)/r^n \geq c_0$, by (2.3).

By a theorem of Gromov [20, 5.3], (M_i, g_i) has a subsequence which converges, in the Hausdorff topology, to a compact length space (M_∞, g_∞). In particular, (M_∞, g_∞) is a connected metric space, and there is a well-defined notion of geodesic balls in M_∞ with the geodesic balls in M_i (sub)converging to geodesic balls in M_∞ in the Hausdorff topology. Further, the notion of length of a curve is well defined and between any two points in M_∞, there is a minimizing geodesic.

Let ε_0 be the constant of Lemma 2.1, determined by the bounds $\operatorname{Ric}_{M_i} = \lambda_i g_i$, $\operatorname{vol}_{M_i} \geq c_1$, $\operatorname{diam}_{M_i} \leq c_2$ on (M_i, g_i). We fix an $r > 0$ but small and let $\{x_k\} = \{x_k^i\}$ be a maximal $\frac{r}{2}$ separated set in M. Thus, the geodesic balls $B_{x_k}(\frac{r}{4})$ are disjoint, while the balls $B_{x_k}(r)$ form a cover of M_i. We let

$$G^r = G_i^r = \bigcup \left\{ B_{x_k}(r) \colon \int_{B_{x_k}(2r)} |R|^{n/2} < \varepsilon_0 \right\},$$

where $R = R_i$ is the curvature tensor of (M_i, g_i). Similarly, set

$$B^r = B_i^r = \bigcup \left\{ B_{x_k}(r) \colon \int_{B_{x_k}(2r)} |R|^{n/2} \geq \varepsilon_0 \right\}.$$

Then $M_i = G_i^r \cup B_i^r$. Note that there is a bound on the number of balls Q_i^r in B_i^r, independent of i and r, namely,

$$Q_i^r \le \frac{c_3}{\varepsilon_0} \cdot m, \tag{5.1}$$

where c_3 is the bound (iii) in Theorem C and m is the maximal number of disjoint balls of radius $\frac{r}{2}$ in M_i contained in a ball of radius $2r$. By the (relative) Bishop comparison theorem, there is a uniform bound on m, independent of i, r.

Now, by Lemma 2.1, on each G_i^r there is a uniform curvature bound $|R_i(x)| \le C(\operatorname{dist}(x, B_i^r)) \le C(r^{-1})$. Clearly, $\operatorname{vol} G_i^r$ is uniformly bounded above, since this holds for (M_i, g_i). From the bound above, on the volume growth of small geodesic balls, and from §2.4, it follows that the injectivity radius of M_i at each $x \in G_i^r$ has a uniform lower bound, depending only on r and the bounds on (M_i, g_i). Thus, the regularity result (4.2) implies a uniform bound on the covariant derivatives $|D^k R|(x)$ for $x \in G_i^r$.

By Theorem 2.2, for $r > 0$ fixed, a subsequence of $\{G_i^r\}$ converges, in the $C^{k,\alpha}$ topology on compact sets, to a smooth Riemannian manifold G^r with Einstein metric g_∞^r. In particular, G^r and G_i^r are diffeomorphic for i sufficiently large (in the subsequence) so that there are smooth embeddings $F_i^r \colon G^r \to M_i$ such that $(F_i^r)^* g_i$ converges uniformly in the $C^{k,\alpha}$ topology to g_∞^r on G^r.

We now choose a sequence $\{r_j\} \to 0$ with $r_{j+1} < \frac{1}{2} r_j$ and perform the above construction for each j. Let $G_i(r_l) = \{x \in M_i \colon x \in G_i^j, \text{ for some } j \le l\}$ so that one has inclusions

$$G_i(r_1) \subset G_i(r_2) \subset \cdots \subset M_i. \tag{5.2}$$

By the argument above, each $\{G_i(r_l)\}$, for l fixed, has a smoothly convergent subsequence to a limit $G(r_l)$. Clearly, $G(r_l) \subset G(r_{l+1})$ and we set

$$G = \bigcup_1^\infty G(r_l), \tag{5.3}$$

with the induced Riemannian metric g_∞. In particular, (G, g_∞) is a smooth Einstein manifold. It follows that there are smooth embeddings $F_i^l \colon G(r_l) \to M_i$, for i sufficiently large, such that $(F_i^l)^*(g_i)$ converges, smoothly in the $C^{k,\alpha}$ topology on $G(r_l)$, to the metric g_∞.

Let $\overline{G}$ be the metric completion of G. We claim there is a finite set of points $\{p_i\}$, $i = 1, \dots, Q'$, such that

$$\overline{G} = G \cup \{p_i\}. \tag{5.4}$$

To see this, we return to the sets $B_i^r \subset M_i$. Let $\{r_j\}$ be as above and recall from (5.1) that the cardinality of the number of balls in $B_i^j \subset M_i$ is uniformly bounded in i, j; by passing to subsequences, we may assume it is constant, say Q. Thus, there is a finite set of disjoint geodesic balls $\{B_{x_k}(Q \cdot r_j)\}_{i=1}^{Q'}$, with $Q' \leq Q$, which, together with $G_i(r_j)$, form a cover of M_i. In particular, every point of $M_i - G_i(r_j)$ is contained in a ball of diameter $\leq 2 \cdot Qr_j$. Using the embeddings $F_i^l \colon G(r_l) \to M_i$, we see that for any fixed j, and i sufficiently large, arbitrarily large compact subsets of $G - G(r_j)$ are almost isometrically embedded into Q' disjoint balls of radius $\leq c \cdot r_j$. Taking the limit as $i \to \infty$, it follows that the Hausdorff distance between $(G - G(r_j), g_\infty)$ and the set of Q' distinct points $\{p_i\}$ is $\leq c \cdot r_j$. Letting $j \to \infty$, it follows that the Hausdorff distance between $\partial G = \overline{G} - G$ and $\{p_i\}$ is 0, and thus these spaces are isometric. This gives (5.4).

It is now easy to see that a subsequence of $\{M_i\}$ converges to $\overline{G}$ in the Hausdorff topology. In fact, if $N = \{x_k\}$ is any ε-net in $\overline{G}$, with $x_k \notin \{p_i\}$, then the embeddings $F_i \colon G \to M_i$ give an ε'-net N_i in M_i converging to N in the Lipschitz topology (cf. [20, 3.5]). Thus, $\overline{G} = M_\infty$, so that $\overline{G}$ is a complete length space, with length function g_∞, which restricts to a smooth Einstein metric on G.

Definition. The points $\{p_i\}_1^{Q'}$ are called the *curvature singularities* of M_∞.

We now examine the structure, topological and metric, of the smooth manifold $N_j = B_{p_j}(r) - \{p_j\} \subset M_\infty$ for r small, by essentially studying the tangent cones of M_∞ at p_j. We will drop the subscript j for convenience. First, note that there is a uniform lower bound for the Sobolev constant c_s (2.2) for functions of compact support in N, since on any compact subset the convergence of M_i is smooth and there is a uniform Sobolev constant for $\{M_i\}$. In particular, it follows that $v(s)/s^n = \operatorname{vol} B_x(s)/s^n \geq c_0$, for any geodesic ball $B_x(s) \subset N$.

Now N is an Einstein manifold with $\int_N |R|^{n/2} < \infty$. Let $r(x)$ be the distance function to p in M_∞ and let $N(s,t) = \{t \in N, s \leq r(x) \leq t\}$. It follows that

$$\int_{N(s/2,2s)} |R|^{n/2} \to 0 \quad \text{as } s \to 0$$

so that there is an R_0 such that for $s \geq R_0$, $\int_{N(s/2,2s)} |R|^{n/2} \leq \varepsilon_0$, where ε_0 is the constant from Lemma 2.1. Thus, by Lemma 2.1, there is a function $\mu(s)$, with $\mu(s) \to 0$ as $s \to 0$ such that

$$\sup_{S(s)} |R| \leq \mu(s)/s^2, \tag{5.5}$$

where $S(s) = \{x \in N \colon r(x) = s\}$.

Now we proceed exactly as in the proof of Theorems A and A$'$, but blowing the metric up instead of down. Namely, given any sequence $s_i \to 0$, the Riemannian manifolds $A(s_i/2, 2s_i) = (N(s_i/2, 2s_i), g_\infty/s_i^2)$ have sectional curvatures converging to zero by (5.5). Further, the volumes of geodesic balls in

$A(s_i/2, 2s_i)$ have upper and lower bounds $c_0 \le v(r)/r^n \le c_0^{-1}$ and thus by §2.4, the injectivity radius of points $x \in A(s_i/2, 2s_i)$ have a uniform lower bound. Let us estimate the number of components of $A(s_i/2, 2s_i)$. If δ is a lower bound for the injectivity radius in $A(s_i/2, 2s_i)$, consider a maximal $\frac{\delta}{2}$ separated set $\{x_j\}$ in $A(s_i/2, 2s_i)$. Then the balls $B_{x_j}(\frac{\delta}{2})$ are disjoint and the balls $B_{x_j}(\delta)$ cover $A(s_i/2, 2s_i)$. From the above volume bounds, it follows that the cardinality of $\{x_j\}$ is uniformly bounded, independent of i. Clearly, points which lie in distinct components of $A(s_i/2, 2s_i)$ are (more than) δ separated so that one obtains a uniform upper bound on the number of components of $A(s_i/2, 2s_i)$. As in the proof of Corollary B(1)(a), there are uniform bounds on the covariant derivatives $|D^k R|$ of the curvature of the metric g_∞/s_i^2 on $A(s_i/2, 2s_i)$. It follows from Theorem 2.2 that a subsequence converges smoothly to a flat Riemannian manifold $A_\infty(\frac{1}{2}, 2)$ with a finite number of components.

This process may be carried out for $(N(s_i/n, ns_i), g_\infty/s_i^2)$, for any given n, and gives rise, by passing to a diagonal subsequence, to a flat Riemannian manifold B_∞ with $A_\infty(\frac{1}{n}, n) \subset B_\infty$ for all n. The proof of Theorems A and A$'$ shows that each component of B_∞ is a cone on a spherical space form $S^{n-1}(1)/\Gamma$ and as before, it follows that each component of N is diffeomorphic to $(0, r) \times S^{n-1}/\Gamma$.

We now show that $N = B_p(r) - p$ is connected for all r small. In fact, we will show that $\partial B_p(r) = S_p(r)$ is connected. First, it is clear that $B_p(r)$ is connected. Since $M_i \to M_\infty$ in the Hausdorff topology, there are smooth connected domains $D_i \subset M_i$ with $D_i \to B_p(r)$ in the Hausdorff topology. Further, since the convergence $M_i \to M_\infty$ is smooth away from p, $\partial D_i = S_i$ converges to $S_p(r)$ in the Lipschitz topology. By the results above, $S_p(r)$ is convex, so that each component of S_i is convex, i.e., has positive definite second fundamental form II w.r.t. the inward normal; in fact, II is approximately $\frac{1}{r}$I for i large. We will suppose $S_p(r)$, and thus ∂D_i, is disconnected and we obtain a contradiction.

Let γ be a geodesic in $D = D_i$ realizing the minimum distance between two distinct boundary components C_1, C_2 of $\partial D = \partial D_i$. A standard form of the second variational formula [6] then gives

$$\mathrm{II}_2(E_2, E_2) + \mathrm{II}_1(E_1, E_1) = \int_\gamma |\nabla_T E|^2 - K(E, T), \tag{5.6}$$

where II_i is the second fundamental form of C_i (w.r.t. the inward normal), E is the Jacobi field along γ determined by the boundary conditions $E(0) = E_1$, $E(l(\gamma)) = E_2$, and T is the unit tangent vector to γ. By the basic inequality for Jacobi fields (since there are no focal points of the normal exponential map of ∂D_i along γ),

$$\int_\gamma |\nabla_T E|^2 - K(E, T) \le \int_\gamma |\nabla_T X|^2 - K(X, T) \tag{5.7}$$

for any vector field X along γ with the same boundary conditions. Choose X to be a parallel vector field along γ with initial condition $E = E(0)$ a unit vector, and then sum (5.6) and (5.7) over an orthonormal basis of initial vectors. We then find

$$H_2 + H_1 \leq -l(\gamma) \cdot \mathrm{Ric}_{M_i},$$

where H_i is the mean curvature of C_i at the endpoints of γ. By the facts above, $H_i \approx (n-1)/r$, so that

$$\frac{2(n-1)}{r} \lesssim -l(\gamma) \cdot \mathrm{Ric}_{M_i}$$

for i large. If $\mathrm{Ric}_{M_i} = 0$ or $+1$, this is clearly impossible. If $\mathrm{Ric}_{M_i} = -1$, this is also impossible for r sufficiently small. It follows that $B_p(r) - p \subset M_\infty$ is connected and thus $B_p(r) \simeq B^n(1)/\Gamma$, the quotient of a ball in $\mathbf{R}^n$ by a finite group of orthogonal isometries.

Remark. Of course, it is possible that $\Gamma = \{e\}$, so that there may be curvature singularities on M_∞, as defined in (5.4), which are regular or smooth points of M_∞.

The arguments above prove that M_∞ has the structure of an orbifold with a finite number of curvature singularity points, each having a punctured neighborhood which is diffeomorphic to a punctured cone on a spherical space form. We now examine the metric behavior of these singularities $(N, p) \subset M_\infty$. It follows from the work above that the manifolds $N((s/2, 2s), g_\infty/r^2)$ converge, in the C^k topology, to the flat metric g_0 on $S^{n-1}/\Gamma \times (\frac{1}{2}, 2)$, as $s \to 0$. In particular, there are C^k diffeomorphisms $\phi_s\colon S^{n-1}/\Gamma \times (\frac{1}{2}, 2) \to N(s/2, 2s)$ such that

$$\|\phi_s^*(g_\infty/r^2) - g_0\|_{C^k} = \varepsilon(s), \tag{5.8}$$

where $\varepsilon(s) \to 0$ as $s \to 0$. Further, the distance functions $r = \mathrm{dist}_N(x, p)$ and $r_0 = \mathrm{dist}_{S^{n-1}/\Gamma\times(0,t)}(x, 0)$ converge, in the sense that $(\phi_s^*(r)/r_0)(x) \to 1$, as $s \to 0$. Thus,

$$\phi_s^* g_\infty = (r \circ \phi_s)^2 \cdot g_0 + (r \circ \phi_s)^2 \mu(s) = \frac{(r \circ \phi_s)^2}{r_0^2} r_0^2 \cdot g_0 + (r \circ \phi_s)^2 \mu(s).$$

This says that if we view $\phi_s^* g_\infty$ as a metric on $S^{n-1}/\Gamma \times (s/2, 2s)$, then

$$\phi_s^* g_\infty = \frac{(r \circ \phi_s)^2}{r_0^2} g_0 + (r \circ \phi_s)^2 \mu(s), \tag{5.9}$$

where $\mu(s) \to 0$, as $s \to 0$. Now the diffeomorphisms ϕ_s are not unique. However, we may view $g_s = g_{\infty|N(s/2,2s)}$ as a smooth curve in the space of metrics on a fixed manifold, say $N(s_0/2, 2s_0)$. It is then clear that the diffeomorphisms ϕ_s may be chosen to depend smoothly on s, satisfying (5.8). Now consider the smooth map $\Phi'\colon S^{n-1}/\Gamma \times (0, t) \to N - \{p\}$ defined by

$\Phi(\theta, s) = \phi_s(\theta) \equiv \phi_s(\theta, 1)$. Then Φ' may no longer be a diffeomorphism; however, this may easily be overcome as follows. Let ρ be a smooth approximation to the distance function r on $N - \{p\}$, such that $|\frac{\rho}{r} - 1| \leq r^2$. Note that $\nabla\rho$ is transverse to all the "level hypersurfaces" $H_s = \operatorname{Im}\Phi'(\cdot, s)$. Let $\psi_t(x)$ be the flow of $\nabla\rho$ and consider $\Phi\colon S^{n-1}/\Gamma \to N - \{p\}$ given by $\Phi(\theta, s) = \psi_{f(\theta,s)}(\phi_s(\theta))$. Here $f(\theta, s)$ is uniquely determined by the requirement that $\Phi(\theta, s) \in L_s \equiv \rho^{-1}(s)$. It is then clear that Φ is a diffeomorphism onto its image. Note also that $|f(\theta, s)| \leq \varepsilon(s) \cdot s$. Now it is easily seen that Φ also satisfies (5.9), i.e.,

$$\Phi^* g_\infty = \frac{(r \circ \Phi)^2}{r_0^2} \cdot g + o(r_0^2). \tag{5.10}$$

We may lift the metrics $\Phi^* g_\infty$ and g_0 to the universal cover $B^n - \{0\}$ of N and $C(S^{n-1}/\Gamma) - \{0\}$ and also lift Φ to a Γ-equivariant diffeomorphism Φ of $B^n - \{0\}$. Then (5.10) shows that the metric $\Phi^* g_\infty$ on $B^n - \{0\}$ has a C^0 extension over $\{0\}$. It can actually be arranged that $\Phi^* g_\infty$ has a C^1 extension over $\{0\}$, but we will not do this here.

It follows that $\Phi^* g_\infty$, which we will just call g_∞, is a weak (C^0) solution to the Einstein equation (4.1). At this stage, the regularity theory is not sufficient to imply that g_∞ is smooth. However, we have $\int_N |R|^{n/2} < \infty$, where R is the curvature tensor of g_∞. This situation has been treated by a number of authors, and we basically refer to these. For example, if $n \geq 5$, an elementary method of Sibner [34, Lemma 2.1, Proposition 2.4], which requires only a bound on the Sobolev constant on $N - p$, may easily be seen to apply to the present setting and shows that $R \in L^p$ for some $p > n/2$. If $n = 4$, using (5.5) and the Sobolev bound in $N - p$, one may verify step-by-step that the basic methods of Uhlenbeck [36, Theorem 4.1] remain valid here also, and show $R \in L^p$ for some $p > 2$. It now follows from standard elliptic theory [28, Chapter 6; 36, Theorem 3.6] that g_∞ does extend smoothly across 0. For more details in the proof of this, we refer, for instance, to [2]. This completes the main part of the proof of Theorem C. □

We note that, even if M_∞ is a smooth manifold with smooth Einstein metric, it does not follow from the proof above that $M_i \to M_\infty$ smoothly. In fact, arbitrarily small neighborhoods of the points $x_k(i) \in B_k^i \subset M_i$, with $x_k(i)$ converging to a curvature singularity $p_k \in M_\infty$, will have nontrivial topology (as we shall see later). This topology is "squeezed" or "bubbled" off as $i \to \infty$, regardless of whether M_∞ is a smooth manifold or not.

For each curvature singularity $p \in \{p_k\} \subset M_\infty$, there is a sequence $x_i \in M_i$ such that $x_i \to p$ and $\inf_{r>0} \sup\{|K_i(x)|\colon x \in B_{x_i}(r) \subset M_i\} \to \infty$, as $i \to \infty$. Since the curvature of M_i remains bounded in bounded distances away from x_i, we may assume that x_i realizes the maximum R_i of $|K(x)|$ on $B_{x_i}(r_0)$ for r_0 small. Now consider the pointed connected Riemannian manifolds

$V_i = (B_{x_i}(r_0), x_i, R_i^{1/2}\, ds^2_{M_i})$. We note that the curvature of V_i is uniformly bounded and $|K(x_i)| = 1$. Similarly, $\int_{V_i} |R|^{n/2} \leq C$ and the Sobolev constants for $\{V_i\}$ are uniformly bounded below, since this is true for M_i itself. As in the proof of Theorems A and A$'$, it follows that a subsequence of $\{V_i\}$ converges, smoothly in the C^k topology on compact sets, to a complete connected Riemannian manifold $V(= V_p)$ satisfying

$$\text{(5.11)} \qquad \begin{array}{ll} \text{(i)} & \mathrm{Ric}_V = 0, \\ \text{(ii)} & \mathrm{vol}\, B(r)/r^n \geq C, \\ \text{(iii)} & \displaystyle\int_V |R|^{n/2} \leq K, \\ \text{(iv)} & |K(x_0)| = 1, \text{ for some } x_0 \in V. \end{array}$$

Definition 5.1. A complete connected Riemannian manifold V satisfying (5.11), obtained in the above manner, will be called an EALE (Einstein, asymptotically locally Euclidean) space associated to the curvature singularity $p \in M_\infty$.

We note that, a priori, there may be more than one EALE space associated with $p \in M_\infty$. By construction, if Ω is a domain with smooth boundary in V, then there are smooth domains $\Omega_i \subset M_i$ and diffeomorphisms $\phi_i \colon \Omega \to \Omega_i$ such that $\phi_i^*(r_i g_i|_{\Omega_i})$ converges smoothly to (Ω, g_V), for an appropriate scaling $r_i \to \infty$. Note that V cannot be diffeomorphic to a ball, by Theorem 3.5, so that Ω_i contains nontrivial topology squeezed off in the limit (cf. also Lemma 6.3 for the 4-dimensional case).

Now, to complete the proof of Theorem C, we recall from Theorem 3.5 that in odd dimensions, nontrivial, i.e., nonflat, EALE spaces do not exist, so that it follows that there are no curvature singularities in M_∞. It follows that M_∞ is a smooth manifold and the convergence $M_i \to M_\infty$ is smooth. □

Remark. Some statements of Kobayashi and Todorov [27] indicate that Theorem C is sharp in the sense that they exhibit examples of sequences of Einstein metrics satisfying the bounds (i), (iii) (and apparently (ii)), which converge to an Einstein orbifold in the above sense. More precisely, let $T^4 = \mathbf{C}^2/\mathbf{Z}^4$, where $\mathbf{Z}^4$ is the standard (square) lattice in $\mathbf{C}^2$. The antipodal map $A\colon \mathbf{C}^2 \to \mathbf{C}^2$ preserves the lattice and the quotient $X = T^4/\mathbf{Z}_2$ defines a flat orbifold, with flat singular metric g_X having sixteen singular points, each a cone on $\mathbf{RP}^3$. If these sixteen points are blown up, one obtains a smooth 4-manifold Y which is a K3 surface. Kobayashi and Todorov [27] then indicate that there are sequences of Ricci-flat Kahler metrics g_i on Y such that $g_i \to g_X$ in the Hausdorff distance, with convergence $g_i \to g_\infty$ smooth away from the singular points.

Finally, we note that by (5.1), the number of singular points of M_∞ is bounded above by the bounds (i)–(iii) of Theorem C. Similarly, the order $|\Gamma_j|$ of each group Γ_j associated with a singular point p_j is bounded above. Namely, note that g_∞ lifts to an Einstein metric on $B^n \setminus \{0\}$, which extends, modulo $\mathrm{Diff}(B^n - \{0\})$, to a smooth metric on B^n. Thus, $\mathrm{vol}(B_0(r), g_\infty) \leq c_1 \cdot r^n$, where $B_0(r)$ is the geodesic ball of radius r about 0 in the g_∞ metric. Here we assume $r \leq r_0$, so that $c_1 = \sinh^{n-1} r_0$. Thus, if $B_p(r)$ is the r-ball about $p \in M_\infty$, then $\mathrm{vol}\, B_p(r) \leq c_1 r^n/|\Gamma|$. However, by the bounds (i), (ii) in Theorem C, there is a uniform lower bound for the Sobolev constant and thus for the volume of small geodesic balls in M_i. It follows that $\mathrm{vol}\, B_x(r/4) \geq c_2 r^n$ for $x \in S_p(\frac{r}{2})$. This shows that $|\Gamma|$ is bounded by $c_1 \cdot c_2$.

6. Compactness of moduli spaces

In this section, we consider the question of the compactness, in a smooth topology, of the moduli space of positive Einstein metrics on compact 4-manifolds. Basically, the relation

$$\chi(M) = \frac{1}{8\pi^2} \int_M |R|^2$$

for Einstein metrics in dimension 4 leads to further consequences of Theorem C.

Proposition 6.1. *Let (M_i, g_i) be a sequence of 4-dimensional Einstein manifolds satsfying the conditions of Theorem C. Then*

$$\underline{\lim}\, \chi(M_i) = \underline{\lim}\, \frac{1}{8\pi^2} \int_{M_i} |R|^2 \geq \frac{1}{8\pi^2} \int_{M_\infty} |R|^2 + \sum_1^Q \int_{V_j} |R|^2, \tag{6.1}$$

where Q is the number of curvature singularities of M_∞ (cf. (5.4)) and V_j are associated EALE spaces (cf. Definition 5.1). Further, if $\underline{\lim}\, \chi(M_i) = \frac{1}{8\pi^2} \int_{M_\infty} |R|^2$, then M_∞ is a smooth manifold diffeomorphic to M_i for i sufficiently large, and the convergence $M_i \to M_\infty$ is smooth.

Proof. By Theorem C, a subsequence of (M_i, g_i), also called (M_i, g_i), converges to an Einstein orbifold M_∞. Let G_i^r and B_i^r be defined as in the proof of Theorem C, $M_i = G_i^r \cup B_i^r$. Then

$$\lim_{i\to\infty} \int_{M_i} |R|^2 = \lim_{i\to\infty} \int_{G_i^r} |R|^2 + \lim_{i\to\infty} \int_{B_i^r} |R|^2 = \int_{G^r} |R|^2 + \lim_{i\to\infty} \int_{B_i^r} |R|^2,$$

since the convergence $G_i^r \to G^r$ is smooth. On the other hand, since $\int |R|^2$ is scale-invariant, $\lim_{i\to\infty} \int_{B_i^r} |R|^2 \geq \int_{V_i} |R|^2$, where V_i is an EALE space associated with p_i. Since r is arbitrary, this proves the first statement. For the second statement, the equality implies there are no nontrivial EALE spaces associated to M_∞, and thus no curvature singularities of M_∞. This implies by the proof of Theorem C that $M_i \to M_\infty$ smoothly. □

Since the metric on M_∞ extends (smoothly in the local universal cover) over the singularities, $(1/8\pi^2) \int_{M_\infty} |R|^2$ may be identified with the orbifold Euler characteristic of M_∞. Namely, if we excise small balls $B_{p_i}(r) \subset M_\infty$ around the singular points and let $M_r = M_\infty - \bigcup B_{p_i}$, then the Gauss-Bonnet theorem for manifolds with boundary [17] gives

$$\chi(M_r) = \frac{1}{8\pi^2}\int_{M_r} |R|^2 + \int_{\partial M_r} \omega,$$

where ω is a 3-form on ∂M_r depending on R and the second fundamental form of ∂M_r. Since $|R|$ is bounded on M_∞, $\int_{M_r} |R|^2 \to \int_{M_\infty} |R|^2$ and $\chi(M_r) = \chi(M_0)$, where $M_0 = M_\infty - \{p_i\}$. Using the fact that each ball $B_{p_i}(r)$ is covered by a ball in $\mathbf{R}^n$ with smooth metric, one calculates that

$$\int_{\partial M_r} \omega \to -\sum_1^Q \frac{1}{|\Gamma_j|}$$

as $r \to 0$, where Γ_j is the fundamental group of $B_{p_j}(r) - p_j$. Thus,

$$\frac{1}{8\pi^2}\int_{M_\infty} |R|^2 = \chi(M_0) + \sum_1^Q \frac{1}{|\Gamma_j|} = \chi(M_\infty), \tag{6.2}$$

where $\chi(M_\infty)$ is the orbifold Euler characteristic of M_∞.

Similarly, $\int_V |R|^2$ also admits a topological interpretation. If V_r denotes a geodesic ball of radius r in V, then as before we have

$$\chi(V_r) = \frac{1}{8\pi^2}\int_{V_r} |R|^2 + \int_{\partial V_r} \omega.$$

For r sufficiently large, $\chi(V_r) = \chi(V)$ and clearly, $\int_{V_r} |R|^2 \to \int_V |R|^2$. Since V is asymptotically flat, one may verify that

$$\int_{\partial V_r} \omega \to \frac{1}{|\Theta|},$$

where $|\Theta|$ is the order of the fundamental group of V at infinity. Thus,

$$\frac{1}{8\pi^2}\int_V |R|^2 = \chi(V) - \frac{1}{|\Theta|}. \tag{6.3}$$

Recall from Theorem 3.5 that $|\Theta| = 1$ if and only if V is isometric to $\mathbf{R}^4$. Summarizing, we obtain

Corollary 6.2. *Let (M_i, g_i) be a sequence of Einstein metrics as above. Then*

$$\lim \chi(M_i) \geq \chi(M_0) + \sum_1^Q \frac{1}{|\Gamma_j|} + \sum_1^Q \left[\chi(V_j) - \frac{1}{|\Theta_j|}\right]. \tag{6.4}$$

Lemma 6.3. *If V is a 4-dimensional EALE space, then $\chi(V) \geq 1$, with equality if and only if V is isometric to $\mathbf{R}^4$.*

Proof. Since V is open, $\chi(V) = 1 - b_1 + b_2 - b_3$. Since V is Ricci flat and asymptotically locally Euclidean, $\pi_1(V)$ is a finite group [1], so that $b_1 = 0$. By the exact homology sequence of the pair $(V, \partial V) = (V, S^3/\Gamma)$, one has

$$0 \to H_4(V, \partial V) \to H_3(\partial V) \to H_3(V) \to H_3(V, \partial V).$$

By Poincaré duality, it then follows that $H_3(V) = 0$, so that $\chi(V) = 1 + b_2$. Now suppose $b_2 = 0$. Then a finite cover V' of V is a manifold with boundary, with all $\mathbf{Z}$-homology groups zero (since V' has no torsion in H_2). For convenience, we drop the prime and assume V is an acyclic 4-manifold with boundary $\partial V = S^3/\Gamma$, $\Gamma \subset \mathrm{SO}(4)$. It is easily seen that ∂V is a homology 3-sphere, since, by the exact homology sequence, we have

$$H_1(S^3/\Gamma; \mathbf{Z}) \approx H_2(V, S^3/\Gamma; \mathbf{Z}) \approx H^2(V; \mathbf{Z}) = 0.$$

It is well known [39, p. 181ff.] that the only spherical space forms with perfect π_1 are S^3 and the Poincaré homology sphere S^3/Γ_0, where Γ_0 is the binary icosohedral group of order 120.

There are two methods of proof to rule out the second case. First, by (6.3), we have

$$\frac{1}{8\pi^2}\int_V |R|^2 = \chi(V) - \frac{1}{|\Gamma_0|} = 1 - \frac{1}{|\Gamma_0|}.$$

On the other hand, one has a comparable expression for the signature τ of V, cf., e.g., [15],

$$0 = \tau(V) = \frac{1}{12\pi^2}\int_V |R^+|^2 - |R^-|^2 + \left(-1 + \frac{1}{|\Gamma_0|}\right) + \eta(S^3/\Gamma_0),$$

where $\eta(S^3/\Gamma_0)$ is the eta invariant and R^+ (R^-) are the self-dual (antiself-dual) components of R. One computes [15] that $\eta(S^3/\Gamma_0) = 1079/360$. Thus,

$$\begin{aligned} 1 - \frac{1}{|\Gamma_0|} - \frac{1079}{360} &= \frac{1}{12\pi^2}\int_V |R^+|^2 - |R^-|^2 \\ &\geq -\frac{2}{3}\left(\frac{1}{8\pi^2}\int_V |R|^2\right) = -\frac{2}{3}\left(1 - \frac{1}{|\Gamma_0|}\right), \end{aligned}$$

i.e.,

$$\frac{5}{3}\left(1 - \frac{1}{|\Gamma_0|}\right) \geq \frac{1079}{360}.$$

This is clearly impossible, so that S^3/Γ_0 cannot bound an acyclic EALE space.

A second proof, purely topological, follows from work of Freedman and Donaldson. Namely, the Poincaré homology 3-sphere cannot bound a smooth contractible 4-manifold, since it bounds a smooth 4-manifold W with definite intersection form, equal to the E_8 lattice. In fact, W is obtained by plumbing the disc bundles over S^2 with $c_1 = -2$ according to the Dynkin diagram of E_8. (The interested reader may refer to [18, Problem section] to pursue this.)

It follows that $\partial V = S^3$ and the result follows from Theorem 3.5. □

Remark. Lemma 6.3 allows one to replace the hypothesis $l_M \geq l > 0$ by $\mathscr{A}_M \geq \mathscr{A} > 0$ in Theorem A. Namely, proceeding as before in the proof of Theorem A, one obtains a nontrivial EALE space N. One then needs to show that the hypothesis $\mathscr{A}_M \geq \mathscr{A} > 0$ leads to a contradiction. Now, by Lemma 6.3, $b_2(N) \neq 0$, so that there is an integral 2-cycle $\Sigma \subset N$, with $[\Sigma] \neq 0$ in $H_2(N; \mathbf{R})$

and area $\Sigma = A > 0$. Since N is a smooth limit of rescaled metrics on M_i, it follows that there is a sequence of domains D_i diffeomorphic to N in M_i and integral 2-cycles $[\Sigma_i] \neq 0$ in $H_2(D_i, \mathbf{R})$, but area $\Sigma_i \to 0$, in (M_i, g_i). Writing the Mayer-Vietoris sequence for the pair $(D_i, M_i - D_i)$, noting that $\partial D_i \approx S^{n-1}/\Gamma$, one sees

$$H_2(D_i;\mathbf{R}) \oplus H_2(M_i - D_i;\mathbf{R}) \simeq H_2(M_i;\mathbf{R}),$$

so that $[\Sigma_i]$ is a nonzero integral class in $H_2(M_i;\mathbf{R})$. This gives the required contradiction.

Returning to the discussion at hand, it follows from Lemma 6.3 that if V is a nontrivial EALE space then

$$\chi(V) - 1/|\Theta| \geq 3/2. \tag{6.5}$$

The Eguchi-Hanson metric on TS^2 [13] provides an example where equality is achieved.

Theorem 6.4. *Let M be a compact 4-manifold of Euler characteristic $0 < \chi(M) \leq 3$ (e.g., $M = S^4$ or CP^2). Then the space of $+1$ Einstein metrics g on M such that $\mathrm{vol}_g(M) \geq c > 0$ is compact in the C^k topology. In particular, the components of positive Einstein metrics in the moduli space of Einstein metrics on M are compact.*

Proof. Let $\{g_i\}$ be a sequence of $+1$ Einstein metrics on M such that $\mathrm{vol}_{g_i}(M) \geq c$. It follows from Theorem C that a subsequence, also called $\{g_i\}$, converges to an Einstein orbifold (M_∞, g_∞). If the convergence $(M, g_i) \to (M_\infty, g_\infty)$ is not smooth, then by the proof of Theorem C, there are nontrivial EALE spaces V associated to the curvature singularities of M_∞. By (6.4) and (6.5), one obtains

$$3 \geq \chi(M) \geq \chi(M_0) + \Sigma\frac{1}{|\Gamma|} + Q\cdot\frac{3}{2},$$

where M_0 is M_∞ with the curvature singularities removed and Q is the number of singularities. Since

$$\chi(M_0) + \Sigma\frac{1}{|\Gamma|} = \frac{1}{8\pi^2}\int_{M_\infty} |R|^2 \geq 0,$$

we see that $Q \leq 2$. If $Q = 2$, then for instance $\int_{M_\infty} |R|^2 = 0$, so that M_0 is flat. This is impossible, since the metric on M_0 is a $+1$ Einstein metric. If $Q = 1$, then

$$\chi(M_0) + 1/|\Gamma| \leq 3/2, \tag{6.6}$$

so that $\chi(M_0) \leq 1$. Note that by Theorem C, M_0 is embedded as an open set in M. Further, M_0 has a boundary of the form S^3/Γ. Now since M carries a metric of positive Ricci curvature, $|\pi_1(M)| < \infty$ and the Seifert-Van Kampen theorem, applied to $(M_0, M - M_0)$, shows that $|\pi_1(M_0)| < \infty$ also. In particular, $b_1(M_0) = 0$, so that $\chi(M_0) = 1$. The homology sequence arguments above show that there is a finite cover M_0' of M_0 with acyclic $\mathbf{Z}$-homology and $\partial M_0' = S^3$. Since $\chi(M_0') = 1$ also, and the Euler characteristic is multiplicative under finite covers, we see $M_0 = M_0'$. Thus, $\chi(M_0) = 1$, $\partial M_0 = S^3$, which contradicts (6.6).

Thus, $Q = 0$, so that there is no nontrivial EALE space associated to M_∞. The result now follows from Proposition 6.1. □

Remark. The method above does not work for $\chi(M) = 4$. For instance, let $M = S^2 \times S^2 = TS^2 \cup_{\partial TS^2} -TS^2$, where TS^2 is the tangent bundle to S^2. Then $\chi(M) = 4$ and M admits $+1$ Einstein metrics. We are not able to rule out the existence of a sequence of $+1$ Einstein metrics converging to an orbifold with one singular point (corresponding to a collapse of TS^2 to $C(\mathbf{RP}^3)$).

Finally, we mention one further result along these lines.

Theorem 6.5. *Let M be a compact 4-manifold. Then the space of $+3$ Einstein metrics on M with volume $\mathrm{vol}_M \geq ((1+\varepsilon)/2) \cdot \mathrm{vol}\, S^4(1)$, for a fixed $\varepsilon > 0$, is compact in the C^k topology.*

Proof. Let g_i be a sequence of $+3$ Einstein metrics on M such that $\mathrm{vol}_{g_i}(M) \geq (1+\varepsilon)/2$. By Theorem C, a subsequence converges to an Einstein orbifold M_∞. As above, if the convergence $(M, g_i) \to (M_\infty, g_\infty)$ is not smooth, then there is at least one nontrivial EALE space V associated with $\{g_i\}$. If $v(r)$ is the volume of a geodesic r-ball $B(r)$ about $p_0 \in V$, then $\lim_{r\to\infty} v(r)/r^4 = \mathrm{vol}\, C(S^3(1)/\Gamma) \leq \frac{1}{2} \mathrm{vol}\, B^4(1)$, since $|\Gamma| \geq 2$. Given $\delta > 0$, choose R such that $v(r)/r^4 \leq (1+\delta)/2 \cdot \mathrm{vol}\, B^4(1)$, $\forall r \geq R$. Now by construction, $B(r) \subset V$ is embedded in M, with metric a scaled limit of $\{g_i\}$. Since the ratio $v(r)/r^4$ is scale invariant, there are geodesic balls $B_i = B_{x_i}(\varepsilon_i) \subset (M, g_i)$, with $\varepsilon_i \to 0$, such that

$$\frac{\mathrm{vol}\, B_{x_i}(\varepsilon_i)}{\varepsilon_i^4} \leq \frac{1+\delta}{2} \mathrm{vol}\, B^4(1).$$

Since the Ricci curvature of (M, g_i) is $+3$, the Bishop comparison theorem implies that $\mathrm{vol}\, B_x(r)/v^{+3}(r)$ is monotone nonincreasing, where $v^{+3}(r)$ is the volume of a geodesic r-ball in $S^4(1)$. Now, by Myer's theorem, $\mathrm{diam}_{g_i}(M) \leq \mathrm{diam}(S^4(1)) = \pi$, so we obtain

$$\frac{\mathrm{vol}\, M}{\mathrm{vol}\, S^4(1)} = \frac{\mathrm{vol}\, B_{x_i}(\pi)}{\mathrm{vol}\, S^4(1)} \leq \frac{\mathrm{vol}\, B_{x_i}(\varepsilon_i)}{v^{+3}(\varepsilon_i)} \leq \frac{1+\delta}{2} \mathrm{vol}\, B^4(1) \cdot \frac{\varepsilon_i^4}{v^{+3}(\varepsilon_i)}.$$

Clearly, $v^{+3}(\varepsilon_i)/\varepsilon_i^4 \to \mathrm{vol}\, B^4(1)$ as $\varepsilon_i \to 0$. Since δ is arbitrary, this contradicts the hypothesis, and thus gives the result. □

We remark that the major difficulty in extending the results above to Ricci-flat or negative Einstein metrics is obtaining control on the diameter of sequences of such metrics.

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Inventiones mathematicae

On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth

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§ 1. Introduction and statement of results

There are several ways to define a Riemannian manifold to be asymptotically locally Euclidean or asymptotically flat (cf. e.g. [B̕a, LP, Kr] and the references therein). Such spaces have been studied in general relativity and also they play an important role in differential geometry (e.g., in solving the Yamabe problem in conjunction with the positive mass theorem). In this paper, we call a smooth n-dimensional complete Riemannian manifold (M, g) *asymptotically locally Euclidean* (ALE) of order $\tau > 0$, if there exists a compact subset $K \subset M$ such that $M \setminus K$ has coordinates at infinity; namely there is $R > 0$, $0 < \alpha < 1$, a finite subgroup $\Gamma \subset O(n)$ acting freely on $\mathbb{R}^n \setminus B(0; R)$, and a C^∞-diffeomorphism $\mathscr{Z}$: $M \setminus K \to (\mathbb{R}^n \setminus B(0; R))/\Gamma$ such that $\varphi = \mathscr{Z}^{-1} \circ \mathrm{proj}$ satisfies (where proj is the natural projection of $\mathbb{R}^n$ to $\mathbb{R}^n/\Gamma$)

$$(\varphi^* g)_{ij}(z) = \delta_{ij} + O(|z|^{-\tau}), \qquad \partial_k(\varphi^* g)_{ij}(z) = O(|z|^{-\tau-1}),$$
$$\frac{|\partial_k(\varphi^* g)_{ij}(z) - \partial_k(\varphi^* g)_{ij}(w)|}{|z-w|^\alpha} = O(\min\{|z|, |w|\}^{-\tau-1-\alpha})$$
$$\text{for } z, w \in \mathbb{R}^n \setminus B(0; R).$$

(For simplicity we assume that (M, g) has only one end. The extensions generally require only minor modifications, which we leave to the reader.)

This definition depends on the choice of the coordinates $\mathscr{Z}$. It is not clear that a quantity which is defined using coordinates (for example the mass of (M, g)) is independent of the choice of the coordinates. In [Ba] Bartnik discussed the problem of the "uniqueness" of the coordinates at infinity and proved that the mass is a geometric invariant of (M, g). But he assumed the existence of at least one coordinate system at infinity. The purpose of this paper is to construct coordinates at infinity under certain intrinsic conditions.

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(1.1) **Theorem.** *Let $o \in M$. Suppose (M, g) has only one end, $n \geqq 3$, and satisfies*

(1.2) $|R| \leqq K r^{-(2+\varepsilon)}$ *for sufficiently large $r = d(o, *)$,*
(R denotes the curvature tensor of g)

(1.3) $|\mathrm{Ric}| \leqq K r^{-(2+\eta)}$ *for sufficiently large r,*

(1.4) $\mathrm{vol}(B(o; t)) \geqq V t^n$ *for all $t > 0$*

where $K \geqq 0$, $V > 0$, and $\eta \geqq \varepsilon > 0$. Then (M, g) has coordinates at infinity $\mathscr{X}$ of order μ, where we can take

$$\mu = \begin{cases} \eta & \text{if } \eta < n-1 \text{ and } \eta \neq n-2, \\ \eta' \ (\text{for any } \eta' < \eta) & \text{if } \eta = n-2 \text{ or } n-1, \\ n-1 & \text{if } \eta > n-1, \\ \varepsilon & \text{if } n \geqq 4 \text{ or } \varepsilon \neq 1, \\ \varepsilon' \ (\text{for any } \varepsilon' < 1) & \text{if } n = 3 \text{ and } \varepsilon = 1. \end{cases}$$

In [K2, K3] the second named author studied the geometry of "ends" of manifolds with asymptotically nonnegative curvature. To be precise, a complete Riemannian manifold (M, g) is said to be of *asymptotically nonnegative curvature*, if there exists a monotone nonincreasing function k: $[0, \infty) \to [0, \infty)$ such that the integral $\int^{\infty} t k(t)$ is finite and the sectional curvature K_M of M is bounded from below by $-k(d(o, *))$ everywhere. In our situation the manifold (M, g) satisfies this condition, and the "end" is isometric to $(S^{n-1}/\Gamma, g_{std})$. But the results of [K2] and [K3] only imply that for fixed $0 < a < b < \infty$, the metric space $(B(o; ar) \setminus B(o; br), r^{-1} d)$ converges to $(S^{n-1}/\Gamma \times [a, b], g_{std})$ in the Hausdorff distance (to be precise, in the $C^{1,\alpha}$-norm of metrics). In the proof of Theorem (1.1) we must study the speed of the above convergence.

Analogous problems for manifolds which do not necessarily satisfy the condition (1.4) are interesting (for example $\mathbb{R} \times$ flat torus). In that case the end collapses to a lower dimensional space.

The motivation of this paper is the work of the third named author [N2] which discussed the convergence of Einstein metrics (see § 5). We can prove his conjecture:

(1.5) **Theorem.** *Let (M, g) be an n-dimensional Ricci-flat manifold ($n \geqq 4$) with*

(1.6) $\mathrm{vol}\, B(o; t) \geqq V t^n$ *for some $o \in M$, $V > 0$,*

(1.7) $\int_M |R|^{n/2} dV_g < \infty.$

Then (M, g) is ALE of order $n-1$. If $n = 4$ or g is Kähler, then (M, g) is ALE of order n.

(1.8) *Remarks.* 1) Theorem (1.1) states only that $\varphi^* g$ converges to the standard metric in $C^{1,\alpha}$-topology, but if we assume that

$$|D^k \mathrm{Ric}| \leqq K r^{-2-k-\eta} \quad \text{for all } 0 \leqq k \leqq N,$$

the corresponding statement holds for $C^{N+1,\alpha}$-topology. The conclusion of Theorem (1.5) then becomes

$$D^k \varphi^* g_{ij}(z) - D^k \delta_{ij} = O(|z|^{-k-\tau}) \quad \text{for all } k \geqq 0,$$

where $\tau = n-1$ in general, and $\tau = n$ if $n=4$ or g is Kähler.

2) (cf. [Sc, Proposition 2]) If the dimension n is odd, then Γ is $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$ according as M is orientable or nonorientable. Then the conclusion of Theorem (1.5) becomes that (M, g) is isometric to $(\mathbb{R}^n, g_{std})$. This is true for the case that n is even and M is simply connected at infinity (i.e. for any compact set K of M, there is a compact set K' such that $K \subset K'$ and $M \setminus K'$ is simply connected).

3) Kronheimer [Kr] classified 4-dimensional ALE hyperkähler (hence Ricci-flat) manifolds. We do not have any simply connected non-hyperkähler examples which satisfy the condition of Theorem (1.5). We conjecture that there are no such examples. For Yang-Mills connections there is a famous conjecture on the nonexistence of non-self-dual Yang-Mills connections on $SU(2)$ bundles over S^4 (see [AJ, p. 118] [BL]). Our conjecture is an Einstein metric version of this.

4) Calabi [Ca] constructed examples of Kähler Ricci-flat ALE manifolds M of order n ($n = \dim_{\mathbb{R}} M$).

5) Theorem (1.5) can be considered as an Einstein metric version of the removable singularities theorem for Yang-Mills connections [Uh]. In fact as a corollary of the method of proof, we prove that apparent point singularities of Einstein metrics with finite curvature integral are removable (Theorem (4.1)).

§ 2. Preliminary results

In this section we discuss some preliminary results which will be used in the proof of the main theorems.

First, we recall the definition of the Hausdorff distance (cf. [Gr, Ch. 3]). Given a metric space Z and subsets $A, B \subset Z$, the Hausdorff distance in Z between A and B is defined by

$$d_H^Z(A, B) := \inf \left\{ \varepsilon > 0 \colon \begin{array}{ll} d_Z(a, B) < \varepsilon & \text{for all } a \in A, \\ d_Z(b, A) < \varepsilon & \text{for all } b \in B \end{array} \right\}.$$

Given two metric spaces X, Y, the Hausdorff distance between them is defined by

$$d_H(X, Y) := \inf d_H^Z(f(X), g(Y)) \tag{2.1}$$

where $Z, f\colon X \to Z$, and $g\colon Y \to Z$, respectively, range over all metric spaces, distance preserving maps from X into Z, and ones from Y into Z. The Hausdorff distance d_H defines a distance on the space of all compact metric spaces. For pointed metric spaces we define the pointed Hausdorff distance similarly. For a metric space Z, subsets $A, B \subset Z$, and points $a_0 \in A$, $b_0 \in B$, the pointed Hausdorff distance in Z between (A, a_0) and (B, b_0) is defined by

$$d_{p,H}((A, a_0), (B, b_0))$$
$$:= \inf \left\{ \varepsilon > 0 \colon \begin{array}{ll} d_Z(a_0, b_0) < \varepsilon, & \\ d_Z(a, B_B(b_0; 1/\varepsilon)) < \varepsilon & \text{for all } a \in B_A(a_0; 1/\varepsilon), \text{ and} \\ d_Z(b, B_A(a_0; 1/\varepsilon)) < \varepsilon & \text{for all } b \in B_B(b_0; 1/\varepsilon) \end{array} \right\}$$

where $B_A(a_0; 1/\varepsilon)$ (resp. $B_B(b_0; 1/\varepsilon)$) denotes the metric ball in A (resp. B) centered at a_0 (resp. b_0) with the radius $1/\varepsilon$. The pointed Hausdorff distance between two pointed metric spaces (X, x_0) and (Y, y_0) is defined as (2.1).

We denote by $\mathscr{S}(n, D)$ the family of n-dimensional complete Riemannian manifolds satisfying $\mathrm{Ric} \geqq -(n-1)$, $\mathrm{diam} \leqq D$ $(D < \infty)$, and by $\mathscr{S}(n, \infty)$ the family of pointed n-dimensional complete Riemannian manifolds satisfying $\mathrm{Ric} \geqq -(n-1)$. Gromov's compactness theorem states (cf. [Gr, Theorem (5.3)]).

(2.2) **Fact.** *For $D < \infty$, $\mathscr{S}(n, D)$ is precompact in the space of all compact, locally compact metric spaces of length with respect to the Hausdorff distance. $\mathscr{S}(n, \infty)$ is precompact in the space of all complete, locally compact proper metric spaces of length with respect to the pointed Hausdorff distance.*

Next we state the Laplacian and Hessian comparison theorems which are frequenctly used in the proof of the main theorems. See, e.g., [K1] for the proof. Let M be a complete Riemannian manifold of dimension n, N be a closed embedded hypersurface of M. We denote by A the shape operator of N. Suppose $\sigma \colon [0, l] \to M$ is a unit speed geodesic joining N to x. Let $\bar{K}$, $\underline{K}$ and R be continuous functions on $[0, l]$ such that

$$\underline{K}(t) \leqq \text{the sectional curvature of any plane containing } \sigma'(t) \leqq \bar{K}(t),$$
$$\text{the Ricci curvature in direction } \sigma'(t) \geqq (m-1)\, R(t).$$

Let $\bar{y}(t)$, $\underline{y}(t)$, $\underline{z}(t)$ and $f(t)$ be the solutions of the equations

$$\begin{array}{ll} \bar{y}''(t) + \bar{K}(t)\, \bar{y}(t) = 0 & \text{with } \bar{y}(0) = 1,\ \bar{y}'(0) \leqq \min \lambda_i, \\ \underline{y}''(t) + \underline{K}(t)\, \underline{y}(t) = 0 & \text{with } \underline{y}(0) = 1,\ \underline{y}'(0) \geqq \max \lambda_i, \\ \underline{z}''(t) + \underline{K}(t)\, \underline{z}(t) = 0 & \text{with } \underline{z}(0) = 0,\ \underline{z}'(0) = 1, \\ f''(t) + R(t)\, f(t) = 0 & \text{with } f(0) = 0,\ f'(0) = 1, \end{array}$$

where λ_i $(i = 1, \ldots, \dim N)$ are the eigenvalues of A on $T_{\sigma(0)} N$.

(2.3) **Fact** (cf. [W]). *Let $J(t)$ be an N-Jacobi field along $\sigma(t)$ with the initial condition $J(0) \in T_{\sigma(0)} N$, $\dfrac{\nabla}{dt} J(0) = A(J(0))$. Then*:

1) $|J(t)|/|J(0)| \leqq \underline{y}(t)$,
2) $|J(t)|/|J(0)| \geqq \bar{y}(t)$ *if $\bar{y}(t)$ is positive on* $[0, l]$.

Following [Wu], for a continuous function f defined on an open set U of M, a point $x \in U$, and a tangent vector $X \in T_x M$, we define a real number $Cf(x:X)$ which is an extension of the Hessian $\nabla^2 f(X, X)$ by

$$Cf(x:X) := \liminf_{t \to 0} \frac{1}{t^2} \{f(\gamma(t)) + f(\gamma(-t)) - 2f(\gamma(0))\}$$

where γ is a geodesic with $\gamma(0) = x$ and $\gamma'(0) = X$.

(2.4) **Fact** (Hessian comparison theorem). *Let ρ be the distance function from N. Suppose σ realizes the distance from N namely $d(\sigma(t), N) = t$ on* $[0, l]$. *Then*:

$$C\rho(x:X) \leqq (z'/z)(\rho(x)) \{\|X\|^2 - \langle \sigma'(l), X \rangle^2\} \qquad \textit{for any } X \in T_x M (x = \sigma(l)).$$

This holds even if N is any closed subset.

(2.5) **Fact** (Laplacian comparison theorem). *The distance function ρ from N satisfies*

$$\Delta \rho \leqq (n-1)(f'/f)(\rho(x))$$

weakly in $W^{1,2}(M \setminus N)$. This holds even if N is any closed subset.

We will also use the Bishop volume comparison theorem (see for example [CGT]).

(2.6) **Fact.** *For an n-dimensional complete Riemannian manifold M with* Ric $\geqq (n-1)\lambda$ *for some constant λ,* $\mathrm{vol}(B(o; r))/\mathrm{vol}(B_\lambda(o; r))$ *is nonincreasing in r, where $B_\lambda(o; r)$ is a geodesic ball in the simply-connected space form of constant curvature λ.*

The following local lower estimate on the injectivity radius is due to Cheeger, Gromov and Taylor [CGT, Theorem (4.3)]. Let M be a complete Riemannian manifold with $\lambda \leqq K_M \leqq \kappa$ for some constants λ, κ. Let $B(p; r)$ be a metric ball in M such that $r \leqq \pi/\sqrt{\kappa}$ (r arbitrary if $\kappa \leqq 0$).

(2.7) **Fact.** *Let r be as above. Choose r_0 and s such that $r_0 + s \leqq r$, $r_0 \leqq r/4$. Then the injectivity radius i_p of M at p satisfies*

$$i_p \geqq \frac{r_0}{2} \{1 + \mathrm{vol}(B_\lambda(o; r_0 + s))/\mathrm{vol}(B(p; s))\}^{-1}$$

where $B_\lambda(o; r_0 + s)$ is as in (2.6).

We recall the definition and some properties of harmonic coordinates. We refer the reader to [Jo]. Let o be a point in a Riemannian manifold M. For a unit tangent vector u at o, we define the almost linear function $L_u(x)$ associated with u as follows: Let

$$r(x) := d_M(o, x), \qquad p(x) := \exp_o(r(x)\, u), \qquad q(x) := \exp_o(-r(x)\, u)$$

and then set

$$L_u(x) := \frac{1}{4r(x)} \{d_M(x, q(x))^2 - d_M(x, p(x))^2\}.$$

Suppose M satisfies $|K_M| \leqq \Lambda^2$ for a constant Λ, and the injectivity radius at o, $i_0 \geqq I$ for a positive constant I. Let R be a number less than $\min(I, \pi/2\Lambda)$ and $\tilde{u}(x)$ be the vector at $x \in B(o; R)$ obtained by the parallel translation along the radial geodesic between o and x. Then we have

(2.8) **Fact.**

(1) $|L_u(x)| \leqq r(x)$

(2) $|DL_u(x) - \tilde{u}(x)| \leqq 2\Lambda^2 \dfrac{\sinh(2\Lambda r(x))}{\sin(2\Lambda r(x))} r^2(x)$,

(3) $|D^2 L_u(x)| \leqq 9\Lambda^2 \dfrac{\sinh(2\Lambda r(x))}{\sin(2\Lambda r(x))} \Lambda r(x) \coth(\Lambda r(x))\, r(x)$.

Let us take an orthonormal basis $\{u_1, \ldots, u_n\}$ of $T_o M$. Let h_i be the solution of the Dirichlet problem;

$$\Delta h_i = 0 \quad \text{in } B(o; R)$$
$$h_i = L_{u_i} \quad \text{on } \partial B(o; R).$$

Define $\mathbb{H}$: $B(o; R) \to \mathbb{R}^n$ by $\mathbb{H}(x) = (h_1(x) - h_1(o), \ldots, h_n(x) - h_n(o))$. Then we have the following

(2.9) **Fact.** *There exist constants $R_0 = R_0(m, \Lambda, I)$ and $C = C(m, \Lambda, I)$ such that for any $R \leqq R_0$ the above harmonic map $\mathbb{H}$: $B(o; R) \to \mathbb{R}^n$ defines a coordinate system around o and satisfies the following properties*:

1) *Let $g_{ij} (= \mathbb{H}_* g_{ij})$ be the coefficient of the metric g of M in terms of the coordinates $\mathbb{H}$. Then*

$$\sum_{i,j} g^{ij} \Gamma_{ij}^k = 0$$
$$\Delta g_{ij} = -2\,\mathrm{Ric}_{ij} + Q_{ij}(g, \partial g)$$

where (g^{ij}) is the inverse matrix of (g_{ij}), and $\{\Gamma_{ij}^k\}$ are the Christoffel symbols, and

$$Q_{ij}(g, \partial g) = -\sum_{k,l,p,q} g^{pq} g_{lj} \partial_p g_{ik} \partial_q g^{kl} - \sum_{k,l,p,q,r,s} 2 g_{ik} g_{lj} g^{pq} g^{rs} \Gamma_{pr}^k \Gamma_{qs}^l.$$

2) $(1 + C)^{-1} r(x) \leqq |\mathbb{H}(x)| \leqq (1 + C)\, r(x)$,

3) $|Dh_i(x) - \tilde{u}_i(x)| \leqq C\Lambda^2 R^2$.

4) (cf. [GT, Theorem (8.32)]). *The metric tensor g_{ij} satisfies the following estimates on $B(o; R/2)$*

$$|g_{ij} - \delta_{ij}| \leqq C\Lambda^2 R^2$$
$$|dg_{ij}| \leqq C\Lambda^2 R$$
$$\frac{|dg_{ij}(x) - dg_{ij}(y)|}{|x - y|^\alpha} \leqq C_1(n, \Lambda, I, \alpha)\, \Lambda^2 R^{1-\alpha} \quad \text{for } \alpha \in (0, 1).$$

5) *Moreover any harmonic function h on $B(o; R)$ satisfies* (cf. [GT, 6.1])

$$\|h\circ \mathbb{H}^{-1}\|_{C^{2,\alpha}(B(o;R/2))} \leqq C_2(m, \Lambda, I, \alpha) \sup_{B(o,R)} |h|.$$

Harmonic coordinates suit the regularity problem of metrics as was pointed out in [DK]. *If a metric $g \in C^{k,\alpha}$ $(1 \leqq k \leqq \infty)$ in some coordinate system, then it is also of class $C^{k,\alpha}$ in harmonic coordinates, while it is of at least class $C^{k-2,\alpha}$ in geodesic normal coordinates.*

We need a local form the Gromov convergence theorem (see [GW, Pe, K3]).

(2.10) **Fact.** *Let $\{M_i\} = \{(M_i, g_i, o_i)\}$ be a sequence of pointed complete Riemannian manifolds which converges, as $i \to \infty$, to a pointed metric space $X_\infty = (X_\infty, d_\infty, o_\infty)$ with respect to the pointed Hausdorff distance. Suppose that for any pair of positive numbers a, b with $a < b$, the sectional curvature K_{M_i} of M_i satisfies*

$$|K_{M_i}| \leqq \Lambda^2$$

on $A_i = \{p \in M_i\colon a \leqq d(p, o_i) \leqq b\}$ for some constant Λ, and moreover the injectivity radius of M_i is bounded uniformly by a positive constant I on A_i. Then $X_\infty \setminus \{o_\infty\}$ is a smooth manifold with $C^{1,\alpha}$-Riemannian metric g_∞ $(0<\alpha<1)$ which is compatible with the distance d_∞, and for any compact domain $D \subset X\setminus\{o_\infty\}$, there exists an into diffeomorphism $F_i\colon D \to M_i$ such that $F_i^* g_i$ converges as $i \to \infty$ to g_∞ in the $C^{1,\alpha}$-topology on D.

We recall the definitions of the weighted Sobolev spaces and the weighted Hölder spaces on $\mathbb{R}^n$ (see, e.g., [Ba, LP] for details). The weighted Lebesgue spaces L^p_a $(p \geqq 1)$ with weight $a \in \mathbb{R}$ are the spaces of functions in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the norms $\|\cdot\|_{L^p_a}$ defined by

$$\|f\|_{L^p_a} = \Big\{\int_{\mathbb{R}^n} |f|^p \sigma^{-ap-n} dV\Big\}^{1/p} \qquad \sigma = (1+|x|^2)^{1/2}$$

are finite. The weighted Sobolev norms $\|\cdot\|_{W^{k,p}_a}$ (k is a nonnegative integer) are now defined by

$$\|f\|_{W^{k,p}_a} = \sum_{j=1}^{k} \|D^j f\|_{L^p_{a-j}}.$$

The weighted Sobolev spaces $W^{k,p}_a$ are the spaces of functions in $W^{k,p}_{\mathrm{loc}}$ such that the weighted Sobolev norms are finite. The weighted Hölder spaces $C^{k,\alpha}_a$ $(0<\alpha<1)$ with weight $a \in \mathbb{R}$ are the spaces of functions in $C^{k,\alpha}(\mathbb{R}^n)$ such that the weighted Hölder norms $\|\cdot\|_{C^{k,\alpha}_a}$ defined by

$$\|f\|_{C^{k,\alpha}_a} = \sum_{j=0}^{k} \sup_{\mathbb{R}^n} \sigma^{-a+j} |D^j f| + \sup_{x \neq y} \min(\sigma(x), \sigma(y))^{-a+k+\alpha} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha}$$

are finite. In an obvious manner, we define the weighted Sobolev spaces $W^{k,p}_a(\Omega)$ and the weighted Hölder spaces $C^{k,\alpha}_a(\Omega)$ for a domain Ω in $\mathbb{R}^n$. Moreover if we have coordinates $\Phi=(x^1, \ldots, x^n)$ at infinity on M, then we can define the weighted Sobolev spaces $W_a^{k,p}(\Phi; D(R))$ and the weighted Hölder spaces $C_a^{k,\alpha}(\Phi; D(R))$ through the coordinates Φ, where $D(R)=\{x\in\mathbb{R}^n: |x|>R\}$.

§ 3. Proof of Theorem (1.1)

Let (M, g) be as in Theorem (1.1). Take $o\in M$, and set $r=d(o, *)$.

Throughout the proof, we use $C_1, C_2, \ldots$ to denote positive constants depending on K, V and n.

Before starting the proof of Theorem (1.1), we recall some facts on M in the following

(3.1) **Lemma** (cf. [K2, § 2]). 1) *The injectivity radius of M at x is greater than $C_1\, r(x)$.*

2) *For fixed positive numbers a, b with $a<1<b$, and for any large R, there exists a diffeomorphism Π_R from a neighborhood of $B(o; bR)\setminus B(o; aR)$ into the flat cone $\mathscr{C}(S^{n-1}/\Gamma)$ over the space form S^{n-1}/Γ of constant curvature* 1 (i.e. $\mathscr{C}(S^{n-1}/\Gamma)=[0,\infty)\times_{t^2} S^{n-1}/\Gamma=([0,\infty)\times S^{n-1}/\Gamma,\ dt^2+t^2 g_{std})$) *such that as R goes to infinity, $\Pi_R(B(o; bR)\setminus B(o; aR))$ converges to $[a,b]\times_{t^2} S^{n-1}/\Gamma$ as sets,* i.e.,

$$\inf\left\{\varepsilon:\ \begin{array}{l} d(\Pi_R(B(o; bR)\setminus B(o; aR)), x)<\varepsilon,\ x\in[a,b]\times S^{n-1}/\Gamma, \\ d(y, [a,b]\times S^{n-1}/\Gamma)<\varepsilon,\ y\in\Pi_R(B(o; bR)\setminus B(o; aR)) \end{array}\right\}$$

goes to zero as $R\to\infty$, and moreover $R^{-2}(\Pi_R^{-1})^* g$ converges to the metric of $\mathscr{C}(S^{n-1}/\Gamma)$ in $C^{1,\alpha}$-topology $(0<\alpha<1)$. *In particular, if we set $S_R=\Pi_R^{-1}(\{1\}\times S^{n-1}/\Gamma)$ and $\tilde{B}_R=$ the (relatively compact) domain bounded by S_R, then*

$$R^{-1} d(S_R, \partial B(o; R))\leqq\varepsilon(R),$$
$$\|RA_R+Id\|\leqq\varepsilon(R),$$

where A_R stands for the shape operator of S_R, and $\varepsilon(R)$ goes to 0 *as $R\to\infty$.*

The lemma just mentioned says that in a sense M is "tangent at infinity" to the flat cone $\mathscr{C}(S^{n-1}/\Gamma)$, and the lemma may be considered as a weak version of Theorem (1.1). To improve the lemma and show Theorem (1.1), we shall make some further observations on the behavior of geodesics tending to infinity and the metric g of M along them, and then construct coordinates at infinity of certain order by using harmonic coordinates instead of normal coordinates (cf. Steps $1\sim4$ below). Finally we apply the theory of weighted Sobolev or Hölder spaces to obtain coordinates at infinity of order μ stated in Theorem (1.1). Harmonic coordinates play an important role through the proof of Theorem (1.1), because we can apply the regularity theory to the metric g and estimate the $C^{1,\alpha}$-norm of g in terms of the curvature of M.

For the sake of simplicity, we assume M is simply connected at infinity, i.e., Γ in the above lemma is trivial.

Step 1. *If R is sufficiently large,*

$$\Phi_R\colon\ (x,t)\in S_R\times[R,\infty)\mapsto \exp(t-R)\,v_R(x)\in M-\tilde{B}_R$$

gives a diffeomorphism where v_R is the outer unit normal vector of S_R.

To verify the claim just above, we use some known facts on Jacobi fields along geodesics with initial vectors normal to a submanifold and its cut locus (cf. e.g. [BC, GKM, Ng, W] etc.).

Let σ: $[R,\infty)\to M$ be a geodesic with $\sigma(R)\in S_R$ and $\sigma'(R)=v_R(\sigma(R))$, and J a (nontrivial) Jacobi field along σ with $J(R)\in T_{\sigma(r)}S_R$, $J(R)=A_R(J(R))$. We first observe that for large R, $J(t)$ never vanishes as long as $d(\sigma(t),S_R)=t-R$. In fact, let $y(t)$ be the solution of the ordinary differential equation:

$$\begin{aligned} &y''(t)+C_2\,t^{-2-\varepsilon}y(t)=0 \qquad \text{for } t\in[R,\infty),\\ &y(R)=1,\qquad y'(R)=R^{-1}(1-\varepsilon(R)) \end{aligned}$$

where a constant C_2 is taken so that the sectional curvature of M at $\sigma(t)$ with $d(\sigma(t),S_R)=t-R$ is bounded from above by $C_2\,t^{-2-\varepsilon}$. If $y(t)$ is positive on $[R,L]$ $(R<L)$, then $y'(t)\leqq R^{-1}(1-\varepsilon(R))$, since $y''(t)\leqq 0$ there. Thus

$$\begin{aligned} y'(t)&=R^{-1}(1-\varepsilon(R))-C_2\int_R^t s^{-2-\varepsilon}y(s)\,ds\\ &\geqq R^{-1}(1-\varepsilon(R))-C_2\int_R^t s^{-2-\varepsilon}\left\{1+\frac{s}{R}(1-\varepsilon(R))\right\}ds. \end{aligned}$$

Therefore if we choose sufficiently large R, $y'(t)$ is positive on $[R,\infty)$ and hence so is $y(t)$. Then applying the comparison theorem on Jacobi fields (Fact (2.3)) to our situation, we have

$$y(t)\leqq |J(t)|/|J(R)| \qquad \text{as long as } d(\sigma(t),S_R)=t-R.$$

This shows S_R has no focal points as long as $d(\sigma(t),S_R)=t-R$.

Suppose S_R has the cut locus $\mathscr{C}_R^+$ outside $\tilde{B}_R$. Let q be a point of $\mathscr{C}_R^+$ which is closest to S_R (i.e., $d(q,S_R)=d(\mathscr{C}_R^+,S_R)$) and σ_R: $[R,\infty)\to M$ a geodesic such that $d(\sigma_R(t),S_R)=t-R$ on $[R,R+L]$ $(L=d(q,S_R))$ and $\sigma_R(R+L)=q$. Then since S_R has no focal points along $\sigma_R|_{[R,R+L]}$, it turns out from the argument in [GKM, Lemma 2, p. 226] that σ_R satisfies:

$$d(\sigma_R(t),S_R)=R+2L-t \qquad \text{on } [R+L,R+2L].$$

(In [GKM], they consider only the case that S_R is a point. However it is not hard to see that the same result is true for the case: S_R is a submanifold (cf. e.g., [Ng, Chap. 4, Theorem (1.5)]).)

Let us now show that $\mathscr{C}_R^+$ must be empty for sufficiently large R. In fact, if there exists a divergent sequence $\{R_i\}$ such that $\mathscr{C}_{R_i}^+$ is not empty for any i, then we can take geodesics σ_{R_i}: $[R_i, R_i+2L_i]\to M\setminus\tilde{B}_i$ $(L_i=d(S_{R_i},\mathscr{C}_{R_i}^+))$ as above. We note that $L_i\geqq C_3\,R_i^{1+\varepsilon}$ (cf. [K3, § 2.2]). Then the metric space $(\{x\in M$:

$\frac{1}{2} \leqq (L_i+R_i)^{-1} d(x, o) \leqq 2\}$, $(L_i+R_i)^{-1} d)$ converges to $(\{v \in \mathbb{R}^n\colon \frac{1}{2} \leqq |v| \leqq 2\}, d_{std})$ in $C^{1,\alpha}$-topology as $i \to \infty$, and the geodesic σ_{R_i} converges to a line γ in $\mathbb{R}^n$ from v_0 with $|v_0| = 1/2$ to v_0' with $|v_0'| = 1/2$ passing through a point v with $|v| = 1$. This is absurd. Thus we have shown that $\mathscr{C}_R^+$ is empty for large R.

Now we rescale the metric g to $R^{-2} g$, and then we assume $R=1$. We denote S_R, A_R by S, A (respectively). Let

$$\Phi^* g = dt + t^2 g_t \qquad \text{for } (x, t) \in S \times [1, \infty).$$

Step 2. *g_t converges to a C^0-metric g_∞ on S, and*

$$|g_t - g_\infty| = O(t^{-\varepsilon'})$$

where $\varepsilon' = \min\{\varepsilon, 1\}$ if $\varepsilon \neq 1$ and ε' is any positive number less than 1 *if $\varepsilon = 1$.*

We fix a point $x \in S$. We take $X \in T_x S$. Let $X(t)$ be the Jacobi field along $\sigma(t) = \Phi(x, t)$ with $X(1) = X$, $\frac{\nabla}{dt} x(1) = A(X)$. Then we have

$$d\Phi_{(x,t)} X = X(t) \qquad \text{for all } t \in [1, \infty).$$

We take a parallel orthonormal frame $\{E_1(t), \ldots, E_n(t)\}$ along γ. The Jacobi equation is written as

$$(X^i)''(t) + \sum X^j(t) K_j^i(t) = 0$$

where $X^i(t) = (X(t), E_i(t))$, $K_j^i(t) = (R(E_j(t), \gamma'(t)) \gamma'(t), E_i(t))$. Thus

$$X^i(t) = X^i(1) + (X^i)'(1)(t-1) - \int_1^t \int_1^s \sum X^j(u) K_j^i(u)\, du\, ds.$$

Using $|K_j^i(t)| \leqq K t^{-2-\varepsilon}$ and Fact (2.3), we have

$$\left| t^{-1} X^i(t) - \left\{ (X^i)'(1) - \int_1^\infty \sum X^j K_j^i \right\} \right| = O(t^{-\varepsilon'}), \tag{3.1}$$

where $\varepsilon' = \min\{\varepsilon, 1\}$ if $\varepsilon \neq 1$ and ε' is any positive number less than 1 if $\varepsilon = 1$. This means

$$|g_t - g_\infty| = O(t^{-\varepsilon'})$$

where $g_\infty = \lim_{t \to \infty} g_t$.

The above claim in Step 2 says that (S, g_t) converges to (S, g_∞) in the Hausdorff distance (cf. [Gr]). Since the shape operator $\bar{A}_t$ of the hypersurface $\{x\colon d(x, S) = t\}$ satisfies

$$\| t \bar{A}_t + \mathrm{Id} \| \to 0 \qquad \text{as } t \to \infty,$$

(S, g_∞) is isometric to (S^{n-1}, g_{std}). In the following we shall identify (S, g_∞) and (S^{n-1}, g_{std}). Then the assertions of Step 2 mean that the map $\Phi^{-1}\colon M \setminus \tilde{B} \to S$

$\times [1, \infty) \cong \mathbb{R}^n \backslash B(0; 1)$ defines coordinates at infinity of order ε' up to the level of C^0-norm. In particular distance and volume agree with the Euclidean case up to $1 + O(r^{-\varepsilon'})$. In the following (Steps 3, 4), using harmonic coordinates instead of geodesic coordinates, we construct coordinates at infinity of order ε' up to the level of $C^{1,\alpha}$-norm.

We set here $\Phi_t(y, \xi) = \Phi(y, t\xi)$ for $y \in S$ and $\xi \in (1/2, 3/2)$. Note that $\Phi_t(B((y, 1); \lambda\delta)) \subset B(\Phi_t(y, 1); \delta) \subset \Phi_t(B((y, 1); \lambda^{-1}\delta))$ for some positive constant $\lambda < 1$ and a sufficiently small constant $\delta > 0$ (cf. the above claim). Let $\mathbb{L}_t$ be almost linear functions defined over $B(\gamma(t); 2\delta t)$ associated with the linear frame $\{E_1(t), \ldots, E_n(t)\}$ at $\gamma(t)$ and $\mathbb{H}_t$ the harmonic coordinates defined by $\mathbb{L}_t$, where $\mathbb{H}_t(\gamma(t))$ is assumed to be the origin $0 \in \mathbb{R}^n$. In what follows, we use some basic facts on harmonic coordinates (see Facts (2.8) and (2.9)).

Step 3.

$$|t^{-1}\mathbb{H}_t \circ \Phi_t - \mathbb{H}_\infty| = O(t^{-\varepsilon'})$$

on a neighborhood of $(x, 1) \in S \times (1/2, 3/2)$ *where* $\mathbb{H}_\infty$ *is a restriction of an affine function on* $\mathbb{R}^n = \mathscr{C}(S^{n-1})$.

Let ω_t be the radially parallel 1-form on $B(\gamma(t); 2\delta t)$ with valued in $\mathbb{R}^n$ such that $\omega_t(\gamma(t))$ is the dual frame of $\{E_1(t), \ldots, E_n(t)\}$. By apriori estimates for harmonic coordinates (Fact (2.9) 3))

$$|d\mathbb{H}_t - \omega_t| \leqq C_3\, t^{-\varepsilon} \quad \text{on } B(\gamma(t); \lambda\delta t).$$

This shows

$$(3.2) \quad |d(t^{-1}\mathbb{H}_t \circ \Phi_t) - t^{-1}\Phi_t^*\, \omega_t| \leqq C_3\, t^{-\varepsilon} \quad \text{on } B((x, 1); \delta) \subset S \times (1/2, 3/2).$$

From (3.1) at the center x, $t^{-1}\Phi_t^*\, \omega_t(x, \xi)$ converges, and

$$|t^{-1}\Phi_t^*\, \omega_t(x, \xi) - s^{-1}\Phi_s^*\, \omega_s(x, \xi)| = O(t^{-\varepsilon'}) \quad \text{for } t < s.$$

On the other hand for $y \in B(x; \delta) \cap S$, we take the parallel translation $\bar{\omega}_t$ of $\omega_t(\Phi(y, t))$ along $\sigma(s) = \Phi(y, s)$. Since the effect of parallel translation along a null homotopic closed path can be estimated by curvature times the area of a surface which bounds the path and the area of surface is estimated in Step 2, we have

$$|\bar{\omega}_t(\Phi(y, s)) - \omega_s(\Phi(y, s))| \leqq C_4 \int_t^s u^{-\varepsilon-1}\, du = O(t^{-\varepsilon}) \quad \text{for } t < s.$$

But the same argument as above shows

$$|s^{-1}\Phi_s^*\, \bar{\omega}_t(y, \xi) - t^{-1}\Phi_t^*\, \omega_t(y, \xi)| = O(t^{-\varepsilon'}).$$

Since $\varepsilon \geqq \varepsilon'$,

$$|t^{-1}\Phi_t^*\, \omega_t(y, \xi) - s^{-1}\Phi_s^*\, \omega_s(y, \xi)| = O(t^{-\varepsilon'}) \quad \text{for } t < s,\ y \in B(x; \lambda\delta) \cap S.$$

Combining this with (3.2), we have

$$|d(t^{-1}\mathbb{H}_t \circ \Phi_t) - d(s^{-1}\mathbb{H}_s \circ \Phi_s)| = O(t^{-\varepsilon'}).$$

Since $\mathbb{H}_t \circ \Phi_t(x, 1) = \mathbb{H}_s \circ \Phi_s(x, 1) = 0$, we get

$$|t^{-1}\mathbb{H}_t \circ \Phi_t - s^{-1}\mathbb{H}_s \circ \Phi_s| = O(t^{-\varepsilon'}).$$

Thus $t^{-1}\mathbb{H}_t \circ \Phi_t$ converges to $\mathbb{H}_\infty$, and

$$|t^{-1}\mathbb{H}_t \circ \Phi_t - \mathbb{H}_\infty| = O(t^{-\varepsilon'}).$$

The metric g_t converges to g_{std} in C^0-topology, and $(t^{-1}\mathbb{H}_t\, \Phi_t)_* \, g_t = (\mathbb{H}_t)_* \, g$ converges to g_{std} in $C^{1,\alpha}$-topology. Hence $\mathbb{H}_\infty \in \mathrm{Iso}(\mathbb{R}^n)$ such that $\mathbb{H}_\infty(x) = 0$ (i.e. $\mathbb{H}_\infty$ are the restriction of affine functions to the neighborhood of x).

Step 4. *There exists coordinates at infinity $\Psi\colon M \setminus \tilde{B} \to S \times [1, \infty)$ of order ε'.*

By Step 2, we can take geodesics $\gamma_1, \ldots, \gamma_N (\gamma_a(t) = \Phi(x_a, t),\ x_a \in S)$ so that $\{B(\gamma_a(\beta^j);\ \delta\beta^j)\}_{a=1,\ldots,N,\, j=1,\ldots}$ cover $M \setminus \tilde{B}$ for some $\beta > 1$. We denote $B(\gamma_a(\beta^j);\ \delta\beta^j)$ by $B_{a,j}$. For all a and j we take harmonic coordinates $\mathbb{H}_{a,j}\colon B(\gamma_a(\beta^j);\ \delta\beta^j) \to \mathbb{R}^n$ as above. When the intersection $B_{a,j} \cap B_{b,k}$ of $B_{a,j}$ and $B_{b,k}$ is not empty, we may assume that $\mathbb{H}_{a,j}$ (resp. $\mathbb{H}_{b,k}$) is defined on $B(\gamma_b(\beta^k); 2\delta\beta^k)$ (resp. $B(\gamma_a(\beta^j); 2\delta\beta^j)$) and $k - j$ is bounded (cf. Step 2). Moreover by setting $\mathbb{H}_{a,j}(\gamma_a(\beta^j)) = \beta^j x_a$, changing the orthonormal frame along γ_a and adding a constant vector, we may assume that $\beta^{-j}\mathbb{H}_{a,j} \circ \Phi_{\beta^j}$ converges to the restriction of the standard coordinate functions on $\mathbb{R}^n$, since every flat bundle is trivial on S which is simply-connected. This means that the limit is independent of a.

Now we claim that if $B_{a,j}$ intersects with $B_{b,k}$, then $\beta^{j(\varepsilon'-1)}|\mathbb{H}_{a,j} - \mathbb{H}_{b,k}|$ is bounded by a constant independent of a, j, b and k. In fact,

$$\begin{aligned}
&\beta^{j(\varepsilon'-1)}|\mathbb{H}_{a,j}(\Phi(x, \xi\beta^j)) - \mathbb{H}_{b,k}(\Phi(x, \xi\beta^j))| \\
&\quad \leqq \beta^{j(\varepsilon'-1)}|\mathbb{H}_{a,j}(\Phi(x, \xi\beta^j)) - \mathbb{H}_{b,j}(\Phi(x, \xi\beta^j))| \\
&\qquad + \beta^{j(\varepsilon'-1)}|\mathbb{H}_{b,j}(\Phi(x, \xi\beta^j)) - \mathbb{H}_{b,k}(\Phi(x, \xi\beta^j))|.
\end{aligned}$$

By Step 3, $\beta^{-j}\mathbb{H}_{a,j} \circ \Phi_{\beta^j}$ converges to $\mathbb{H}_{a,\infty}$ with order $O(\beta^{-j\varepsilon'})$, and $\mathbb{H}_{a,\infty} = \mathbb{H}_{b,\infty}$, the first term is bounded by a constant independent of j, k. From $\mathbb{H}_{b,k}(\Phi(x, \xi\beta^j)) = \mathbb{H}_{b,k}(\Phi_{\beta^k}(x, \xi\beta^{j-k}))$, $\beta^{-j}\mathbb{H}_{b,k}(x, \xi\beta^j))$ converges to

$$\beta^{-N}\mathbb{H}_{b,\infty}(x, \xi\beta^N) = \mathbb{H}_{b,\infty}(x, \xi)$$

when $j, k \to \infty$ keeping $j - k = N$ (constant). Thus the second term is also bounded by a constant independent of j, k.

By apriori estimates for harmonic functions (Fact (2.9) 4)), for fixed $0 < \alpha < 1$, there exists a constant C_5 independent of j, k such that

$$\| \mathbb{H}_{a,j} \circ \mathbb{H}_{b,k}^{-1} - \mathrm{Id} \|_{C^{2,\alpha}_{1-\varepsilon'}(\mathbb{H}_{b,k}(B_{a,j} \cap B_{b,k}))} \leqq C_5$$

where the weighted Hölder norms $\|\cdot\|_{C^{2,\alpha}_{1-\varepsilon'}(\Omega)}$ on a domain Ω of $\mathbb{H}_{b,k}(B_{b,k})$ (in $\mathbb{R}^n$) are described at the end of § 2.

We take a partition of unity $\{\rho_{a,j}\}$ associated to the covering $\{B_{a,j}\}$ so that if $B_{a,j}$ intersects with $B_{b,k}$,

$$\|\rho_{b,k}\circ \mathbb{H}_{a,j}^{-1}\|_{C^{2,\alpha}_{1-\varepsilon'}(\mathbb{H}_{a,j}(B_{a,j}))}\leqq C_6.$$

In fact, let ρ: $[0,\infty)\to[0,\infty)$ be a smooth function such that $\rho=1$ on $[0,\delta_1]$ and $\operatorname{supp}(\rho)\subset[0,\delta_2]$ $(0<\delta_1<\delta_2)$. Then for sufficiently small constants δ_1, δ_2, $\tilde\rho_{a,j}:=\rho(|\mathbb{H}_{a,j}-\beta^j x_a|/\beta^j)$ is well defined for any a, j and hence we set $\rho_{a,j}=\tilde\rho_{a,j}/\sum_{b,k}\tilde\rho_{b,k}$.

Now we define a smooth map Ψ: $M\setminus\tilde B\to\mathbb{R}^n$ by

$$\Psi(x)=\sum_{a,j}\rho_{a,j}(x)\,\mathbb{H}_{a,j}(x).$$

Then we have

$$\begin{aligned}&\|\Psi\circ\mathbb{H}_{a,j}^{-1}-\mathrm{Id}\|_{C^{2,\alpha}_{1-\varepsilon'}(\mathbb{H}_{a,j}(B_{a,j}))}\\&\quad=\|\sum_{b,k}\rho_{b,k}\circ\mathbb{H}_{a,j}^{-1}(\mathbb{H}_{b,k}\circ\mathbb{H}_{a,j}^{-1}-\mathrm{Id})\|_{C^{2,\alpha}_{1-\varepsilon'}(\mathbb{H}_{a,j}(B_{a,j}))}\leqq C_7.\end{aligned}$$

Therefore it turns out that Ψ induces a diffeomorphims from the outside of a compact set in M onto $S\times[a,\infty)$ for some $a>0$, and in fact Ψ defines coordinates at infinity of order ε'.

Now we change coordinates at infinity to improve the order. In the above construction we can not expect the precise order by the following reason: In the definition of ALE manifolds, the speed of convergence is estimated by the absolute value of coordinates $|z|$ which may be different from the distance r. In the above construction we use r. The difference between $|z|$ and r is the cause of the failure of the order. For example, if a metric g is written as $\Psi_* g=dr^2+(r-a)^2 d\theta^2$ where $d\theta^2$ is the standard metric on S^{n-1}, the order is only 1 (if $a\neq 0$), although $\mathscr{X}_* g_{ij}=\delta_{ij}$ in a suitable coordinates $\mathscr{X}$.

Let $\Psi=(x^1,\ldots,x^n)$ be coordinates at infinity of order ε' constructed as above. We fix ν: $\max\{0,1-\varepsilon'\}<\nu<1$, and p: $p>n$. Since $\Delta x_i\in C^{0,\alpha}_{-1-\varepsilon'}(\Psi;\,D(R))$ $(i=1,\ldots,n)$, from [Ba, 3.1] we can take $u^i\in C^{2,\alpha}_\nu(\Psi;\,D(R))$ such that

$$\Delta u^i=\Delta x^i\qquad(i=1,\ldots,n).$$

Then $\mathscr{X}=(x^1-u^1,\ldots,x^n-u^n)$ defines a new harmonic coordinate system at infinity such that

$$\Delta\mathscr{X}=0,\qquad g_{ij}-\delta_{ij}\in C^{1,\alpha}_{-1+\nu}(\mathscr{X};\,D(R'))$$

for some large R' where we write g_{ij} for $(\mathscr{X}_* g)_{ij}$.

We shall study four cases 1) $\eta<n-2$, 2) $n-2<\eta<n-1$, 3) $n-1<\eta$ and 4) $n-1\leqq\varepsilon$ or $\varepsilon=n-2$ separately. The methods are different in each cases.

Case 1. *If* $\eta<n-2$, $g_{ij}-\delta_{ij}\in C^{1,\alpha}_{-\eta}(\mathscr{X};\,D(R'))$.

In fact $u=g_{ij}-\delta_{ij}$ satisfies

$$(3.3)\qquad \Delta u=-2\,\mathrm{Ric}_{ij}+Q_{ij}(g,\partial g)$$

where $Q_{ij}(g, \partial g)$ is quadratic in ∂g (cf. Fact (2.9) 5)). Thus we have

$$|\Delta u| \leqq C_8(|z|^{-2-\eta} + |z|^{-2(2-\nu)}) \leqq C_9 |z|^{-2-\eta'}$$

for $\eta' = \min(\eta, 2-2\nu)$. We set $r_0 = |z|$. Then

$$\begin{aligned}\Delta r_0^{-\beta} &= \sum g^{ij} \partial_i \partial_j r_0^{-\beta} = \sum (g^{ij} - \delta^{ij}) \partial_i \partial_j r_0^{-\beta} + \sum \partial_i \partial_i r_0^{-\beta} \\ &= \beta(\beta - n + 2) r_0^{-\beta-2} + O(|z|^{-\beta-3+\nu}).\end{aligned}$$

Thus for any constant A,

$$\Delta(A r_0^{-\eta'} \pm u) \leqq (A\eta'(\eta' - n + 2) + C_9 + C_{10} r_0^{-1+\nu}) r_0^{-\eta'-2}.$$

If $\eta' + 2 < n$, we take A sufficiently large so that

$$\begin{aligned}&\Delta(A r_0^{-\eta'} \pm u) \leqq 0 \\ &A r_0^{-\eta'} \pm u \geqq 0 \quad \text{on } r_0 = 1.\end{aligned}$$

By the strong maximal principle, we have

$$|g_{ij} - \delta_{ij}| = |u| \leqq A r_0^{-\eta'} = A |z|^{-\eta'}.$$

By the apriori estimates [GT, Theorem (8.32)]

$$g_{ij} - \delta_{ij} \in C^{1;\alpha}_{-\eta'}(\mathscr{X}; D(R')).$$

We substitute the above to (3.3),

$$|\Delta u| \leqq C_{11}(|z|^{-2-\eta} + |z|^{-2-2\eta'}) \leqq C_{12} |z|^{-2-\eta''},$$

where $\eta'' = \min(2\eta', \eta)$. If $\eta'' < n-2$, we repeat the above argument to get

$$g_{ij} - \delta_{ij} \in C^{1;\alpha}_{-\eta''}(\mathscr{X}; D(R')).$$

Inductively we can repeat the argument until we get

$$g_{ij} - \delta_{ij} \in C^{1;\alpha}_{-\eta}(\mathscr{X}; D(R')).$$

Case 2. *If* $n-1 < \eta$, $g_{ij} - \delta_{ij} \in C^{1;\alpha}_{1-n}(\mathscr{X}; D(R'))$.

By Case 1 and [Ba, Proposition (1.6)],

$$u = g_{ij} - \delta_{ij} \in W^{2;p}_{-a}(\mathscr{X}; D(R')) \qquad \text{for all } a < n-2.$$

We denote by Δ_0 the standard Laplacian of $\mathbb{R}^n$. Then $\Delta_0 u \in L^p_{-2-\nu}(\mathscr{X}; D(R'))$ where $\nu = \min(2a, \eta)$. To use the result of Bartnik [Ba], we recall the notion of exceptional values which correspond to the order of harmonic functions. The weighting parameter $a \in \mathbb{R}$ is said to be *exceptional* if $a \in \{k \in \mathbb{Z}\colon k \neq -1, -2, \ldots, 3-n\}$, *non-exceptional* if it is not exceptional. By [Ba, Theorem (1.10)] for all nonexceptional $l > -\nu$, there exists $v^l_{ij} \in W^{2,p}_l$ such that

$$\Delta_0 v^l_{ij} = \Delta_0(g_{ij} - \delta_{ij}) \qquad \text{on } |z| \geqq R'.$$

The expansion for harmonic functions shows that

$$g_{ij} - \delta_{ij} = v^l_{ij} + A_{ij}|z|^{2-n} + O(|z|^{1-n})$$

for some constant A_{ij}. But since $\mathscr{Z}$ are harmonic coordinates, it satisfies

$$g^{ij}\Gamma^k_{ij} = 0.$$

This implies that $A_{ij} = 0$.

In case $\eta > n-1$, we can take l so that $-\eta < l < 1-n$. If we take sufficiently large p, the Sobolev embedding theorem [Ba, Theorem (1.2)] implies

$$u = g_{ij} - \delta_{ij} \in C^{1,\alpha}_{1-n}(\mathscr{Z}; D(R')).$$

Case 3. *If* $n-2 < \eta < n-1$, $g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\eta}(\mathscr{Z}; D(R'))$.

We have $\nu = \min(2a, \eta) = \eta$ in this case. In the argument of Case 2, we can take l so that $-\eta < l < 2-n$. Thus we have

$$u = g_{ij} - \delta_{ij} \in W^{2;p}_{-\mu'}(\mathscr{Z}; D(R')) \qquad \text{for all } \mu' < \eta.$$

There exists a Green function G on M (see for example [MSY]) such that

$$C_{13}\, d(x,y)^{2-n} \leqq G(x,y) \leqq C_{14}\, d(x,y)^{2-n},$$
$$|\nabla G(x,y)| \leqq C_{15}\, d(x,y)^{1-n}.$$

Since $|u| = O(r^{-\mu'})$ and $|du| = O(r^{-1-\mu'})$ by the Sobolev embedding theorem (cf. [Ba, Theorem (1.2)]), u can be represented as

$$u(x) = \int_M G(x,y) f(y)\, dy$$

for $f = -\Delta u = O(r^{-2-\eta})$ (cf. (3.3)), which implies $f \in L^1$. Moreover we have

$$u(x)\, G(x,o)^{-1} \to 0$$
$$|G(x,o)^{-1} \int_{B(x;r(x)/2)} G(x,y) f(y)\, dy| \leqq C_{16}\, r(x)^n \sup_{B(x;r(x)/2)} |f| \to 0$$

as $x \to \infty$. Hence we have

$$G(x,o)^{-1} \int_{M \setminus B(x;r(x)/2)} G(x,y) f(y)\, dy \to 0. \tag{3.4}$$

Since we can join o and y by a pass $\sigma(t)$ such that

$$d(\sigma(t), x) \geqq C_{17}^{-1}\, d(x,y),$$
$$\operatorname{length}(\sigma) \leqq C_{17}\, r(y).$$

We have

$$|G(x,y) - G(x,o)| \leqq C_{18}\, r(y)\, d(x,y)^{1-n}. \tag{3.5}$$

So $G(x, o)^{-1} G(x, y)$ converges to 1 as $x \to \infty$. Since $G(x, o)^{-1} G(x, y)$ is bounded by a constant on $M \setminus B(x; r(x)/2)$, we apply the Lebesgue dominant convergence theorem to (3.4) to have

$$\int_M f(y)\, dy = 0.$$

Thus we get

$$u(x) = \int_M \{G(x, y) - G(x, o)\} f(y)\, dy.$$

Substituting (3.5),

$$\begin{aligned} r(x)^\eta |u(x)| &\leqq C_{19}\, r(x)^\eta \int_M d(x, y)^{1-n} r(y)^{-1-\eta} dy \\ &\leqq C_{20}\, r(x)^{-1} \int_{B(x;\, r(x)/2)} d(x, y)^{1-n} dy \\ &\quad + C_{20}\, r(x)^{\eta+1-n} \int_{B(o;\, 2r(x)) \setminus B(x;\, r(x)/2)} r(y)^{-1-\eta} dy \\ &\quad + C_{20}\, r(x)^\eta \int_{M \setminus B(o;\, 2r(x))} r(y)^{-\eta-n} dy \\ &\leqq C_{21}. \end{aligned}$$

We remark that the second term is bounded since $-1-\eta > -n$. By the apriori estimates (cf. [GT, Theorem (8.32)]) we have $u \in C^{1,\alpha}_{-\eta}(\mathscr{X}; D(R'))$.

Case 4. *There exist coordinates at infinity* Ψ *of order* ε *if* $n \geqq 4$ *or* $\varepsilon \neq 1$, *and of order* ε' (*for any* $\varepsilon' < 1$) *if* $n = 3$ *and* $\varepsilon = 1$.

If we insist on harmonic coordinates, we can not improve the order as is observed in Case 2. We return back to the construction of coordinates at infinity in Steps $1 \sim 4$. In Steps $1 \sim 4$, we have fixed a hypersurface S_R and constructed coordinates using outer normal geodesics. In this case we use inner normal geodesics from "end".

For the proof of Theorem (1.1) of the remaining case, we may assume $\varepsilon = n-2$ or $\varepsilon \geqq n-1$. By case 1, we have coordinates at infinity $\mathscr{X}$ of order μ for some μ which is determined by ε and may be assumed to be greater than 1. In what follows, identifying the end of M with the end of $\mathbb{R}^n$ through the coordinates $\mathscr{X} = (x^1, \ldots, x^n)$, and extending the metric $g = \sum g_{ij}\, dx^i\, dx^j$ to the whole of $\mathbb{R}^n$, we consider M as $\mathbb{R}^n$ with the metric g satisfying $g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\mu}$ $(0 < \alpha < 1)$. Let $B_R = \{x \in \mathbb{R}^n \colon |x| < R\}$ and $S_R = \partial B_R$. Then the second fundamental form A_R of S_R satisfies

$$|RA_R + \mathrm{Id}| \leqq C_{22}\, R^{-\mu}. \tag{3.6}$$

Let us consider a map

$$\Phi_R \colon (x, t) \in S_1 \times [K, R] \mapsto \exp(R-t)\, v_R(Rx) \in M$$

where ν_R is the *inner* normal vector of S_R. Then we claim first that Φ_R induces a diffeomorphism from $S_1 \times [K, R]$ onto a domain $A_{R,K}$ of $\bar{B}_R$ for some fixed $K>0$ which is independent of R. In fact it follows from the comparison theorem on Jacobi fields as in Step 1 that for some C_{23} independent of R, S_R has no focal points along any geodesics $\sigma_R(t)=\Phi_R(x,t)$ $(t\in[C_{23}, R])$ as long as σ_R realizes the distance to S_R, i.e., $d(S_R, \sigma_R(t))=R-t$. Let $\mathscr{C}_R^-$ be the cut locus of S_R within the ball B_R and q a point of $\mathscr{C}_R^-$ which is nearest to S_R. Suppose $R-L>C_{23}$ where $L=d(q, S_R)=d(\mathscr{C}_R^-, S_R)$. Then by the same reason as in Step 1, we have a geodesic γ: $[0, 2L]\to\bar{B}_R$ through q such that $d(\gamma(t), S_R)=t$ for $t\in[0, L]$ and $d(\gamma(t), S_R)=2L-t$ for $t\in[L, 2L]$. If we denote by r_0 the Euclidean distance to the origin $0\in\mathbb{R}^n$, then the hessian $\frac{1}{2}\nabla^2 r_0^2$ of $\frac{1}{2}r_0^2$ (with respect to g) can be written as

$$\tfrac{1}{2}\nabla^2 r_0=\sum_{i=1}^{n} dx^{i2}+\sum_{i,j,k}\Gamma_{ij}^k x^k dx^i dx^j,$$

where Γ_{ij}^k is the Christoffel symbol of the metric g with respect to the coordinates $x=(x^1,\dots,x^n)\in\mathbb{R}^n$. We observe from the assumption: $g_{ij}-\delta_{ij}\in C^{1,\alpha}_{-\mu}$ $(\mu>1)$ that $|\langle\nabla r_0, \gamma'(0)\rangle+1|\leqq C_{24}R^{-\mu}$, $|\langle\nabla r_0, \gamma'(2L)\rangle-1|\leqq C_{24}R^{-\mu}$ and $r_0(\gamma(t))\geqq C_{24}(R-d(\gamma(t), S_R))$. Therefore we have

$$\begin{aligned}
2R-C_{25}&\leqq\tfrac{1}{2}r_0^2(\gamma(t))'|_{t=2L}-\tfrac{1}{2}r_0^2(\gamma(t))'|_{t=0}\\
&=\int_0^{2L}\tfrac{1}{2}r_0^2(\gamma(t))''\,dt\\
&\leqq\int_0^{2L}(1+C_{26}\,r_0(\gamma(t))^{-\mu})\,dt\\
&\leqq 2L+C_{27}\int_0^{2L}(R-d(\gamma(t), S_R))^{-\mu}dt\\
&\leqq 2L+C_{27}\int_C^{\infty}u^{-\mu}du\\
&\leqq 2L+C_{28}.
\end{aligned}$$

Hence it follows that

$$R-L\leqq C_{29}.$$

Thus by setting $K=\max\{C_{23}, C_{29}\}$, we have shown the claim. We note here that for large R, the extension of Φ_R: $S_1\times[k,\infty)\ni(x,t)\mapsto\exp(R-t)\nu_R(x)\in\mathbb{R}^n$ also induces a diffeomorphism from $S_1\times[k,\infty)$ onto $\hat{A}_{R,K}=A_{R,K}\cup(\mathbb{R}^n\setminus B_R)$ (cf. Step 1, or the same argument as above).

Before constructing coordinates at infinity of order ε, we shall make some further observations. For a point $x\in S^{n-1}$, we take a parallel vector fields $E_1(t),\dots,E_n(t)$ along $\sigma_R(t)=\Phi_R(x,t)$, where $E_i(R)=\partial/\partial x^i$. For $k=1,\dots,n$ let $X_{R,k}$ be a Jacobi fields along σ_R with $X_{R,k}(R)=RE_k(R)$, $\frac{\nabla}{dt}X_{R,k}^{\perp}(R)=$

$-RA_R(E_k^\perp(R))$, and $\frac{\nabla}{dt}(X_{R,k}-X_{R,k}^\perp)(R)=E_R(R)-E_R^\perp(R)$, where for a vector X at $\sigma_R(t)$, $X^\perp$ denotes the component of X perpendicular to $\sigma'(t)$. Set $X_{R,k}^i(t)=(X_{R,k}(t), E_i(t))$. Then we have

$$X_{R,k}^i(t)=X_{R,k}^i(R)+(X_{R,k}^i)'(R)(t-R)+\int_t^R\int_s^R \sum X_{R,k}^j(u)\,K_j^i(u)\,du\,ds$$

for $t\in[K,R]$, where $K_j^i(t)=(R(E_j(t),\sigma_R'(t))\sigma_R'(t), E_i(t))$. Noting that $|X_{R,K}^i(t)|\leqq C_{30}\,t$ (cf. Fact (2.3)) and $|K_j^i(t)|\leqq C_{30}\,t^{-2-\varepsilon}$ for $t\in[K,R]$, we get by (3.6)

$$|t^{-1}X_{R,k}^i(t)-\delta_{ik}|\leqq C_{31}(R^{-\mu}+t^{-\varepsilon}). \tag{3.7}$$

Let us take here the harmonic coordinates $\mathbb{H}_{x,t}$ on $B(\sigma_R(t);\delta t)$ associated with $\{E_1(t),\dots,E_n(t)\}$, where it is assumed that $\mathbb{H}_{x,t}(\sigma_R(t))=tx$, and compare $\mathbb{H}_{x,t}$ with the map $\hat{\Phi}_R$: $\hat{A}_{R,K}\to\mathbb{R}^n$ defined by $\hat{\Phi}_R(\exp(R-t)\nu_R(x))=tx$. By the definition, $\mathbb{H}_{x,t}(\sigma_R(t))=\hat{\Phi}_R(t))=tx$. Moreover, it follows from (3.7) and the same argument as in Step 3 that

$$\|D\mathbb{H}_{x,t}-D\hat{\Phi}_R\|\leqq C_{32}(R^{-\mu}+t^{-\varepsilon})$$

on $B(\sigma_R(t);\delta t)$. This shows that

$$|\mathbb{H}_{x,t}-\hat{\Phi}_R|\leqq C_{33}(R^{-\mu}t+t^{1-\varepsilon}) \tag{3.8}$$

on $B(\sigma_R(t);\delta t)$. Since $g_{ij}-\delta_{ij}\in C_{-\mu}^{1,\alpha}$, it is not hard to see that

$$\begin{aligned}|(\hat{\Phi}_R-\mathrm{Id})(\exp(R-t)\,\nu_R(x))|&\leqq C_{34}((R-t)\,R^{-\mu}+t^{1-\mu}+R^{1-\mu})\\&\leqq C_{34}(t^{1-\mu}+R^{1-\mu}).\end{aligned} \tag{3.9}$$

Let us now construct coordinates at infinity of order ε using the same arguments as in Step 4. We take geodesics $\gamma_1,\dots,\gamma_N(\gamma_a(t)=\Phi_R(x_a,t),\ x_a\in S^{n-1})$, the parallel frame $\{E_a^1(t),\dots,E_a^n(t)\}$ along $\gamma_a(t)$ with $E_a^i(R)=\partial/\partial z^i$, and harmonic coordinates $\mathbb{H}_{a,j}$ on $B_{a,j}=B(\gamma_a(\beta^j);\delta\beta^j)$ associated with $\{E_a^1(\beta^j),\dots,E_a^n(\beta^j)\}$ where N, x_a, $\beta>1$, and $\delta>0$ are chosen appropriately as in Step 4, and it is assumed that $\mathbb{H}_{a,j}(\gamma_a(\beta^j))=\hat{\Psi}(\gamma_a(\beta^j))=\beta^j a$. If $B_{a,j}$ intersects with $B_{b,k}$, then on $B_{a,j}\cap B_{b,k}$, we have by (3.8)

$$\begin{aligned}|\mathbb{H}_{a,j}-\mathbb{H}_{b,k}|&\leqq|\mathbb{H}_{a,j}-\hat{\Phi}_R|+|\mathbb{H}_{b,k}-\hat{\Phi}_R|\\&\leqq C_{35}(R^{-\mu}\beta^j+\beta^{j(1-\varepsilon)}).\end{aligned}$$

This implies that

$$\|\mathbb{H}_{b,k}\circ\mathbb{H}_{a,j}^{-1}-\mathrm{Id}\|_{C_0^{2,\alpha}(\mathbb{H}_{a,j}(B_{a,j}))}\leqq C_{36}(R^{-\mu}\beta^j+\beta^{j(1-\varepsilon)})$$

(cf. Fact (2.9)). By taking sufficiently large K if necessarily, we define coordinates Ψ_R: $B_R\setminus B_K\to\mathbb{R}^n$ as in Step 4. Then it follows from (3.9), (3.10) and Fact (2.9) that

$$\begin{aligned}&|\Psi_R-\mathrm{Id}|\leqq C_{37}(R^{1-\mu}+\beta^{j(1-\varepsilon)}) \quad \text{on } B_{a,j},\\&\|(\Psi_{R*}\,g)_{ij}-\delta_{ij}\|_{C_0^{1,\alpha}(\Psi_R(B_{a,j}))}\leqq C_{38}(R^{-\mu}+\beta^{-j\varepsilon}).\end{aligned}$$

Since it is assumed that $\mu > 1$, $R^{1-\mu}$ goes to 0 as $R \to \infty$. Taking some divergent sequence $\{R_k\}$ and letting $R_k \to \infty$, we obtain coordinates $\Psi\colon \mathbb{R}^n \setminus B_K \to \mathbb{R}^n$ such that

$$(\Psi_* g)_{ij} - \delta_{ij} \in C^{1;\alpha}_{-\varepsilon}(\Psi(\mathbb{R}^n \setminus B_K)) = C^{1;\alpha}_{-\varepsilon}(\Psi; D(K')).$$

for some $K' > 0$. This completes the proof of Theorem (1.1).

§ 4. Proof of Theorem (1.5)

In this section we prove Theorem (1.5). First we prove several general apriori estimates for a non-negative function u which satisfies

$$\Delta u \geqq -fu \quad \text{on } M$$

with a non-negative function f, where M is a complete, noncompact Riemannian manifold such that

$$\Big\{\int_M v^{2\gamma}\Big\}^{1/\gamma} \leqq S \int_M |Dv|^2 \qquad \text{for all } v \in C^1_c(M),$$

$$\operatorname{vol}(B(o;t)) \leqq V t^n \qquad \text{for all } t > 0$$

with some positive constants S, V and $\gamma = n/(n-2)$. (It will be turned out that the manifolds as in Theorem (1.5) fulfill the above conditions.) We denote $M \setminus B(o;r)$ by $D(r)$.

(4.1) **Lemma.** *Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > 1$. Then $u \in L^q$ for all $q \geqq p$, and there exists $\varepsilon_1 = \varepsilon_1(S, V, p_0, p_1) > 0$ such that if*

$$\Big\{\int_{D(r)} f^{n/2}\Big\}^{2/n} \leqq \varepsilon_1,$$

then

$$\Big\{\int_{D(2r)} u^{p\gamma}\Big\}^{1/\gamma} \leqq C_1 r^{-2} \int_{D(r)\setminus D(2r)} u^p$$

where $C_1 = C_1(S, V, p_0)$.

(4.2) **Lemma.** *Suppose $f \in L^{n/2}$, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > \gamma$. Then there exists $\varepsilon_2 = \varepsilon_2(S, V, p_0, p_1) > 0$ such that if*

$$\Big\{\int_{D(r)} f^{n/2}\Big\}^{2/n} \leqq \varepsilon_2,$$

then

$$\int_{D(2r)} u^p \leqq C_2 \int_{D(r)\setminus D(2r)} u^p,$$

$$\int_{D(r)} u^p = O(r^{-\varepsilon})$$

where $C_2 = C_2(S, V, p_0)$, $\varepsilon = \varepsilon(S, V, p_0)$.

Proof. For a nonnegative function φ with a compact support, we have

$$\begin{aligned}\int f\varphi^2 u^p &\geqq \int \varphi^2 u^{p-1}(-\Delta u)\\ &\geqq 4(p-1)p^{-2}\int|\varphi\nabla(u^{p/2})|^2+4p^{-1}\int(u^{p/2}\nabla\varphi, \varphi\nabla(u^{p/2})).\end{aligned}$$

We apply Schwarz inequality to the second term and get

$$2(p-1)p^{-2}\int|\varphi\nabla(u^{p/2})|^2 \leqq \int f\varphi^2 u^p+2(p-1)^{-1}\int|\nabla\varphi|^2 u^p.$$

Using the Sobolev inequality we obtain

$$\begin{aligned}\{\int|\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma} &\leqq C_3\{p^2(p-1)^{-1}\int f\varphi^2 u^p+(2p^2(p-1)^{-2}+2)\int|\nabla\varphi|^2 u^p\}\\ &\leqq C_3 p^2(p-1)^{-1}\{\int_{\operatorname{supp}\varphi} f^{n/2}\}^{2/n}\{\int|\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma}\\ &\quad + C_3(2p^2(p-1)^{-2}+2)\int|\nabla\varphi|^2 u^p.\end{aligned} \tag{4.3}$$

Let φ be a function such that $0\leqq\varphi\leqq 1$, $\varphi=0$ in $B(o;r)\setminus D(2r')$, $\varphi=1$ in $D(2r)\setminus D(r')$ with $|D\varphi|\leqq C_4(r^{-1}+r'^{-1})$ for $r<r'$. We take $\varepsilon_1=C_3^{-1}p^{-2}(p-1)/2$. By the assumption we have

$$\{\int_{D(r)} f^{n/2}\}^{2/n}\leqq C_3^{-1}p^{-2}(p-1)/2.$$

Substituting this into (4.3) we have

$$\{\int|\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma}\leqq 2C_3(2p^2(p-1)^{-2}+2)\int|\nabla\varphi|^2 u^p. \tag{4.4}$$

Letting $r'\to\infty$, we get (4.1).

From the Hölder inequality we obtain

$$\{\int|\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma}\leqq 2C_3(2p^2(p-1)^{-2}+2)\{\int|\nabla\varphi|^n\}^{2/n}\{\int_{\operatorname{supp}|\nabla\varphi|} u^{p\gamma}\}^{1/\gamma}. \tag{4.5}$$

Letting $r'\to\infty$, we get (4.2). □

(4.6) **Lemma.** *If $f\in L^q$ for some $q>n/2$, $u\in L^p$ for $p>1$, and it holds*

$$\int_{D(r)} f^q\leqq A r^{-(2q-n)}$$

for some constant A, then

$$\sup_{D(2r)} u^p\leqq C_5 r^{-n}\int_{D(r)} u^p$$

where $C_5=C_5(A, S, V, p)$.

Proof. We shall use the Moser iteration technique. Let $\beta\leqq p$. From (4.3) we have

$$\{\int(\varphi u^{\beta/2})^{2\gamma}\}^{1/\gamma}\leqq C_6\{\beta\int f\varphi^2 u^\beta+\int|\nabla\varphi|^2 u^\beta\} \tag{4.7}$$

where $C_6 = C_6(S, V, p)$. From the Hölder inequality we obtain

$$\beta \int f \varphi^2 u^\beta \leqq \beta \{ \int_{\operatorname{supp}\varphi} f^q \}^{1/q} \{ \int |\varphi u^{\beta/2}|^{2\gamma} \}^{1/s} \{ \int \varphi^2 u^\beta \}^{1/t}$$

where $1/q + 1/s + 1/t = 1$, $\gamma/s + 1/t = 1$. For $\delta > 0$ we have

$$\beta \int f \varphi^2 u^\beta \leqq \delta \{ \int |\varphi u^{\beta/2}|^{2\gamma} \}^{1/\gamma} + \delta^{-st/\gamma} \beta^t \{ \int_{\operatorname{supp}\varphi} f^q \}^{t/q} \int \varphi^2 u^\beta.$$

Taking δ so that $C_6 \delta = 1/2$, we substitute this into (4.7) to get

$$\{ \int |\varphi u^{\beta/2}|^{2\gamma} \}^{1/\gamma} \leqq C_7 [\beta^t \{ \int_{\operatorname{supp}\varphi} f^q \}^{t/q} \int \varphi^2 u^\beta + \int |\nabla \varphi|^2 u^\beta].$$

For $r_1 < r_2 < r_3 < r_4$ with $r_1 - r_2 = r_3 - r_4$, we take φ so that $0 \leqq \varphi \leqq 1$, $\varphi = 0$ in $B(0; r_1) \cup D(r_4)$, $\varphi = 1$ in $D(r_2) \setminus D(r_3)$ with $|D\varphi| \leqq C_8 (r_2 - r_1)^{-1}$. Then

$$\{ \int_{D(r_2)\setminus D(r_3)} u^{\beta\gamma} \}^{1/\gamma} \leqq C_9 \{ \beta^t r_1^{-2} + (r_2 - r_1)^{-2} \} \int_{D(r_1)\setminus D(r_4)} u^\beta.$$

We set

$$\Phi(\beta, r, r') = \{ \int_{D(r)\setminus D(r')} u^\beta \}^{1/\beta}.$$

We have

$$\Phi(\beta\gamma, r_2, r_3) \leqq C_{10} (\beta^t r_1^{-2} + (r_2 - r_1)^{-2} \}^{1/\beta} \Phi(\beta, r_1, r_4).$$

Taking $r_{1,m} = (1 - 2^{-m}) r$, $r_{2,m} = r_{1,m+1}$, $r_{4,m} = (2 + 2^{-m}) r$, $r_{3,m} = r_{4,m+1}$, $\beta_m = p\gamma^m$, we obtain

$$\Phi(\beta_{m+1}, r_{1,m+1}, r_{4,m+1}) \leqq \{ C_{11} 2^{-2m} r^{-2} \gamma^{mt} \}^{1/p\gamma^m} \Phi(\beta_m, r_{1,m}, r_{4,m})$$

where $C_{11} = C_{11}(A, S, V, p)$. Inductively we have

$$\Phi(\beta_m, r_{1,m}, r_{4,m}) \leqq C_{12}^{\Sigma m\gamma^{-m}} r^{-2\Sigma\gamma^{-m}/p} (p, r/2, 5r/2).$$

Since $\Sigma m\gamma^{-m} < \infty$ and $\Sigma\gamma^{-m} = n/2$, letting $m \to \infty$, we have

$$\sup_{D(r)\setminus D(2r)} u \leqq C_{13} r^{-n/p} \{ \int_{D(r/2)} u^p \}. \quad \square$$

(4.8) **Proposition.** 1) *Suppose $f \in L^{n/2}$, and u, f satisfy the conditions of Lemma (4.4). Then $u = O(r^{-\alpha})$ for any $\alpha < n-2$.*

2) *If $u \in L^{n/2}$ satisfies $\Delta u \geqq -au^2$ with a constant a, then*

$$\int_{D(r)} u^{\gamma n/2} = O(r^{-2\gamma}),$$

$$u = O(r^{-\alpha}) \quad \textit{for any } \alpha < n-2.$$

Proof. 1) By Lemma (4.6) $u=O(r^{-\alpha})$ for some $\alpha>0$. Let α_0 be the supremum of such α. Suppose $\alpha_0<n-2$. For any $p>n/\alpha_0>\gamma$, we have $u\in L^p$ by the definition of α_0. By Lemma (4.2) we have $\varepsilon>0$ which is independent of p such that

$$\int_{D(r)} u^p = O(r^{-\varepsilon}).$$

Substituting this into Lemma (4.6) we get $u=O(r^{-(n+\varepsilon)/p})$. Since we can take p arbitrarily close to n/α_0, this contradicts to the definition of α_0.

2) We apply Lemma (4.1) with $p=n/2$ to get $u\in L^q$ for $q=n\gamma/2$, and

$$\int_{D(r)} u^q = O(r^{-2\gamma}).$$

Since $2\gamma=2q-n$, $f=au$ satisfies the conditions of Lemma (4.6). By 1) we have $u=O(r^{-\alpha})$ for any $\alpha<n-2$. $\square$

(4.9) **Lemma.** *Suppose S, T are tensors having the same symmetry as the curvature tensor R, and the covariant derivative ∇R of the curvature tensor of the Einstein metric g respectively. Then there exists $\delta=\delta(n)$ such that*

$$(1+\delta)\,|(S,T)|^2 \leqq |S|^2\,|T|^2,$$

where (S,T) is a 1-form defined by $(S,T)(X)=(S,T(X))$ for a tangent vector X. Moreover if g is Kähler, we can take $\delta=4/(n+2)$. If $n=4$, and g is self-dual or anti-self-dual, we can take $\delta=2/3$.

We shall prove this lemma in the appendix.

(4.10) **Corollary.** *If (M,g) is an Einstein manifold,*

$$\Delta|R|^{1-\delta} \geqq -C_{14}|R|^{2-\delta}$$

where $C_{14}=C_{14}(n)$, and δ is as in Lemma (4.7). Moreover we can take $\delta=2/3$ in the case $n=4$, even if g is neither self-dual or anti-self-dual.

Proof. From Bochner-Weitzenböck formula [BL] we have

$$\Delta|R|^2 = -2(\Delta R, R) + 2|\nabla R|^2 - (Q(R), R)$$

where $Q(R)$ is quadratic in R. Since M is Einstein, $\Delta R=0$. Hence we have

$$\Delta|R|^2 \geqq 2|\nabla R|^2 - C_{15}|R|^3.$$

Using Lemma (4.9) we have

$$|\nabla R|^2 \geqq (1+\delta)\,|(\nabla R, R)|^2\,|R|^{-2} = (1+\delta)\,|\nabla|R||^2.$$

Hence we have

$$\begin{aligned}\Delta|R|^{1-\delta} &\geqq -(1-\delta)(1+\delta)\,|R|^{-1-\delta}|\nabla|R||^2 + (1-\delta)\,|\nabla R|^2|R|^{-1-\delta} - C_{16}|R|^{2-\delta}\\ &\geqq -C_{16}|R|^{2-\delta}.\end{aligned}$$

In $n=4$, we decompose the curvature tensor to the self-dual part and anti-self dual part $R=R_+ + R_-$. Since R_+, ∇R_+ have the symmetry as the curvature tensor R and the covariant derivative of curvature tensor ∇R of self-dual metric g, we apply Lemma (4.9) to get

$$\Delta|R_+|^{1-\delta} \geqq -C_{14}|R||R_+|^{1-\delta}, \qquad \Delta|R_-|^{1-\delta} \geqq -C_{14}|R||R_-|^{1-\delta}.$$

Adding this we obtain

$$\Delta|R|^{1-\delta} \geqq -C_{17}|R|^{2-\delta}. \quad \square$$

Now we are in the position to prove Theorem (1.5). Let (M, g) be a Riemannian manifold which satisfies the assumption of Theorem (1.5). Since (M, g) is Ricci-flat and satisfies the condition (1.6), we see by the Cheeger-Gromoll splitting theorem [CG] that M has one end and also by the result of Croke [Cr] that the Sobolev inequality holds on (M, g). Since $|R|$ satisfies

$$\Delta|R| \geqq -C_{18}|R|^2,$$

we can apply Proposition (4.8) 2) to get

$$\int_{D(r)} |R|^{\gamma n/2} = O(r^{-2\gamma}).$$

Thus we can apply Proposition (4.8) 1) for $\Delta|R|^{1-\delta} \geqq -C_{18}|R||R|^{1-\delta}$. We get $|R|^{1-\delta} = O(r^{-\alpha})$ for any $\alpha < n-2$. Thus we have $|R| = O(r^{-3-\varepsilon})$ for some $\varepsilon > 0$ for $n \geqq 5$. For $n=4$, we have $|R| = O(r^{-\alpha})$ for any $\alpha < 6$. In both cases we can apply Theorem (1.1). Moreover if $n=4$ or g is Kähler, we already know that (M, g) is ALE of order $n-1$. We can then apply the method in § 3 Case 2 to $\Delta|R|^{1-\delta} \geqq -C|R|^{2-\delta}$ to get $|R|^{1-\delta} = O(r^{-(n-2)})$. This means $|R| = O(r^{-(n+2)})$.

$\square$

§ 5. Removable singularities theorem for Einstein metrics

In [Uh] Uhlenbeck proved that an apparent point singularity of a Yang-Mills connection with curvature in L^2 over a 4-dimensional manifold can be removed by a gauge transformation. Later it was extended by Sibner [Si] to a higher dimension n under the assumption that the connection has curvature in $L^{n/2}$. In this section we prove the corresponding results for the case of Einstein metrics.

(5.1) **Theorem.** *Let $B = B(o; 1)$ be the unit ball centered at a point o in a complete, locally compact metric space of length (X, d). Suppose $B \setminus \{o\}$ is a locally connected* (i.e. *for any open set $U \ni o$, there is an open set $V \ni o$ such that $V \subset U$, $V \setminus \{o\}$ is connected) n-dimensional C^∞ manifold with an Einstein metric g satisfying*

(5.2) $$\int_{B\setminus\{o\}} |R|^{n/2} < \infty,$$

(5.3) $$\Big\{ \int_{B\setminus\{o\}} |v|^{\sigma} \Big\}^{1/\sigma} \leqq S \int_{B\setminus\{o\}} |Dv| \quad \textit{for all } v \in C_0^1(B \setminus \{o\}),$$

(5.4) $$\mathrm{vol}(B(o; t)) \leqq V t^n \quad \textit{for all } t > 0$$

with positive constants S, V and $\sigma = n/(n-1)$. Then the metric g extends smoothly to B as an orbifold metric B.

The third author has shown the removable singularities theorem for Yang-Mills connection with sufficiently small L^2 norm [N1] (see also [OS]). That theorem relied on a monotonicity formula for Yang-Mills connections (see [Pr]). We are unable to prove the corresponding result for Einstein manifolds because we do not have such a monotonicity formula.

Combining Theorem (1.5) and Theorem (5.1) with the results of the third author [N2] we obtain

(5.5) **Theorem.** *Let (X_i, g_i) be a sequence of n-dimensional smooth manifolds and Einstein metrics on them with Einstein constants $k(\pm 1$, or $0)$ satisfying*

$$\operatorname{diam}(X_i, g_i) \leqq D, \qquad \operatorname{vol}(X_i, g_i) \geqq V, \quad \text{and} \quad \int_{X_i} |R_{g_i}|^{n/2} dV_i \leqq R$$

for some positive constants D, V, R. Then there exist a subsequence $\{j\} \subset \{i\}$ and a compact Einstein orbifold (X_∞, g_∞) with a finite singular set $S = \{x_1, \ldots, x_k\} \subset X_\infty$ (possibly empty) for which the following statements hold:

1) *(X_j, g_j) converges to (X_∞, g_∞) in the Hausdorff distance.*

2) *There exists an into diffeomorphism F_j: $X_\infty \backslash S \to X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞-topology on $X_\infty \backslash S$.*

3) *For every $x_a \in S$ $(a = 1, \ldots, k)$ and j, there exist $x_{a,j} \in X_j$ and positive number r_j such that*

3a) *$B(x_{a,j}; \delta)$ converges to $B(x_a; \delta)$ in the Hausdorff distance for all $\delta > 0$,*

3b) $\lim_{j\to\infty} r_j = \infty$,

3c) *$((X_j, r_j g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat n-manifold which is ALE of order $n-1$,*

3d) *there exists an into diffeomorphism G_j: $M_a \to X_j$ such that $G_j^*(r_j g_j)$ converges to h_a in the C^∞-topology on M_a.*

4) *It holds*

$$\lim_{j\to\infty} \int_{X_j} |R_{g_j}|^{n/2} dV_j \geqq \int_{X_\infty} |R_{g_\infty}|^{n/2} dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^{n/2} dV_{h_a}.$$

The same results except for the statement 3c) are obtained by Anderson [An] independently.

It would be conjectured that the equality holds in 4), and that the local fundamental group around x_a and the fundamental group of the end of M_a coincide. However the difficulty in solving it arises from the case that two curvature concentrations happen at the same point.

Proof of Theorem (5.1). Throughout the proof, we use $C_1, C_2, \ldots$ to denote positive constants depending on n, S, V in the Theorem unless otherwise mentioned. We denote $B \backslash \{o\}$ by B^*.

First we shall show $|R|$ is bounded. We denote by $B(r)$ the ball of radius r with center o. We remark that for $v \in C_0^{0,1}(B)$

$$\{\int_B |v|^{2\gamma}\}^{1/\gamma} \leqq 2S \int_B |Dv|^2 \qquad (\gamma = n/(n-2)). \tag{5.6}$$

In fact, if $v \in C_0^1(B^*)$, the above inequality follows substituting $v^{2(n-1)/(n-2)}$ into (5.3). Let φ be a cut-off function such that $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on $B \backslash B(2r')$, $\varphi = 0$ on $B(r')$, and $|D\varphi| \leqq 2r'^{-1}$. Then

$$\{\int_B |\varphi v|^{2\gamma}\}^{1/\gamma} \leqq S \int_B |D(\varphi v)|^2 \leqq 2S \int_B \varphi^2 |Dv|^2 + 2S \{\int_B |D\varphi|^n\}^{2/n} \{\int_{\operatorname{supp}|D\varphi|} |v|^{2\gamma}\}^{1/\gamma}.$$

Letting $r' \to 0$, we have (5.6). (5.6) also holds for $v \in W^{1,2}(B^*)$. In fact let $F_k(t)$ be

$$F_k(t) = \begin{cases} k, & \text{for } t \geqq k \\ t, & \text{for } |t| < k \\ -k, & \text{for } t \leqq -k. \end{cases}$$

Then we have

$$\{\int_B |F_k(v)|^{2\gamma}\}^{1/\gamma} \leqq 2S \int_B |D(F_k(v))|^2 = 2S \int_{|v|<k} |Dv|^2.$$

Letting $k \to \infty$, by Fatou's Lemma we have (5.6) for $v \in W^{1,2}(B^*)$.

Suppose a nonnegative function u satisfies

$$\Delta u \geqq -fu \qquad \text{on } B^*$$

with a nonnegative function f.

(5.8) **Lemma** (cf. Lemma (4.2)). *Suppose $f \in L^{n/2}$, and $u \in L^q$ for some $q \in [q_0, q_1]$ where $q_0 > \gamma$. Then there exist $\varepsilon_1 = \varepsilon_1(S, V, q_0, q_1) > 0$ and $\varepsilon = \varepsilon(S, V, q_0)$ such that if*

$$\{\int_{B(r)} f^{n/2}\}^{2/n} \leqq \varepsilon_1,$$

then

$$\int_{B(r)} u^q = O(r^\varepsilon).$$

Proof. In the inequality (4.5)

$$\{\int |\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma} \leqq C_1 \{\int |\nabla \varphi|^n\}^{2/n} \{\int_{\operatorname{supp}|\nabla\varphi|} u^{p\gamma}\}^{1/\gamma},$$

($C_1=C_1(S, V, q_0)$) we take a cut-off function φ so that $0\leqq\varphi\leqq 1$, $\varphi=0$ in $B(r')\cup(B\backslash B(2r))$, $\varphi=1$ on $B(r)\backslash B(2r')$ with $|D\varphi|\leqq Cr'^{-1}$ for $2r'<r$. Letting $r'\to 0$, we get

$$\{\int_{B(r)} u^q\}^{1/\gamma}\leqq C_2\{\int_{B(2r)\backslash B(r)} u^q\}^{1/\gamma}$$

where $C_2=C_2(S, V, q_0)$. □

(5.9) **Lemma.** *Under the same conditions as Lemma* (5.8), *we have* $u\in L^q$ *for all* q.

Proof. Let $\bar{q}$ be the supremum of q with $u\in L^q$, and suppose $\bar{q}<\infty$. For the inequality (4.4) we use the Hölder inequality to find

$$\{\int|\varphi u^{p/2}|^{2\gamma}\}^{1/\gamma}\leqq C_3\{\int|\nabla\varphi|^{2\gamma'}\}^{1/\gamma'}\{\int_{\operatorname{supp}|\nabla\varphi|} u^{p\gamma''}\}^{1/\gamma''}$$

where $1/\gamma'+1/\gamma''=1$ and $(n-2\gamma')\gamma''/\gamma'=\varepsilon/2$. Taking $q=p\gamma''$ and φ as in Lemma (5.8), we find

$$\{\int_{B(r')}|\nabla\varphi|^{2\gamma'}\}^{1/\gamma'}\{\int_{B(r')} u^q\}^{1/\gamma''}$$

goes to 0 as $r'\to 0$ by Lemma (5.8). This shows that $u\in L^{p\gamma}=L^{q\gamma/\gamma''}$. Since we can take q arbitrarily close to $\bar{q}$, this leads to a contradiction $\bar{q}\geqq\bar{q}\gamma/\gamma''$. □

By Corollary (4.10) $u=|R|^{1-\delta}$ and $f=|R|$ satisfy (5.7). Hence we can use Lemma (5.9) to find $u\in L^q$ for all q. Moreover the proof of Lemma (5.8) (cf. Lemma (4.2)) actually implies $Du\in L^2$. Now we can show that u satisfies $\Delta u\geqq -fu$ in the whole ball B, namely

$$\int_B Du\,D\eta\leqq\int_B fu\eta \quad \text{for all } \eta\in C_0^{0,1}(B). \tag{5.10}$$

In fact let η_k be a cut-off function such that $\eta_k\in C_0^1(B^*)$, $\eta_k\to 1$ almost everywhere, and

$$\int_B|D\eta_k|^2\to 0.$$

Then we find

$$\begin{aligned}\int_B \eta_k\,Du\,D\eta &= \int_B Du\,D(\eta\eta_k)-\int_B \eta Du\,D\eta_k\\ &\leqq\int_B fu\eta\,\eta_k+\sup_B|\eta|\int_B|Du|^2\int_B|D\eta_k|^2.\end{aligned}$$

Letting $k\to\infty$, we get (5.10). Then by using the Moser iteration technique on B, we find that u (hence $|R|$) is bounded on B (cf. Lemma (4.4)). We remark that although we do not know that B is a manifold or not, we can use the Moser iteration technique since it only requires the Sobolev inequality and the volume growth condition.

By the Hopf-Rinow theorem, all closed bounded subsets of B are compact and moreover for any pair $p, q \in B$, there exists a distance minimizing curve joining them which is geodesic in B^*. Since the Sobolev inequality (5.3) implies the isoperimetric inequality, we have the lower volume estimate:

$$\text{vol}(B(p;s)) \geqq C_3 s^n \tag{5.11}$$

for any metric ball $B(p;s)$ around $p \in B^*$ of radius s with $B(p;s) \subset B^*$. Hence by Fact (2.7) (which is valid for this case, although the point $0 \in B$ may be singular), we have the lower injectivity radius estimate:

$$i_p \geqq C_4 r(p) \quad \text{for } r(p) = d(o, p) \tag{5.12}$$

for $p \in B^*$.

Let us now consider a family of pointed metric spaces $B_\varepsilon = (B, \varepsilon^{-1} d, o)$ $(\varepsilon > 0)$. Then it follows from (5.4) and (5.11) that for any $\delta > 0$ and every $R > 0$, the maximum number of disjoint metric balls with radius δ in $B_\varepsilon(o; R)$ is bounded uniformly in ε (we denote by $B_\varepsilon(o; R)$ the metric ball with respect to the distance $\varepsilon^{-1} d$). Hence by Fact (2.2) (which is valid for our case (cf. [Gr, Proposition (5.2)])), we see that for any sequence $\{\varepsilon_i\}$ tending to zero as $i \to \infty$, there exists a subsequence $\{\varepsilon_j\}$ for which the metric spaces B_{ε_j} converge to a complete, locally compact pointed metric space of length $B_0 = (B_0, d^*, o^*)$ with respect to the pointed Hausdorff distance. Thus we have seen that B has a "tangent cone" B_0 at o. In the next stage, we shall show the uniqueness of B_0, and clarify its structure.

(5.13) **Lemma.** 1) *$B_\varepsilon = (B, \varepsilon^{-1} d, o)$ converges, as ε goes to zero, to B_0 with respect to the pointed Hausdorff distance.*

2) *B_0 is isometric to the flat cone $\mathscr{C}(S^{n-1}/\Gamma)$ over the quotient space of unit sphere S^{n-1} in $\mathbb{R}^n$ by some finite group $\Gamma \subset O(n)$.*

3) *Fix two positive numbers a, b with $a < 1 < b$. Then for small ε, there exists a diffeomorphism Π_ε from $B(\varepsilon b) \setminus B(\varepsilon a)$ into $\mathscr{C}(S^{n-1}/\Gamma)$ such that as ε goes to zero, $\Pi_\varepsilon(B(\varepsilon a) \setminus B(\varepsilon b))$ converges to $[a, b] \times S^{n-1}/\Gamma$ as sets, and further the rescaled metric $\varepsilon^{-2} \Pi_{\varepsilon *}$* g *converges to the flat metric* g_{std} *of $\mathscr{C}(S^{n-1}/\Gamma)$ in C^∞-topology.*

Proof. First we show that $\text{vol}(B(t) \setminus \{o\})/t^n$ tends to a positive constant as $t \to 0$. In fact, since the distance between two points p and q of B can be realized by a Lipschitz curve joining them which is geodesic in B^*, we are able to apply the Laplacian comparison theorem to the distance $r = d(o, *)$ (first applying it to the distance function to $\partial B(\varepsilon)$ and letting ε go to zero), and hence we have

$$\Delta r \leqq (n-1) \frac{a \cosh ar}{\sinh ar} \qquad \Big(a = \sqrt{\sup_{B*} |R|}\Big)$$

weakly in $W^{1,2}(B^*)$. We set here

$$F_\varepsilon(t) := \int_\varepsilon^t \int_\varepsilon^s (\sinh au)^{n-1} du / (\sinh as)^{n-1} ds \qquad (\varepsilon > 0).$$

Then $F_\varepsilon \circ r$ satisfies

$$\Delta F_\varepsilon \circ r \leqq 1$$

weakly in $W^{1,2}(B \setminus B(\varepsilon))$. Integrating the both sides of this inequality over $B(t) \setminus B(\varepsilon)$, we get

$$\frac{\int_\varepsilon^t (\sinh as)^{n-1} ds}{(\sinh at)^{n-1}} \mathscr{H}_{n-1}(\partial B(t)) \leqq \operatorname{vol}(B(t) \setminus B(\varepsilon))$$

for almost all ε, t $(0 < \varepsilon < t)$, where $\mathscr{H}_{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. Letting ε go to zero, we have

$$\frac{d}{dt} \log \operatorname{vol}(B(t) \setminus \{o\}) \leqq \frac{d}{dt} \log \int_0^t (\sinh as)^{n-1} ds$$

for almost all $t > 0$. Here we have used the coarea formula (see, e.g., [Sm, § 2])

$$\frac{d}{dt} \operatorname{vol}(B(t) \setminus \{o\}) = \mathscr{H}_{n-1}(\partial B(t))$$

for almost all t. This implies that $\operatorname{vol}(B(t) \setminus \{o\}) \Big/ \int_0^t (\sinh as)^{n-1} ds$ is monotone nonincreasing in $t > 0$ (cf. Fact (2.6)), and thus $\operatorname{vol}(B(t) \setminus \{o\})/t^n$ converges to a positive constant V_0 which is bounded by V in (5.4) as $t \to 0$. In particular, we have

$$\begin{aligned} (5.14) \qquad & \lim_{\varepsilon \to 0} \operatorname{vol}(B_\varepsilon(o; b) \setminus B_\varepsilon(o; a)) \qquad (0 < a < b) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \operatorname{vol}(B(\varepsilon b) \setminus B(\varepsilon a)) \\ &= (b^n - a^n) V_0. \end{aligned}$$

Let us fix two positive numbers a, b with $a < 1 < b$ and consider a sequence $\{\varepsilon_j\}$ tending to zero as $j \to \infty$ for which the sequence of pointed metric spaces $B_{\varepsilon_j} = (B, \varepsilon_j^{-1} d, o)$ converges to B_0 as $j \to \infty$. Let δ be a positive number such that $\delta < \min\{aC_2, a, b-a\}/10$. Since the maximum number of disjoint metric balls of radius δ in $B_{\varepsilon_j}(o; 2b) \setminus B_{\varepsilon_j}(o; a/2)$ is bounded uniformly in ε_j, and moreover the curvature of $B_{\varepsilon_j} \setminus \{o\}$ on $B_{\varepsilon_j}(o; 2b) \setminus B_{\varepsilon_j}(o; a/2)$ goes to zero as $j \to \infty$, we can apply a local version of Gromov convergence theorem (Fact (2.10)) to $B_{\varepsilon_j}(o; b) \setminus B_{\varepsilon_j}(o; a)$ to show that $B_0 \setminus \{o^*\}$ is a flat Riemannian manifold and there exists a diffeomorphism Π_j from a neighborhood of $B_{\varepsilon_j}(o; b) \setminus B_{\varepsilon_j}(o; a)$ into $B_0 \setminus \{o^*\}$ which has the following properties: as $j \to \infty$, the image $\Pi_j(B_{\varepsilon_j}(o; b) \setminus B_{\varepsilon_j}(o; a))$ converges to $B_0(o^*; b) \setminus B_0(o^*; a)$ as sets and the metric $\varepsilon_j^{-2} \Pi_{j*} g$ converges to the flat metric g_0 of $B_0 \setminus \{o^*\}$ in C^∞-topology (in a general situation we only have a $C^{1,\alpha}$-convergence for some $\alpha \in (0, 1)$, but in this case we have C^∞-convergence since the metric g is Einstein (cf. [N2])). We put $r_j = \varepsilon_j^{-1}(r \circ \Pi_{\varepsilon_j})$. Then it follows from the Laplacian comparison theorem that

$$\Delta_j r_j \leqq (n-1) \frac{a\varepsilon_j \cosh a\varepsilon_j r_j}{\sinh a\varepsilon_j r_j}$$

weakly in $W^{1,2}(\Pi_j(B_{\varepsilon_j}(o; b) \setminus B_{\varepsilon_j}(o; a)))$, where Δ_j denotes the Laplacian with respect to the metric $\varepsilon_j^{-2} \Pi_{j*} g$. Since r_j converges uniformly to the distance r_0 in B_0 to o^*, r_0 satisfies

$$\Delta_0 r_0 \leqq \frac{n-1}{r_0}$$

and hence

$$\Delta_0 r_0^2 \leqq 2n$$

weakly in $W^{1,2}(B_0(o; b) \setminus B_0(o; a))$, where Δ_0 is the Laplacian of g_0. Integrating the both sides over $B_0(o; t) \setminus B_0(o; s)$ $(a \leqq s < t \leqq b)$, we have

$$\begin{aligned} \text{(5.15)} \qquad 2n(t^n - s^n) V_0 &= 2n \operatorname{vol}(B_0(o; t) \setminus B_0(o; s)) \qquad \text{(by (5.14))} \\ &\geqq \int_{B_0(o;t) \setminus B_0(o;s)} \Delta r_0^2 \\ &= 2\{t\mathscr{H}_{n-1}(\partial B_0(o; t)) - s\mathscr{H}_{n-1}(\partial B_0(o; s))\} \end{aligned}$$

for almost all t, s as above. On the other hand we have

$$\frac{d}{dt} V_0 t^n = \mathscr{H}_{n-1}(\partial B_0(o; t))$$

for almost all t. Thus the equality holds in (5.15). This shows that $\Delta_0 r_0^2 = 2n$ weakly in $W^{1,2}(B_0(o; b) \setminus B_0(o; a))$, and hence in $W^{1,2}(B_0 \setminus \{o^*\})$, by letting a go to zero and b go to infinity. Now it turns out from the elliptic regularity theory that r_0 is smooth on $B_0 \setminus \{o^*\}$. Moreover applying the Hessian comparison theorem (cf. Fact (2.4)) to r_0, we get

$$\nabla^2 r_0 \leqq \frac{1}{r_0}(g_o - dr_0 \otimes dr_0)$$

and hence

$$\nabla^2 r_0 = \frac{1}{r_0}(g_o - dr_0 \otimes dr_0)$$

on $B_0 \setminus \{o^*\}$, since $\Delta r_0 = (n-1)/r_0$. Thus we have proved that B_0 is isometric to the flat cone $\mathscr{C}(S^{n-1}/\Gamma) = ([0, \infty) \times S^{n-1}/\Gamma,\ dt^2 + t^2 g_{std})$ for some finite group Γ of $O(n)$.

It remains to show the uniqueness of B_0. This can be done by the same argument as in [K2]. In fact, we note that r has no critical points in $B(\varepsilon_0) \setminus \{o\}$ for sufficiently small ε_0. More precisely, if we write $\nabla \cdot r(x)$ $(x \in B^*)$ for the set of unit vectors $v \in T_x B^*$ such that there exists a distance minimizing curve γ:

$[0, r(x)] \to B$ with $\gamma(0)=o$, $\gamma(r(x))=x$, and $\gamma'(r(x))=v$, then it follows from the fact shown just above that $\max\{\measuredangle(u, v)\colon u, v \in \nabla \cdot r(x)\}$ goes to zero as $x \in B^* \to o$. Therefore applying the same argument as in [K2] to the familiy of rescaled metric spheres $\{\varepsilon^{-1}\partial B(\varepsilon)\}_{\varepsilon>0}$, we see that $\varepsilon^{-1}\partial B(\varepsilon)$ converges, as $\varepsilon \to 0$, to a compact, connected, metric space of length (with respect to the Hausdorff distance), which is isometric to S^{n-1}/Γ as above. This completes the proof of Lemma (5.13). □

We are now in a position to complete the proof of Theorem (5.1). In what follows, taking the universal covering of $B(\varepsilon_0)\setminus\{o\}$, we assume $\Gamma=\{e\}$ and construct a coordinate system $\Psi\colon B(\varepsilon_1)\setminus\{o\} \to \mathbb{R}^n$ for some $\varepsilon_1>0$ as in § 2 Case 3.

For small $\varepsilon>0$, we set $S_\varepsilon := \Pi_\varepsilon^{-1}(\{1\} \times S^{n-1})$. Then we have by Lemma (5.13)

$$\varepsilon^{-1} d(S_\varepsilon, \partial B(\varepsilon)) \leqq a(\varepsilon),$$
$$\|\varepsilon A_\varepsilon + \mathrm{Id}\| \leqq a(\varepsilon)$$

where A_ε stands for the shape operator of S_ε, and $a(\varepsilon)$ goes to zero as $\varepsilon \to 0$. Then $\varepsilon^{-1}(S_\varepsilon, g|_{S_\varepsilon})$ converges to S^{n-1} in the Hausdorff distance.

In the proof of Theorem (1.1) we first constructed coordinates at infinity of order ε' and then improved the order. We can not construct coordinates of the precise order directly for the reason that the absolute value of coordinates $|z|$ may be different from the distance r. But in this case they coincide. So the proof is much simpler than one of Theorem (1.1).

For small $\varepsilon>0$, we consider a map

$$\Phi_\varepsilon\colon (x, t) \in S^{n-1} \times [\varepsilon, K] \mapsto \exp(t-\varepsilon)\, v_\varepsilon(x) \in B$$

where $v_\varepsilon(x)$ is the outer unit normal vector of S_ε at x. As in the proof of Theorem (1.1) we can show that Φ_ε is an into-diffeomorphism for some fixed K independent of ε. Taking a point $x \in S_\varepsilon$, we consider parallel orthonormal vector fields $E_1(t), \ldots, E_n(t)$ along $\sigma_\varepsilon(t)=\Phi_\varepsilon(x, t)$. Let $X_\varepsilon(t)$ be a Jacobi field along σ_ε with $X_\varepsilon(\varepsilon)=\varepsilon E_k(\varepsilon)$, $\frac{\nabla}{dt} X_\varepsilon^\perp(\varepsilon) = -\varepsilon A_\varepsilon(E_k^\perp(\varepsilon))$, and $\frac{\nabla}{dt}(X_\varepsilon - X_\varepsilon^\perp)(\varepsilon) = E_\varepsilon(\varepsilon) - E_\varepsilon^\perp(\varepsilon)$ for some $k=1, \ldots, n$. Then $X_\varepsilon^i(t)=(X_\varepsilon(t), E_i(t))$ satisfies

$$X_\varepsilon^i(t) = X_\varepsilon^i(\varepsilon) + (X_\varepsilon^i)'(\varepsilon)(t-\varepsilon) + \int_\varepsilon^t \int_\varepsilon^s \sum X_\varepsilon^j(u)\, K_j^i(u)\, du\, ds$$

where $K_j^i(u) = (R(E_j(u), \sigma_\varepsilon'(u))\, \sigma_\varepsilon'(u), E_i(u))$. Using (5.13) we find

$$|t^{-1} X_\varepsilon^i(t) - \delta_{ik}| = O(\varepsilon t^{-1} + t^2).$$

As in the proof of Theorem (1.1) we construct coordinates $\Psi_\varepsilon\colon S^{n-1} \times [\varepsilon, K] = \{x \in \mathbb{R}^n\colon \varepsilon \leqq |x| \leqq K\} \to B$ such that

$$\Psi_\varepsilon^* g_{ij}(x) - \delta_{ij} = O(\varepsilon|x|^{-1} + |x|^2), \qquad \partial_k \Psi_\varepsilon^* g_{ij}(x) = O(\varepsilon|x|^{-2} + |x|^1).$$

Letting $\varepsilon \to 0$, we get coordinates $\Psi: \{|x| \leqq K\} \to B$ such that

$$\Psi^* g_{ij}(x) - \delta_{ij} = O(|x|^2), \qquad \partial_k \Psi^* g_{ij}(x) = O(|x|^1).$$

This means $\Psi^* g$ is of class $C^{1,1}$. We take a harmonic coordinate Φ for $\Psi^* g$ ([DK, Lemma (1, 2)]). Outside the origin o, Φ is smooth and $u = \Phi^* g_{ij}$ satisfies

$$\Delta u = -2ku + Q_{ij}(g, \partial g) \tag{5.16}$$

where Q_{ij} is quadratic in ∂g and k is the Einstein constant ($\mathrm{Ric} = kg$). Since u is $C^{1,1}$, it is easy to see that u satisfies (5.16) on the whole B in the weak sense. Then the elliptic regularity theory implies $\Phi^* g$ is smooth.

We remark that in Theorem (5.5) the connectedness of $\partial B(x_a; \delta)$ for small $\delta > 0$ is shown as follows: Suppose it is disconnected. Then for sufficiently large j, $\partial B(x_{a,j}; \delta)$ is also disconnected. We take a harmonic function u_j on $B(x_{a,j}; \delta)$ with boundary values 0 on a certain component of $\partial B(x_{a,j}; \delta)$, and 1 on the other components. $u_j \circ F_j$ converges to a bounded harmonic function u_∞ on each component of $B(x_a; \delta) \backslash \{x_a\}$. Since we already know that each component of $B(x_a; \delta) \backslash \{x_a\}$ is a Riemannian orbifold, the usual removable singularities theorem for harmonic functions implies that u_∞ can be extended across x_a on each component. But it has constant boundary values on each component, so it must be 0 on a certain component, 1 on the other components. But by the apriori estimates [CY], we have $|du_j| \leqq C$ on $B(x_{a,j}; \delta)$ for some constant C. This is a contradiction.

§ 6. Manifolds of nonnegative Ricci curvature with maximal volume growth

As we mentioned in Remark (1.8) 2), we can obtain a gap theorem for a class of manifolds as in Theorem (1.5). In this section, we shall consider a complete, noncompact Riemannian manifold (M, g) with nonnegative Ricci curvature and prove a result of this type for such M under certain additional conditions (cf. Corollary (6.7) 2)).

Let (M, g) be as above. Fix a point o of M as a reference point, and consider a family of pointed Riemannian manifolds $M_t = (M, t^{-2} g, o)$ $(t > 0)$. Then by Fact (2.2), for any divergent sequence $\{t_i\}$, there exists a subsequence $\{t_j\}$ of $\{t_i\}$ for which M_{t_j} converges, as $j \to \infty$, to a complete, locally compact pointed metric space of length $M_\infty^* = (M_\infty^*, d^*, o^*)$ with respect to the pointed Hausdorff distance. It would be of interest to study various problems on the structure of M_∞^*, the relations between the geometry of M and that of M_∞^*, etc. For example, in the case that the sectional curvature of M is nonnegative, M_∞^* is independent of the choice of reference points or divergent sequences $\{t_i\}$ and moreover it is isometric to the cone over a compact metric space of length (cf. [K2]). In the case of M as in Theorem (1.5), we have seen that M_∞^* is unique and isometric to the flat cone over a space form of constant curvature 1, from which a gap theorem as in Remark (1.8) 2) follows.

In what follows, we shall make some observations on open manifolds of nonnegative Ricci curvature under certain additional conditions. We begin with the following

(6.1) **Proposition.** *Suppose a complete Riemannian manifold* (M, g) *of dimension* n *satisfies*

$$\mathrm{Ric} \geqq 0, \tag{6.2}$$

$$\mathrm{vol}\, B(o;t) \geqq V t^n \quad \text{for some } V>0, o \in M, \tag{6.3}$$

$$|R| \leqq K r^{-2} \quad \text{if } r \text{ is sufficiently large.} \tag{6.4}$$

Then for any divergent sequence $\{t_i\}$, *there exists a subsequence of* $\{t_i\}$, *denoted again by* $\{t_i\}$, *such that* $\{(M, t_i^{-2} g, o)\}$ *converges to a pointed metric space of length* (M_∞^*, d^*, o) *in the pointed Hausdorff distance, where* $M_\infty^* \backslash \{o^*\}$ *is a smooth manifold with a* $C^{1,\alpha}$*-Riemannian metric* g_∞^*, *and* $r^* = d^*(o^*, *)$ *satisfies*

$$\Delta_\infty^* r^* = (n-1) r^{*-1}.$$

Moreover if

$$|R|\, r^2 \to 0 \quad \text{as } r \to \infty, \tag{6.4$'$}$$

or

$$\int_M |R|^{n/2} dV < \infty, \tag{6.4$''$}$$

M_∞^* *is the flat cone over* S^{n-1}/Γ *for some finite subgroup* $\Gamma \subset O(n)$.

In [BK] the first named author and R. Kobayashi constructed examples of manifolds which satisfy conditions (6.2)$\sim$(6.4), but not (6.4)$''$. They have constructed Ricci-flat Kähler metrics on $X \backslash D$, where X is a Fano manifold (i.e. $c_1(X) > 0$), and D is a smooth reduced divisor on X with $c_1(X) = \alpha[D]$ for some $\alpha > 1$, and D admits a Kähler-Einstein metric of positive scalar curvature. For example, let $X = \mathbb{P}^n$, $D = F_d$ (Fermat hypersurface of degree $d = n$ or $n-1$). The existence of a Kähler-Einstein metric on D was recently proved by Siu [Su] and Tian [Ti]. In this case the tangent cone M_∞^* is isometric to the warped product $[0, \infty) \times_{t^2} S_\infty$, where S_∞ is a circle bundle over D. On the other hand, when $X = \mathbb{P}^n$, $D = \mathbb{P}^{n-1}$, we obtain the standard metric on $\mathbb{C}^n = X \backslash D$.

We do not know that the conditions (1.2), (1.3) in Theorem (1.1) can be weakened to the condition which implies that a tangent cone at infinity is isomorphic to the flat cone $\mathscr{C}(S^{n-1}/\Gamma)$ for some $\Gamma \subset O(n)$ (for example (6.4)$'$ or (6.4)$''$). We can not estimate the speed of the convergence as in the proof of Theorem (1.1).

Proof. Since the Sobolev constant of M is positive (cf. [Cr]), and the injectivity radius of M at $x \in M$ is greater than $C_1\, d(o, x)$ (cf. Fact (2.7)), we can apply the theory of $C^{1,\alpha}$-convergence for Riemannian manifolds (cf. Fact (2.10)) as in the proof of Theorem (5.1) to prove the first assertion of Proposition (6.1). As for the case that (6.4)$'$ or (6.4)$''$ is satisfied, the limit space (M_∞^*, g_∞^*) is flat, so that by the same reason as in the proof of Theorem (5.1), M_∞^* turns out to be isometric to the flat cone $\mathscr{C}(S^{n-1}/\Gamma)$ for some finite subgroup $\Gamma \subset O(n)$. This completes the proof of Proposition (6.1). □

As a consequence from Proposition (6.1), we have the following

(6.7) **Corollary.** *Let M be a complete, non-compact Riemannian manifolds of dimension $n \geqq 3$ satisfying conditions* (6.2)∼(6.4) *Then*

1) *There exists a smooth function $\hat{r}$ on M such that $\hat{r}/r$ ($r = d(o, *)$) converges to* 1, *and $|\nabla \hat{r}|$ tends to* 1, *as $r \to \infty$. In particular, M is diffeomorphic to the sublevel set $\{x \in M : \hat{r}(x) < R\}$ of $\hat{r}$ for large R.*

2) *Moreover suppose M satisfies* (6.4)′ *or* (6.4)″. *Then M is isometric to Euclidean space $\mathbb{R}^n$ if n is odd or if n is even and M is simply connected at infinity.*

Proof. We define $\hat{r}$ by using the Riemannian convolution smoothing of r

$$\hat{r}(x) := \frac{1}{\varepsilon^n} \int_{v \in T_x M} \rho\left(\frac{|v|}{\varepsilon}\right) r(\exp_x(v))$$

where the integration is with respect to the measure induced on the tangent space $T_x M$ at x by the Riemannian metric of M, ε is a sufficiently small positive constant, and $\rho: \mathbb{R} \to \mathbb{R}$ is a nonnegative smooth function that has support contained in $[-1, 1]$, is constant in a neighborhood of 0, and has $\int \rho = 1$. Then since r has no critical points outside a compact set, $\hat{r}$ enjoys the property stated in the theorem (cf. [K2]). The second assertion is an immediate consequence of the Bishop's comparison theorem. This completes the proof of Corollary (6.7). □

(6.8) *Remarks.* 1) It is not clear that in Proposition (6.1), r_∞^* satisfies $\nabla_\infty^2 r^* = r^{*-1}(g_\infty^* - dr^* \otimes dr^*)$ (namely, M_∞^* is the warped product $[0, \infty) \times_{t^2} S_\infty$, where $S_\infty = \{r^* = 1\}$. However this is true if, in addition to (6.2)∼(6.4), we assume that

$$|\nabla \operatorname{Ric}| \leqq C r^{-3}$$

for a constant C. In fact in this case, the metric g_∞^* on $M_\infty^* \setminus \{o^*\}$ is of class $C^{2,\alpha}$ and the sectional curvature of g_∞^* vanishes for any plane tangent to the gradient $\nabla_\infty r^*$ (cf. Examples given after the statement of Proposition (6.1)).

2) It is an open question whether the first assertion of Corollary (6.7) is true or not, without condition (6.4).

3) It is interesting to investigate Kähler manifolds satisfying conditions (6.2)∼(6.4). We refer to [Mo] for the related topics.

Appendix

We prove Lemma (4.9). The basic idea is the trick of Schoen, Simon, and Yau [SSY].

(4.9) **Lemma.** *Suppose S, T are tensors having the same symmetry as the curvature tensor R, and the covariant derivative ∇R of the curvature tensor of the Einstein metric g respectively. Then there exists $\delta=\delta(n)$ such that*

$$(1+\delta)\,|(S,T)|^2 \leqq |S|^2\,|T|^2,$$

where (S,T) is a 1-form defined by $(S,T)(X)=(S,T(X))$ for a tangent vector X. Moreover if g is Kähler, we can take $\delta=4/(n+2)$. If $n=4$, and g is self-dual or anti self-dual, we can take $\delta=2/3$.

We use the index notation for tensors. The summation convention will always hold.

First suppose the metric g is Kähler. Let n be the complex dimension of the Kähler manifold M. We calculate the maximum of the function $|(S,T)|^2$ under the constraint $|S|=|T|=1$ by Lagrange's method of indeterminate coefficients. If the maximum value is achieved at S, T, then there exist real numbers a, b such that

$$\text{(A.1)} \qquad \begin{aligned} &((S,T),(\bar{S},T))=a(\bar{S},S),\\ &((S,T),(S,\bar{T})=b(\bar{T},T) \qquad \text{for any } \bar{S},\bar{T}. \end{aligned}$$

Hence we have $a=b=|(S,T)|^2$. We define a 1-form λ by $\lambda=a^{-1}(S,T)$. From (A.1) we have

$$\text{(A.2)} \qquad (\bar{S},(\lambda,T))=(\bar{S},S),\ (\bar{T},\lambda\otimes S)=(\bar{T},T) \qquad \text{for any } \bar{S},\bar{T}.$$

We denote by p_1, p_2 the projections onto the space of the tensors having the same symmetry as S and T respectively. (A.2) means

$$S=p_1((\lambda,T)), \qquad T=p_2(\lambda\otimes S).$$

Thus we have

$$\begin{aligned} T=T_{i\bar{j}k\bar{l},p}&=p_2(S_{i\bar{j}k\bar{l}}\lambda_p)\\ &=\frac{1}{3}\Big\{(S_{i\bar{j}k\bar{l}}\lambda_p+S_{p\bar{j}k\bar{l}}\lambda_i+S_{i\bar{j}p\bar{l}}\lambda_k)-\frac{1}{n+3}(S_{i\bar{j}p\bar{q}}\lambda_q\delta_{k\bar{l}}+S_{p\bar{q}k\bar{l}}\lambda_q\delta_{i\bar{j}}\\ &\qquad +S_{i\bar{l}p\bar{q}}\lambda_q\delta_{k\bar{j}}+S_{k\bar{j}p\bar{q}}\lambda_q\delta_{i\bar{l}}+S_{i\bar{j}k\bar{q}}\lambda_q\delta_{p\bar{l}}+S_{i\bar{q}k\bar{l}}\lambda_q\delta_{p\bar{j}})\Big\}. \end{aligned}$$

We may assume that $\lambda_i=\lambda\delta_{i1}$ for some positive λ. Hence we have

$$\begin{aligned} S=S_{i\bar{j}k\bar{l}}&=p_1(\lambda_p T_{i\bar{j}k\bar{l},p})\\ &=\tfrac{1}{4}(\lambda_p T_{i\bar{j}k\bar{l},p}-\lambda_p T_{\bar{j}ik\bar{l},p}-\lambda_p T_{i\bar{j}\bar{l}k,p}+\lambda_p T_{\bar{j}i\bar{l},p})\\ &=\frac{\lambda^2}{3}\{S_{i\bar{j}k\bar{l}}+\tfrac{1}{2}(\delta_{i\bar{1}}S_{1\bar{j}k\bar{l}}+\delta_{1\bar{j}}S_{i\bar{1}k\bar{l}}+\delta_{k\bar{1}}S_{i\bar{j}1\bar{l}}+\delta_{1\bar{l}}S_{i\bar{j}k\bar{1}})\}\\ &\qquad -\frac{\lambda^2}{3(n+3)}\{S_{i\bar{j}1\bar{1}}\delta_{k\bar{l}}+S_{1\bar{1}k\bar{l}}\delta_{i\bar{j}}+S_{k\bar{j}1\bar{1}}\delta_{i\bar{l}}+S_{i\bar{l}1\bar{1}}\delta_{k\bar{j}}\\ &\qquad +\tfrac{1}{2}(S_{i\bar{j}k\bar{1}}\delta_{1\bar{l}}+S_{i\bar{j}1\bar{l}}\delta_{k\bar{1}}+S_{i\bar{1}k\bar{l}}\delta_{1\bar{j}}+S_{1\bar{j}k\bar{l}}\delta_{i\bar{1}})\}. \end{aligned}$$

We show that the minimum of λ for which the above equation has a non-zero solution is equal to $(n+3)/(n+1)$.

If $S_{i\bar{j}k\bar{l}}=0$ in case that at least one of $i, \bar{j}, k, \bar{l}$ is equal to 1, we have $\lambda^2=3>(n+3)/(n+1)$. Hence we may assume $S_{i\bar{j}\bar{k}1}\neq 0$ for some $i, \bar{j}, k$. We have

$$\begin{aligned} S_{i\bar{j}k\bar{1}}=\lambda^2\Big\{&\left(\frac{1}{2}-\frac{1}{6(n+3)}\right)S_{i\bar{j}k\bar{1}}+\left(\frac{1}{6}-\frac{1}{2(n+3)}\right)\delta_{i\bar{1}}S_{1\bar{j}k\bar{1}}\\ &+\left(\frac{1}{6}-\frac{1}{2(n+3)}\right)\delta_{k\bar{1}}S_{i\bar{j}1\bar{1}}+\left(\frac{1}{6}-\frac{1}{6(n+3)}\right)\delta_{1\bar{j}}S_{\bar{i}1k\bar{1}}\\ &-\frac{1}{3(n+3)}\delta_{i\bar{j}}S_{1\bar{1}k\bar{l}}-\frac{1}{3(n+3)}\delta_{k\bar{j}}S_{i\bar{1}1\bar{1}}\Big\}. \end{aligned}$$

The minimum of λ for which the above equation has a non-trivial solution is $(n+3)/(n+1)$.

Next suppose $n=4$, and the metric g is self-dual. We may suppose 1-form $(S, T)=S_{ijkl}\,T_{ijkl,p}$ is non-zero only if $p=1$. From the Schwarz inequality we have

$$|(S, T)|^2\leqq|S|^2\,|T_{****,1}|^2 \tag{A.3}$$

where $T_{****,1}$ means $T(\partial/\partial x_1)$. Changing coordinates x_2, x_3, x_4 we may assume $T_{1i1j,1}=0$ if $i\neq j$. From the self-duality equation we have

$$|T_{****,p}|^2=4\sum_{i,j\geqq 2}T^2_{1i1j,p}.$$

We assume $i, j, k\geqq 2$ have the orientation in this order. From the self-duality equation we have

$$T_{1i1i,1}=T_{1ijk,1}=-T_{1ik1,j}-T_{1i1j,k}=T_{1i1k,j}-T_{1i1j,k}.$$

We denote this by a_j-a_k. Then we have

$$\begin{aligned}\sum_{i,j\geqq 2}T^2_{1i1j,1}&=(a_2-a_3)^2+(a_3-a_4)^2+(a_4-a_2)^2\\ &\leqq 3(a_2^2+a_3^2+a_4^2)\leqq\tfrac{3}{8}(|T_{****,2}|^2+|T_{****,3}|^2+|T_{****,4}|^2).\end{aligned}$$

Hence we get

$$|T_{****,1}|^2\leqq\tfrac{3}{5}|T|^2.$$

Substituting this into (A.3) we have verified the assertion.

We do not give the proof for general cases since we do not calculate the precise value. But it is easy to see that there exists a positive constant δ such that $(1+\delta)\,|(S,T)|^2 \leqq |S|^2\,|T|^2$.

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Lower bounds on Ricci curvature and the almost rigidity of warped products

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0. Introduction

The basic rigidity theorems for manifolds of nonnegative or positive Ricci curvature are the "volume cone implies metric cone" theorem, the maximal diameter theorem, [Cg], and the splitting theorem, [CG]. Each asserts that if a certain geometric quantity (volume or diameter) is as large as possible relative to the pertinent lower bound on Ricci curvature, then the metric on the manifold in question is a warped product metric of a particular type.

In this paper we provide quantitative generalizations of the above mentioned results. Among the applications are the splitting theorem for Gromov-Hausdorff limit spaces X, where $M_i^n \to X, \mathrm{Ric}_{M_i^n} \geq -\varepsilon_i$, and $\varepsilon_i \to 0$, as well as Gromov's conjecture that manifolds of almost nonnegative Ricci curvature have almost nilpotent fundamental groups; see [FY]. Other applications include the assertion that for complete manifolds, M^n, with $\mathrm{Ric}_{M^n} \geq 0$ and Euclidean volume growth, all tangent cones at infinity are metric cones; compare [BKN], [CT], [P1].

Via rescaling arguments, there are also strong consequences for the local structure of manifolds whose Ricci curvature satisfies a fixed lower bound and for their Gromov-Hausdorff limits. Some of these are announced in [CCo1]; for a more detailed discussion see [CCo2], [CCo3], [CCo4].

Our work further develops and significantly extends techniques which were introduced in [Co1], [Co2] and significantly extended in [Co3], in order to prove certain "stability" conjectures of Anderson-Cheeger, Gromov and Perelman. The results of [Co1]–[Co3] were announced in [Co4]. We briefly review some of those results.

Let d_{GH} denote the Gromov-Hausdorff distance between metric spaces; see [GLP]. Let S_1^n denote the unit sphere and recall that S_1^n is the unique complete

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riemannian manifold satisfying $\mathrm{Ric}_{M^n} \geq (n-1)$, $\mathrm{Vol}(M^n) = \mathrm{Vol}(S_1^n)$; compare [Cg].

As conjectured by Anderson-Cheeger and Perelman, it is shown in [Co1] that for all $\varepsilon > 0$, there exists $\delta(n, \varepsilon) > 0$, such that $\mathrm{Ric}_{M^n} \geq (n-1)$ and $\mathrm{Vol}(M^n) \geq (1 - \delta(n, \varepsilon)) \mathrm{Vol}(S_1^n)$ implies $d_{GH}(M^n, S_1^n) < \varepsilon$.

Conversely, in [Co2] it is shown that $\mathrm{Ric}_{M^n} \geq (n-1)$ and $d_{GH}(M^n, S_1^n) < \delta(n, \varepsilon)$ implies $\mathrm{Vol}(M^n) \geq (1-\varepsilon) \mathrm{Vol}(S_1^n)$. By [P2], it follows that for $\varepsilon < \varepsilon(n)$, M^n is homeomorphic to a sphere in each of these cases; in fact by [CCo2], it is diffeomorphic to a sphere.

The result of [Co2] is actually a special case of a much more general conjecture of Anderson-Cheeger, proved in [Co3]. They conjectured that for M^n smooth, the assumptions $\mathrm{Ric}_{M_i^n} \geq \Lambda > -\infty$ and $d_{GH}(M_i^n, M^n) \to 0$, imply $\mathrm{Vol}(M_i^n) \to \mathrm{Vol}(M^n)$. They also noted that this would imply that M_i^n is homeomorphic to M^n for i sufficiently large, given the result which was proved later in [P2]; see Appendix 1 of [CCo2] for the implication, "diffeomorphic".

Observe that in the theorems discussed so far, the model space is *unique*, or *fixed in advance*, and the manifold in question is shown to have almost the same quantitative structure as the given model. Hence, in this context, we use the term "stability"; compare [G3].

In the present paper by contrast, in each of our theorems, a certain *class* of warped product spaces, $(a, b) \times_f N^{n-1}$, for some fixed f, serves as the collection of *smooth* rigid models. Here the cross-section, N^{n-1}, is subjected only to a fixed lower Ricci curvature bound. Thus, in each case, the totality of model spaces is infinite.

As is suggested by the rigidity theorems for spaces of nonnegative Ricci curvature which were mentioned at the beginning of this section, the spaces $(a, b) \times_f N^{n-1}$ admit (among others) the following two characterizations:

- When suitably normalized, the volume is maximal relative to the behavior of the Ricci curvature.

- If $f(a) = f(b) = 0$, the diameter is maximal relative to the behavior of the Ricci curvature.

Here, we show more generally that if for a manifold, M^n, the volume or diameter is *almost* maximal, then M^n, is *close* in the Gromov-Hausdorff sense, to a space, $(a, b) \times_f X$, with the same warping function as would obtain in the corresponding rigid case. As a consequence, it follows that the rigidity theorems themselves extend to Gromov-Hausdorff limits of spaces satisfying the appropriate lower Ricci curvature bounds.

Clearly, the cross-section, X, depends on the particular manifold, M^n. It is partly for this reason that we employ the term "almost rigidity" rather than "stability"; compare [G1].

There are several additional features of our situation which have fairly close counterparts to the one considered in [G1]. First of all, the topology of M^n need *not* be that of a product, no matter how slightly the geometric and curvature conditions are relaxed; see e.g. [A1], [P2]. Additionally, even though M^n is assumed smooth, the cross-sections, X, which arise *naturally*, include metric spaces which are not homotopy equivalent to manifolds, and hence, which satisfy the relevant lower bound on Ricci curvature only in a generalized sense. Finally, since we do not assume a definite lower bound on the volume of M^n, the Hausdorff dimension of such a cross-section can be strictly less than $n-1$.

Note that although the cross-section, X, is not uniquely determined, the issue of the existence of cross-sections with preferred properties is nonetheless a significant one. Although in proving our theorems we do construct explicit cross-sections, their properties are not studied in detail in the present paper; compare however Section 7 and see [CCo2], [CCoT].

The remainder of the paper is divided into three parts and eight sections as follows.

I. Integral estimates on Hessians imply almost rigidity

1. Warped products and Hessians
2. Integral estimates; Hessians, distances, angles
3. Gromov-Hausdorff approximations

II. Almost maximality implies integral estimates on Hessians

4. Volume
5. Finite diameter
6. Infinite diameter; the splitting theorem

III. Applications

7. The structure at infinity of manifolds with $\mathrm{Ric}_{M^n} \geq 0$
8. Almost nonnegative Ricci curvature and the fundamental group

Our results were announced in [CCo1], where additional applications to the structure of Gromov-Hausdorff limits of spaces with Ricci curvature satisfying a fixed lower bound were described as well. These and others will be treated in [CCo2], [CCo3], [CCo4] and elsewhere.

We are indebted to Mike Anderson, Grisha Perelman and Gang Tian for helpful conversations.

I. Integral estimates on Hessians imply almost rigidity

1. Warped products and Hessians

The smooth warped product spaces that we consider are riemannian manifolds, $(a,b)\times_f N^{n-1}$, where $(a,b)\times N^{n-1}$ is the underlying smooth manifold and the riemannian metric, $\underline{g}$, is given by

$$\underline{g} = dr^2 + f^2(r)\tilde{g}, \tag{1.1}$$

with $\tilde{g}$ the riemannian metric on N^{n-1}. Note that $(a,b)\times_f N^{n-1}$ need not be geodesically convex. Sometimes, when we need to refer to some arbitrary model space with warping function, f, we will choose $(a,b)\times_f \mathbb{R}$, since $\mathbb{R}$ is complete and $\overline{x_1,x_2}$ takes arbitrary nonnegative values. The following is well known (and easily seen).

Let $c:[0,l]\to N^{n-1}$ have length $L[c]=l$. If $|c'|\equiv 1$ and $k:[0,l]\to(a,b)$, then $(k,c)\subset(a,b)\times_f N^{n-1}$ has length, $L[(k,c)] = \int_0^l (1+f^2(k(t))(k'(t))^2)^{\frac{1}{2}}\,dt$. As a consequence, the distance, $\overline{(r_1,x_1),(r_2,x_2)}$, between $(r_1,x_1),(r_2,x_2)\in (a,b)\times_f N^{n-1}$ is given by function $\rho_f(r_1,r_2,\overline{x_1,x_2})$, where $\overline{x_1,x_2}$ denotes distance in N^{n-1}. Moreover, it is clear that $L[(k,c)]\geq \rho_f(r_1,r_2,l)$.

For X an arbitrary metric space, we define the metric space, $(a,b)\times_f X$, to be the space, $(a,b)\times X$, with metric,

$$\overline{(r_1,x_1),(r_2,x_2)} = \rho_f(r_1,r_2,\overline{x_1,x_2}). \tag{1.2}$$

This degree of generality is required for the statement of our main results.

For any fixed x, the radial curves, $t\to(t,x)$, are geodesics. Moreover, if $X=N^{n-1}$ is a riemannian manifold, the second fundamental form of the level surface, $r^{-1}(a)$, of the function, r, is given by

$$II_{r^{-1}(a)} = -\frac{f'(a)}{f(a)}\tilde{g}\otimes\frac{\partial}{\partial r}. \tag{1.3}$$

With the aid of (1.3), it is easy to check that the function,

$$\mathcal{F}(r) = -\int_r^b f(u)\,du, \tag{1.4}$$

satisfies

$$\mathrm{Hess}_{\mathcal{F}} = \mathcal{F}''(r)\underline{g} = f'(r)\underline{g}. \tag{1.5}$$

Conversely, the spaces $(a,b)\times_f N^{n-1}$ are essentially characterized by the existence of a function, h, whose Hessian is some function, k, times the metric tensor.

To see this, let M^n be a riemannian manifold such that for functions, h,k,

$$\mathrm{Hess}_h = kg. \tag{1.6}$$

Let ∇h denote the gradient of h. Then by (1.6),

$$d(|\nabla h|^2) = 2\,\mathrm{Hess}_h(\ ,\nabla h) = 2k\,dh. \tag{1.7}$$

Thus,

$$d(|\nabla h|^2) \wedge dh = 0, \tag{1.8}$$

which implies that for $|dh| \neq 0$,

$$d\left(\frac{dh}{|dh|}\right) = 0. \tag{1.9}$$

Therefore, we can let r satisfy

$$dr = \frac{dh}{|dh|}, \tag{1.10}$$

and regard h as a function of r. Putting $\nabla r = \frac{\partial}{\partial r}$, we get (with slight abuse of notation)

$$\nabla h = h'(r)\frac{\partial}{\partial r}. \tag{1.11}$$

It follows easily from (1.10) that

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0. \tag{1.12}$$

By (1.6), we have

$$\begin{aligned} k &= \mathrm{Hess}_h\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \\ &= \frac{\partial^2 h}{\partial r^2} - \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}(h) = h''(r). \end{aligned} \tag{1.13}$$

Let L denote the Lie derivative. Then

$$L_{\nabla h} g = 2\ \mathrm{Hess}_h\,. \tag{1.14}$$

Also, if

$$\left[\frac{\partial}{\partial r}, U\right] = \left[\frac{\partial}{\partial r}, V\right] = 0, \tag{1.15}$$

then

$$\begin{aligned} [\nabla h, U] &= -U(h')\frac{\partial}{\partial r}, \\ [\nabla h, V] &= -V(h')\frac{\partial}{\partial r}. \end{aligned} \tag{1.16}$$

Hence,

$$h'\frac{\partial}{\partial r}(g(U,V)) = \nabla h(g(U,V)) \tag{1.17}$$
$$[4pt] = (L_{\nabla h}g)(U,V) + g([\nabla h, U], V) + g(U,[\nabla h, V])$$

and with (1.13), (1.14), (1.16), we get,

$$\frac{\partial}{\partial r}(g(U,V)) = \frac{2h''}{h'}\left\{g(U,V) - g\left(\frac{\partial}{\partial r}, U\right) g\left(\frac{\partial}{\partial r}, V\right)\right\}. \tag{1.18}$$

Let $a < c < d < b$ and let $r^{-1}([c,d])$ be complete in the induced metric. By integrating (1.18), we find that the restriction of g to $r^{-1}((c,d))$ is a warped product metric, given in product coordinates associated to the function r, by

$$g = dr^2 + (h'(r))^2\tilde{g}. \tag{1.19}$$

Here, $h', k, \tilde{g}$ are determined up to multiplicative constants. Once these have been fixed, h is determined up to an additive constant.

Relation (1.6) implies that (1.14) is equivalent to the assertion that the vector field, ∇h, is *conformal.* The flow generated by this field is $(x, r) \to (x, \Phi_t(r))$, where

$$t = \int_r^{\Phi_t(r)} \frac{du}{f(u)}. \tag{1.20}$$

The following three examples are the most important ones.

Example 1.21. (Metric cones) Here, $(a,b) = (0,\infty)$, $f = r$, $h = \frac{1}{2}r^2$, $\Phi_t(r) = e^t r$ and in particular, $\mathbb{R}^n \setminus 0 = (0,\infty) \times_r S_1^{n-1}$.

Example 1.22. (Metric suspensions) Here, $(a,b) = (0,\pi)$, $f = \sin r$, $h = (2\sin\frac{1}{2}r)^2$, $\Phi_t(r) = 2\tan^{-1}(e^t \tan\frac{1}{2}r)$ and in particular, S_1^n is the completion of $(0,\pi) \times_{sin\, r} S_1^{n-1}$.

Example 1.23. (Products) Here, $(a,b) = (-\infty,\infty), f = 1, h = r, \Phi_t(r) = r + t$ and in particular, $\mathbb{R}^n = (-\infty,\infty) \times_1 \mathbb{R}^{n-1}$.

The preceding discussion suggests that if (1.6) holds only approximately, then perhaps in the presence of suitable additional assumptions, (1.19) should continue to hold in some weakened sense. This is verified in Sections 2 and 3, where "approximately" is taken to mean "in the L_1- sense" and "weakened" is taken to mean "in the Gromov-Hausdorff sense".

In order to give an indication of our approach and to record some required preliminaries, we now show how, from a standpoint somewhat different from that which was just explained, the function, $\mathcal{F}(r)$, satisfying (1.5), determines the metric.

Let $\mathcal{F}$ be defined as in (1.4). Since $\mathcal{F}' = f > 0$, we can define a function, H, by

$$H = \mathcal{F}'' \circ \mathcal{F}^{-1}. \tag{1.24}$$

Let $\underline{\gamma}(s)$ be a geodesic of length ℓ in $(a, b) \times_f N^{n-1}$, with $|\underline{\gamma}'(s)| \equiv 1$. Set

$$\underline{r}(s) = r(\underline{\gamma}(s)), \tag{1.25}$$

$$\underline{\theta}(s) = \angle(\underline{\gamma}'(s), \frac{\partial}{\partial r}), \qquad 0 \leq \underline{\theta} \leq \pi, \tag{1.26}$$

where $\angle$ denotes "angle".

PROPOSITION 1.27. *If*

$$\ell < \pi/\sqrt{K}, \tag{1.28}$$

where

$$K = \max_r \left(0, -\frac{f''(r)}{f(r)}\right), \tag{1.29}$$

and

$$a + \ell < r_0 < b - \ell, \tag{1.30}$$

$$r_0 - \ell \leq r_\ell \leq r_0 + \ell, \tag{1.31}$$

then the differential equation,

$$\underline{\mathcal{U}}''(s) = H(\underline{\mathcal{U}}(s)), \tag{1.32}$$

has a unique solution subject to the conditions

$$\underline{\mathcal{U}}(0) = \mathcal{F}(r_0), \tag{1.33}$$

$$\underline{\mathcal{U}}(\ell) = \mathcal{F}(r_\ell), \tag{1.34}$$

$$|\underline{\mathcal{U}}'(0)| \leq f(r_0). \tag{1.35}$$

For this solution, on the interval $[0,\ell]$,

$$\underline{r}(s) = \mathcal{F}^{-1}(\underline{\mathcal{U}}(s)), \tag{1.36}$$

$$\cos \underline{\theta}(s) = \frac{1}{f(r(s))}\underline{\mathcal{U}}'(s). \tag{1.37}$$

Proof. We can rewrite (1.5) as

$$\mathrm{Hess}_{\mathcal{F}} = H(\mathcal{F})\underline{g}, \tag{1.38}$$

from which it follows immediately that if we put

$$\underline{\mathcal{U}}(s) = \mathcal{F}(\underline{\gamma}(s)), \tag{1.39}$$

then $\underline{\mathcal{U}}(s)$ satisfies (1.32)–(1.34). Also,

$$\begin{aligned} \underline{\mathcal{U}}'(s) &= \langle \nabla \mathcal{F}, \gamma'(s)\rangle \\ &= f(r(s)) \cos \underline{\theta}(s), \end{aligned} \tag{1.40}$$

which gives (1.37). In particular, this $\underline{\mathcal{U}}(s)$ is the unique solution of (1.32) satisfying the initial conditions, (1.33) and

$$\underline{\mathcal{U}}'(0) = f(r(0)) \cos \underline{\theta}(0). \tag{1.41}$$

To complete the proof, it suffices to check the following. Given c, with

$$|c| \leq f(r(0)), \tag{1.42}$$

there exists a solution, $\underline{\underline{\mathcal{U}}}(s)$ (necessarily unique) of (1.32) on $[0, \ell]$, satisfying (1.33) and

$$\underline{\underline{\mathcal{U}}}'(0) = c. \tag{1.43}$$

In addition, the map

$$\underline{\underline{\mathcal{U}}}'(0) \to \underline{\underline{\mathcal{U}}}(\ell), \tag{1.44}$$

is injective.

But from (1.30), it is clear that we can take

$$\underline{\underline{\mathcal{U}}}(s) = \mathcal{F}(\underline{\underline{\gamma}}(s)), \tag{1.45}$$

for suitable $\underline{\underline{\gamma}}(s)$ satisfying

$$\underline{\underline{\gamma}}(0) = \underline{\gamma}(0). \tag{1.46}$$

For such solutions, the injectivity is a direct consequence of the first variation formula. Indeed $\angle(\underline{\underline{\gamma}}'(0), \frac{\partial}{\partial r}) \neq 0, \pi$ implies $\angle(\underline{\underline{\gamma}}'(\ell), \frac{\partial}{\partial r}) \neq 0, \pi$. Thus, the derivative of $\underline{\underline{r}}(\ell)$ with respect to $\underline{\underline{\theta}}(0)$ is nonvanishing, $0 < \underline{\underline{\theta}}(0) < \pi$, provided $\underline{\underline{\gamma}}(\ell)$ is not conjugate to $\underline{\underline{\gamma}}(0)$ along γ. This is guaranteed by (1.28), (1.29). $\square$

Remark 1.47. It is important to note that in proving Proposition 1.27, we only used the existence of a function, $\mathcal{F}(r)$ satisfying (1.38), for which r is a distance function. As a consequence, for *any* space on which such a function exists, the functions $\underline{r}(s), \underline{\theta}(s)$ of (1.25), (1.26) coincide with those of any model space, for example, the space $(a, b) \times_f \mathbb{R}$.

In view of (1.33), (1.34), (1.37) there are well-defined functions, $\underline{\Theta}, \underline{\mathcal{O}}$, such that

$$\underline{\theta}(s) = \underline{\Theta}(r(\gamma(0)), r(\gamma(\ell)), \ell, s), \tag{1.48}$$

$$\underline{\theta}(\ell) = \underline{\mathcal{O}}(r(\gamma(0)), r(\gamma(\ell)), \ell). \tag{1.49}$$

If $\underline{\sigma}$ is a radial geodesic such that $\underline{\sigma}(0) = \underline{z}$ and $\underline{x}$ is a point close to $\underline{\sigma}$, put

$$\ell(t) = \overline{\underline{x}, \underline{\sigma}(t)}. \tag{1.50}$$

Let $\underline{\gamma}_t$ be minimal from $\underline{x}$ to $\underline{\sigma}(t)$. By the first variation formula, $\ell(t)$ satisfies the differential equation

$$\ell'(t) = \cos \mathcal{O}(r(\underline{x}), t + r(\underline{z}), \ell(t)), \tag{1.51}$$

with initial condition,

$$\ell(0) = \overline{\underline{x}, \underline{z}}. \tag{1.52}$$

Let x_1, x_2, z_1, z_2 (sufficiently close to each other) satisfy

$$r(\underline{x}_2) - r(\underline{x}_1) = \overline{\underline{x}_1, \underline{x}_2}, \tag{1.53}$$

$$r(\underline{z}_2) - r(\underline{z}_1) = \overline{\underline{z}_1, \underline{z}_2}. \tag{1.54}$$

It follows easily from (1.50), (1.51) that for some function Q determined by $\mathcal{F}$,

$$\overline{\underline{x}_2, \underline{z}_2} = Q(r(\underline{x}_1), r(\underline{x}_2), r(\underline{z}_1), r(\underline{z}_2), \overline{\underline{x}_1, \underline{z}_1}). \tag{1.55}$$

If we specialize to the case $r(\underline{x}_1) = r(\underline{z}_1) = c$ and let $\underline{z}_1 \to \underline{x}_1$, then the extrinsic distance can be replaced by the intrinsic distance measured on the level surface $r^{-1}(c)$. This easily suffices to determine the function ρ_f.

Remark 1.56. In the situation considered in Sections 2 and 3 below, the equalities above e.g. (1.55), are only approximate and we will not be able to pass to the limit in the final step. Thus, very small scale information can be lost.

2. Integral estimates; Hessians, distances, angles

In this section we show that if on an annular region, (1.6) almost holds in the integral sense, then for most pairs of sufficiently close points y_1, y_2, the behavior of the minimal geodesic, γ_{y_1,y_2}, from y_1 to y_2, is almost described by the functions in (1.25), (1.26) (see (1.36), (1.37), Remark 1.47, (2.48), (2.49)). This is shown to yield a corresponding approximate version of (1.55), for most quadruples of points which are close to one another.

The main technical result of this section is Theorem 2.11. What has been accomplished in the context of "almost rigidity" is summarized in Proposition 2.80.

Let M^n be a complete riemannian manifold and let $K \subset M^n$ be a compact subset. Let $r(x) = \overline{x, K}$ denote the distance function from K and for $0 < a < b$, put $A_{a,b} = r^{-1}((a, b))$.

Let $f, \mathcal{F}$ be as in (1.4). In what follows, we regard $\mathcal{F} = \mathcal{F}(r(x))$ as a function on $A_{a,b}$.

Let $\underset{\sim}{\mathcal{F}} : A_{a,b} \to \mathbb{R}$ satisfy

$$\text{range } \underset{\sim}{\mathcal{F}} \subset \text{range } \mathcal{F}, \tag{2.1}$$

$$|\underset{\sim}{\mathcal{F}} - \mathcal{F}| \leq \delta, \tag{2.2}$$

$$\frac{1}{\mathrm{Vol}(A_{a,b})} \int_{A_{a,b}.} |\nabla \underset{\sim}{\mathcal{F}} - \nabla \mathcal{F}| \leq \delta \tag{2.3}$$

$$\frac{1}{\mathrm{Vol}(A_{a,b})} \int_{A_{a,b}} |\operatorname{Hess}_{\underset{\sim}{\mathcal{F}}} - H(\underset{\sim}{\mathcal{F}}) g| \leq \delta \tag{2.4}$$

In (2.3), (2.4) and elsewhere, when integrating over a subset of a riemannian manifold, the *natural measure associated to the riemannian metric* will be understood and no symbol such as "$d\,$vol" will be included to indicate the measure.

Remark 2.5. In the context of Sections 4 and 5 of Part II, the annular domain, $A_{a,b}$, is the appropriate one to consider. The main effort there is devoted to *obtaining* (2.1)–(2.4). But, in Section 6 of Part II, the domain $A_{a,b}$ must be replaced by a ball. This circumstance makes the analog of (2.4) more difficult to obtain. However, in the present Part I we are concerned with *consequences* of (2.1)–(2.4). Since, the relevant arguments can easily be adapted to the set-up of Section 6, we will not give a separate treatment for that case.

Our next result, Theorem 2.11, will allow us to convert estimates like the ones in (2.3) or (2.4) into corresponding estimates along a collection of minimal geodesics. Theorem 2.11 replaces the technique of integration over the unit sphere bundle, used in [Col]–[Co3] for similar purposes. The present version provides an improvement which enables us to handle situations in which the volume of M^n has no fixed lower bound, i.e., the "collapsed" case.

Let Y^n be a riemannian manifold with

$$\mathrm{Ric}_{Y^n} \geq (n-1)\Lambda \qquad (\Lambda > -\infty). \tag{2.6}$$

Let $A_1, A_2 \subset Y^m$ be open sets and assume that for all $y_1 \in A_1, y_2 \in A_2$, there is a minimal geodesic, γ_{y_1,y_2}, from y_1 to y_2, such that for some open set, W,

$$\bigcup_{y_1,y_2} \gamma_{y_1,y_2} \subset W. \tag{2.7}$$

If v_i is a tangent vector at y_i, $i = 1,2$, and $|v_i| = 1$, set

$$I(y_i, v_i) = \{t \mid \gamma(t) \in A_{i+1}, \gamma|[0,t] \text{ is minimal}, \ \gamma'(0) = v_i\}. \tag{2.8}$$

Here $A_{2+1} := A_1$.

Let $|I(y_i, v_i)|$ denote the measure of $I(y_i, v_i)$ and put

$$\mathcal{D}(A_i, A_{i+1}) = \sup_{y_i, v_i} |I(y_i, v_i)|. \tag{2.9}$$

The set, B, of points, $(y_1, y_2) \in A_1 \times A_2$, for which there is a unique minimal geodesic from y_1 to y_2 has full measure. Below, we keep the more suggestive notation, $A_1 \times A_2$, where in actuality, we mean B.

Let M^n_Λ denote the simply connected n-dimensional space of constant curvature, Λ.

For $\underline{p} \in M^n_\Lambda$, put

$$\mathcal{A}^n_\Lambda(u) = \operatorname{Vol}(\partial B_u(\underline{p})). \tag{2.10}$$

THEOREM 2.11. *Let e be a nonnegative integrable function on W. Let $\max \overline{y_1, y_2} = D$. Then*

$$\int_{A_1 \times A_2} \int_0^{\overline{y_1,y_2}} e(\gamma_{y_1,y_2}(s))\, ds \leq c(n,\Lambda,D)[\mathcal{D}(A_1,A_2) \operatorname{Vol}(A_1) + \mathcal{D}(A_2,A_1) \operatorname{Vol}(A_2)] \times \int_W e, \tag{2.12}$$

where $c(n,\Lambda,D) = \sup_{0<s/2\leq u \leq s} A^n_\lambda(s)/A^n_\lambda(u)$.

Proof. We can assume $(y_1, y_2) \in B$. Set

$$E(y_1, y_2) = \int_0^{\overline{y_1,y_2}} e\left(\gamma_{y_1,y_2}(s)\right) ds, \tag{2.13}$$

$$E_1(y_1, y_2) = \int_{\frac{1}{2}\overline{y_1,y_2}}^{\overline{y_1,y_2}} e\left(\gamma_{y_1,y_2}(s)\right) ds, \tag{2.14}$$

$$E_2(y_1, y_2) = \int_0^{\frac{1}{2}\overline{y_1,y_2}} e(\gamma_{y_1,y_2}(s))\, ds. \tag{2.15}$$

Then

$$E = E_1 + E_2, \tag{2.16}$$

and by an obvious symmetry argument, it suffices to bound

$$\int_{A_1 \times A_2} E_1 = \int_{A_1} \int_{A_2} E_1. \tag{2.17}$$

Fix y_1 and a unit tangent vector, v_1, at y_1. Let $\gamma'(0) = v_1$. Along γ, write the volume element of Y^n in geodesic polar coordinates as

$$ds \wedge \mathcal{A}(s). \tag{2.18}$$

Then for $s = \overline{y_1, y_2}$, by the Bishop-Gromov inequality,

$$\begin{aligned} E_1(y_1, y_2)\mathcal{A}(s) &= \mathcal{A}(s) \int_{\frac{1}{2}s}^{s} e(\gamma(u))\, du \\ &\leq c(n, \Lambda, D) \int_{\frac{1}{2}s}^{s} e(\gamma(u))\mathcal{A}(u)\, du. \end{aligned} \tag{2.19}$$

Thus, by (2.10)

$$\int_{I(y_1,v_1)} E_1(y_1,y_2)\mathcal{A}(s)\,ds \leq c(n,\Lambda,D)\mathcal{D}(A_1,A_2)\int_0^{T(v_1)} e(\gamma(u))\mathcal{A}(u)\,du, \tag{2.20}$$

where $T(v_1)$ is the supremum of t such that $t \in I(y_1,v_1)$. Integrating relation (2.20) over the unit sphere in the tangent space at y_1 gives

$$\int_{A_2} E_1(y_1,y_2) \leq c(n,\Lambda,D)\mathcal{D}(A_1,\, A_2)\int_W e. \tag{2.21}$$

If we then integrate (2.21) over A_1, the resulting estimate, together with the corresponding one in which the roles of A_1 and A_2 are interchanged, gives (2.12). □

Let $A_{a,b}$ be as at the beginning of this section and assume

$$\mathrm{Ric}_{M^n} \geq (n-1)\Lambda, \tag{2.22}$$

so that Theorem 2.11 applies. Put

$$\mathcal{V}(u) = \inf \frac{\mathrm{Vol}(B_u(q))}{\mathrm{Vol}(A_{a,b})}, \tag{2.23}$$

where for fixed $u > 0$, the infimum is taken over all $q \in A_{a,b}$ with $u \leq \min(b - r(q), r(q) - a)$.

The following immediate consequence of the Bishop-Gromov inequality will suffice for our subsequent applications.

PROPOSITION 2.24. *If (2.22) holds and for some $B_{2\mathcal{R}}(z)$, with $\overline{B_{2\mathcal{R}}(z)}$ complete, $A_{a,b} \subset B_{\mathcal{R}}(z) \subset M^n$, then*

$$\mathcal{V}(u) \geq c(n,\Lambda,\mathcal{R})u^n. \tag{2.25}$$

Assumption. From now on, we assume that (2.1)–(2.4) hold.

Fix attention on some p, with

$$B_{4R}(p) \subset A_{a,b}. \tag{2.26}$$

By (2.1), (2.4), and Theorem 2.11, we get:

COROLLARY 2.27.

$$\frac{1}{[\mathrm{Vol}(B_R(p))]^2}\int_{B_R(p)\times B_R(p)}\int_0^{\overline{y_1,y_2}} |\,\mathrm{Hess}_{\underset{\sim}{\mathcal{F}}} - H(\underset{\sim}{\mathcal{F}})g| \circ \gamma_{y_1,y_2}(s)\,ds \leq 2c(n,\Lambda,R)R\mathcal{V}^{-1}(R)\delta = c(n,\Lambda,R,\mathcal{V})\delta. \tag{2.28}$$

For fixed y_1, y_2, put

$$\underset{\sim}{\mathcal{F}}(\gamma_{y_1,y_2}(s)) = \mathcal{U}(y_1, y_2, s). \tag{2.29}$$

We will just write $\mathcal{U}(s)$ for $\mathcal{U}(y_1, y_2, s)$ and $\mathcal{U}', \mathcal{U}''$ for $\frac{\partial \mathcal{U}}{\partial s}, \frac{\partial^2 \mathcal{U}}{\partial s^2}$.

Since

$$\mathrm{Hess}_{\underset{\sim}{\mathcal{F}}}\left(\gamma'_{y_1,y_2}(s), \gamma'_{y_1,y_2}(s)\right) = \mathcal{U}''(s), \tag{2.30}$$

by (2.28), we get

$$\frac{1}{[\mathrm{Vol}(B_R(p))]^2} \int_{B_R(p)\times B_R(p)} \int_0^{\overline{y_1,y_2}} |\mathcal{U}'' - H(\mathcal{U})|\, ds \le c(n, R, \Lambda, \mathcal{V})\delta. \tag{2.31}$$

Let $D_\varepsilon(y_1)$ denote the set of points, $y_2 \in B_R(p)$, such that $\frac{\partial}{\partial r}$ is uniquely defined at $\gamma_{y_1,y_2}(s)$, for almost all s, and such that

$$\int_0^{\overline{y_1,y_2}} |\mathcal{U}'' - H(\mathcal{U})|\, ds \quad < \quad \varepsilon, \tag{2.32}$$

$$\int_0^{\overline{y_1,y_2}} |\nabla \underset{\sim}{\mathcal{F}} - \nabla \mathcal{F}| \circ \gamma_{y_1,y_2}(s)\, ds \quad < \quad \varepsilon, \tag{2.33}$$

$$|\nabla \underset{\sim}{\mathcal{F}}(y_2) - \nabla \mathcal{F}(y_2)| \quad < \quad \varepsilon. \tag{2.34}$$

Also, put

(2.35)

$$Q_\varepsilon = \left\{ y_1 \middle| \, |\nabla \underset{\sim}{\mathcal{F}}(y_1) - \nabla \mathcal{F}(y_1)| < \varepsilon, \ \mathrm{Vol}(D_\varepsilon(y_1)) \ge (1-\varepsilon)\ \mathrm{Vol}(B_R(p)) \right\}.$$

From (2.3), (2.12) applied to the function $|\nabla \underset{\sim}{\mathcal{F}} - \nabla \mathcal{F}|$ and (2.31), we obtain:

COROLLARY 2.36. *There exists* $\tau = \tau(\varepsilon, n, \Lambda, \mathcal{V}, R)$, *such that if in* (2.3), (2.4), $\tau > \delta > 0$, *then*

$$\mathrm{Vol}(Q_\varepsilon) \ge (1-\varepsilon)\ \mathrm{Vol}(B_R(p)). \tag{2.37}$$

From now on, we put $\ell = \overline{y_1, y_2}$.

In order to use the information in (2.32), we need the following standard lemma. Assume that $\mathcal{U}(s)$ is defined on $[0, \ell]$, that $H(\underline{\mathcal{U}}(s))$ is defined and that

$$\underline{\mathcal{U}}'' \quad = \quad H(\underline{\mathcal{U}}), \tag{2.38}$$

$$|\underline{\mathcal{U}}(0) - \mathcal{U}(0)| \quad \le \quad \varepsilon_0, \tag{2.39}$$

$$|\underline{\mathcal{U}}'(0) - \mathcal{U}'(0)| \quad \le \quad \varepsilon_1. \tag{2.40}$$

LEMMA 2.41. *Let* (2.32) *hold. Then for* $s \in [0,\ell]$,

$$|\mathcal{U}(s) - \underline{\mathcal{U}}(s)| \leq \varepsilon_0 \cosh\sqrt{K}s + [\varepsilon_1 + \varepsilon]\frac{\sinh\sqrt{K}s}{\sqrt{K}}, \tag{2.42}$$

$$|\mathcal{U}'(s) - \underline{\mathcal{U}}'(s)| \leq \varepsilon_1 + \varepsilon + K\int_0^s |\mathcal{U}(u) - \underline{\mathcal{U}}(u)|\,du, \tag{2.43}$$

where

$$K = \sup|H'| = \max_{r\in[a,b]} \frac{|f''|}{|f|}. \tag{2.44}$$

Note that Lemma 2.41 applies if, in particular, $y_1 \in D_\varepsilon(y_2)$.

Let $\underline{\mathcal{U}}(s)$ be the solution defined in (1.40). From now on, write

$$\underline{r}(r(\gamma(0)), r(\gamma(\ell)), \ell, s), \tag{2.45}$$

for the function defined in (1.36). Let the functions $\underline{\Theta}, \underline{\mathcal{O}}$ be defined as in (1.48), (1.49).

Given $y_1, y_2, \gamma_{y_1,y_2}$ as above, put

$$r(y_1, y_2, s) = r(\gamma_{y_1,y_2}(s)), \tag{2.46}$$

and at points where $\frac{\partial}{\partial r}$ is uniquely defined, put

$$\theta(y_1, y_2, s) = \angle(\gamma'_{y_1,y_2}(s), \frac{\partial}{\partial r}). \tag{2.47}$$

COROLLARY 2.48. *Let* $\varepsilon > 0$. *The number,* τ, *in Corollary* 2.36 *can be chosen such that if in* (2.2)–(2.4), $\tau > \delta > 0$, *then for* $y_1 \in Q_\varepsilon$, $y_2 \in D_\varepsilon(y_1)$,

$$|r(y_1,y_2,s) - \underline{r}(r(y_1),r(y_2),\ell,s)| < \varepsilon, \tag{2.49}$$

$$|\Theta(y_1,y_2,\ell) - \underline{\mathcal{O}}(r(y_1),r(y_2),\ell)| < \varepsilon, \tag{2.50}$$

$$\int_0^{\overline{y_1,y_2}} |\cos\theta(y_1,y_2,s) - \cos\underline{\Theta}(r(y_1),r(y_2),\ell,s)| < c(f)\varepsilon R. \tag{2.51}$$

Proof. Let $\varepsilon_0, \varepsilon_1$ in (2.38), (2.39) be given. By (2.28),

$$\mathcal{U}'(s) = \langle\nabla\underset{\sim}{\mathcal{F}}, \gamma'(s)\rangle. \tag{2.52}$$

Also, for any $\underline{\mathcal{U}}(s)$,

$$\underline{\mathcal{U}}'(s) = \langle\nabla\mathcal{F}, \gamma'(s)\rangle = f(r(s))\cos\underline{\theta}(s). \tag{2.53}$$

Thus, there exists $0 < \tau_1 = \tau_1(\varepsilon, \varepsilon_0, \varepsilon_1, n, \Lambda, R, \mathcal{V})$ such that if in (2.2)–(2.4), $\delta < \tau_1$, then there exists $\underline{\mathcal{U}}(s)$ as in (1.40), such that (2.39), (2.40) hold. This follows from (2.34) and (2.2). Below, we take $\varepsilon_0, \varepsilon_1$ sufficiently small, depending on ε. Then we take $\tau = \tau_1$.

By (2.42) (for $s = \ell$, where ℓ satisfies (1.28)) together with (2.2), we see that $\underline{\mathcal{U}}(s)|[0,\ell]$ can also be viewed as the solution, $\mathcal{F}(\underline{\gamma}(s))$, where rather than controlling the initial values, we specify instead that $|r(\underline{\gamma}(0))-r(y_1)|$, $|r(\underline{\gamma}(\ell))-r(y_2)|$ are small, i.e., as small as we like if $\tau_1 > 0$ is sufficiently small.

Thus, if $\underline{\underline{\mathcal{U}}}(s)$ is the solution on $[0,\ell]$ of the form $\mathcal{F}(\underline{\underline{r}}(s))$ with $\underline{\underline{\gamma}}(s) \subset (a,b)\times_f N^{n-1}$,

$$\begin{aligned} r(\gamma(0)) &= r(\underline{\underline{\gamma}}(0)), \\ r(\gamma(\ell)) &= r(\underline{\underline{\gamma}}(\ell)), \end{aligned} \tag{2.54}$$

we will have $|\mathcal{U}(0)-\underline{\underline{\mathcal{U}}}(0)|, |\mathcal{U}'(0)-\underline{\underline{\mathcal{U}}}'(0)|$ as small as we like, provided $\tau_1 > 0$ is sufficiently small; see (1.44) and its proof. Note that the existence of $\underline{\underline{\mathcal{U}}}(s)$ follows from (2.26). By (2.42), (2.43), this suffices to complete the proof. □

Relations (2.49)–(2.51) generalize the information contained in Remark 1.47. We will now consider the corresponding generalization of relation (1.55) and the consequences thereof.

Given $x_1, z_1, x_2, z_2 \in B_R(p)$, with

$$r(z_1) - r(x_1) = \overline{x_1, z_1}, \tag{2.55}$$

$$r(z_2) - r(x_2) = \overline{x_1, z_2}, \tag{2.56}$$

we wish to estimate

$$|\overline{z_1, z_2} - Q(r(x_1), r(z_1), r(x_2), r(z_2), \overline{x_1, x_2})|\,. \tag{2.57}$$

The restriction, $x_1, z_1, x_2, z_2 \in B_R(p)$, will be removed in the next section.

If $x_1 = z_1 = x$, we put

$$L\left(r(x), r(x_2), r(z_2), \overline{x, x_2}\right) = Q\left(r(x), r(x), r(x_2), r(z_2), \overline{x, x_2}\right). \tag{2.58}$$

Clearly, in order to estimate the quantity in (2.57), it suffices to estimate

$$|\overline{x, z_2} - L\left(r(x), r(x_2), r(z_2), \overline{x, x_2}\right)|\,. \tag{2.59}$$

Fix a small number, $1 > \eta > 0$. We can assume

$$\overline{x_2, z_2} > \eta, \tag{2.60}$$

since otherwise, effectively, there is nothing to prove. By choosing ε of Corollary 2.36 appropriately, we can assume that if the hypothesis of Corollary 2.36 holds, i.e., $\delta < \tau$, then there exists

$$y \in B_{\eta^3}(x) \cap Q_{\eta^3}. \tag{2.61}$$

Similarly, we can assume that there exist

$$\begin{aligned} q &\in B_{\eta^3}(x_2), \\ w &\in B_{\eta^3}(z_2) \end{aligned} \tag{2.62}$$

with

$$w \in D_{\eta^3}(q). \tag{2.63}$$

Put

$$\gamma_{q,w} = \lambda, \tag{2.64}$$

$$\overline{y, \lambda(s)} = \ell(s), \tag{2.65}$$

$$\overline{q, w} = d. \tag{2.66}$$

By applying Theorem 2.11 to the function $\theta(y, \lambda(s), \ell(s))$ of (2.47), we can assume in addition that

$$\int_0^d |\cos\theta(y, \lambda(s), \ell(s)) - \cos\underline{\theta}(r(y), r(\lambda(s)), \ell(s))| \, ds < \eta^2. \tag{2.67}$$

In view of (2.56), (2.62), we have (in (2.67))

$$|r(\lambda(s)) - s| < 2\eta^3 \tag{2.68}$$

and with (2.63), we have for some constant,

$$C = C\left(f, \overline{\partial B_{3R}(p), \partial A_{a,b}}\right), \tag{2.69}$$

$$\int_0^d |\theta(q, w, s)| \, ds < C\eta. \tag{2.70}$$

Then for

$$\alpha(s) = \angle\left(\gamma'_{y,\lambda(s)}\left(\ell(s)\right), \lambda'(s)\right), \tag{2.71}$$

from (2.67), (2.68), (2.69), we get

$$\int_0^d |\cos\alpha(s) - \cos\underline{\theta}(r(s), s, \ell(s))| \, ds < C\eta^2. \tag{2.72}$$

In the model space, $(a, b) \times_f \mathbb{R}$, let $\underline{y}, \underline{w}_1, \underline{w}_2$ satisfy

$$r(\underline{y}) = r(y), \tag{2.73}$$

$$r(\underline{w}) = r(w), \tag{2.74}$$

$$r(\underline{q}) = r(q),$$

$$\overline{\underline{q}, \underline{w}} = \overline{q, w} = d, \tag{2.75}$$

$$\overline{\underline{y}, \underline{q}} = \overline{y, q} = \ell(0). \tag{2.76}$$

Then for $\underline{\lambda}$, $\underline{\gamma}_{\underline{y},\underline{\lambda}(s)}$, $\underline{\alpha}(s)$ defined as above, it is clear that as in (2.72),

$$\int_0^d |\cos\underline{\alpha}(s) - \cos\underline{\theta}(r(y), s, \underline{\ell}(x))| \le C\eta^2. \tag{2.77}$$

By the first variation formula, for almost all s,

$$(\ell(s) - \underline{\ell}(s))' = \cos\alpha(s) - \cos\underline{\alpha}(s). \tag{2.78}$$

Since $\ell(0) = \underline{\ell}(0)$, from (2.72), (2.77), (2.78), we easily find that

$$|\ell(d) - \underline{\ell}(d)| \le C\eta^2(e^d - 1). \tag{2.79}$$

Let $B_{3R}(p) \subset A_{a,b}$. From (2.59), (2.61), (2.62), (2.69), we obtain the following result (which we state using the constant ε rather than η).

PROPOSITION 2.80. *Given $\varepsilon > 0$, there exists $\zeta = \zeta(\varepsilon,n,\Lambda,\mathcal{V},r(p)-a,b-r(p))$, such that if in* (2.2)–(2.4), *$\delta \le \zeta$, then for x_1,z_1,x_2,z_2 as in* (2.55), (2.56),

$$|\overline{z_1,z_2} - Q(r(x_1),r(z_1),r(x_2),r(z_2),\overline{x_1,z_2})| < \varepsilon. \tag{2.81}$$

Remark 2.82. Theorem 2.11 leads directly to a lower bound on the smallest nonzero eigenvalue of the Laplacian and to Poincaré-type inequalities. Although the constants are not quite sharp, these inequalities are essentially the known ones due to Gromov and Li-Yau; see [G3] and [LY]. The derivation via Theorem 2.11 is similar to Gromov's approach. For instance, if in Theorem 2.11 we take $A_1 = A_2 = M^n$ and $e = |\nabla f|^2$, where $\int_{M^n} f = 0$, then from (2.12) and the Schwarz inequality, we obtain

$$\lambda_1 \ge [c(n,\Lambda,\operatorname{diam}(M^n))\operatorname{diam}(M^n)]^{-2}, \tag{2.83}$$

where for $\Lambda \ge 0$, the constant $c = c(n,\Lambda)$ is actually independent of $\operatorname{diam}(M^n)$. Clearly, there exist analogs of (2.12) (and of the implication that (2.12) yields (2.83)) in other contexts, e.g., in the context of graph theory.

3. Gromov-Hausdorff approximations

In this section, we will show that for $\alpha > 0$, an annulus, $A_{a+\alpha,b-\alpha}$ is close (in a precise sense specified below) to a warped product, $(a+\alpha, b-\alpha) \times_f X$, provided the number ε in (2.81) is sufficiently small and an additional technical condition, of almost maximality, (3.8), holds. This technical condition is satisfied in the applications.

As mentioned in the introduction, the choice of X cannot be made completely canonical. However, a specific choice will be employed in the proof.

In what follows, let $d^{\alpha'}$ denote the metric of $A_{a+\alpha',b-\alpha'}$. Thus, for $y,\hat{y} \in A_{a+\alpha',b-\alpha'}$, $d^{\alpha'}(y,\hat{y})$ is the infimum of lengths of curves, c, from y to $\hat{y}$ such that $c \subset A_{a+\alpha',b-\alpha'}$. For $0 \le \alpha' \le \alpha$, let $d^{\alpha',\alpha}$ denote the restriction of $d^{\alpha'}$ to $A_{a+\alpha,b-\alpha} \subset A_{a+\alpha',b-\alpha'}$.

Given a warped product $(a,b) \times_f X$, we also denote by $\underline{d}^{\alpha'}, \underline{d}^{\alpha',\alpha}$ respectively, the metric on $(a+\alpha', b-\alpha') \times_f X$ and its restriction to the subset, $(a+\alpha, b-\alpha) \times X$.

Before stating the main result of this section, we note some elementary properties of the metrics $\underline{d}^{\alpha'}, \underline{d}^{\alpha',\alpha}$.

Put

$$m(a,b,\alpha,\alpha',f) = \min\left(\inf_{\alpha'<r-a<\alpha}\frac{f(r)}{f(a+\alpha')}, \inf_{\alpha'<b-r<\alpha}\frac{f(r)}{f(b-\alpha')}\right).$$

Then

$$\underline{d}^{\alpha',\alpha} \leq \underline{d}^{\alpha} \leq \frac{1}{m(a,b,\alpha,\alpha',f})\underline{d}^{\alpha',\alpha}. \tag{3.1}$$

The first inequality in (3.1) is obvious. To see the second, let $(k(s), c(s))$ be a curve in $(a-\alpha', b-\alpha')\times_f X$. Replace any segments of (k,c) for which $a+\alpha' < k(s) < \alpha'$ (respectively $b-\alpha' < k(s) < b-\alpha'$) by segments of the form $(a+\alpha', c(s))$ (respectively $(b-\alpha', c(s))$). Then the length of the resulting curve is at most $\frac{1}{m(a,b,\alpha,\alpha',f})L[(c,k)]$ (where $L[\ ]$ denotes length). Thus, the second inequality follows.

Recall that the distance function on $(a,b)\times_f X$ is determined by a function, $\rho(r_1, r_2, \overline{x_1, x_2})$ (where the dependence on (a,b) is suppressed). Let $\rho_{a+\alpha',b-\alpha';f}(r_i, r_2, \overline{x_1, x_2})$ denote the corresponding function, for the distance function, $\underline{d}^{\alpha'}$, on $(a+\alpha', b-\alpha')\times_f X$.

As in (3.1), if $\rho_{a+\alpha'-\chi,b-\alpha'+\chi;f}(a+\alpha', a+\alpha', v) < \chi$, then

$$\begin{aligned}\frac{1}{f(a+\alpha')}&\rho_{a+\alpha'-\chi,b-\alpha'+\chi;f}(a+\alpha', a+\alpha', v)\\ &\leq \frac{1}{m_1 f(a+\alpha')}\rho_{a+\alpha'-\chi,b-\alpha'+\chi;f}(a+\alpha', a+\alpha', v),\end{aligned} \tag{3.2}$$

where $m_1 = m_1(a,\alpha',\chi,f) = \inf_{-\chi/2\leq r-a-\alpha'\leq\chi/2}\frac{f(r)}{f(a+\alpha')}$. Note additionally, that for $v_1, v_2 \geq 0$, the following is clear:

$$\begin{aligned}\rho_{a+\alpha',b-\alpha';f}(r_1,r_2,v_1) &\leq \rho_{a+\alpha',b-\alpha';f}(r_1,r_2,v_1+v_2)\\ &\leq \rho_{a+\alpha;b-\alpha;f}(r_1,r_2,v_1) + v_2\min(f(r_1)), f(r_2)).\end{aligned} \tag{3.3}$$

Also, for all $r_0, r_1, \ldots, r_N$ between $a+\alpha$ and $b-\alpha$, and all nonnegative $v_1, \ldots v_N$,

$$\sum_{i=1}^{N}\rho_{a+\alpha',b-\alpha';f}(r_{i-1}, r_i, v_i) \geq \rho_{a+\alpha',b-\alpha';f}\left(r_0, r_N, \sum_{i=1}^{N} v_i\right). \tag{3.4}$$

Moreover, for fixed $r_0, r_N, \{v_i\}$, there exist $r_1, \ldots, r_{N-1}$ such that (3.4) is an equality.

Let (k,c) be a curve in $\overline{(a+\alpha', b-\alpha')\times_f X}$ (the closure of $(a+\alpha', b-\alpha')\times_f X$). Assume that $L[(k,c)]$ equals the distance with respect to $\underline{d}^{\alpha'}$, between the end points of (k,c). An elementary argument based on the first variation formula shows that if $(k,c)\cap\partial(a+\alpha', b-\alpha'\times_f X)$ is nonempty, then (k,c) is tangent to the boundary at those points at which it enters or leaves

$\partial((a+\alpha', b-\alpha') \times_f X)$. The remaining segments of (k, c) are geodesics lying in $(a+\alpha', b-\alpha') \times X$.

Recall that if $\underline{\gamma}$ is a geodesic in $(a, b) \times_f X$ and $\underline{\theta}(s) = \angle(\underline{\gamma}'(s), \frac{\partial}{\partial r})$, then by the classical theorem of Clairaut, the function, $f(\underline{\gamma}(s)) \cdot \sin \underline{\theta}(s)$ is constant on $\underline{\gamma}$.

Let $(k, c) : [0, l] \to \overline{(a+\alpha'; b-\alpha') \times_f X}$ satisfy

$$L[(k, c)] = \underline{d}^{\alpha'}((k(0), c(0)), (k(\ell), c(\ell))).$$

Then the above mentioned facts have the following direct consequence.

For all $\varepsilon > 0$, there exists $\underline{\delta}(\varepsilon, a, b, f) > 0$, such that at least one of the following holds.

- The curve, (k, c), may contain a segment lying in $\partial((a+\alpha', b-\alpha') \times_f X)$. However, $\angle((k'(s), c'(s)), \frac{\partial}{\partial r}) \geq \underline{\delta}$, for all $s \in [0, \ell]$.

•• The curve, (k, c), consists of a single geodesic segment whose interior is contained in $(a+\alpha', b-\alpha') \times_f X$. Moreover, $|k(0) - k(\ell)| - L[(k, c)] < \varepsilon$.

Let $\Psi(u_1, \ldots, u_k|., \ldots)$ denote a nonnegative function depending on the numbers, $u_1, \ldots u_k$, and some additional parameters, such that when these additional parameters are fixed, we have

$$\lim_{u_1, \ldots u_k \to 0} \Psi(u_1, \ldots, u_k|., \ldots) = 0. \tag{3.5}$$

We can now state the main result of this section.

THEOREM 3.6. *Let $0 < \alpha' < \alpha$ and $\alpha - \alpha' > \xi$. Assume that for the metric $d^{\alpha',\alpha}$,*

$$\mathrm{diam}(A_{a+\alpha, b-\alpha)} \leq D. \tag{3.7}$$

Assume, in addition, that for all $x \in r^{-1}(a+\alpha')$, there exists $y \in r^{-1}(b-\alpha')$ with

$$d^{\alpha'-\zeta}(x,y) \leq b - a - 2\alpha' + \zeta. \tag{3.8}$$

Finally, assume that for x_1,z_1,x_2,$z_2 \in A_{a+\alpha', b-\alpha'}$ satisfying the hypothesis of Proposition 2.80, *the number, δ, in Proposition* 2.80 *satisfies $\delta < \zeta$. Then there exists a metric space X, with* $\mathrm{diam}(X) \leq c(a,b,\alpha',f,D)$, *such that for the metrics $d^{\alpha',\alpha}$, $\underline{d}^{\alpha',\alpha}$,*

$$d_{GH}(A_{a+\alpha, b-\alpha},(a+\alpha, b-\alpha) \times_f X) \leq \Psi(\zeta|\alpha', \xi, n, f, D). \tag{3.9}$$

Remark 3.10. Note that for x, y as in (3.8), a minimal geodesic, γ, from x to y, might contain some points $\gamma(s)$ (where $s < \zeta/2$) for which $r(\gamma(s)) < a+\alpha'$. It is because of this possibility (which cannot be ruled out in the applications in Sections 4 and 6) that we must consider the metrics $d^{\alpha',\alpha}, \underline{d}^{\alpha',\alpha}$ (where $\alpha - \alpha' \geq \xi$) rather than just the metrics $d^{\alpha}, \underline{d}^{\alpha}$.

Proof of Theorem 3.6. First we specify the metric space, X. Choose $\chi < \alpha'$ (χ to be further specified later) and define a metric, l_χ, on $r^{-1}(a+\alpha')$ as follows. Given $x, \hat{x} \in r^{-1}(a+\alpha')$, consider all sequences of points, $x = x_0, x_1, \ldots x_N = \hat{x}$, such that $d^{\alpha'-\chi}(x_i, x_{i+1}) \leq \chi$, for all i. Put

$$\ell_\chi(x,\hat{x}) = \frac{1}{f(a+\alpha')} \inf_{x_0,\ldots x_N} \sum_i d^{\alpha'-\chi}(x_{i-1}, x_i). \tag{3.11}$$

Let X be the space, $r^{-1}(a+\alpha')$ with the metric, l_χ.

For all $y \in A_{\alpha'+a,b-\alpha'}$, choose a point, $z \in r^{-1}(a+\alpha')$, closest to y. Put

$$\pi(y) = z, \tag{3.12}$$

$$\beta(y) = (r(y), \pi(y)). \tag{3.13}$$

We will show that $\beta | A_{a+\alpha,b-\alpha}$ is the desired Gromov-Hausdorff equivalence. For this we need an extension of Proposition 2.80.

Let $\phi : [0,\ell] \to \overline{A_{a+\alpha',b-\alpha'}}$ be parametrized by arclength.

Claim. There exists $N = N(\zeta, a, b, \alpha', f, \ell)$ with the following property. Let $t_i = \frac{i}{N}\ell, i = 0, 1, \ldots N$. Let γ_i be minimal from $\phi(t_i)$ to $\pi(\phi(t_i))$. On each γ_i, insert N equally spaced points, $\gamma(s_{i,j}), j = 0 \ldots N$, where $s_{i,0} = 0$. Then each quadruple, $\gamma_{i-1}(s_{i-1j-1}), \gamma_{i-1}(s_{i-1,j}), \gamma_i(s_{i,j-1}), \gamma_i(s_{i,j})$ satisfies the hypothesis of Proposition 2.80.

It is clear that N as above can be chosen such that the quadruples, $\gamma_{i-1}(s_{i-1,0}), \gamma_{i-1}(s_{i-1,1}), \gamma_i(s_{i,0}), \gamma_i(s_{i,1})$ satisfy the hypothesis of Proposition 2.80. Then our claim follows from Proposition 2.80 by induction.

Let $y, \hat{y} \in A_{a+\alpha,b-\alpha}$ and let ϕ as above join y to $\hat{y}$, with $L[\phi] = d^{\alpha',\alpha}(y,\hat{y})$. From the claim which was just established, it is clear that there exist $\lambda(\chi, N_1)$, $N_1 = N_1(\zeta, \chi, a, b, \alpha', f, D)$ such that if $\tau \leq \lambda(\chi, N_1)$, then the points, $\{\pi(\phi(t_i))\}$, where $i = 0, \ldots, N_1$, satisfy

$$d^{\alpha'-\chi}(\pi(\phi(t_{i-1})), \pi(\phi(t_i))) \leq \chi. \tag{3.14}$$

Moreover, for $\Psi = \Psi(\tau|\alpha', a, b, n, f)$, we have by induction,

$$\begin{aligned} &d^{\alpha-\chi}(\phi(t_{i-1}), \phi(t_i)) \\ &\quad = Q(\pi(\phi(t_{i-1})), \phi(t_{i-1}), \pi(\phi(t_i)), \pi(\phi(t_i)), \underline{d}^{\alpha'-\chi}(\pi(\phi(t_{i-1})), \pi(\phi(t_i))) + \Psi. \end{aligned} \tag{3.15}$$

From (3.15) together with (3.2)–(3.4), we easily obtain for $\zeta \leq \lambda(\chi, N_1)$,

$$\begin{aligned} d^{\alpha'-\chi}(y,\hat{y}) \;\geq\; &\rho_{a+\alpha'-\chi,b-\alpha'-\chi;f}(r(y), r(\hat{y}), l_\chi(\pi(y), \pi(\hat{y})) \\ &+ \Psi(s, \chi|a, b, \alpha', n, f, D). \end{aligned} \tag{3.16}$$

Since $d^{\alpha'}(y,\hat{y}) \geq d^{\alpha'-\chi}(y,\hat{y})$, it follows that (3.16) provides a lower bound on $d^{\alpha',\alpha}$.

Let $x \in r^{-1}(a+\alpha')$. Given $c \in (a+\alpha', b-\alpha')$, it follows from (3.8) that there exists $y \in A_{a+\alpha,b-\alpha'}$, with

$$d^{\alpha'-\zeta}(x,y) - (r(y) - r(x)) \leq \zeta. \tag{3.17}$$

By applying (3.16) with $y = y$, $\hat{y} = x$ we find

$$\ell\chi(x, \pi(y) \leq \Psi(\zeta | a, b, \alpha', n, f). \tag{3.18}$$

This shows that range β is Ψ–dense, with Ψ as in (3.16).

From the Ψ–density of range β together with (3.16), it is clear that we get the bound, $\operatorname{diam}(X) \leq c(a,b,\alpha',f,D)$.

Finally, we must obtain an upper bound for $d^{\alpha',\alpha}$ which converges to the one in (3.16) as ζ, χ tend to zero, with $\zeta < \lambda(\chi, N_1)$. Here, for the first time, the assumption, $\alpha - \alpha' > \xi$ enters.

Let $y, \hat{y} \in A_{a+\alpha,b-\alpha}$. Let $\pi(y) = x_0, x_1, \ldots, x_{N_2} = \pi(\hat{y})$ be a sequence such that $x_i \in r^{-1}(a+\alpha')$ and $d^{\alpha'-\chi}(x_{i-1}, x_i) \leq \chi$. Assume in addition that

$$\sum_{i=1}^{N_1} d^{\alpha'-\chi}(x_{i-1}, x_i) = \ell\chi(\pi(y), \pi(\hat{y})). \tag{3.19}$$

Clearly, (3.19) implies that $d^{\alpha'-\chi}(x_{i-1}, x_i) + d^{\alpha'-\chi}(x_i x_{i+1}) \leq \chi$ holds for all i. This gives

$$N_2 \leq \frac{2\operatorname{diam}(X)}{\chi}. \tag{3.20}$$

Let $(c,k) \subset \overline{(a+\alpha'+\eta, b-\alpha'-\eta) \times_f X}$ have length $\underline{d}^{\alpha'+\eta,\alpha}(\beta(y), \beta(\hat{y}))$ (where $\eta < \frac{\alpha'}{4}, \eta < \xi$). Fix $\varepsilon > 0$, and let $\underline{\delta}$ be as in • which precedes Theorem 3.6. For $\chi < \lambda_1(\underline{\delta}, \alpha', \eta, a, b, f)$, if the alternative • holds, we have for all i,

$$\underline{d}^{\alpha'}((k(t_{i-1}), c(t_{i-1})), (k(t_i), c(t_i))) < \frac{1}{2}\eta, \tag{3.21}$$

where

$$t_i = \sum_{j=0}^{i} d^{\alpha'-\chi}(x_{j-1}, x_j). \tag{3.22}$$

Since range β is Ψ–dense, there exist, $y = y_0, y_1, \ldots, y_{N_2} = \hat{y}$, such that

$$\underline{d}^{\alpha'}(y_i, (k(t_i), c(t_i))) < \Psi. \tag{3.23}$$

Then for ζ sufficiently small, it is clear that

$$d^{\alpha'}(y_{i-1}, y_i) < \eta. \tag{3.24}$$

Thus, in particular, the curve formed by joining the points $\{y_i\}$ consecutively by minimal geodesic segments, lies in $A_{a+\alpha',b-\alpha'}$. By arguing as above, we get

$$\begin{aligned}(3.25)\qquad d^{\alpha',\alpha}(y,\hat{y}) &\leq \underline{d}^{\alpha'+\eta,\alpha}(\beta(y),\beta(\hat{y})) \\ &\quad + \Psi(\zeta|a,b,\alpha',\xi,\chi,\eta,n,f).\end{aligned}$$

Now, by letting $\chi,\eta \to 0$, and using (3.1) to compare the estimates (3.16), (3.25), we obtain Theorem 3.6 in this case.

If the second alternative, ••, holds we have

$$(3.26)\qquad d^{\alpha'/2}(\pi(y),\pi(\hat{y})) \leq \Psi(\varepsilon|a,b,f).$$

Let $\gamma,\hat{\gamma}$ be minimal from $\pi(y)$ to y and from $\pi(\hat{y})$ to $\hat{y}$ respectively. Assume say $r(\hat{y}) \leq r(y)$. Then

$$(3.27)\qquad d^{\alpha'}(y,\hat{y}) \leq r(y) - r(\hat{y}) + d^{\alpha'}(\gamma(r(\hat{y})),\hat{\gamma}(r(\hat{y}))).$$

By using (3.26) and the claim above to estimate the second term on the right-hand side of (3.27), we get Theorem 3.6 in this case as well. □

We now make some further observations which are needed for the application in Section 4 to the case in which the annulus, $A_{a,b}$, has almost maximal volume. In what follows, we no longer assume that $A_{a,b}$ is connected.

Let $\mathcal{V}$ be as in (2.23).

LEMMA 3.28. *Given α', $\zeta > 0$, there exists $\omega = \omega(\zeta,\alpha',n,a,b,f,\mathcal{V})$, such that if*

$$(3.29)\qquad \frac{\mathrm{Vol}(A_{a,b})}{\mathrm{Vol}(r^{-1}(a))} \geq (1-\omega)\frac{\int_a^b f^{n-1}(u)\,du}{f^{n-1}(a))},$$

then (3.6) *holds.*

Proof. Note that (3.29) together with the Bishop-Gromov technique implies that for $a \leq c \leq b$,

$$(3.30)\qquad \left|\frac{\mathrm{Vol}(r^{-1}(a))}{\mathrm{Vol}(r^{-1}(c))} - 1\right| \leq \Psi(c-a,\omega|n,a,b,f).$$

Assume that (3.6) fails to hold for some $x \in r^{-1}(a+\alpha')$. Then (3.6) also fails on $B_{\zeta/2}(x)$, provided we replace ζ by $\zeta/2$ and assume (without loss of generality) $\zeta < \alpha'$.

By the co-area formula, for some $a+\alpha-\frac{\zeta}{2} \leq r_0 \leq a+\alpha+\frac{\zeta}{2}$,

$$\begin{aligned}(3.31)\qquad \mathrm{Vol}(r^{-1}(r_0)\cap B_{(\zeta/2)}) &\geq \frac{1}{\zeta}\mathrm{Vol}(B_{(\zeta/2)}) \\ &\geq \frac{1}{\zeta}\mathcal{V}(\zeta/2)\mathrm{Vol}(A_{a,b})\end{aligned}$$

$$\geq \frac{1}{\zeta}\mathcal{V}(\zeta/2)(1 \pm \Psi(\alpha', \omega | n, a, b, f))$$
$$\cdot\ (1-\omega)\mathrm{Vol}(r^{-1}(r_0)).$$

Since (3.6) (with ζ replaced by $\zeta/2$) fails on $r^{-1}(r_0) \cap B_{\zeta/2}(x)$, using (3.31), we easily contradict (3.29), if $\omega(\zeta, \alpha', a, b, f, \mathcal{V})$ is sufficiently small. □

Given $a < c < b$, put $\underline{r} = \min(c-a, b-c)$. In general it is clear that $A_{a,b}$ has at most $\frac{1}{\mathcal{V}(\underline{r})}$ components whose intersection with $r^{-1}(c)$ is nonempty. Moreover, from arguments like those given above, we obtain the following proposition whose straightforward proof we omit.

PROPOSITION 3.32. *Given* $0 < \alpha' < \alpha$, $\alpha' - \alpha > \xi > 0$, *there exists* $0 < \omega = \omega(\alpha',\xi,n,a,b,f,\mathcal{V})$ *such that if* (3.29) *holds for such* ω, *then for the metric* $d^{\alpha',\alpha}$, *the annulus* $A_{a+\alpha,b-\alpha}$, *has at most* $\#(n,a,b,f,\mathcal{V})$ *components,* X_i. *Moreover,* $\mathrm{diam}(X_i) \leq D(n,a,b,f,\mathcal{V})$ *for all* i.

Proposition 3.32, implies that the assumptions (3.7), (3.8) in Theorem 3.6 are unnecessary, provided that one weakens the conclusion to $\mathrm{diam}(X_i) \leq D(n,a,b,f,\mathcal{V})$ for each of the at most $\#(n,a,b,f,\mathcal{V})$ components of X. In Section 4, we will show that the assumption concerning Proposition 2.80 is unnecessary as well.

As in the proof of Gromov's compactness theorem, [GLP], one also obtains:

PROPOSITION 3.33. *If* (3.29) *holds for* $\omega = \omega(\alpha',\xi,n,a,b,f,\mathcal{V})$, *then the space* X, *in Theorem* 3.6 *can be chosen to be a length space.*

II. Almost maximality implies integral estimates on Hessians

4. Volume

In this section and the two which follow, we prove our main results on *almost rigidity* by showing that almost maximal volume or diameter implies the existence of a function $\underset{\sim}{\mathcal{F}}$, satisfying the hypotheses of Section 2. Our arguments are inspired by those of [Co1]–[Co3].

As in previous sections, we consider a metric annulus, $A_{a,b}$, associated to a distance function r.

We say that the mean curvature, m, of the hypersurface, $r^{-1}(a) \subset M^n$ is $\leq \kappa$ and write

(4.1) $$m \leq \kappa \qquad (\text{on } r^{-1}(a))$$

if in the barrier sense,

(4.2) $$\Delta r \leq \kappa \qquad (\text{on } r^{-1}(a)).$$

Let $f\colon [a,b] \to \mathbb{R}^+$ be as in Part I. Note that for a warped product, $(a,b)\times_f N^{n-1}$,

$$\underline{m} = (n-1)\frac{f'}{f}, \tag{4.3}$$

where we use the underline to denote quantities associated to such a model space. Moreover,

$$\underline{\mathrm{Ric}}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) = -(n-1)\frac{f''}{f}. \tag{4.4}$$

Suppose that for M^n as above,

$$m \leq (n-1)\frac{f'(a)}{f(a)} \quad (\text{on } r^{-1}(a)) \tag{4.5}$$

and for all r, with $a<r<b$,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\frac{f''(r)}{f(r)}. \tag{4.6}$$

Then by standard comparison arguments, for $a<r<b$,

$$m \leq \underline{m} = (n-1)\frac{f'}{f}, \tag{4.7}$$

$$\frac{\mathrm{Vol}(A_{a,b})}{\mathrm{Vol}(r^{-1}(a))} \leq \frac{\int_c^d f^{n-1}(r)\,dr}{f^{n-1}(a)}. \tag{4.8}$$

In case the inequality in (4.8) is an equality, it follows directly from the Bishop-Gromov inequality together with an easy analysis based on the Riccati equation, that the annulus, $A_{a,b}$, is isometric to a warped product $(a,b)\times_f X$. We call this the "volume annulus implies metric annulus" theorem or sometimes the "volume cone implies metric cone" theorem.

In this paper, our concern is with the situation in which the inequality in (4.8) is *almost* an equality. We begin by noting the most direct consequence of this assumption.

Let $k=k(r)$ be regarded as a function on $A_{a,b}$.

PROPOSITION 4.9. *Let* (4.5), (4.6) *hold. Then*

$$\frac{\mathrm{Vol}(A_{a,b})}{\mathrm{Vol}(r^{-1}(a))}(1-\omega)\frac{\int_a^b f^{n-1}(r)\,dr}{f^{n-1}(a)} \tag{4.10}$$

implies that for $a\leq d\leq b$

$$\frac{\mathrm{Vol}(r^{-1}(a))}{f^{n-1}(a)} \geq \frac{\mathrm{Vol}(r^{-1}(d))}{f^{n-1}(d)} \geq \left[1-\omega\frac{\int_a^b f^{n-1}\,dr}{\int_d^b f^{n-1}\,dr}\right]\frac{\mathrm{Vol}(r^{-1}(a))}{f^{n-1}(a)} \tag{4.11}$$

Proof. This is a direct consequence of the Bishop-Gromov technique. $\square$

We will also need the following lemma which is the counterpart, in our situation, of the result of [LS]. Fix d, with $a < d < b$.

Set

$$\mathcal{G}(r) = \int_r^d f^{1-n}(u)\,du. \tag{4.12}$$

Then

$$\underline{\Delta}\mathcal{G} = -(1-n)f^{-n}f' - (n-1)\frac{f'}{f}f^{1-n} = 0. \tag{4.13}$$

(If $a = 0$ and $r \sim f(r)$ as $r \to 0$, then up to a constant, $\mathcal{G}$ is a multiple of the Green's function with singularity at $r = 0$). Thus, for $\mu > 1$, $a < r < d$,

$$\underline{\Delta}\mathcal{G}^\mu = \mu(\mu-1)\mathcal{G}^{\mu-2}(\mathcal{G}')^2 \geq c_2(a,b,\mu) > 0. \tag{4.14}$$

Note also that

$$-c_1(a,b,\mu) \leq (\mathcal{G}^\mu)' < 0. \tag{4.15}$$

Below, we choose μ so as to maximize the ratio $\frac{c_2(a,b,\mu)}{c_1(a,b,\mu)}$ and for this choice we just write $\frac{c_2}{c_1}$.

LEMMA 4.16. *Let* (4.5), (4.6) *hold. Then for Dirichlet bounded conditions on $A_{a,b}$, the smallest eigenvalue, λ_1, of the Laplacian, $-\Delta$, satisfies*

$$\lambda_1 \geq \left(\frac{c_2}{2c_1}\right)^2. \tag{4.17}$$

Proof. By Laplacian comparison, (4.14), (4.15) imply that if $\mathcal{G}^\mu$ is regarded as a function on $A_{a,d}$, then in the distribution sense,

$$\Delta(\mathcal{G}^\mu) \geq c_2. \tag{4.18}$$

Thus, if h is smooth and vanishes on $\partial A_{a,d}$,

$$\begin{aligned} c_2\int_{A_{a,d}} h^2 &\leq \int_{A_{a,b}} h^2\Delta(\mathcal{G}^\mu) \\ &\leq -\int_{A_{a,d}} \langle\nabla(h^2),\nabla(\mathcal{G}^\mu)\rangle \\ &\leq 2\int_{A_{a,d}} |h|\cdot|\nabla h|\cdot|\nabla\mathcal{G}^\mu| \\ &\leq 2c_1\left(\int_{A_{a,d}} h^2\right)^{\frac{1}{2}}\left(\int_{A_{a,d}} |\nabla h|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.19}$$

By squaring this inequality, we get (4.17). $\square$

Our construction of the function, $\underset{\sim}{\mathcal{F}}$ (with bounded gradient) whose Hessian is close in the integral sense to being a function times the metric, will of necessity, be slightly indirect. In this way, we avoid having to deal with the equation $\Delta\underset{\sim}{\mathcal{F}} = nH(\underset{\sim}{\mathcal{F}})$; compare (1.26), (1.38). However, for the key examples (cone, suspension, product) mentioned in Section 1, the function H is actually linear. So in these cases, it is possible to proceed by employing the above equation directly; compare Section 6.

We point out that the gradient bound on $\underset{\sim}{\mathcal{F}}$ mentioned in the previous paragraph, is *not* part of the hypothesis of Proposition 2.80. Rather, the gradient bound is used in showing that the specific function, $\underset{\sim}{\mathcal{F}}$, which we construct in this section, satisfies relation (2.4). Relation (2.4) *is* part of the hypothesis of Proposition 2.80.

Let $\mathcal{F}(r)$ be the function defined in (1.4), whose Hessian in the warped product case is a function times the metric; see (1.5). Since for $\mathcal{G}$ as in (4.12), we have $\mathcal{G}' > 0$, there exists F, such that

$$\mathcal{F} = F(\mathcal{G}). \tag{4.20}$$

Regard $\mathcal{G}$ as a function on $A_{a,b}$ and let $\underset{\sim}{\mathcal{G}}$ satisfy

$$\Delta\underset{\sim}{\mathcal{G}} = 0, \tag{4.21}$$

$$\underset{\sim}{\mathcal{G}}\,|\partial A_{a,b} = \mathcal{G}|\,\partial A_{a,b}. \tag{4.22}$$

Our primary interest is in the function

$$\underset{\sim}{\mathcal{F}} = F(\underset{\sim}{\mathcal{G}}). \tag{4.23}$$

Note that by (4.21), (4.23),

$$\nabla\underset{\sim}{\mathcal{F}} = F'(\underset{\sim}{\mathcal{G}})\nabla\underset{\sim}{\mathcal{G}}, \tag{4.24}$$

$$\Delta\underset{\sim}{\mathcal{F}} = F''(\underset{\sim}{\mathcal{G}})|\nabla\underset{\sim}{\mathcal{G}}|^2. \tag{4.25}$$

Also, the chain rule, together with (1.4), (4.12), gives

$$F'(\mathcal{G}) = f^n, \tag{4.26}$$

$$F''(\mathcal{G}) = nf^{2n-2}f'. \tag{4.27}$$

Thus, to control $\nabla\underset{\sim}{\mathcal{F}}$ and $\Delta\underset{\sim}{\mathcal{F}}$, it suffices to control $\underset{\sim}{\mathcal{G}}$ and $\nabla\underset{\sim}{\mathcal{G}}$.

LEMMA 4.28.

$$\inf_{\partial A_{a,d}} \mathcal{G} \le \underset{\sim}{\mathcal{G}} \le \sup_{\partial A_{a,d}} \mathcal{G}. \tag{4.29}$$

Proof. This is just the maximum principle. □

Put

$$K = \sup_{\partial A_{a,d}} \mathcal{G} - \inf_{\partial A_{a,d}} \mathcal{G}. \tag{4.30}$$

Since $\mathcal{G}' < 0$, from (4.7), (4.13),

$$\Delta \mathcal{G} \geq 0. \tag{4.31}$$

By Stokes' Theorem and a standard regularization argument, we have

$$\begin{aligned} \int_{A_{a,d}} \Delta \mathcal{G} &= \int_{\partial A_{a,d}} \mathcal{G}' \\ &= f^{1-n}(a)\mathrm{Vol}(r^{-1}(a)) - f^{1-n}(d)\mathrm{Vol}(r^{-1}(d)). \end{aligned} \tag{4.32}$$

From (4.31), (4.32) together with Proposition 4.9, we immediately get the following crucial relation (compare [Co3] Lemma 1.10).

PROPOSITION 4.33.

$$\frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} |\Delta \mathcal{G}| \leq \frac{\omega}{1-\omega} \left[\frac{1}{\int_a^d f^{n-1}\,dr} + \frac{1}{\int_d^b f^{n-1}\,dr} \right]. \tag{4.34}$$

Now we easily obtain:

PROPOSITION 4.35.

$$\frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} |\nabla(\mathcal{G} - \underset{\sim}{\mathcal{G}})|^2 \leq \Psi(\omega | n, f, a, b-d). \tag{4.36}$$

Proof. For K as in (4.30),

$$\begin{aligned} \frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} |\nabla(\mathcal{G} - \underset{\sim}{\mathcal{G}})|^2 &= \frac{1}{\mathrm{Vol}(A_{a,d})} \left| \int_{A_{a,d}} \Delta(\mathcal{G} - \underset{\sim}{\mathcal{G}})(\mathcal{G} - \underset{\sim}{\mathcal{G}}) \right| \\ &\leq \frac{K}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} |\Delta \mathcal{G}|. \end{aligned} \tag{4.37}$$

Then (4.36) follows from (4.34). □

PROPOSITION 4.38.

$$\frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} (\mathcal{G} - \underset{\sim}{\mathcal{G}})^2 \leq \Psi(\omega | n, f, a, b-d), \tag{4.39}$$

$$\frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} (\mathcal{F} - \underset{\sim}{\mathcal{F}})^2 \leq \Psi(\omega | n, f, a, b-d). \tag{4.40}$$

Proof. From (4.22), (4.37) together with Lemma 4.16, we get (4.39). Since by (4.20), (4.23),

$$|\mathcal{F} - \underset{\sim}{\mathcal{F}}| \leq |\mathcal{G} - \underset{\sim}{\mathcal{G}}| \sup_{A_{a,d}} F', \tag{4.41}$$

from (4.26) and (4.39) we also get (4.40). □

COROLLARY 4.42.

$$\frac{1}{\mathrm{Vol}(A_{a,d})} \int_{A_{a,d}} |\nabla \mathcal{F} - \nabla \underset{\sim}{\mathcal{F}}|^2 \leq \Psi(\omega|n,f,a,b-d). \tag{4.43}$$

Proof. Since

$$\nabla \mathcal{F} - \nabla \underset{\sim}{\mathcal{F}} = F'(\mathcal{G})(\nabla \mathcal{F} - \nabla \underset{\sim}{\mathcal{G}}) + [F'(\mathcal{G}) - F'(\underset{\sim}{\mathcal{G}})]\nabla \mathcal{G}, \tag{4.44}$$

relation (4.43) follows from (4.12), (4.26), (4.27), together with (4.36), (4.39). □

LEMMA 4.45. *Given a_1,b_1, with $a < a_1 < b_1 < d$, there exists $C = C(n,f,a,a_1,b,b_1)$, such that*

$$|\nabla \underset{\sim}{\mathcal{G}}| \leq CK \qquad (on\ A_{a_1,b_1}), \tag{4.46}$$

$$|\nabla \underset{\sim}{\mathcal{F}}| \leq CK \sup_{A_{a_1,b_1}} f^n \qquad (on\ A_{a_1,b_1}), \tag{4.47}$$

$$|\Delta \underset{\sim}{\mathcal{F}}| \leq (CK)^2 \sup_{A_{a_1,b_1}} nf^{2n-2}|f'|. \tag{4.48}$$

Proof. Given (4.6), (4.24)–(4.27) and (4.29), this is just the gradient estimate of [CgY]. □

Actually, we will have to shrink the annulus A_{a_1,b_1} several times. But by Proposition 4.9, with no loss of generality, we can assume that ω has been chosen such that for all i,

$$\mathrm{Vol}(A_{a_i,b_i}) \geq \frac{1}{2} \mathrm{Vol}(A_{a,b}). \tag{4.49}$$

Then we may replace $A_{a,d}$ by A_{a_i,b_i} in Propositions 4.33, 4.35, 4.38 and in Corollary 4.42.

Let the function $\mathcal{V}$ be as in (2.23) for the annulus $A_{a,b}$. Then for $a_1 < a_2 < b_2 < b_1$ as above, Proposition 4.38 and Lemma 4.45 give:

PROPOSITION 4.50. *On A_{a_2,b_2}*

$$|\mathcal{G} - \underset{\sim}{\mathcal{G}}| \leq \Psi(\omega|n,f,a,a_1,a_2,b,b_1,b_2,\mathcal{V}), \tag{4.51}$$

$$|\mathcal{F} - \underset{\sim}{\mathcal{F}}| \leq \Psi(\omega|n,f,a,a_1,a_2,b,b_1,b_2,\mathcal{V}). \tag{4.52}$$

Remark 4.53. Relations (4.29), (4.52), (4.43) correspond to (2.1), (2.2), (2.3) respectively. (To get (2.3) from (4.43) we use the Schwarz inequality.)

Remark 4.54. So far, it would have sufficed to assume that (4.6) holds only in the radial direction, provided we assumed in addition some definite lower bound on the Ricci curvature, in order to obtain Lemma 4.45 (the gradient estimate of [CgY]). However, in what follows, the stronger hypothesis, (4.6), is actually required.

In order to control $\mathrm{Hess}_{\underset{\sim}{\mathcal{F}}}$, we will use the Bochner formula,

$$\frac{1}{2}\Delta(|\nabla\underset{\sim}{\mathcal{F}}|^2) = |\,\mathrm{Hess}_{\underset{\sim}{\mathcal{F}}}|^2 + \mathrm{Ric}(\nabla\underset{\sim}{\mathcal{F}}, \nabla\underset{\sim}{\mathcal{F}}) + \langle\nabla\Delta\underset{\sim}{\mathcal{F}}, \nabla\underset{\sim}{\mathcal{F}}\rangle. \tag{4.55}$$

We rewrite (4.55) as

$$\begin{aligned} \frac{1}{2}\Delta(|\nabla\underset{\sim}{\mathcal{F}}|^2) - \langle\nabla\Delta\underset{\sim}{\mathcal{F}}, \nabla\underset{\sim}{\mathcal{F}}\rangle \ & - \frac{1}{n}(\Delta\underset{\sim}{\mathcal{F}})^2 + (n-1)\frac{f''}{f}|\nabla\mathcal{F}|^2 \\ & + (n-1)\frac{f''}{f}[|\nabla\underset{\sim}{\mathcal{F}}|^2 - |\nabla\mathcal{F}|^2] \\ = & \left|\mathrm{Hess}_{\underset{\sim}{\mathcal{F}}} - \frac{1}{n}\Delta\underset{\sim}{\mathcal{F}}g\right|^2 \\ & + [\mathrm{Ric}(\nabla\underset{\sim}{\mathcal{F}}, \nabla\underset{\sim}{\mathcal{F}}) + (n-1)\frac{f''}{f}|\nabla\underset{\sim}{\mathcal{F}}|^2], \end{aligned} \tag{4.56}$$

and note that both terms on the right-hand side of (4.56) are nonnegative.

We are going to multiply both sides of (4.56) by a cut-off function, integrate over A_{a_1,b_2} and apply integration by parts (Stokes' Theorem) to two of the terms on the resulting left-hand side. The integration by parts produces terms involving the gradient and Laplacian of the cut-off function. In order to be able to control the latter of these, we will use a special cut-off function defined in terms of $\underset{\sim}{\mathcal{G}}$; alternatively, we could use the cut-off function constructed in Theorem 6.33.

Let $a_2 < a_3 < b_3 < b_2$, and put $\beta = \frac{1}{6}(a_3 - a_2) = \frac{1}{6}(b_2 - b_3)$ (where we assume $a_3 - a_2 = b_2 - b_3$). Let $\phi(r)$ satisfy

$$\phi \geq 0, \tag{4.57}$$

$$\phi|[a_3 - 2\beta, b_3 + 2\beta] \equiv 1, \tag{4.58}$$

$$\phi|[a_2, a_2 + 2\beta] \cup [b_2 - 2\beta, b_2] \equiv 0. \tag{4.59}$$

Write

$$\phi = \Phi(\underset{\sim}{\mathcal{G}}), \tag{4.60}$$

and put

$$\underset{\sim}{\phi} = \Phi(\underset{\sim}{\mathcal{G}}). \tag{4.61}$$

By letting Ψ in (4.51) depend in addition on a_3, b_3, we can assume that ω is so small that

$$\underset{\sim}{\phi} |\, A_{a_3-\beta, b_3+\beta} \equiv 1, \tag{4.62}$$

$$\underset{\sim}{\phi} | \{A_{a_2,b_2} \setminus A_{a_2+\beta, b_2-\beta}\} \equiv 0. \tag{4.63}$$

Similarly, for the function Ψ in (4.40), (4.43), from now on we understand

$$\Psi = \Psi(\omega | n, f, a, a_1, a_2, a_3, b, b_1, b_2, b_3, \mathcal{V}). \tag{4.64}$$

As in Lemma 4.45 we have from (4.61),

LEMMA 4.65.

$$|\nabla \underset{\sim}{\phi}| \;\leq\; C_1 K, \tag{4.66}$$

$$|\Delta \underset{\sim}{\phi}| \;\leq\; C_2 K^2, \tag{4.67}$$

where $C_i = C_i(n,f,a,a_1,a_2,a_3,b,b_1,b_2,b_3)$, $i = 1{,}2$.

Now, by multiplying both sides of (4.56) by $\underset{\sim}{\phi}$ and applying Stokes' Theorem, we find that

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}(A_{a_2,b_2})} \int_{A_{a_2,b_2}} \left| \mathrm{Hess}_{\underset{\sim}{\mathcal{F}}} - \frac{1}{n} \Delta \underset{\sim}{\mathcal{F}} g \right|^2 \\
&\qquad + \left[\mathrm{Ric}\,(\nabla \underset{\sim}{\mathcal{F}}, \nabla \underset{\sim}{\mathcal{F}}) + (n-1)\frac{f''}{f} |\nabla \underset{\sim}{\mathcal{F}}|^2 \right] \\
&= \frac{1}{\mathrm{Vol}(A_{a_2,b_2})} \int_{A_{a_2,b_2}} \left\{ \frac{1}{2} \Delta \underset{\sim}{\phi} |\nabla \underset{\sim}{\mathcal{F}}|^2 + \Delta \underset{\sim}{\mathcal{F}} (\nabla \underset{\sim}{\phi}, \nabla \underset{\sim}{\mathcal{F}}) + \underset{\sim}{\phi} (\Delta \mathcal{F})^2 \right. \\
&\qquad\qquad - \frac{1}{n} \underset{\sim}{\phi} (\Delta \mathcal{F})^2 + (n-1) \underset{\sim}{\phi} \frac{f''}{f} |\nabla \mathcal{F}|^2 \\
&\qquad\qquad \left. + (n-1) \underset{\sim}{\phi} \frac{f''}{f} ([|\nabla \mathcal{F}|^2 - |\nabla \underset{\sim}{F}|^2] \right\}.
\end{aligned} \tag{4.68}$$

We claim that up to a negligible error, we can replace all functions on the right-hand side of (4.68) which are underlined with a tilde, by the corresponding functions of the variable r. Explicitly,

$$\nabla \underset{\sim}{\mathcal{F}} \to F'(\mathcal{G}) \nabla \mathcal{F}, \tag{4.69}$$

$$\Delta \underset{\sim}{\mathcal{F}} \to F''(\mathcal{G})|\nabla \mathcal{G}|^2, \tag{4.70}$$

$$\nabla \underset{\sim}{\phi} \to \Phi'(\mathcal{G})\nabla \mathcal{G}, \tag{4.71}$$

$$\Delta \underset{\sim}{\phi} \to \Phi''(\mathcal{G})|\nabla \mathcal{G}|^2, \tag{4.72}$$

where (a.e.)

$$\nabla \mathcal{G} = f^{1-n}(r)\frac{\partial}{\partial r}. \tag{4.73}$$

To justify the above substitutions we use the following standard elementary lemma.

LEMMA 4.74. *Let $k_1, \ldots k_N$, $\widehat{k}_1 \ldots \widehat{k}_N$ be functions on a measure space, U, such that for all i,*

$$\sup_U |k_i| + |\widehat{k}_i| \le C, \tag{4.75}$$

$$\int_U |k_i - \widehat{k}_i| \le \varepsilon. \tag{4.76}$$

Then

$$\int_U |k_1 \cdots k_N - \widehat{k}_1 \cdots \widehat{k}_N| \le NC^{N-1}\varepsilon. \tag{4.77}$$

Proof. If we write

$$\begin{aligned} k_1 \cdots k_N - \widehat{k}_1 \cdots \widehat{k}_N \quad &= \quad (k_1 - \widehat{k}_1)k_2 \cdots k_N \\ &\quad + \sum_{i=1}^{N-2} \widehat{k}_1 \cdots \widehat{k}_i(k_{i+1} - \widehat{k}_{i+1})k_{i+2} \cdots k_N \\ &\quad + \widehat{k}_1 \cdots \widehat{k}_{N-1}(k_n - \widehat{k}_N), \end{aligned} \tag{4.78}$$

the conclusion is obvious. □

Let $T(r)$ be the function of r resulting from the replacements, (4.69)–(4.72), everywhere in the integrand on the right-hand side of (4.68). In view of Lemma 4.74, together with (4.43), (4.52) and the pointwise bounds which follow from (4.46), (4.51), the right-hand side of (4.68) can be replaced by

$$T(r) + \Psi, \tag{4.79}$$

where Ψ is as in (4.64).

We claim that

$$\int_{a_2}^{b_2} T(r) f^{n-1}(r)\, dr = 0. \tag{4.80}$$

To see this note that when we apply our considerations to an annulus contained in some $(a,b)\times_f N^{n-1}$ (N^{n-1} compact smooth) then both sides of (4.56) vanish. Hence the right-hand side of (4.68) (which is obtained from (4.56) by integration by parts) vanishes in this case as well. Since for an annulus contained in $(a,b)\times_f N^{n-1}$, the right-hand side of (4.68) is a constant multiple of the left-hand side of (4.80), the latter also vanishes.

From (4.49), (4.79), (4.80), with (4.11):

PROPOSITION 4.81. *Let Ψ be as in (4.64). Then*

$$\frac{1}{\operatorname{Vol}(A_{a_3,b_3})}\int_{A_{a_3,b_3}}\left|\operatorname{Hess}_{\underset{\sim}{\mathcal{F}}}-\frac{1}{n}\Delta\underset{\sim}{\mathcal{F}}g\right|^2 + \left[\operatorname{Ric}(\nabla\underset{\sim}{\mathcal{F}},\nabla\underset{\sim}{\mathcal{F}})+(n-1)\frac{f''}{f}|\nabla\underset{\sim}{\mathcal{F}}|^2\right]\leq\Psi. \tag{4.82}$$

For the sake of consistency with (2.4), we note that (4.82), together with the Schwarz inequality, gives:

COROLLARY 4.83. *Let Ψ be as in (4.64). Then*

$$\frac{1}{\operatorname{Vol}(A_{a_3,b_3})}\int_{A_{a_3,b_3}}\left|\operatorname{Hess}_{\underset{\sim}{\mathcal{F}}}-\frac{1}{n}\Delta\underset{\sim}{\mathcal{F}}g\right|\leq\Psi. \tag{4.84}$$

Now we are ready for our main result in the almost maximal volume case. For the case $\omega=0$, it is just the "volume annulus implies metric annulus" theorem. Let $\mathcal{V}$ be as in (2.23).

THEOREM 4.85. *Let $0<\alpha'<\alpha$, $\alpha-\alpha'>\xi>0$. Assume that*

$$\operatorname{Ric}_{M^n} \;\geq\; -(n-1)\frac{f''(a)}{f(a)} \qquad (\text{on } r^{-1}(a)). \tag{4.86}$$

$$m \;\leq\; (n-1)\frac{f'(a)}{f(a)} \qquad (\text{on } r^{-1}(a)), \tag{4.87}$$

$$\frac{\operatorname{Vol}(A_{a,b})}{\operatorname{Vol}(r^{-1}(a))} \;\geq\; (1-\omega)\frac{\int_a^b f^{n-1}(r)\,dr}{f^{n-1}(a)}. \tag{4.88}$$

Then there exists a length space X, with at most $\#\,(a,b,f,\mathcal{V})$ components X_i, satisfying

$$\operatorname{diam}(X_i)\leq D(a,b,f,\mathcal{V}), \tag{4.89}$$

such that for the metrics $d^{\alpha',\alpha}$, $\underline{d}^{\alpha',\alpha}$,

$$d_{GH}(A_{a+\alpha,b-\alpha},(c+\alpha,d-\alpha)\times_f X)\leq\Psi(\omega\mid n,f,a,b,\alpha',\xi,\mathcal{V}). \tag{4.90}$$

Proof. By Remark 4.54, together with (4.84), we have (2.1)–(2.4), where for δ in (2.2)–(2.4), we have $\delta \leq \Psi$. Thus, we can apply Proposition 2.80. Then, by using Theorem 3.8, together with the Propositions 3.32, 3.33 we complete the proof. □

As previously mentioned, for our geometric applications, Proposition 2.24 will suffice to control the function $\mathcal{V}$. At this point we will give only the most basic application of Theorem 4.85 to the description of the local structure of manifolds with Ricci curvature bounded below. For further consequences, see Section 7 and [CCo2], [CCo3].

Recall that in Theorem 4.85 it was necessary to use the metrics $d^{\alpha',\alpha}, \underline{d}^{\alpha',\alpha}$. Thus, in Theorem 4.91 below, we will understand that the metric on the annulus, $A_{\underline{r},\Omega\underline{r}}$ is induced by restricting the metric on a slightly larger annulus, say $A_{(1-\eta)r,(1+\eta)r\Omega}$. Similarly, the metric on $(\underline{r},\Omega\underline{r}) \times_r X$ is induced from $((1-\eta)\underline{r},(1+\eta)\Omega\underline{r}) \times_r X$.

THEOREM 4.91. *Let M^n be complete, with*

$$\mathrm{Ric}_{M^n} \geq -(n-1). \tag{4.92}$$

Assume that for some $p \in M^n$ and $v > 0$,

$$\mathrm{Vol}(B_D(p)) \geq v. \tag{4.93}$$

Given $\varepsilon,\eta > 0, \Omega > 1$, there exists $\# = \#(\varepsilon,\eta,\Omega,n,v,D)$, with the following property. If $\{r_j\}$ satisfies $0 < \Omega r_j \leq r_{j+1}, j = 1,\ldots N-1$, and $N > \#$, then for some j, there exists a length space X_j, with diam$(X_j) \leq \pi$, such that

$$d_{GH}(A_{r_j,\Omega r_j},(r_j,\Omega r_j) \times_r X_j) < \varepsilon r_j. \tag{4.94}$$

Proof. The assertion that X_j can be taken to be a length space with $\mathrm{diam}(X_j) \leq \pi$, follows as in the proof of Theorem 5.12 below. Thus, we will omit the argument here. □

Given a sequence, $\{r_j\}$, as above, put $s_i = r_{2i+1}$. Then $\Omega^2 s_i \leq s_{i+1}, 1 \leq i \leq N'$, where $N' = [\frac{1}{2}(N-1)]$. With no loss of generality, we can assume $(1-\eta)(1-\eta) \leq \frac{1}{2}\Omega^2$.

We claim that for fixed $\omega > 0$, there exists $\#'(\omega,\Omega,n,v,D)$ such that if for all i,

$$\frac{\mathrm{Vol}(A_{s_i,\Omega^2 s_i})}{\mathrm{Vol}(\partial B_{s_i}(p))} \leq e^{-\omega}\frac{\int_{s_i}^{\Omega^2 s_i}\sinh^{n-1} u\, du}{\sinh^{n-1} s_i}, \tag{4.95}$$

then $N' \leq \#'(\omega,\Omega,v,D)$. Note that given this assertion, the proposition follows from Theorem 4.85 and the fact that $\lim_{s\to 0}\sinh s/s = 1$.

To see the existence of N', note that by the Bishop-Gromov inequality,

$$\frac{\operatorname{Vol}(\partial B_{s_i}(p))}{\sinh^{n-1} s_i} \geq \frac{\operatorname{Vol}(A_{s_i,\Omega^2 s_i})}{\int_{s_i}^{\Omega^2 s_i} \sinh^{n-1} u \, du} \geq \frac{\operatorname{Vol}(\partial B_{\Omega^2_{s_i}}(p))}{\sinh^{n-1} \Omega^2 s_i}, \tag{4.96}$$

and similarly, with (4.95),

$$\frac{\operatorname{Vol}(\partial B_{\Omega^2 s_i}(p))}{\sinh^{n-1} \Omega^2 s_i} \geq \frac{\operatorname{Vol}(\partial B_{s_{i+1}}(p))}{\sinh^{n-1} s_{i+1}}. \tag{4.97}$$

From (4.95)–(4.97) together with (4.93), the existence of $\#'$ as above easily follows.

Remark 4.98. Examples of Perelman, [P3], show that the annuli in Theorem 4.85 (or Theorem 4.91) need not have the topology of a product no matter how small the constant ω is chosen.

Remark 4.99. An immediate consequence of Theorem 4.91 is the assertion that for spaces which are Gromov-Hausdorff limits of manifolds whose Ricci curvatures and volumes are uniformly bounded from below, any tangent cone is a metric cone, $C(X)$, with $\operatorname{diam}(X) \leq \pi$. See [CCo2] and compare Theorems 5.12, 5.14.

Remark 4.100. In [CCo2], we will show that Theorem 4.85 leads in the noncollapsing case to a generalization to Gromov-Hausdorff limits of the "volume cone implies metric cone" theorem; compare also [CCo3].

Remark 4.101. A key reason why Theorem 4.85 holds is that it can be supplemented by an L_2-Toponogov Theorem in the spirit of those proved in [Co1]–[Co3]. Since the details are similar to those given in the context of the splitting theorem (see Theorem 6.130) we will not provide further discussion here.

Remark 4.102. In view of Lemma 1.8 of [Co3] and Lemma 1.4 of [Co1] or Lemma 1.12 of [Co3] or Propositions 4.33 and 4.81 of this section, it is natural to ask if an estimate on the L_2–norm of $|\Delta r|$, or equivalently of $|\operatorname{Hess}(r)|$ can be obtained. However, in the presence of conjugate points of order 1 (for example on complex projective space with its canonical metric) the function $|\Delta r|$ will not be square integrable (even if one omits a neighborhood of the origin). Therefore, as in [Co1]–[Co3], it is crucial to approximate the distance function by a function with nicer properties.

5. Finite diameter

In this section, we consider a set-up analogous to the one in Section 4, but with volume replaced by diameter. It turns out, however, that in this case, the volumes of annuli are almost maximal in the sense of Section 4. Thus, our results on the almost warped product structure follow from those of that section.

Let $(0,b), f$ be as in previous sections (here, we make the normalization, $a=0$). Assume in addition, that

$$f(r) \sim r - \frac{K_0}{3}r^3 \qquad (r \to 0) \tag{5.1}$$

$$f(r) \sim (b-r) - \frac{K_b}{3}(b-r)^3 \qquad (r \to b). \tag{5.2}$$

Let M^n be complete, connected and suppose that for some $p \in M^n$,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\frac{f''(r)}{f(r)}. \tag{5.3}$$

Then by the second variation formula,

$$\sup_x \overline{p,x} \leq b. \tag{5.4}$$

Put $\sup_x \overline{p,x} = d_p$.

By Laplacian comparison,

$$m \geq (n-1)\frac{f'(c)}{f(c)} \qquad (\text{on } r^{-1}(c)), \tag{5.5}$$

(compare (4.5)).

Assume in addition that for some $q \in M^n$, with

$$\overline{p,q} = b - \mu, \tag{5.6}$$

$$\mathrm{Ric}_{M^n} \geq -(n-1)\frac{f''(b-r)}{f(b-r)}. \tag{5.7}$$

Then, by relative volume comparison,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(M^n)} \geq \frac{\int_0^r f^{n-1}(u)\,du}{\int_0^b f^{n-1}(u)\,du}, \tag{5.8}$$

$$\frac{\mathrm{Vol}(B_s(q))}{\mathrm{Vol}(M^n)} \geq \frac{\int_{b-s}^b f^{n-1}(u)\,du}{\int_0^b f^{n-1}(u)\,du}. \tag{5.9}$$

Note that (5.1), (5.2), (5.4) imply that for some $\Lambda = \Lambda(f)$,

$$\mathrm{Ric}_{M^n} \geq (n-1)\Lambda. \tag{5.10}$$

Then by a standard argument based on the relative volume comparison theorem, given $\eta > 0$, for all x, we have

$$\overline{p,x} + \overline{q,x} - \overline{p,q} < \eta, \tag{5.11}$$

provided $\mu \leq \tau(\eta, n, f))$; compare [E], [GP].

THEOREM 5.12. *Let* (5.1)–(5.3) *and* (5.7) *hold. Then for all* $\varepsilon > 0$, *there exists* $\xi(\varepsilon,n,f) > 0$, *such that if in* (5.6) $\mu < \xi$, *then for some length space*, X, *with* $\operatorname{diam}(X) \le \pi$,

$$d_{GH}(M^n,(0,b) \times_f X) < \varepsilon. \tag{5.13}$$

Proof. We will in effect reduce our considerations to those of Section 4. First of all note that by (5.8), (5.9), and (5.11) it follows that given ω, (4.34) holds for any c,d with $a < c < d < b$, provided $\eta = \eta(\omega,n,f)$ is sufficiently small. Also, by (5.4), (5.10), the function $\mathcal{V}(u)$ of Proposition 2.24 has a definite lower bound. As a consequence, we can apply Theorem 3.6 to an annulus $A_{c,d}$. The hypothesis (3.8) of Theorem 3.6 is an obvious consequence of (5.11).

The only remaining point is to show that the cross-section, X, is connected, with $\operatorname{diam}(X) \le \pi$. If not, arguing by contradiction, we obtain from Gromov's compactness theorem, a space $(0,b) \times_f X$, which is the Gromov-Hausdorff limit of connected manifolds, M_i^n, with $\operatorname{Ric}_{M_i^n} \ge (n-1)\Lambda$. Moreover, either X has at least two components or $\operatorname{diam}(X) > \pi$. Since $f(r) \sim r$ at $r = 0$, by applying a second sequence of rescalings, we obtain the metric cone, $C(X)$, as a pointed Gromov-Hausdorff limit, of manifolds, whose Ricci curvatures are bounded below by $-\varepsilon_i \to 0$ (where $\varepsilon_i > 0$). Now by the Abresch-Gromoll theorem, [AG] (which obviously passes to Gromov-Hausdorff limit spaces as above) it follows immediately that either X is connected with $\operatorname{diam}(X) \le \pi$, or X consists, precisely, of two points. In all other cases, $C(X)$ contains a line through the vertex, but does not satisfy the Abresch-Gromoll inequality ($E \equiv 0$) on the excess function, E, associated to this line.

If X consists of two points, $(0,b) \times_f X$ is a circle, S^1_{2b}, of circumference $2b$. For $d_{GH}(M_i^n,S^1_{2b}) < \frac{b}{4}$, the natural map, $\pi_1(M_i^n) \to \pi_1(S^1_{2b}) = \mathbb{Z}$ is surjective. This is a particular instance of a well known fact concerning Gromov-Hausdorff limits. Thus, for such i the universal covering spaces, $\tilde{M}_i^n$, are noncompact. But for the pull-back metric, $\tilde{M}_i^n$ clearly satisfies the diameter bound (5.4) (with respect to some point $\tilde{p}_i$). This is a contradiction. □

As an immediate consequence of Theorem 5.12, we obtain:

THEOREM 5.14. *Let* Y *be the Gromov-Hausdorff limit of complete riemannian manifolds* M_i^n, *satisfying* (5.3), (5.7), (5.10). *If for points* $p_i,q_i \in M_i$, $p_i \to p$, $q_i \to q$ *where* $p,q \in Y$ *and*

$$\overline{p,q} = b, \tag{5.15}$$

then for some length space X, *with* $\operatorname{diam}(X) \le \pi$, *the space* Y *is* (*the completion of*) *a warped product*,

$$Y = \overline{(0,b) \times_f X}. \tag{5.16}$$

Probably the most interesting case of Theorems 5.12 and 5.14 is the one of *metric suspensions*, in which $f = \sin x$, $b = \pi$; compare [Cg], [Co1], [Co2]. In this case, examples of Anderson show that X need not have the topology of the sphere; see [A1].

Remark 5.17. Note that since in this section, the space which we consider is a closed manifold, M^n, the counterpart of (4.82) can in fact be obtained directly from Stokes' theorem, without recourse to the special cut-off function, $\underset{\sim}{\phi}$ of (4.61). In Section 6, however, cut-off functions are actually required and a general construction is given in Theorem 6.33.

Remark 5.18. Theorem 5.14 can be supplemented by an L_2-Toponogov Theorem; compare Remark 4.101, Theorem 6.80 and [Co1]–[Co3].

6. Infinite diameter; the splitting theorem

In this section, we consider the most important analog for infinite diameter, of the finite diameter case considered in Section 5. This leads to the generalization to Gromov-Hausdorff limits of the splitting theorem of [CG], [T]; see Theorems 6.62, 6.64.

Since the model case in our situation is an isometric product, $(-\infty,\infty)\times_1 N$, we have $\mathcal{F} = r$, the Hessian of which vanishes identically. For this reason, initially we will be able to proceed somewhat more directly than in Sections 4 and 5.

However, a serious point arises constructing the analog of the cut-off function, $\underset{\sim}{\phi}$, of (4.61). This reflects the fact that in the present context *the level surfaces of the function, $\mathcal{F} = r$, need not be compact.* Thus, we are forced to work on a ball, $B_R(p)$. The required cut-off function is constructed in Theorem 6.33.

As an application of the generalized splitting theorem we will state a result on the local structure of spaces with Ricci curvature bounded below (see Theorem 6.68). This theorem has significant implications for the structure of Gromov-Hausdorff limits. These are further elaborated in [CCo2], [CCo3]. For the proof of Theorem 6.68, see [CCo2].

Let M^n be complete. Fix $q_+,q_- \in M^n$ and put

$$E(x) = \overline{x,q_+} + \overline{x,q_-} - \overline{q_+,q_-}. \tag{6.1}$$

The following slight generalization of the Abresch-Gromoll inequality will only be stated in qualitative form, since this is all that we require; compare [AG], [Che], [CCoY].

PROPOSITION 6.2. *Given $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon,n) > 0$, $L = L(\varepsilon,n) < \infty$, such that if for $p \in M^n$ and $R > 0$,*

$$\min(\overline{p,q_+}, \overline{p,q_-}) \geq LR, \tag{6.3}$$

$$E(p) \leq \tau R, \tag{6.4}$$

$$\mathrm{Ric}_{M^n} \geq -(n-1)\tau R^{-2} \qquad (\text{on } B_{2LR}(p)), \tag{6.5}$$

then

$$\sup_{B_R(p)} E \leq \varepsilon R. \tag{6.6}$$

Proof. Since the statement is scale invariant it suffices to assume $R = 1$.

The only difference between our hypotheses and those of [AG] is that rather than assuming $E(p) = 0$, we have assumed (6.4).

As in [AG], given $x \in B_1(p)$, we can construct for some small $\psi > 0$, a function G on $B_{1+\psi}(x) \setminus x$, such that for $s(y) = \overline{y,x}$:

$$G = G(s), \tag{6.7}$$

$$\frac{\partial G}{\partial s} < 0, \tag{6.8}$$

$$\Delta(E - G) < 0 \qquad (\text{in the barrier sense}). \tag{6.9}$$

By (6.9), $E - G$ has no interior minimum on $B_{1+\psi}(x) \setminus x$. By (6.8), $G(1) - G(1+\psi) > 0$. For any annulus $\overline{B_{1+\psi}(x) \setminus B_\eta(x)}$, the function, $E - G$, cannot take its minimum on $\partial B_{1+\psi}(x) \subset \partial(\overline{B_{1+\psi}(x)} \setminus B_\eta(x))$ if

$$\tau R < G(1) - G(1+\psi). \tag{6.10}$$

Thus, the proof can be completed as in [AG]. The only difference is that we cannot let $\psi \to 0$. But this is not required for (6.6) nor is it required for the estimate of [AG] if some sharpness is sacrificed. □

Remark 6.11. The reason for insisting on (6.4) rather than assuming that p lies on a minimal segment from q_+ to q_-, is that the latter assumption is not general enough for our eventual application to the splitting theorem for Gromov-Hausdorff limits; see Theorems 6.62 and 6.64.

As explained above, of necessity, we continue to work on a ball, $B_R(p)$, rather than on an annulus, as in Sections 4 and 5. We keep the normalization, $R = 1$, and continue to assume that (6.3)–(6.5) are in force, until we state otherwise. We also assume say $\tau \leq 1$, so that the Ricci curvature has a definite lower bound.

Put

$$b_\pm(x) = \overline{x,q_\pm} - \overline{p,q_\pm} \tag{6.12}$$

and let $\mathbf{b}$ denote the function on $B_1(p)$ such that

$$\Delta \mathbf{b} = 0, \tag{6.13}$$

$$\mathbf{b}|\partial B_1(p) = b_+|\partial B_1(p). \tag{6.14}$$

Let the function Ψ be as in (3.5).

LEMMA 6.15.

$$|\mathbf{b} - b_+| \leq \Psi(\tau, L^{-1}|n) \qquad (\text{on } B_1(p)). \tag{6.16}$$

Proof. As in [LS], [AG] (and (6.7)–(6.9)) there exists a smooth function, G, on $B_1(p)$ such that

$$\Delta G > 1, \tag{6.17}$$

$$0 < G \leq c(n)\Delta G. \tag{6.18}$$

Since

$$\Delta b_\pm \leq \Psi(\tau, L^{-1}|n), \tag{6.19}$$

we have

$$\Delta(b_+ - \Psi G) < 0, \tag{6.20}$$

$$\Delta(-b_- + \Psi G) > 0. \tag{6.21}$$

Then the lemma follows from the maximum principle, together with Proposition 6.2. □

LEMMA 6.22.

$$\frac{1}{\mathrm{Vol}(B_1(p))}\int_{B_1(p)} |\Delta b_+| \leq c(n). \tag{6.23}$$

Proof. By Stokes' Theorem and the fact that $|\nabla b_+| \equiv 1$ (where defined),

$$\begin{aligned} \left|\int_{B_1(p)} \Delta b_+\right| &= \left|\int_{\partial B_1(p)} *db_+\right| \\ &\leq \mathrm{Vol}(\partial B_1(p)) \leq c(n)\,\mathrm{Vol}(B_1(p)). \end{aligned} \tag{6.24}$$

Then (6.23) follows from (6.24) together with Laplacian comparison. □

LEMMA 6.25.

$$\frac{1}{\mathrm{Vol}(B_1(p))}\int_{B_1(p)} |\nabla(\mathbf{b} - b_+)|^2 \leq \Psi(\tau, L^{-1}|n). \tag{6.26}$$

Proof. By Stokes' Theorem,

$$\int_{B_1(p)} |\nabla(\mathbf{b} - b_+)|^2 = -\int_{B_1(p)} \Delta(\mathbf{b} - b_+)(\mathbf{b} - b_+). \tag{6.27}$$

Now the claim follows from (6.16) and (6.23). □

By (6.14) and (6.16), the gradient estimate of [CgY] gives

$$\sup_{B_{\frac{1}{2}}(p)} |\nabla \mathbf{b}| \leq c(n). \tag{6.28}$$

In light of the arguments given in previous sections, it is clear that to prove a generalized splitting theorem (see Theorems 6.62 and 6.64) it remains to show that the L_2-norm of $\mathrm{Hess}_{\mathbf{b}}$ (normalized by the volume) can be bounded by $\Psi(\tau, L^{-1}|n)$.

Since $\mathbf{b}$ is harmonic, Bochner's formula gives

$$\frac{1}{2}\Delta|\nabla \mathbf{b}|^2 = |\,\mathrm{Hess}_{\mathbf{b}}\,|^2 + \mathrm{Ric}(\nabla \mathbf{b}, \nabla \mathbf{b}), \tag{6.29}$$

or

$$\frac{1}{2}\Delta|\nabla \mathbf{b}|^2 + (n-1)\tau|\nabla \mathbf{b}|^2 = |\,\mathrm{Hess}_{\mathbf{b}}\,|^2 + [\mathrm{Ric}(\nabla \mathbf{b}, \nabla \mathbf{b}) + (n-1)\tau|\nabla \mathbf{b}|^2], \tag{6.30}$$

the right-hand side of which is nonnegative; see (6.5).

Let ϕ: $B_1(p) \to [0,1]$ be such that

$$\phi \mid B_{\frac{1}{2}}(p) \equiv 1, \qquad \mathrm{supp}\,\phi \subset B_1(p). \tag{6.31}$$

If we multiply both sides of (6.30) by ϕ, we get

$$\begin{aligned} &\frac{1}{2}\int_{B_1(p)} \phi(\Delta|\nabla \mathbf{b}|^2 + (n-1)\tau|\nabla \mathbf{b}|^2) \\ &\qquad = \frac{1}{2}\int_{B_1(p)} \Delta\phi(|\nabla \mathbf{b}|^2 - 1) + (n-1)\phi[\mathrm{Ric}(\nabla \mathbf{b}, \nabla \mathbf{b}) + \tau|\nabla \mathbf{b}|^2]. \end{aligned} \tag{6.32}$$

By (6.30), the integrand on the left-hand side of (6.32) is nonnegative. Thus, by (6.32), Lemma 6.25, and (6.31), to estimate the integral over $B_{\frac{1}{2}}(p)$ of the quantity in (6.30), it suffices to know that ϕ can be chosen such that $|\Delta\phi|$ has a definite bound, $c(n)$, on its pointwise norm; compare [SY].

THEOREM 6.33. *Given $R > 0$, there exists $c(n,R)$ such that if* $\mathrm{Ric}_{M^n} \geq -(n-1)$ *and $p \in M^n$, then there exists ϕ: $M^n \to [0,1]$, such that*

$$\phi|B_{\frac{1}{2}R}(p) \equiv 1, \tag{6.34}$$

$$\mathrm{supp}\,\phi \subset B_R(p), \tag{6.35}$$

$$|\nabla\phi| \leq c(n,R), \tag{6.36}$$

$$|\Delta\phi| \leq c(n,R). \tag{6.37}$$

Proof. By scaling, we can assume $R \geq 2$.

Let $\delta = \delta(n,R) > 0$ be such that there exists $\mathcal{G}$: $(0,\infty) \to (0,\infty)$ (singular at $r = 0$) such that

$$\mathcal{G} \downarrow \quad (\text{on } (0,R)), \tag{6.38}$$

$$\mathcal{G}(1) = 1, \tag{6.39}$$

$$\mathcal{G}(R) = 0, \tag{6.40}$$

$$\mathcal{G}'' + (n-1)\coth r \cdot \mathcal{G}' = \delta. \tag{6.41}$$

Essentially, $\mathcal{G}$ is the function of Proposition 6.2.

If we set $r(x) = \overline{x,p}$, then by Laplacian comparison, it follows that

$$\Delta \mathcal{G}(r) \geq \delta. \tag{6.42}$$

Let the function, k: $\overline{B_R(p)} \setminus B_1(p) \to \mathbb{R}$, satisfy

$$k|\partial B_1(p) \equiv 1, \tag{6.43}$$

$$k|\partial B_R(p) \equiv 0, \tag{6.44}$$

$$\Delta k = \delta. \tag{6.45}$$

By applying the minimum principle to $k - \mathcal{G}$, we get

$$k \geq \mathcal{G} \qquad (\text{on } \overline{B_R(p)} \setminus B_1(p)). \tag{6.46}$$

Let $\mathcal{K}$: $(0,\infty) \to (0,\infty)$ satisfy

$$\mathcal{K} \uparrow, \tag{6.47}$$

$$\mathcal{K}(0) = 0, \tag{6.48}$$

$$\mathcal{K}'' + (n-1)\coth r \cdot \mathcal{K}' = 1. \tag{6.49}$$

Put $s(y) = \overline{y,x}$, where $x \in B_R(p)$. By Laplacian comparison, it follows that

$$\Delta \mathcal{K}(s) \leq 1, \tag{6.50}$$

and so,

$$\Delta(k - \delta\mathcal{K}(s)) \geq 0. \tag{6.51}$$

By applying the maximum principle to $k - \delta\mathcal{K}$, we get

$$1 - \delta\mathcal{K}(r(x) - 1) \geq k(x). \tag{6.52}$$

Choose $\eta = \eta(n,R) > 0$ such that

$$\mathcal{G}(1+\eta) > 1 - \delta\mathcal{K}((1-\eta)R). \tag{6.53}$$

Let ψ: [0,1] $\to$ [0,1] satisfy

$$\psi|[\mathcal{G}(1+\eta),1] \equiv 1, \tag{6.54}$$

$$\psi|[0,1-\delta\mathcal{K}(1-\eta)R] \equiv 0. \tag{6.55}$$

Then, $\phi = \psi \circ k$ satisfies

$$\phi|[1,1+\eta] \equiv 1, \tag{6.56}$$

$$\phi|[(1-\eta)R,R] \equiv 0, \tag{6.57}$$

$$\nabla\phi = \psi'\nabla k, \tag{6.58}$$

$$\Delta\phi = \psi''|\nabla k|^2 + \psi'\delta. \tag{6.59}$$

Thus, $\operatorname{supp}\nabla\phi, \operatorname{supp}\Delta\phi \subset \overline{B_{R(1-\eta)}(p)} \setminus B_{1+\eta}(p)$. By the Cheng-Yau gradient estimate, the theorem follows. □

From Theorem 6.33 and (6.32) we get:

PROPOSITION 6.60.

$$\frac{1}{\operatorname{Vol}(B_1(p))}\int_{B_1(p)} |\operatorname{Hess}_{\mathbf{b}}|^2 \le \Psi(\tau,L^{-1}|n). \tag{6.61}$$

Let $B_R(w) \subset W^n$. Assume that $B_r(w)$ is compact for $r < R$.

From Theorem 6.33, Proposition 6.60, the arguments preceding it and those of previous sections, we get the main results of this section.

THEOREM 6.62. *Let* (6.3)–(6.5) *hold. Then there exists* $X,\underline{p} \in \mathbb{R}\times_1 X$, *such that*

$$d_{GH}(B_R(0),B_R(\underline{p})) \le \varepsilon R. \tag{6.63}$$

In view of Gromov's compactness theorem, Theorem 6.62 is equivalent to the following result which was conjectured in [FY]. Let $B_{R_i}(p_i)$ be a sequence of balls, $B_{R_i}(p_i) \subset M_i^n$, such that $B_r(p_i)$ has compact closure, for all $r < R_i$. Assume $R_i \to \infty$.

THEOREM 6.64. *Let* (Y,y) *be such that in the pointed Gromov-Hausdorff sense,*

$$d_{GH}((B_{R_i}(p_i),p_i),(Y,y)) \to 0 \qquad (R_i \to \infty). \tag{6.65}$$

If for $\varepsilon_i > 0$,

$$\operatorname{Ric}_{B_{R_i}(p_i)} \ge -(n-1)\varepsilon_i \qquad (\varepsilon_i \to 0), \tag{6.66}$$

and Y *contains a line, then* Y *splits, isometrically,*

$$Y = \mathbb{R}\times_1 X. \tag{6.67}$$

We now state an application of Theorem 6.62 to the local structure of spaces, M^n, with $\mathrm{Ric}_{M^n} \geq -(n-1)$. For the proof of Theorem 6.68 see [CCo2].

THEOREM 6.68. *Given $\varepsilon > 0$, there exists a disjoint union of balls, $\cup_{i=1}^{N_\varepsilon} B_{r_i}(y_i) = U_\varepsilon \subset B_R(w)$, such that*

$$\mathrm{Vol}(U_\varepsilon) \;\geq\; (1-\varepsilon)\,\mathrm{Vol}(B_R(w)), \tag{6.69}$$

$$r_i \;\geq\; \lambda(\varepsilon,n) > 0, \tag{6.70}$$

and such that for all i, there exists, k_i, such that for $B_{r_i}(0) \subset \mathbb{R}^{k_i}$,

$$d_{GH}(B_{r_i}(y_i),B_{r_i}(0)) < \varepsilon r_i. \tag{6.71}$$

Moreover, there exist harmonic functions, $\mathbf{b}_{1,i},\dots,\mathbf{b}_{k_{i,i}}$, on $B_{r_i}(y_i)$ and an εr_i-Gromov-Hausdorff equivalence, $\beta_i : B_{r_i}(y_i) \to B_{r_i}(0)$, such that if $\underline{\mathbf{b}}_{j,i}$ denotes the j-coordinate function on $\mathbb{R}^{k_i}$,

$$|\mathbf{b}_{j,i} - \underline{\mathbf{b}}_{j,i} \circ \beta_i| < \varepsilon r_i. \tag{6.72}$$

We now note some supplements to Theorem 6.62.

PROPOSITION 6.73. *There exists $P_\varepsilon \subset B_R(p)$ such that if $z \in P_\varepsilon$ there are unique minimal geodesics σ_z^+,σ_z^- from z to q_+,q_- such that*

$$\mathrm{Vol}(P_\varepsilon) \geq (1-\varepsilon)\,\mathrm{Vol}(B_R(p)), \tag{6.74}$$

$$|\angle(\sigma_+'(0),\sigma_-'(0)) - \pi| < \varepsilon. \tag{6.75}$$

Proof. Note that we can obtain Lemma 6.25 for b_- and $\mathbf{b}_-$ as well as for b_+ and $\mathbf{b}(=\mathbf{b}_+)$. By the Abresch-Gromoll inequality and the gradient estimate, $|\nabla(\mathbf{b}^+ - \mathbf{b}^-)|$ is uniformly small on a ball slightly smaller that $B_R(p)$ (and centered at p). From this the claim easily follows. □

Remark 6.76. The vectorfield of Proposition 6.74, which is nonvanishing on a set of almost full measure, is virtually all one can hope for, since examples show that there is not always a topological splitting; see [A2], [P3].

Remark 6.77. In the situation of Theorem 6.62 it is not difficult to see that there exists a *connected* subset, Q_ε of $B_R(p)$, having almost full relative measure, such that $||\nabla \mathbf{b}_+| - 1| < \varepsilon$ (and similarly for $\mathbf{b}$). Indeed, by Lemma 6.25 the above estimate holds on a set, Q_ε', of almost full measure. If one applies Proposition 6.60 together with Theorem 2.11, one finds that for almost all points, x, in Q_ε', for almost all y, the integral of $|\mathrm{Hess}|$ along $\gamma_{x,y}$ is small. Clearly, this implies our assertion.

Remark 6.78. Particularly in the collapsed case, the splittings obtained in Theorem 6.68 have strong implications for the structure of Gromov-Hausdorff limits of spaces with Ricci curvature bounded below; see [CCo1]–[CCo4] and compare [F], [Y], [BGP]. In the noncollapsed case, even stronger conclusions can be obtained by using as a starting point, the results of Section 4.

We close this section by stating one of several (equivalent) integral versions of Toponogov's theorem which hold in our context. Given what has already been established, the proof is strictly analogous to the corresponding results in [Co1]–[Co3]. Hence, it will be omitted.

Let the assumptions be as in Proposition 6.2. Let B denote the subset of $B_R(p) \times B_R(p)$ consisting of points (y_1,y_2) for which there is a unique minimal geodesic, γ_{y_1,y_2}, from y_1 to y_2. We equip B with the measure induced from the product measure on $B_R(p) \times B_R(p)$. As in Theorem 2.11, in the integrals in (6.80) and (6.81) below, rather than writing B for the domain, we use the more suggestive notation $B_R(p) \times B_R(p)$.

Put $s = s(y_1,y_2) = \overline{y_1,y_2}$ and $b_+(x) = \overline{x,q_+}$.

Let $\angle_\xi(y_1,y_2) = \angle(-\nabla b_+(\gamma_{y_1,y_2}(\xi s)),\gamma'_{y_1,y_2}(\xi s))$. Let $\underline{\angle}_\xi(y_1,y_2)$ be the corresponding angle in the triangle in $\mathbb{R}^2$, with the same edge length as the triangle with vertices q_+,y_1,y_2.

THEOREM 6.79. *The functions $\tau(\varepsilon,n),\lambda(\varepsilon,n)$ of Proposition 6.2 can be chosen such that for all $0 \le \xi \le 1$,*

$$\frac{1}{[\mathrm{Vol}(B_R(p))]^2}\int_{B_R(p)\times B_R(p)} |b_+(\gamma_{y_1,y_2}(\xi s)) - (1-\xi)b_+(y_1) - \xi b_+(y_2)| \le \varepsilon\xi R. \tag{6.80}$$

and

$$\frac{1}{[\mathrm{Vol}(B_R(p))]^2}\int_{B_R(p)\times B_R(p)} \left|\angle_\xi(y_1,y_2) - \underline{\angle}_\xi(y_1,y_2)\right| \le \varepsilon\xi R. \tag{6.81}$$

Remark 6.82. As in [Co3], one can also obtain a version of Theorem 6.79 for collections of small disjoint balls whose union has almost full relative measure.

III. Applications

7. The structure at infinity of manifolds with $\mathrm{Ric}_{M^n} \ge 0$

Let M^n be complete, noncompact, with

$$\mathrm{Ric}_{M^n} \ge 0. \tag{7.1}$$

By Gromov's compactness theorem, [GLP], given $p \in M^n$ and any sequence $r_i \to \infty$, there is a subsequence, $r_j \to \infty$, such that

$$(M^n,p,r_j^{-2}g) \to M_\infty, \tag{7.2}$$

where M_∞ is some length space and the convergence is in the pointed Gromov-Hausdorff sense.

If M_∞ is independent of the sequences, $\{r_i\}$,$\{r_j\}$, then it follows directly that M_∞ has a 1-parameter family of homotheties fixing the base point. However, an example of Perelman shows that the tangent cones need not be unique, even if all of them are metric cones; see also [CCo2] for further discussion.

Example 7.3. ([P]) For $n \geq 4$, there exists a complete metric with $\mathrm{Ric}_{\mathbb{R}^n} > 0$ on $\mathbb{R}^n$, such that for some $v > 0$,

$$\frac{\mathrm{Vol}(B_r(p))}{r^n} > v, \tag{7.4}$$

but for which M_∞ is not unique. In addition, for $n = 4$,

$$|K_{\mathbb{R}^4}| \leq \frac{c}{r^2}. \tag{7.5}$$

Note that in (7.5), K denotes sectional curvature.

In Example 7.3, all M_∞ are metric cones. More generally, the following immediate consequence of Theorem 4.85 (compare Theorem 4.91) confirms that in the Euclidean volume growth situation, (7.4), this is always the case; compare [BKN], [CT].

THEOREM 7.6. *Let M^n be complete, noncompact, and assume that* (7.1), (7.4) *hold. Then every M_∞ is a metric cone, $C(X)$, where* $\mathrm{diam}(X) \leq \pi$.

Proof. By the Bishop-Gromov inequality,

$$\frac{\mathrm{Vol}(\partial B_r(p))}{r^{n-1}} \downarrow . \tag{7.7}$$

Thus, given $\omega > 0$, $\Omega > 1$, there exists $R(\omega,\Omega)$ such that for $r \geq R$, on $A_{r,\Omega r}$, with rescaled metric $r^{-2}g$, (4.89) of Theorem 4.85 holds. Since $\mathrm{Ric}_{M^n} \geq 0$, apart from the diameter bound on X, our claim follows directly from Theorem 4.85, together with Gromov's compactness theorem. The diameter bound follows as in the proof of Theorem 5.12. The possibility that X consists of two points is ruled out by (7.7). □

Remark 7.8. Let $f(r)$ denote the decreasing function in (7.7). Even if $\lim_{r\to\infty} f(r) = 0$, we may conclude that every tangent cone is a metric cone (of dimension $\leq n-1$) provided that for $k > 0$, the function, f, satisfies $\lim_{r\to\infty} \frac{f(kr)}{f(r)} = 1$. This holds, for example, if $f \approx (\log r)^{-a}$, for some $a > 0$. Such examples (having infinite topological type) can be constructed by employing the Gibbons-Hawking ansatz, as in [AKL].

The results of Section 6 also give information in case the growth of M^n is "slow" in a suitable sense.

Fix $p \in M^n$. For $r > 0$ let $q_1,q_2 \in \partial B_r(p)$ and set

$$\mathcal{D}(r) = \max_{q,q_2} \overline{q_1,q_2}. \tag{7.9}$$

Suppose $\mathrm{Ric}_{M^n} \geq 0$. For each r, let $q_r \in \partial B_r(p)$. Let $r_i \to \infty$. By Gromov's compactness theorem, for some subsequence, $r_j \to \infty$,

$$(M^n, q_{r_j}, \mathcal{D}^{-2}(r_j)g) \to P_\infty, \tag{7.10}$$

in the Gromov-Hausdorff sense, for some complete length space P_∞. Presumably, P_∞ need not be unique in general.

THEOREM 7.11. *If*

$$\lim_{r\to\infty} \frac{\mathcal{D}(r)}{r} = 0, \tag{7.12}$$

then every P_∞ splits isometrically,

$$P_\infty = \mathbb{R} \times_1 Y \tag{7.13}$$

where Y is some compact length space.

Proof. As is well known, M^n contains at least one ray γ emanating from p. It is clear from (7.12) that γ determines a line in P_∞. Thus, by (an easy case of) the splitting theorem, Theorem 6.64, P_∞ splits as in (7.13). From the definition of P_∞ it is now clear that

$$\mathrm{diam}(Y) = 1. \tag{7.14}$$

□

8. Almost nonnegative curvature and the fundamental group

Let M^n be complete with

$$\mathrm{Ric}_{M^n} \geq (n-1)H \qquad (H > 0). \tag{8.1}$$

Recall that by Meyers' Theorem, M^n is compact. Since for the pull-back metric, $\mathrm{Ric}_{\tilde{M}^n} \geq (n-1)H$, it follows that $\tilde{M}^n$ is compact as well. Thus, $\pi_1(M^n)$ is finite.

By means of the splitting theorem, these results, and the related estimate of Bochner, [B], on $b_1(M^n)$, were extended in [CG]. There it was shown that if M^n is compact and $\mathrm{Ric}_{M^n} \geq 0$, then $\tilde{M}^n$ splits isometrically as $\tilde{M} = \mathbb{R}^k \times \underline{M}^{n-k}$ where $\underline{M}^{n-k}$ is compact. Moreover, for finite groups, F_1,F_2 and some Bieberbach group, B, there are exact sequences,

$$1 \to F_1 \to \pi_1(M^n) \to B \to 1, \tag{8.2}$$

$$1 \to \mathbb{Z}^k \to B \to F_2 \to 1. \tag{8.3}$$

For M^n complete, noncompact, with $\mathrm{Ric}_{M^n} \geq 0$, Milnor showed that finitely generated subgroups of $\pi_1(M^n)$ have polynomial growth; [Mi]. Later, Gromov proved his famous theorem to the effect that groups of polynomial growth are *almost nilpotent*; [G2]. In other words, any such group has a nilpotent subgroup of finite index.

In [GLP] Gromov conjectured that for M^n compact, there exists $\varepsilon(n) > 0$ such that if

$$\mathrm{diam}^2(M^n)\,\mathrm{Ric}_{M^n} \geq -(n-1)\varepsilon(n)$$

and $b_1(M) = n$ then M is homeomorphic to the torus, T^n. This can be viewed as a generalization of Bochner's theorem. This conjecture was proved in [Co3], using in part, a conjecture of Anderson-Cheeger on topological convergence, which was proved there as well; see also [Y] where Yamaguchi proved this conjecture of Gromov under the stronger assumption of $\mathrm{diam}^2(M^n)K_{M^n} \geq -\varepsilon(n)$ and [CCo2] for the case of diffeomorphism.

Gromov conjectured further that for M^n compact, there exists $\varepsilon(n) > 0$ such that if

$$\mathrm{diam}^2(M^n)\,\mathrm{Ric}_{M^n} \geq -(n-1)\varepsilon(n) \tag{8.4}$$

then $\pi_1(M^n)$ is almost nilpotent.

In [FY], Fukaya-Yamaguchi proved Gromov's second conjecture under the stronger assumption,

$$\mathrm{diam}^2(M^n)K_{M^n} \geq -\varepsilon(n), \tag{8.5}$$

where K_{M^n} denotes sectional curvature. They also observed that their proof would go through in the case of almost nonnegative Ricci curvature, provided that two conjectures could be established.

One of their conjectures was:

If $\mathrm{Ric}_{M_i^n} \geq -(n-1)$ and $M_i^n \to M^n$ in the Gromov-Hausdorff sense, then there exists $\varepsilon > 0$, such that for all i and $p_i \in M_i^n$, the image of $\pi_1(B_\varepsilon(p_i))$ in $\pi_1(M^n)$ (under the natural map) is trivial.

This conjecture is implied by the above mentioned conjecture of Anderson-Cheeger on topological convergence which was proved in [Co3].

The second conjecture of [FY] is precisely our Theorem 6.64. Thus, we get:

THEOREM 8.6 (Gromov's Conjecture). *There exists $\varepsilon(n) > 0$ such that $\mathrm{diam}^2(M^n)\,\mathrm{Ric}_{M^n} \geq -(n-1)\varepsilon(n)$ implies $\pi_1(M^n)$ is almost nilpotent.*

We refer the reader to [FY] for further consequences of their work, the result of [Co3] mentioned above and Theorem 6.64; see for example [FY, Th. 0.6]. As a particular instance, we have the following generalized Margulis lemma.

THEOREM 8.7. *There exists $\varepsilon(n) > 0$, such that if M^n is complete with* $\mathrm{Ric}_{M^n} \geq -(n-1)$ *then for $r < \varepsilon(n)$, the image under the inclusion homomorphism, $\pi_1(B_r(p)) \to \pi_1(M^n)$, is almost nilpotent, for all $p \in M^n$.*

Remark 8.8. Note that by an obvious scaling argument, Theorem 8.7 actually implies Theorem 8.6.

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CURVATURE AND INJECTIVITY RADIUS ESTIMATES FOR EINSTEIN 4-MANIFOLDS

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0. Statement of main results

It is of fundamental interest to study the geometric and analytic properties of compact Einstein manifolds and their moduli. In dimension 2 these problems are well understood. A 2-dimensional Einstein manifold, (M^2, g), has constant curvature, which after normalization, can be taken to be -1, 0 or 1. Thus, (M^2, g) is the quotient of a space form and the metric, g, is completely determined by the conformal structure. For fixed M^2, the moduli space of all such g admits a natural compactification, the Deligne-Mumford compactification, which has played a crucial role in geometry and topology in the last two decades, e.g. in establishing Gromov-Witten theory in symplectic and algebraic geometry.

In dimension 3, it remains true that Einstein manifolds have constant sectional curvature and hence are quotients of space forms. An essential portion of Thurston's geometrization program can be viewed as the problem of determining which 3-manifolds admit Einstein metrics. The moduli space of Einstein metrics on a 3-dimensional manifold is also well understood. As a consequence of Mostow rigidity, the situation is actually simpler than in two-dimensions.

In dimension 4 however, the class of Einstein metrics is significantly more general than that of metrics of constant curvature. For example, almost all complex surfaces with definite first Chern class admit Kähler-Einstein metrics. Still, the existence of an Einstein metric does impose strong constraints on the underlying 4-manifold. Hence, it is natural to look for sufficient conditions for a closed 4-manifold to admit an Einstein metric. Any approach to this existence problem by geometric analytic methods, e.g. by Ricci flow, will lead to the question of how, in limiting cases, solutions to the Einstein equation can develop singularities, or equivalently, how Einstein metrics can degenerate.

On the other hand, most Einstein 4-manifolds have nontrivial moduli spaces. These moduli spaces and their natural compactifications are differentiable invariants of underlying smooth 4-manifolds. Thus, one wants to understand the geometry of such moduli spaces and their compactifications. Here one can normalize the Einstein constant, λ, to be $-3, 0$ or 3, and in the (scale invariant) case, $\lambda = 0$,

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add the additional normalization so that the volume is equal to 1. Specifically, one would like to know the properties of a natural compactification analogous to the Deligne-Mumford compactification in the 2-dimensional case. For this purpose, once again, one must understand how Einstein metrics can degenerate.

In somewhat more concrete terms, we wish to describe the geometric structure of metric spaces, Y, which arise as limits of sequences, (M^4, g_i), of Einstein manifolds (M^4, g_i) with fixed topology and Einstein constant. After passing to a subsequence, if necessary, such limits always exist in a suitable weak geometric sense, the pointed Gromov-Hausdorff sense. These limit spaces can be thought of as Einstein manifolds with singularities, although a priori they might not have any manifold points whatsoever. Indeed, the first result on the existence of manifold points in the collapsed case, that in which $\dim Y < 4$, is a consequence of Theorem 0.8 of the present paper; compare [ChCo3]. For the structure of noncollapsed limit spaces in dimension 4, see [An1], [An3], [Na], [Ti]; for a structure theory in the noncollapsed case in higher dimensions, see [ChCo0]–[ChCo3], [ChCoTi2], [Ch2].

The present paper constitutes the first step in our program, the ultimate goal of which is to obtain a complete understanding of how Einstein metrics on 4-manifolds can degenerate.

According to the finiteness/compactness theory, [Ch1], [GvLP], for manifolds with bounded sectional curvature, there are precisely three mechanisms (which can occur in combination) which can cause a sequence, (M^n, g_i), to degenerate; namely, the diameter can go to ∞, the volumes of unit balls can go uniformly to 0 at all points, or the curvature can go to $\pm\infty$ at certain points. Thus, in our situation, we have the following issues.

1. For $\lambda = -3, 0$, the diameter, $\operatorname{diam}(M^4, g_i)$ need not remain uniformly bounded. What are the noncompact limits?

2. For $\lambda = 0$, the sequence can collapse; i.e., the volumes of all unit balls can go uniformly to 0. Is $\lambda = 0$ the only case in which collapse can take place?

3. For $\lambda = -3, 0, 3$, the sectional curvature need not remain uniformly bounded. In the noncollapsing case, this phenomenon is well understood. How badly can the curvature blow up in the collapsing case?

Note that in each of the above instances, we wish to understand both the structure of limiting objects and the detailed nature of the convergence to the limit.

In this paper, we prove a number of analytic estimates for Einstein 4-manifolds with finite L^2-norm of curvature. In particular, we solve the third problem (see Theorem 0.14 and Theorem 9.1) and the second problem under the additional assumption that the global volume stays bounded below; see Theorem 0.14. Our results shed light on the first problem as well; see Theorems 7.10, 10.3, 10.5; other applications willl be discussed elsewhere.

In view of [An1], [An3], [Na], [Ti], we only need to consider the case of Einstein 4-manifolds which are sufficiently collapsed. However, this case is technically much more difficult than the noncollapsed case and additional new techniques are required.

Suitably formulated, our results continue to hold for 4-manifolds which are sufficiently Ricci pinched, or whose Ricci tensor has a definite 2-sided bound. We will give detailed arguments in the Einstein case; to avoid nonessential complications in the exposition, we indicate the above-mentioned generalizations in a number of remarks.

Three main results. The following theorem states that if an Einstein 4-manifold is sufficiently collapsed and there is a definite bound on the L_2-norm of the curvature, then in the L_2 sense, almost all of the curvature is concentrated very near at most a definite number of points; for a generalization, see the proof of Theorem 10.5.

Theorem 0.1 (Collapse implies L_2 concentration of curvature)**.** *There exists $v > 0$, β, c, such that the following holds. Let M^4 denote a complete Einstein 4-manifold satisfying*

$$|\lambda| \leq 3\,, \tag{0.2}$$

$$\int_{M^4} |R|^2 \leq C\,, \tag{0.3}$$

and for all p and some $s \leq 1$,

$$\frac{\mathrm{Vol}(B_s(p))}{s^4} \leq v\,. \tag{0.4}$$

Then there exist $p_1, \ldots, p_N$, with

$$N \leq \beta \cdot C\,, \tag{0.5}$$

such that

$$\int_{M^4 \setminus \left(\bigcup_i B_s(p_i)\right)} |R|^2 \leq c \cdot \left(\sum_i \frac{\mathrm{Vol}(B_s(p_i))}{s^4} + \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} \right). \tag{0.6}$$

If $\lambda \neq 0$, then $\mathrm{Vol}(M^4) < \infty$ and, in particular,

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4} = 0\,. \tag{0.7}$$

If in Theorem 0.1, the manifold, M^4, is compact, then the bound, (0.3), on the L_2-norm of the curvature, can be replaced by a bound on the Euler characteristic; see (1.3). Of course, in this case, the term, $\lim_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^4}$, in (0.6) vanishes.

Our next result is an ϵ-regularity theorem whose significant feature is the *absence* of the assumption that the L_2-norm of the curvature is sufficiently small *with respect to the collapsing*; compare (1.12), (1.15).

Theorem 0.8 (ϵ-regularity)**.** *There exists $\epsilon > 0$, c, such that the following holds. Let M^4 denote an Einstein 4-manifold satisfying* (0.2) *and let $r \leq 1$. If $B_s(p)$ has compact closure for all $s \leq r$ and*

$$\int_{B_r(p)} |R|^2 \leq \epsilon\,, \tag{0.9}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq c \cdot r^{-2}\,. \tag{0.10}$$

If $\lambda = 0$ and the assumption, $r \leq 1$, is dropped, then (0.10) *holds.*

Remark 0.11. If in Theorem 0.1, we take $v \leq \epsilon \cdot (\beta \cdot C)^{-1}$, then by Theorem 0.8, the curvature bound, (0.10), is valid for all $B_r(p) \subset M^4 \setminus \bigcup_i B_s(p_i)$.

Remark 0.12. Theorem 0.8 should be compared with the ϵ-regularity theorems for the cases of 4-dimensional Yang-Mills fields, [Uh], and harmonic maps, [Mor], [SchUh]. The Yang-Mills equation and harmonic map equation are uniformly elliptic modulo gauge transformations. In the Einstein equations however, the nonlinearity is much stronger, since the coefficients of the highest derivatives which occur depend on the solutions. So initially, one must decide if the equation is uniformly elliptic modulo gauge transformations, or equivalently, uniformly elliptic in suitable local coordinates. Although harmonic coordinates will suffice for this purpose, one cannot ensure the existence of such local coordinate systems on metric balls of a definite size. In actuality, the topology of the ball, $B_r(p)$, occurring in the statement of Theorem 0.8 need not be that of a Euclidean ball and there may be no global coordinate sytem at all on such $B_r(p)$.

Indeed, it is a *consequence* of the curvature bound in our ϵ-regularity theorem, together with [ChFuGv], that the point, p, does have a neighborhood of a definite size with known topology. Namely, there exists such a neighborhood which is quasi-isometric (with a definite constant) to either a Euclidean ball or to a tube around a nilmanifold; see Theorem 1.7 of [ChFuGv] and Appendix 1 of [ChFuGv].

Remark 0.13. The ϵ-regularity theorems for Yang-Mills and harmonic maps can be proved by Moser iteration. This requires a bound on the Sobolev constant of the domain. Since in these cases the domain is effectively a standard ball, such a bound is available.

In [An1], [An3], [Na], [Ti], the Moser iteration argument was extended to n-dimensional Einstein manifolds yielding a pointwise curvature bound as in (0.10). In these works, in order to apply Moser iteration, it is assumed that the $L_{\frac{n}{2}}$-norm of the curvature is sufficiently small *with respect to the Sobolov constant.* The latter can be bounded in terms of a lower bound on the collapsing.

According to Theorem 0.8, in dimension 4, to obtain the pointwise curvature bound in (0.10), it merely suffices to assume that the L_2-norm of the curvature is sufficiently small. The proof is accomplished by showing that once the L_2-norm of the curvature is sufficiently small, then on a smaller concentric ball of a definite radius, the L_2-norm will *automatically* be so small with respect to the collapsing that the hypothesis of the ϵ-regularity theorem of [An3] will be verified.

As previously mentioned, the proof of Theorem 0.8 is considerably more difficult than those of the earlier ϵ-regularity theorems and employs entirely different techniques. Neither Moser iteration nor the Sobolev inequality enter directly in the argument. For an outline of the proof, including the role of the assumption, $n = 4$, see Section 1.

Next, we state a theorem on the noncollapsing of Einstein 4-manifolds. If $\lambda = \pm 3$, then $|R|^2$ has the pointwise lower bound $|R|^2 \geq 6$. Substituting this into the left-hand side of (0.6) and using (0.5) on the right-hand side gives the following.

Theorem 0.14 (Lower bound on collapse). *There exists $w > 0$, such that if M^4 denotes a complete 4-dimensional Einstein manifold satisfying* (0.3) *and*

$$\lambda = \pm 3\,, \tag{0.15}$$

then for some $p \in M^4$,

$$\mathrm{Vol}(B_1(p)) \geq w \cdot \frac{\mathrm{Vol}(M^4)}{C}\,. \tag{0.16}$$

For $\lambda = 3$, Myers' theorem together with the Bishop-Gromov inequality provides stronger information than (0.16). The interesting case is $\lambda = -3$, in which relation (0.16) can be viewed as a partial replacement for the Heintze-Margulis theorem. The latter gives a lower bound for the collapse, for compact manifolds with negative *sectional* curvature, $-1 \leq K_{M^n} < 0$.

If M^4 is compact, Kähler-Einstein, with $\lambda = \pm 3$, then $\mathrm{Vol}(M^4) = 2\pi^2 \cdot c_1^2(M^4)$, where c_1 denotes the first Chern class. By (1.1) below, we can take $C = 8\pi^2 \cdot \chi(M^4)$. The topological invariants, $c_1^2(M^4)$, $\chi(M^4)$, are positive integers. In dimension 4, Seiberg-Witten theory provides lower volume bounds under more general assumptions, e.g. if M^4 admits a symplectic structure; see [LeBru1], [LeBru2], [Tau], [Wi].

1. Preliminary discussion of proofs

In this section we describe the various techniques which enter in the proofs of our main results and give a brief indication of how they are used.

Underlying the proofs of our results is an extension of the "equivariant good chopping" theorem of [ChGv3], valid for manifolds with locally bounded curvature, i.e., no global curvature bound is assumed. According to this theorem, a compact domain, K, with "rough" boundary can be approximated from the outside by a smooth submanifold, Z^n, with nonempty boundary, for which the norm of the second fundamental form of the boundary, $|II_{\partial Z}|$, is controlled. The term "equivariant" refers to the possibility of choosing Z to be invariant under the group of isometries which leaves K invariant.

The bound on $|II_{\partial Z}|$ involves the reciprocal of the local scale $r_{|R|}(p)$, for p in a suitable neighborhood of ∂K. By definition, $r_{|R|}(p)$ is the supremum of those r, such that if the metric is rescaled, $g \to r^{-2} g$, then the ball, $B_r(p)$, becomes a unit ball on which the norm of the curvature, $|R|$, is bounded by 1.

The chopping theorem can be used to control the boundary term in the Chern-Gauss-Bonnet formula for manifolds with boundary, thereby yielding information on the interior term; compare [ChGv3]. Moreover, for Einstein manifolds with $L_{\frac{n}{2}}$ curvature bounds, the appearance of the local scale can be removed by employing an inequality which bounds $|(r_{|R|})^{-1}|_{L_{n-1}}$ in terms of $(|R|_{L_{\frac{n}{2}}})^{\frac{n-1}{2}}$, where both norms are computed over a neighborhood of ∂K.

The eventual restriction, $n = 4$, in our *main* results stems from the relation between the Chern-Gauss-Bonnet form, P_χ, and the L_2-norm of the curvature in that dimension.

If M^4 is Einstein, then

$$P_\chi = \frac{1}{8\pi^2} \cdot |R|^2 \cdot \mathrm{Vol}(\,\cdot\,)\,, \tag{1.1}$$

where $\mathrm{Vol}(\,\cdot\,)$ denotes the local choice of volume form corresponding to the choice of local orientation used in defining P_χ; see p. 161 of [Be] and (1.2) below.

There is an alternate route to our main results which bypasses chopping, using in its stead the existence of an essentially canonical transgression form, $\mathcal{T}P_\chi$, satisfying $d\mathcal{T}P_\chi = P_\chi$ and $|\mathcal{T}P_\chi| \leq c(n) \cdot (r_{|R|}(p))^{-(n-1)}$, on subsets of Riemannian manifolds which are sufficiently collapsed with locally bounded curvature. In actuality, our first proofs used this approach. The construction of $\mathcal{T}P_\chi$ is briefly indicated in Section 12.

The Chern-Gauss-Bonnet form in dimension 4**.** Let $W, \overset{\circ}{r}, s$ denote the Weyl tensor, traceless Ricci tensor and scalar curvature respectively. By definition, $\overset{\circ}{r} = \mathrm{Ric} - \frac{s}{n} \cdot g$, where $n = \dim M^n$.

For $n = 4$, the Chern-Gauss-Bonnet form satisfies

$$P_\chi = \frac{1}{8\pi^2} \cdot \left[|W|^2 - |\overset{\circ}{r}|^2 + \frac{1}{24} s^2 \right] \cdot \mathrm{Vol}(\,\cdot\,)\,; \tag{1.2}$$

see [Be].

If M^4 is Einstein, then $\overset{\circ}{r} = 0$ and we get (1.1). It follows from (1.1) that if M^4 is closed, then the quantity on the left-hand side of (0.3) (the square of the global L_2-norm of curvature) has the well-known topological interpretation,

$$\frac{1}{8\pi^2} \int_{M^4} |R|^2 = \chi(M^4)\,, \tag{1.3}$$

where $\chi(\,\cdot\,)$ denotes the Euler characteristic. As previously indicated, in the present paper it is also crucial to consider manifolds with nonempty boundary.

Remark 1.4. Relation (1.2) implies that if the Ricci tensor of M^4 is only sufficiently pinched, then $P_\chi \geq \eta \cdot |R|^2 \cdot \mathrm{Vol}(\,\cdot\,)$, where the constant, $\eta > 0$, depends on the pinching. In addition, if M^4 is arbitrary, with Ricci tensor satisfying $|\mathrm{Ric}_{M^4}| \leq 3$, then $|R|^2$ has a definite bound at points at which P_χ is bounded above by any definite (positive or negative) constant times the 4-form $\mathrm{Vol}(\,\cdot\,)$. From these observations and some additional technicalities, it follows that Theorem 0.1 and its consequences, such as Theorem 0.14, are valid for 4-manifolds which are sufficiently Ricci pinched, while Theorem 0.8 and its consequences have extensions for manifolds with bounded Ricci curvature. This will be explained at greater length in subsequent sections.

Collapse. A subset, $U \subset M^n$, such that for all $p \in U$,

$$\sup_{B_1(p)} \mathrm{Ric}_{M^n} \geq -(n-1)\,, \tag{1.5}$$

is called v-*collapsed* if for all $p \in U$,

$$\mathrm{Vol}(B_1(p)) \leq v\,, \tag{1.6}$$

and v-*noncollapsed* if (1.6) holds for no $p \in U$.

Note that the assertion that U is *not* v-collapsed is weaker than the assertion that U is v-noncollapsed.

The local scale $r_{|R|}(p)$. Let M^n denote an arbitrary Riemannian manifold. Let $r_{|R|}(p) > 0$ denote the supremum of those r such that $B_s(p)$ is compact for $s < r$ and

$$\sup_{B_r(p)} |R| \leq r^{-2}\,. \tag{1.7}$$

In particular,

$$|R(p)| \leq (r_{|R|}(p))^{-2}\,. \tag{1.8}$$

The quantity, $r_{|R|}(p)$, will be called the *local scale at* p. Rescaling the metric, $g \to (r_{|R|}(p))^{-2} \cdot g$, converts the ball, $B_{r_{|R|}(p)}(p)$, to a ball of unit radius on which the norm of the curvature is bounded by 1.

Clearly, either $r_{|R|} \equiv \infty$ and $R \equiv 0$, or $r_{|R|}(p)$ is locally Lipschitz, with local Lipschitz constant,

$$\operatorname{Lip} r_{|R|} \leq 1\,. \tag{1.9}$$

Collapse with locally bounded curvature. Although the term "collapse with locally bounded curvature" does not enter in the statements of Theorems 0.1, 0.8, 0.14, this notion plays a central role in the proofs.

We say that U is *v-collapsed with locally bounded curvature* if for all $p \in U$,

$$\operatorname{Vol}(B_{r_{|R|}(p)}(p)) \leq v \cdot (r_{|R|}(p))^n\,, \tag{1.10}$$

and that U is *(v,a)-collapsed with locally bounded curvature* if, in addition, for all p with $r_{|R|}(p) \geq a$,

$$\operatorname{Vol}(B_a(p)) \leq v \cdot a^n\,.$$

We say that U is *v-collapsed with bounded curvature* if U is v-collapsed and $r_{|R|}(p) \geq 1$ for all $p \in U$.

F-structures and N-structures. Theorem 0.1 of [ChGv2] (see in particular (0.2)) asserts the existence of a constant, $t = t(n) \leq 1$, such that if M^n is v-collapsed with locally bounded curvature, where $v \leq t$, then M^n carries a topological structure called an *F-structure of positive rank*. This concept generalizes that of a torus action for which all orbits have positive dimension. The local action of the F-structure is isometric for a metric close to the given one, provided v is sufficiently small. The orbits of the structure represent the collapsed directions on the *scale of the injectivity radius*.

If M^n admits an F-structure of positive rank, it is not difficult to construct a locally finite covering, $M^n = \bigcup_i U_i$, such that (a finite covering space of) every nonempty intersection, $U_{i_1} \cap \cdots \cap U_{i_j}$, is invariant under the flow of a nonvanishing vector field; see [ChGv2]. Hence, $\chi(U_{i_1} \cap \cdots \cap U_{i_j}) = 0$. Since $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, this implies $\chi(M^n) = 0$, *provided the covering is finite*.

In case the curvature is bounded, $|R| \leq 1$, a description of the geometry of M^n on a *fixed scale*, $r(n)$, is given in [ChFuGv], in terms of an essentially canonical *nilpotent Killing structure of positive rank*, a structure based on nilpotent Lie groups, rather than tori. From now on, we will refer to such a structure as an N-structure.

The discussion of [ChFuGv] has an essentially obvious extension to the case of collapse with locally bounded curvature. This provides a description of the geometry on the scale $r(n) \cdot r_{|R|}(p)$. N-structures produced by the construction of [ChFuGv] and its extension to the case of locally bounded curvature have some significant properties, several of which are relevant in connection with the results on equivariant good chopping proved in Section 3. Structures with these properties will be referred to as *standard*; for details, see Section 2.

Each orbit, $\mathcal{O}_p$, of the N-structure is the union of orbits of an associated F-structure. A subset which is the union of the orbits of its points (with respect to either structure) is called *saturated*. Any (sufficiently regular) saturated subset has vanishing Euler characteristic. The smallest saturated subset containing a given set is called its *saturation*. Without loss of generality, we can assume that $t = t(n)$ above has been chosen such that if M^n is (t,a)-collapsed with locally bounded curvature, then all orbits, $\mathcal{O}_p$, have extrinsic diameter $\leq \frac{1}{8}\min(r_{|R|}(p), a)$.

Equivariant good chopping. For $K \subset M^n$, $r > 0$, put

$$T_r(K) = \{p \in M^n \mid \overline{p, K} < r\},$$

and for $0 \leq r_1 < r_2$,

$$A_{r_1,r_2}(K) = T_{r_2}(K) \setminus \overline{T_{r_1}(K)}.$$

Let TP_χ denote the integrand in the boundary term of the Chern-Gauss-Bonnet formula for manifolds with boundary. Thus, TP_χ is a homogeneous invariant polynomial of degree $n-1$ in the second fundamental form of the boundary and the curvature, where second fundamental form terms are regarded as having degree 1 and curvature terms as having degree 2.

Let M^n denote a complete Riemannian manifold. Assume $K \subset M^n$ is compact and that $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, for some $r \leq 1$. It follows from the equivariant good chopping theorem, Theorem 3.13, that there is a saturated submanifold, Z^n, with smooth boundary, satisfying $T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{3}r}(K)$, such that the boundary term in the Chern-Gauss-Bonnet formula satisfies

$$\left|\int_{\partial Z^n} TP_\chi\right| \leq c(n)\cdot r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right).$$

Since $\chi(Z^n) = 0$, by the Chern-Gauss-Bonnet formula, we get

$$\left|\int_{Z^n} P_\chi\right| \leq c(n)\cdot r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(n-1)} + (r_{|R|})^{-(n-1)}\right). \tag{1.11}$$

Note that in the case of bounded curvature, $|R| \leq 1$, the above estimate reduces to the one given in [ChGv3].

In the next two subsections, we indicate how for Einstein manifolds, under additional collapsing assumptions, collapsed regions with locally bounded curvature can be located, and how (absent any additional collapsing assumptions) the right-hand side of (1.11) can be bounded in terms of $|R|_{L_{\frac{n}{2}}}$.

ϵ-Regularity and collapse with locally bounded curvature. In the context of Einstein manifolds, a condition for a set, U, to be $(v,1)$-collapsed with locally bounded curvature is given in Section 5 of [An3]; see Theorem 5.1. By appealing to [ChGv2], an F-structure on (a slight fattening of) U is obtained.

Anderson's results are based on an ϵ-regularity theorem, Theorem 4.4 of [An3]. This ϵ-regularity theorem is valid for arbitrary n, but when specialized to $n=4$, the assumptions in its hypothesis are stronger than those of Theorem 0.8; compare (0.9) versus (1.12).

Let M^n_H denote the simply connected space of constant curvature H. In what follows, for fixed r, we consider $M^n_{-r^{-2}}$ and $\underline{p} \in M^n_{-r^{-2}}$.

According to Theorem 4.4 of [An3], there exists $\tau = \tau(n) > 0$, such that if

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\cdot \int_{B_r(p)} |R|^{\frac{n}{2}} \leq \tau, \tag{1.12}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq c\cdot r^{-2}\cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\cdot \int_{B_r(p)} |R|^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{1.13}$$

Without loss of generality, one can assume $c \cdot \tau^{\frac{2}{n}} \leq 4$, so that

$$\sup_{B_{\frac{1}{2}r}(p)} |R| \leq 4r^{-2}\,. \tag{1.14}$$

Relation (1.12) can be rewritten as

$$\int_{B_r(p)} |R|^{\frac{n}{2}} \leq \tau \cdot \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}\,, \tag{1.15}$$

which specifies directly that the $L_{\frac{n}{2}}$-norm of the curvature is sufficiently small with respect to the collapsing.

Note that by relative volume comparison, [GvLP], the expression inside the parentheses on the right-hand side of (1.13) is a monotonically nondecreasing function of r, which vanishes at $r = 0$.

Now assume

$$|\lambda| \leq n-1\,. \tag{1.16}$$

Let $\underline{p} \in \underline{M}^n_{-1}$ and define $\theta = \theta(n)$ by

$$\theta = \frac{1}{2^n \mathrm{Vol}(B_1(\underline{p}))}\,. \tag{1.17}$$

Recall that collapse with locally bounded curvature is defined via condition (1.10). If U is $(\theta \cdot v)$-collapsed with $v \leq 1$ and if for all $p \in U$,

$$\int_{B_1(p)} |R|^{\frac{n}{2}} \leq \theta \cdot v \cdot \tau\,, \tag{1.18}$$

then U is v-collapsed with locally bounded curvature, where in particular, one can take $v = t = t(n)$, where t-collapse with locally bounded curvature implies the existence of a standard N-structure; see Theorem 5.1 of [An3] and compare also [Ya]. (Both of the above references deal with F-structures.)

For completeness, we give the argument for the case $v = t$. (For v arbitrary, the argument is the same.)

Modulo the choice of normalizing constant, the following definition is taken from (4.21) of [An3]. The notation is as in (1.12)–(1.18).

If $p \in M^n$ satisfies

$$\frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_1(p))} \cdot \int_{B_1(p)} |R|^{\frac{n}{2}} \leq \tau\,, \tag{1.19}$$

put $\rho(p) = 1$. Otherwise, define $\rho(p)$ to be the (largest) solution of

$$\frac{\mathrm{Vol}(B_{\rho(p)}(\underline{p}))}{\mathrm{Vol}(B_{\rho(p)}(p))} \cdot \int_{B_{\rho(p)}(p)} |R|^{\frac{n}{2}} = \tau\,, \tag{1.20}$$

where, since the left-hand side of (1.20) is a nondecreasing function of r, we have $\rho(p) < 1$.

Relation (1.13) gives

$$\frac{1}{2}\rho(p) \leq r_{|R|}(p)\,. \tag{1.21}$$

If $\rho(p)$ is defined by (1.19), then we have $\sup_{B_{\frac{1}{2}}(p)} |R| \leq 4$, and since U is $(\theta \cdot t)$-collapsed, it follows that (1.10) holds with $v = t$.

If $\rho(p)$ is defined by (1.20), then from (1.18) (with $v = t$) we have

$$\frac{\mathrm{Vol}(B_{\rho(p)}(p))}{\mathrm{Vol}(B_{\rho(p)}(\underline{p}))} \leq \theta \cdot t\,,$$

which by (1.17) and relative volume comparison implies that (1.10) holds with $v = t$.

Remark 1.22. The proof of Theorem 4.4 of [An3], as well as that of an earlier ϵ-regularity theorem proved (independently) in [An1], [Na], [Ti], is based on Moser iteration, which leads to an estimate in which the factor, $\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}$, in (1.12), (1.13), is replaced by s^n, with s a suitable Sobolev constant. In the proof, the Sobolev inequality is applied only to functions which are supported in $B_r(p)$.

In [An1], [Na], [Ti], a global Sobolev constant is employed, while [An3] uses the Sobolev constant, $s(r,p)$, of $B_r(p)$; i.e., for f supported in $B_r(p)$,

$$\left(\int_{B_r(p)} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq s(r,p) \cdot \int_{B_r(p)} |df|\,.$$

Modulo this difference, the analytical details of the proofs are identical.

According to Theorem 4.1 of [An3], there exists c such that

$$s(r,p) \leq c \cdot \left(\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}\right)^{\frac{1}{n}}\ ;$$

compare the closely related estimate for $s(r,p)$ in [ChGvTa]; see also [Cr] and [ChengLiYau]. Estimation of the global Sobolev constant requires a global diameter bound.

Remark 1.23. The application to collapse with locally bounded curvature only uses the bound on $s(r,p)$ for balls which satisfy $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq \theta \cdot t$ (for which the estimate of [ChGvTa] suffices); compare Remark 1.31. To see this, modify the definition of $\rho(p)$ by replacing $\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))}$ by $s(r,p)^n$ in (1.19), (1.20) and proceed, mutatis mutandis, as above.

Bounds on $(r_{|R|}(p))^{-1}$. To obtain a bound on $(r_{|R|}(p))^{-1}$, we first recast the discussion in terms of the maximal function. For $c = c(n)$, we get for any $r \leq 1$,

$$(r_{|R|}(p))^{-(n-1)} \leq c \cdot (r^{-(n-1)} + (M_{|R|^{\frac{n}{2}}}(p,r))^{\frac{n-1}{n}})\,, \tag{1.24}$$

where $M_{|R|^{\frac{n}{2}}}(p,r)$ denotes the maximal function over balls of radius $\leq r$, of the function, $|R|^{\frac{n}{2}}$, evaluated at the point p; for details, see Section 4.

For $\alpha < 1$, the normalized L_α-norm of the maximal function, M_f, of f can be bounded in terms of the L_1-norm of f; see (4.2). From this, together with (1.24), it follows that the integral of the right-hand side of (1.11) can be bounded in terms of $|R|_{L_{\frac{n}{2}}}$. Thus, if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, for some $r \leq 1$, then for any $\Omega \geq \mathrm{Vol}(A_{\frac{1}{4}r,\frac{3}{4}r}(K))$, we get

$$\left|\int_{Z^n} P_\chi\right| \leq c(n) \cdot \Omega \cdot r^{-1}\left(r^{-(n-1)} + \left(\Omega^{-1} \cdot \int_{A_{\frac{1}{4}r,\frac{3}{4}r}(K)} |R|^{\frac{n}{2}}\right)^{\frac{n-1}{n}}\right)\,. \tag{1.25}$$

(The detailed proof of (1.25) is concluded after (4.4).)

The key estimate; $n = 4$. Let $E \subset M^4$ denote a bounded open subset such that $T_1(E)$ is $(\theta \cdot t)$-collapsed and (1.18) holds for all $p \in T_1(E)$. Replace E by the saturation of $\overline{E}$ for some standard N-structure and apply the discussion leading to (1.25). Since we assume $n = 4$, on the left-hand side of (1.25), P_χ can be replaced by $\frac{1}{8\pi^2}|R|^2$ and the domain of integration can be changed from Z^n to E. The resulting relation provides an estimate for $(|R|_{L_2})^2$, on the set E in terms of $(|R|_{L_2})^{\frac{3}{2}}$ on the set $A_{\frac{1}{4}r,\frac{3}{4}r}(E)$.

The reduction in the exponent, $2 \to \frac{3}{2}$, leads to an iteration argument yielding the following key estimate.

Theorem 1.26. *There exists $\delta > 0$, $c > 0$, such that the following holds. Let M^4 denote a complete Einstein manifold satisfying* (0.2), (0.3), *and let $E \subset M^4$ denote a bounded open subset such that $T_1(E)$ is t-collapsed with*

$$\int_{B_1(p)} |R|^2 < \delta \qquad (\textit{for all } p \in T_1(E))\,. \tag{1.27}$$

Then

$$\int_E |R|^2 \leq c \cdot \mathrm{Vol}(A_{0,1}(E))\,. \tag{1.28}$$

Remark 1.29. In the iteration (or self-improvement) argument, the exponent, $\frac{n-1}{n} = \frac{3}{4} < 1$, in (1.25), plays a role analogous to that which this same exponent plays in Moser iteration. There, it enters via the Sobolev inequality (which is not used in the iteration argument occurring in the proof of Theorem 1.26). As previously mentioned, the ϵ-regularity theorem of [An1], [Na], [Ti] and Theorem 4.4 of [An3] are proved by Moser iteration.

Remark 1.30. Suppose in Theorem 1.26, we drop the assumption that E is bounded. If $\lambda \neq 0$, then by (0.3), we have $\mathrm{Vol}(M^4) < \infty$ and by an obvious exhaustion argument, (1.28) continues to hold. If $\lambda = 0$, an exhaustion argument together with scaling leads easily to (0.6).

Implementation of the key estimate. Apart from the ϵ-regularity theorem, Theorem 0.8, all of our results in dimension 4 are relatively simple consequences of the key estimate.

In proving Theorem 0.1, we use (1.18) and a standard covering argument to choose the balls, $B_s(p_i)$, such that we can take the set, E, to be the set $M^4 \setminus \bigcup_i B_s(p_i)$ with suitably rescaled metric. Then the conclusion, (0.6), reduces to (1.28).

In proving Theorem 0.8, we take $\epsilon = \theta \cdot t \cdot \tau$. If

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq \theta^2 \cdot t\,,$$

then the ϵ-regularity Theorem 4.4 of [An3] can be applied directly. Otherwise, we can take E in Theorem 1.26 to be (a suitable rescaling of) $B_{\frac{1}{2}r}(p)$. This gives, for c an absolute constant,

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2 \leq c\,.$$

If we knew $c \leq \tau$, then the hypothesis, (1.12), of the ϵ-regularity theorem (Theorem 4.4 of [An3]) would be verified. Since this does *not* seem to be clear, our argument employs a second nontrivial step. Namely, we show that once we pass to a smaller concentric ball of a definite radius, the hypothesis of Theorem 4.4 of [An3] is verified.

Remark 1.31. The statement of Theorem 4.4 of [An3] requires only that the ball, $B_r(p)$, satisfies (1.12) and makes no assumption concerning a lower bound for $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}$ or for r. However, in all previous applications, e.g. to collapse with locally bounded curvature, in order to *verify* (1.12), a definite lower bound, $\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))} \geq v$, on the collapsing of $B_r(p)$, has been used; compare Remark 1.23.

Organization of remaining sections. In Section 2, we review N-structures and their construction, in the context of collapsed manifolds with bounded curvature; see [ChFuGv]. We observe that mutatis mutandis, the discussion extends naturally to collapse with locally bounded curvature.

In Section 3, we prove an extension of the equivariant good chopping theorem of [ChGv3], in which no global bound on curvature is assumed. In the conclusion, the bound on the norm of the second fundamental form is in terms of the reciprocal of the local scale $r_{|R|}$.

In Section 4, we consider Einstein manifolds with $L_{\frac{n}{2}}$ curvature bounds. We note the estimate, (1.24), which provides a lower bound on the local scale. This, together with a universal inequality for the maximal function, leads to a stronger chopping theorem in this more restricted context. An immediate application is the integrality of the geometric Euler characteristic in arbitrary dimension; compare [ChGv3].

In Section 5, we prove Theorem 1.26, the key estimate in dimension 4.

In Section 6, we prove Theorem 0.1, our main global result on collapse in dimension 4.

In Section 7, we consider the case of a negative Einstein constant in dimension 4. We prove our results on noncollapse (Theorems 0.14, 7.8). We also show that for Einstein 4-manifolds with negative Einstein constant and finite L_2-norm of curvature, the volume decreases exponentially at infinity.

In Section 8, we prove Theorem 0.8, the ϵ-regularity theorem in dimension 4.

In Section 9, we note some consequences of Theorem 0.8.

In Section 10, we discuss the implications of our main theorems for Gromov-Hausdorff limit spaces, using the language of compactifications of moduli spaces of Einstein metrics on 4-manifolds.

In Section 11, we speculate on some possible extensions of our main results for anti-self-dual metrics, Kähler metrics of constant scalar curavture, entire solutions of the Ricci flow and higher-dimensional Einstein manifolds.

In the Appendix, Section 12, we indicate the construction of the transgression form $\mathcal{T}P_\chi$. In proving Theorem 1.26, the form, $\mathcal{T}P_\chi$, can be used in place of the good chopping theorem.

2. Collapse with locally bounded curvature; N-structures

An N-structure on a manifold, N^n, is a sheaf of nilpotent Lie algebras of vector fields, which are Killing fields for some Riemannian metric and for which certain additional properties hold; for details see [ChFuGv]. The N-structure decomposes the manifold as a disjoint union of compact orbits, each of which has a canonical affine flat structure isomorphic to that of a nilmanifold. The *rank* of the structure is the dimension of an orbit of smallest dimension. If the rank is positive, then any sufficiently nice saturated subset (e.g. a submanifold, U^n, with piecewise smooth boundary) has vanishing Euler characteristic.

Standard N-structures and invariant metrics. Let (N^n, g) denote a v-collapsed, manifold with $v \leq t(n)$, and bounded curvature, $|R| \leq 1$. The construction given in [ChFuGv] associates to (N^n, g) an essentially canonical N-structure of positive rank and an invariant metric, $\widetilde{g}$, with the following significant properties:

i) (**Local structure on a fixed scale**) There exist $c = c(n)$, $r = r(n)$, such that for all $p \in M^n$, there is an orbit, $\mathcal{O}_q$, with second fundamental form satisfying $|II_{\mathcal{O}_q}| \leq c$ and normal injectivity radius at least $3r$, such that $B_r(p)$ is contained in the tube $T_{3r}(\mathcal{O}_q)$.

ii) (**Invariant metric**) There is a metric, $\widetilde{g}$, for which the local nilpotent actions associated to the structure are isometric. In particular, with respect to $\widetilde{g}$, any tubular neighborhood of an orbit is a saturated subset.

Moreover, for all $\epsilon > 0$, k, C, there exist $\eta = \eta(n, \epsilon)$, $\delta = \delta(\epsilon, k, C) > 0$, such that:

iii) (**Orbits with small diameter**) If in addition, $v \leq \eta$, then every orbit, $\mathcal{O}$, has extrinsic diameter satisfying $\mathrm{diam}(\mathcal{O}) \leq \epsilon$.

iv) (**Closeness of invariant metric**) If $|\nabla^i R| \leq C$, for $i = 0, \ldots, k$, and $v \leq \delta$, then $\widetilde{g}$ can be chosen such that $|\nabla^i(R - \widetilde{R})| \leq \epsilon$, $i = 1, \ldots k-1$.

Remark 2.1. The chopping theorems, Theorems 3.1, 3.13, depend on properties ii)–iv) above. The estimate, (12.2), for the transgression form, $\mathcal{T}P_\chi$, namely, $|\mathcal{T}P_\chi| \leq c(n)(r_{|R|})^{-(n-1)}$, requires property i) as well.

Fix $0 < a \leq 1$ and put

$$\ell_a = \min(r_{|R|}, a)\,. \tag{2.2}$$

Let N^n be an arbitrary complete Riemannian manifold and let $\mathcal{N}$ denote an N-structure on N^n. We say that $\mathcal{N}$ is *a-standard* if for all p, its restriction to $B_{\ell_a(p)}(p)$ has properties i)–iv) with respect to the rescaled metric, $g \to (\ell_a(p))^{-2} \cdot g$.

We have $\mathrm{Lip}\,\ell_a \leq 1$; see (1.9). As a consequence, when restricted to any ball, $B_{\zeta\cdot\ell_a(p)}(p)$, with $\zeta < 1$, the positive function, ℓ_a, satisfies a Harnack inequality with constant, $\frac{1+\zeta}{1-\zeta}$. Thus, *the function, ℓ_a, varies moderately on its own scale.* This implies, for example, that *locally*, coverings by balls of the form, $\{B_{\ell_a(p_i)}(p_i)\}$, behave essentially like coverings by balls of a fixed radius. As a consequence, for the case of locally bounded curvature discussed below, constructions involving such coverings can be reduced to the case of bounded curvature by local scaling arguments.

Existence of standard N-structures. Due to its essential locality, the construction of [ChFuGv] extends directly to the case of sufficiently collapsed manifolds with locally bounded curvature, thereby yielding a structure satisfying the scaled version of i)–iv).

Theorem 2.3. *There exists $t = t(n) > 0$ such that if M^n is complete and $W \subset M^n$ is (t, a)-collapsed with locally bounded curvature, then there exists an a-standard N-structure on a subset containing W.*

Proof. We will require two simple facts pertaining to coverings. These generalize corresponding statements in the case of Ricci curvature bounded below.

Fix a small constant, $\zeta > 0$, and let $\{p_\alpha\}$ denote a maximal set of points such that

$$\overline{p_{\alpha_1}, p_{\alpha_2}} \geq \zeta \cdot \min(\ell_a(p_{\alpha_1}), \ell_a(p_{\alpha_2})) \qquad (\alpha_1 \neq \alpha_2)\,.$$

If the metric on $B_{\ell_a(p)}(p_i)$ is rescaled, $g \to (\ell_a(p_i))^{-2} \cdot g$, the resulting metric has bounded curvature $|R| \leq 1$. By an obvious variant of the corresponding argument in the case of bounded curvature, it follows that $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$ is a covering, with multiplicity $\leq N(n)$.

The covering, $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$, can be partitioned into at most $N(n)$ disjoint subcollections, S_i, of mutually nonintersecting balls, $B_{2\zeta \cdot \ell_a(p_{i,j})}(p_{i,j})$, such that a given member of any such subcollection intersects at most one member of any other such subcollection; see Lemma 2.2 of [ChGv3] and (6.4.1)–(6.4.5) of [ChGv3]. In addition, if

$$B_{2\zeta \cdot \ell_a(p_{i_1,j_1})}(p_{i_1,j_1}) \cap B_{2\zeta \cdot \ell_a(p_{i_2,j_2})}(p_{i_2,j_2}) \neq \emptyset\,,$$

then

$$(1-\zeta) \cdot \ell_a(p_{i_1,j_1}) \leq \ell_a(p_{i_2,j_2}) \leq (1+\zeta) \cdot \ell_a(p_{i_1,j_1})\,.$$

In [Ab], [Ya], procedures are given for regularizing a metric with bounded curvature, $|R| \leq 1$. Let ∇ denote the Riemannian connection of g. Given $\eta > 0$, we can arrange that the regularized metric, $\widehat{g}$, and its connection, $\widehat{\nabla}$, satisfy

$$\begin{aligned} (1+\eta)^{-2} g \leq \widehat{g} &\leq (1+\eta)^2 g\,, \\ |\nabla - \widehat{\nabla}|_g &\leq c(n)\eta\,, \\ |\widehat{\nabla}^k \widehat{R}|_{\widehat{g}} &\leq c(n,k,\eta)\,. \end{aligned}$$

We extend this to our situation as follows. (If for example, we are dealing with Einstein manifolds, this step is actually unnecessary since Einstein metrics are already regular in the appropriate sense.)

Fix η. On each ball, $B_{r_{|R|}(p_\alpha)}(p_\alpha)$, we apply the regularization procedure of [Ab] to the rescaled metric, $(\ell_a(p_\alpha))^{-2} g$, the norm of whose curvature tensor is bounded by 1. We denote the resulting regularized metric of bounded curvature by $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$.

Next, as in [ChGv3], we regularize the distance function of the metric, $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$, to obtain a smooth function with definite bounds on all covariant derivatives with respect to the metric $(\ell_a(p_\alpha))^{-2}\widehat{g_\alpha}$.

By composing the regularized distance functions with standard bump functions, we obtain a partition of unity, $\{\phi_\alpha\}$, subordinate to the cover $\{B_{2\zeta \cdot \ell_a(p_\alpha)}(p_\alpha)\}$.

The metric, $\widetilde{g}$, defined by

$$\widetilde{g} = \sum_\alpha \phi_\alpha \widehat{g_\alpha}\,,$$

satisfies

$$(1+\eta)^{-2} g \leq \widetilde{g} \leq (1+\eta)^2 g\,, \tag{2.4}$$

$$|\nabla - \widetilde{\nabla}|_g \leq c(n)\eta \ell_a^{-1}\,, \tag{2.5}$$

$$|\widetilde{\nabla}^k \widetilde{R}|_{\widetilde{g}} \leq c(n,k,\eta)(\ell_a)^{-(k+2)}\,; \tag{2.6}$$

compare Theorem 1.12 of [ChFuGv].

As constructed above, the metric, $\widetilde{g}$, need not be invariant under the isometries of g. However, this can be arranged by a simple modification of the construction; compare the corresponding remark in [ChGv3] (in the context of bounded curvature). Namely, we replace the points, p_α, by a corresponding maximal collection of orbits, $\mathcal{O}_\alpha$, under the isometry group, and the balls, $B_{\ell_a(p_\alpha)}(p_\alpha)$, by tubular neighborhoods, $T_{2\zeta\cdot\ell_a(p_\alpha)}(\mathcal{O}_{p_\alpha})$.

The remainder of the construction of [ChFuGv] is local in nature. Hence, by straightforward scaling arguments, the case of locally bounded curvature can be reduced to the case of bounded curvature.

Specifically, in [ChFuGv], one constructs a series of almost mutually compatible $\mathrm{O}(n)$-equivariant local fibrations of the inverse image in the frame bundle of (slightly fattened) members of a covering by balls of fixed radius; see Sections 2–5 of [ChFuGv].

Using the subcollections, S_i, the fibrations are modified so as to become mutually compatible; see Section 6 of [ChFuGv]. Flat affine structures are specified on the fibres and the fibrations are modified again so as to make the flat affine structures compatible as well; see Section 7 of [ChFuGv]. Finally, an invariant metric close to the original one is constructed by a local averaging process; see Section 8 of [ChFuGv].

In the case of collapse with locally bounded curvature, one begins with the covering, $\{B_{2\zeta\cdot\ell_a(p_\alpha)}(p_\alpha)\}$, constructed above. Since for the collection of fibrations described above, the selection process is local, it can be carried out in the context of locally bounded curvature by making the local rescaling of the metric $g \to (\ell_a(p_\alpha))^{-2}\cdot g$. The modification process uses the subcollections, S_i, $i = 1, \dots, N(n)$. As above, the fibration corresponding to each ball in S_i is modified at the i-stage so as to fit the now mutually compatible fibrations corresponding to (slight shrinkings of) the at most $i-1$ balls in $S_1, \dots, S_{i-1}$ with which it has nonempty intersection. Again, since the modification process in [ChFuGv] is local, it follows that, by scaling, the case of local bounded curvature can be reduced to the case of bounded curvature.

The remainder of the proof can be completed by using scaling arguments such as those we have just described. □

Remark 2.7. Suppose that the assumption that M^n is (t,a)-collapsed with locally bounded curvature is weakened in the following way. Rather than assuming that for the metric, $(\ell_a(p))^{-2}g$, on each ball, $B_{\ell_a(p)}(p)$, the curvature satisfies $|R| \le 1$, we assume that the metric on the universal covering space of this ball satisfies definite $C^{1,\alpha}$-bounds, for all $\alpha < 1$, and $L_{2,q}$-bounds in harmonic coordinates, for all $q < \infty$. In this case, the theory of N-structures (and the subsequent theorem on equivariant choppings) continue to hold. As above, a key point is to construct a suitable regularization of the metric; see [Ya].

3. Equivariant good chopping with local curvature bounds

In this section, we prove Theorems 3.1, 3.13, which generalize the equivariant chopping theorem of [ChGv3] to the case of locally bounded curvature.

Let M^n denote a complete Riemannian manifold and $K \subset M^n$ a closed subset. For $N^k \subset M^n$ a smooth submanifold without boundary, we denote by II_{N^k} the second fundamental form of N^k.

For ℓ_a as in (2.2), put

$$S_a(K) = K \cup \left(\bigcup_{p \in \partial K} B_{\ell_a(p)}(p) \right) .$$

Let $t = t(n)$ be as in previous sections.

Theorem 3.1. *There exists $c = c(n) < \infty$ and a smooth manifold with boundary, Z^n, satisfying*

$$K \subset Z^n \subset K \cup S_a(K) \,, \tag{3.2}$$

$$|II_{\partial Z}| \leq c \cdot \ell_a^{-1} \,, \tag{3.3}$$

and for all $k_1, k_2 > 0$,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} \cdot |R|^{k_2} \leq c \cdot \int_{S_a(K)} {\ell_a}^{-(k_1+1)} (r_{|R|})^{-2k_2} \,. \tag{3.4}$$

In particular,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} |R|^{k_2} \leq c \int_{S_a(K)} \left(a^{-(k_1+2k_2+1)} + (r_{|R|})^{-(k_1+2k_2+1)} \right) . \tag{3.5}$$

Moreover, if $S_a(K)$ is (t, a)-collapsed with locally bounded curvature, then Z^n can be chosen to be saturated for some standard N-structure.

Proof. Given the formulation, the proof is a relatively straightforward generalization of that of the chopping theorem of [ChGv3], which deals with the case of the special case in which M^n has bounded curvature $|R| \leq 1$. We will recall the argument in that case, indicate the required modifications and refer to [ChGv3] for additional details.

The main technical result of [ChGv3] asserts the existence of constants $0 < \delta(n) \leq 1$, $0 < \epsilon(n)$, $c(n)$, such that if $f : M^n \to \mathbf{R}$ satisfies $\mathrm{Lip}\, f \leq L$, then for all $r \leq 1$, there exists $F : M^n \to \mathbf{R}$, with

$$F \leq f \leq (1 + \delta(n)) r F \,,$$
$$|\nabla F| \leq 2L \,,$$
$$|\mathrm{Hess}_F| \leq c(n) L r^{-1} \,,$$
$$|\nabla F(x)| \geq \epsilon(n) L \qquad \big(x \in F^{-1}([0, \delta(n) r L]) \big) \,.$$

In addition, $F \,|\, F^{-1}((-\infty, \delta(n) r L])$ can be chosen to be invariant under the isometries of $f^{-1}((-\infty, rL])$ which fix $f \,|\, f^{-1}((-\infty, rL])$.

The submanifold with boundary, Z, is constructed as a certain sublevel set, $F^{-1}(\underline{y})$, with $\underline{y} \in [0, \delta(n) r]$, where as above, F is associated to the distance function $f = \rho_K(x) = \overline{x, K}$. The second fundamental form is estimated by means of the relation,

$$\langle \nabla_V W, N \rangle = - \frac{\mathrm{Hess}_F(V, W)}{|\nabla F|} \,, \tag{3.6}$$

where $N = \nabla F / |\nabla F|$ and V, W are tangent to $F^{-1}(\underline{y})$.

At the outset of the construction of the function, F, given in [ChGv3], the metric, g, and the function, f, are regularized. This permits subsequent application of a quantitative version due to Yomdin, of the A.P. Morse lemma, yielding on each ball of a cover, $\{B_1(p_\alpha)\}$, an interval of a definite size, on the inverse image of which the gradient has a definite lower bound; for a discussion of Yomdin's theorem, see [Gv2]. The remainder of the construction consists of a sequence of modifications of the regularized function, eventually yielding the function, F, for which the above mentioned intervals can be chosen independently of the particular ball $B_1(p_\alpha)$.

To see the need for regularization, recall that the classical Morse-Sard theorem makes the (qualitative) assertion that if $f : M^{n_1} \to M^{n_2}$ and $f \in C^k$, with $k-1 \geq \max(n_1 - n_2, 0)$, then almost all values are regular.

Since the gradient and Hessian of f with respect to the *original* metric must be controlled, one must employ a regularization of the metric such that the Riemannian connections of the initial and regularized metrics are close; compare the discussion in Section 2 and see below.

In the present case, which is more general than that considered in [ChGv3], given $\mathrm{Lip}\, f \leq L$, we will construct F satisfying the above invariance property and

$$F \leq \ell_a^{-1} f \leq (1+\delta(n))rF\,, \tag{3.7}$$

$$|\nabla F| \leq 2L\ell_a^{-1}\,, \tag{3.8}$$

$$|\mathrm{Hess}_F| \leq c(n)Lr^{-1}\ell_a^{-2}\,, \tag{3.9}$$

$$|\nabla F(x)| \geq \epsilon(n)L\ell_a^{-1} \quad \big(x \in F^{-1}([0,\delta(n)rL])\big)\,. \tag{3.10}$$

In the application to chopping, we again choose $f = \rho_K$, or, in the equivariant case, the distance function from the saturation of K and realize Z as $F^{-1}(\underline{y})$, with $\underline{y} \in [0, \delta(n)r]$. Then (3.2) is clear, while (3.3) follows from (3.6), (3.9), (3.10). Finally, (3.4) follows from (3.3), (3.8) and the coarea formula applied to the sublevel set $F^{-1}([0,\delta(n)rL])$.

Note that in applying the coarea formula, we multiply the integrand on the left-hand side of (3.4) by $|\nabla F|_{\widetilde{g}}$, which satisfies the bound (3.8). This accounts for why the exponent in (3.4) is $-(k_1+2k_2+1)$, rather than $-(k_1+2k_2)$. The scale invariant relation, (3.4), leads to the exponent, $-(k_1+2k_2)$, in (3.16), which is crucial for our subsequent applications; compare the derivation of (3.16).

We now describe the construction of the function, F.

By scaling, we can assume $\mathrm{Lip}\, f \leq 1$.

We begin by regularizing the metric locally on the scale, ℓ_a, as in Section 2. Given $\eta > 0$, we can arrange that the metric, $\widetilde{g}$, so obtained, satisfies (2.4)–(2.6); compare (1.2)–(1.4) of [ChGv3].

From now on we choose $(1+\eta)^2 = (\frac{6}{5})^2$.

Next we choose a covering, $\{B_{\ell_a(p_\alpha)}(p_\alpha)\}$, and partition it into at most $N(n)$ mutually disjoint subcollections, S_i, of mutually nonintersecting balls, as in the proof of Theorem 2.3, on the existence of standard N-structures. We put $W_i = \bigcup_j B_{\frac{1}{2}\ell_a(p_{i,j})}(p_{i,j})$.

As in Section 2, we construct a partition of unity, $\{\phi_{i,j}\}$, satisfying

$$|\widetilde{\nabla}^k \phi_{i,j}| \leq c(n,k)(\ell_a(p_{i,j}))^{-k}\,;$$

compare (1.20), (1.21) of [ChGv3].

On each ball, $B_{\ell_a(p_{i,j})}(p_{i,j})$, all covariant derivatives of curvature are bounded for the metric $\ell_a^{-2}(p_{i,j})\widetilde{g}$. Hence, as in[ChGv3], for all i,j, we can smooth the restriction of the function, $\ell_a^{-1} f - \frac{1}{8}$, to the ball $B_{\ell_a(p_{i,j})}(p_{i,j})$. Combining these functions by means of the partition of unity, $\{\phi_{i,j}\}$, yields a function, F_0, satisfying

$$F_0 \le \ell_a^{-1} f \le F_0 + \frac{1}{4}\,,$$
$$|\nabla F_0|_{\widetilde{g}} \le \frac{4}{3}\ell_a^{-1}\,,$$
$$|\widetilde{\nabla}^k F_0|_{\widetilde{g}} \le c(n,k)\ell_a^{-k} \qquad (k \ge 2)\,;$$

compare (1.16)–(1.18) of [ChGv3].

Let $p_{i,j} \in S_i$ and consider the function, $(\ell_a(p_{i,j}))^{-1} f$, on the ball, $B_{\ell_a(p_{i,j})}(p_{i,j})$, with rescaled metric, $\ell_a^{-2}(p_{i,j})g$, for which we have

$$|\nabla F_0|_{\ell_a^{-2}(p_{i,j})\widetilde{g}} \le \frac{4}{3}\,, \tag{3.11}$$

$$|\widetilde{\nabla}^k F_0|_{\ell_a^{-2}(p_{i,j})\widetilde{g}} \le c(n,k) \qquad (k \ge 2)\,. \tag{3.12}$$

Since for the metric $\ell_a^{-2}(p_{i,j})\widetilde{g}$, the curvature and its covariant derivatives satisfy the bounds in (2.4)–(2.6), as in [ChGv3], we can apply the quantitative version of the A.P. Morse Lemma due to Yomdin, to obtain by induction: a sequence of functions, $F_{N(n)} \le F_{N(n)-1}, \ldots$, constants, $0 < \delta_{N(n)}(n) < \delta_{N(n)-1}(n), \ldots$, $0 < \epsilon_{N(n)} < \epsilon_{N(n)-1} < \ldots$ and $c(n,k)$, such that for $1 \le m \le N(n)$,

$$F_m \le \ell_a^{-1} f \le F_m + \frac{1}{4} + \frac{1}{4N(n)}\,,$$
$$|\widetilde{\nabla} F_m|_{\widetilde{g}} \le \left(\frac{4}{3} + \frac{m}{3N(n)}\right)\ell_a^{-1}\,,$$
$$|\widetilde{\nabla}^k F_m|_{\widetilde{g}} \le c_m(n,k)\ell_a^{-k}\,,$$

and for $x \in F_m^{-1}([0,\delta_m(n)]) \cap (W_1 \cup \ldots \cup W_m)$,

$$|\widetilde{\nabla} F_m|_{\widetilde{g}} \ge \epsilon_m(n) > 0\,;$$

compare (1.22)–(1.25) of [ChGv3].

If we put $F = F_{N(n)}$, then (3.7)–(3.10) hold. □

Let K be as above. Theorem 3.1 easily implies:

Theorem 3.13. *There exists $c = c(n) < \infty$ and a smooth manifold with boundary, Z^n, satisfying*

$$T_{\frac{1}{3}r}(K) \subset Z^n \subset T_{\frac{2}{3}r}(K)\,, \tag{3.14}$$

$$|II_{\partial Z}| \le c \cdot \left(r^{-1} + (r_{|R|})^{-1}\right)\,, \tag{3.15}$$

and for all $k_1, k_2 > 0$,

$$\int_{\partial Z} |II_{\partial Z}|^{k_1} |R|^{k_2} \le \frac{c}{r}\int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(r^{-(k_1+2k_2)} + (r_{|R|})^{-(k_1+2k_2)}\right). \tag{3.16}$$

Moreover, if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, then Z^n can be chosen to be saturated for some standard N-structure.

Proof. As in the proof of Theorem 3.1 in the equivariant case, we replace K by its saturation. By scaling, we can suppose $r = 1$.

Let ℓ_a be as in Theorem 3.1 and choose $a = \frac{1}{10}$. For each $s \in [\frac{2}{5}, \frac{3}{5}]$, we apply Theorem 3.1 with the set K replaced by $T_s(K)$. This yields a set, Z_s, for which (3.15), (3.14) hold.

To obtain a value of s for which (3.16) holds as well, we will estimate the integral with respect to s of the function which assigns to each s, the right-hand side of the version of (3.4) obtained by replacing K by $T_s(K)$. From this integral estimate, it will follow immediately that (3.16) holds for some $s \in [\frac{2}{5}, \frac{3}{5}]$. This will suffice to complete the proof.

Let $\{p_\alpha\}$ denote a maximal subset of $A_{\frac{1}{3},\frac{2}{3}}(K)$ such that

$$\overline{p_{\alpha_1}, p_{\alpha_2}} \geq \frac{1}{8} \cdot \min(\ell_a(p_{\alpha_1}), \ell_a(p_{\alpha_2})) \qquad (\alpha_1 \neq \alpha_2)\,.$$

By relative volume comparison together with rescaling, $\{B_{\frac{1}{8}\cdot \ell_a(p_i)}(p_i)\}$ is a covering of $A_{\frac{2}{5},\frac{3}{5}}(K)$, with multiplicity $\leq N$, a definite constant.

Let s_α be such that $p_\alpha \in \partial T_{s_\alpha}(K)$. Let $A \subset T_1(K) \times [0,1]$ be defined by

$$A = \{(x,s) \,|\, x \in S_a(T_s(K)\}\,.$$

We claim that

$$A \subset \bigcup_\alpha B_{\frac{1}{8}\cdot \ell_a(p_\alpha)}(p_\alpha) \times [s_i - \frac{1}{2}\ell_a(p_\alpha), s_\alpha]\,,$$

and in addition, that for $x \in B_{\frac{1}{8}\ell_a(p_i)}(p_i)$, we have

$$\ell_a(x) \geq \frac{7}{8} \cdot \ell_a(p_\alpha)\,.$$

This implies

$$\int_A (\ell_a)^{-(k_1+1)} (r_{|R|})^{-2k_2} \leq c \cdot \sum \int_{B_{\frac{1}{8}\ell_a(p_i)}(p_i)} (\ell_a)^{-(k_1)} (r_{|R|})^{-2k_2}\,.$$

As indicated above, the theorem follows.

To verify the claim, let $x \in S(T_s(K)) \cap B_{\frac{1}{8}\ell_a(p_\alpha)}(p_\alpha)$ for some s. Since $\operatorname{Lip} \ell_a \leq 1$, we have $\ell_a(x) \geq \frac{7}{8} \cdot \ell_a(p_\alpha)$. Moreover, $x \in B_{\frac{1}{8}\ell_a(q)}(q) \cap B_{\frac{1}{8}\ell_a(p_i)}(p_\alpha)$, for some $q \in \partial T_s(K)$. Since $\operatorname{Lip} \ell_a \leq 1$, it follows easily that $B_{\frac{1}{8}\ell_a(q)}(q) \subset B_{\frac{1}{2}\ell_a(p_\alpha)}(p_\alpha)$, which implies $s \geq s_\alpha - \frac{1}{2}\ell_a(p_\alpha)$. □

4. $L_{\frac{n}{2}}$-CURVATURE BOUNDS

In this section, for Einstein manifolds with $L_{\frac{n}{2}}$-curvature bounds, we give a pointwise bound on the scale, $r_{|R|}$, in terms of the maximal function $M_{|R|^{\frac{n}{2}}}$. This gives rise to an estimate on the boundary term of the Chern-Gauss-Bonnet formula applied to a good chopping, in terms of $(|R|_{\frac{n}{2}})^{\frac{n-1}{n}}$. We require some preliminaries concerning maximal functions.

Maximal functions. For (X,μ) a metric measure space, with μ a finite Radon measure, and $f \in L_1$, put

$$⨍_A |f| = \frac{1}{\mu(A)} \int_A |f|\,.$$

Define the maximal function for balls of radius at most r by

$$M_f(x,r) = \sup_{s \leq r} ⨍_{B_s(x)} |f|\,.$$

Let $W \subset X$ denote a measurable subset.

Lemma 4.1. *If every ball, $B_s(x)$, with $x \in W$, $s \leq 4r$, satisfies*

$$\mu(B_{2s}(x)) \leq 2^\kappa \mu(B_s(x)) \,,$$

then for all $\Omega \geq \mu(W)$, $\alpha < 1$,

$$\left(\frac{1}{\Omega}\int_W (M_f(x,r))^\alpha \, d\mu\right)^{\frac{1}{\alpha}} \leq \frac{c(\kappa,\alpha)}{\Omega}\int_{T_{6r}(W)} |f|\, d\mu. \tag{4.2}$$

Proof. Put

$$W_b = \{x \in W \mid M_f(x,r) \geq b\}\,.$$

By the weak-type $(1,1)$ inequality,

$$\mu(W_b) \leq 2^{3\kappa} \cdot b^{-1} \cdot \int_{T_{6r}(W)} |f|\, d\mu\,.$$

Fix a to be determined below. By writing

$$W = W_a \cup \left(\bigcup_{i=0}^{\infty}(W_{a\cdot 2^{i+1}} \setminus W_{a\cdot 2^i}\right)$$

and using the weak-type $(1,1)$ inequality, we get

$$\int_W (M_f(x,r)^\alpha)\, d\mu \leq a^\alpha \mu(W) + 2^{3\kappa}\cdot \sum_{i=0}^{\infty}(a\cdot 2^{(i+1)})^\alpha \cdot (a\cdot 2^i)^{-1} \cdot \int_{T_{6r}(W)} |f|\, d\mu\,.$$

Choosing

$$a = \frac{1}{\mu(W)}\cdot \int_{T_{6r}(W)} |f|\, d\mu$$

and summing the above geometric series gives (4.2) with $\Omega = \mu(W)$. Since $\alpha < 1$, this implies (4.2) for all $\Omega \geq \mu(W)$ as well. □

Bounding the local scale from below. For the remainder of this paper, we make the convention $\underline{p} \in M^n_{-1}$.

For $s \leq 1$, we have $\mathrm{Vol}(B_s(\underline{p})) \leq c(n)\cdot s^n$. Thus, from (1.19), (1.20), we get for any $0 < s \leq 1$ and $c = c(n)$,

$$\rho(p)^{-1} \leq c \cdot \max((M_{|R|^{\frac{n}{2}}}(p,s))^{\frac{1}{n}}, s^{-1})\,,$$

which together with (1.21), gives (1.24), namely,

$$(r_{|R|}(p))^{-(n-1)} \leq c\cdot (s^{-(n-1)} + (M_{|R|^{\frac{n}{2}}}(p,s))^{\frac{n-1}{n}})\,.$$

Chern-Gauss-Bonnet and the proof of relation (1.25). Let $s \leq r \leq 1$. By Theorem 3.13, we can approximate a compact subset, K, from the outside, by a submanifold with boundary, Z, with $K \subset Z \subset T_r(K)$, where the boundary term in the Chern-Gauss-Bonnet formula for Z satisfies the estimate (3.16). With (1.24), this gives for $c = c(n)$,

$$\left|\int_{\partial Z^n} TP_\chi\right| \leq c\cdot r^{-1}\cdot \int_{A_{\frac{1}{3}r,\frac{2}{3}r}(K)} \left(s^{-(n-1)} + \left(M_{|R|^{\frac{n}{2}}}(\,\cdot\,,s)\right)^{\frac{n-1}{n}}\right)\,. \tag{4.3}$$

By choosing $s = \frac{1}{512}r$ and employing (4.2) of Lemma 4.1, we get

$$\begin{aligned}(4.4)\qquad &\mathrm{Vol}(A_{0,r}(K))^{-1}\left|\int_{\partial Z^n} TP_\chi\right| \\ &\leq c\cdot\left(r^{-n} + r^{-1}\left(\frac{1}{\mathrm{Vol}(A_{0,r}(K))}\int_{A_{\frac{1}{4}r,\frac{3}{4}r}(K)} |R|^{\frac{n}{2}}\right)^{\frac{n-1}{n}}\right).\end{aligned}$$

From (4.4) and the Chern-Gauss-Bonnet formula, it follows that if $T_r(K)$ is (t,r)-collapsed with locally bounded curvature, then (1.25) holds.

Integrality of the geometric Euler characteristic. By an exhaustion argument as in [ChGv3], our generalized chopping theorem implies the following generalization of an application given in that paper.

Theorem 4.5. *Let M^n be a complete Einstein manifold with bounded Ricci curvature, finite volume and finite $L_{\frac{n}{2}}$-norm of curvature. Then*

$$(4.6)\qquad \int_{M^n} P_\chi \in \mathbf{Z}\,.$$

Proof. Let $K_0 \subset K_1 \subset \cdots$ denote an exhaustion of M^n. Apply (4.4) to each K_i, with $r = 1$ and note that the right-hand side goes to 0 as $i \to \infty$. By the Chern-Gauss-Bonnet formula, for i sufficiently large, we have

$$\int_{M^n} P_\chi = \chi(Z_i) \in \mathbf{Z}\,.$$

□

Remark 4.7. In [An3], it is asserted that (5.25) of that paper follows from the chopping theorem of [ChGv3]. However, since no proof is given that the curvature is uniformly bounded outside a compact subset, the chopping theorem of [ChGv3] cannot be applied. In actuality, (5.25) of [An3] is a special case of our Theorem 4.5 (valid in all dimensions) which does not require a global curvature bound. On the other hand, in the 4-dimensional case considered in [An3], the existence of a global curvature bound outside a compact subset does follow from our Theorem 9.1, a main result of the present paper; compare also Remark 10.10.

Remark 4.8. Suppose that the assumption that M^n is Einstein is weakened to

$$(4.9)\qquad |\mathrm{Ric}_{M^n}| \leq n-1\,.$$

Then the discussion can still be carried out. In particular, (4.4) and Theorem 4.5 continue to hold. The crucial point is the *local bounded covering geometry* in the sense of [ChFuGv]. Specifically, each point p has a neighborhood, U_p, containing a ball, $B_{r(n)}(p)$, of a definite size, such that the universal covering space, $\tilde{U}$, has $C^{1,\alpha}$-bounded geometry and $L_{2,p}$-bounded geometry, for all $p < \infty$; compare Remark 2.7. This follows from the existence of metric g_1, which is at bounded distance from g in the C^0-norm (i.e., bi-Lipschitz g with controlled bi-Lipschitz constant) such that g_1 has a definite bound on its curvature. Such a metric, g_1, is constructed in [Ya] via local Ricci flow; see also [Ab].

5. The key estimate in dimension 4

In this section we prove Theorem 1.26, the main new estimate on which our results in dimension 4 are based.

Lemma on sequences.

Lemma 5.1. *Let $0 \le \alpha < 1$. For $i = 0, 1, \ldots$, let a_i, b_i, x_i be nonnegative real numbers satisfying*

$$x_i \le a_i + b_i \cdot x_{i+1}^{\alpha} \,, \tag{5.2}$$

$$\liminf_{i \to \infty} x_i^{\alpha^i} = 1 \,. \tag{5.3}$$

Then

$$x_0 \le \max(2a_0, C_1, C_2) \,, \tag{5.4}$$

where

$$C_1 = \limsup_{i \ge 1} \left(\prod_{j=1}^{i-1} (2b_j)^{\alpha^j} \right) \cdot (2a_i)^{\alpha^i} \,, \tag{5.5}$$

$$C_2 = \limsup_{i \ge 0} \prod_{j=0}^{i} (2b_j)^{\alpha^j}. \tag{5.6}$$

Proof. By (5.2), we have

$$x_i \le \max(2a_i, 2b_i \cdot x_{i+1}^{\alpha}) \,.$$

Thus, for $i \ge 1$,

$$x_i^{\alpha^i} \le \max\left((2a_i)^{\alpha^i}, (2b_i)^{\alpha^i} \cdot x_{i+1}^{\alpha^{i+1}} \right) \,.$$

By induction, we get for all i,

$$x_0 \le \max(2a_0, C_{1,i}, C_{2,i}) \,,$$

where

$$C_{1,i} = \left(\prod_{j=0}^{i-1} (2b_j)^{\alpha^j} \right) \cdot (2a_i)^{\alpha^i} \,,$$

$$C_{2,i} = \left(\prod_{j=0}^{i} (2b_j)^{\alpha^j} \right) \cdot x_{i+1}^{\alpha^{i+1}} \,.$$

In view of (5.3), this suffices to complete the proof. □

If in particular, the sequence, $\{x_i\}$, is bounded, say $x_i \le C$, then the hypothesis, (5.3), is satisfied, and the lemma provides a bound on the initial term, x_0, which is *independent of* C. This is the situation in the application below where the following special case of Lemma 5.1 will suffice.

If for some constant $K \ge 1$,

$$\max(a_i, b_i, x_i) \le c \cdot K^i \,, \tag{5.7}$$

then we have the (nonsharp) bound

$$x_0 \le (2c)^{2(1+\alpha+\alpha^2+\cdots)} \cdot (2K)^{2(1+1\cdot\alpha+2\cdot\alpha^2+\cdots)} < \infty \,. \tag{5.8}$$

Proof of Theorem 1.26; an iteration argument.

Proof. By assumption, $T_1(E)$ is t-collapsed. Thus, we have (1.25) (proved in Section 4). We will apply (1.25) in each step of an iteration argument.

Since $n = 4$ we can use (1.1) to replace P_χ by $\frac{1}{8\pi^2}|R|^2$ on the left-hand side of (1.25), to get for some constant, c (independent of M^4)

$$\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(A_{0,1}(K))} ⨍_E |R|^2 \leq c \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))}\int_{A_{\frac{1}{4},\frac{3}{4}}(E)} |R|^2\right)^{\frac{3}{4}}\right). \tag{5.9}$$

For $i = 2, 3, \ldots$, put

$$D_i = \{p \in A_{2^{-i},1-2^{-i}}(E) \,|\, r_{|R|}(p) \leq 2^{-(i+1)}\}\,,$$
$$F_i = A_{2^{-i},1-2^{-i}}(E) \setminus D_i\,.$$

We have

$$T_{2^{-(i+1)}}(D_i) \subset A_{2^{-(i+1)},1-2^{-(i+1)}}(E)\,.$$

Moreover, $\mathrm{Lip}\, r_{|R|} \leq 1$ implies

$$\sup_{T_{2^{-(i+1)}}(D_i)} r_{|R|} \leq 2^{-i}\,.$$

Since $A_{0,1}$ is t-collapsed with locally bounded curvature, it follows that $T_{2^{-(i+1)}}(D_i)$ is $(t, 2^{-(i+1)})$-collapsed with locally bounded curvature. Hence, (1.25) implies that (5.9) holds with E replaced by D_i. By splitting the integral on the left-hand side of (5.10) below into a sum of integrals over D_i and F_i, and applying (5.9) to the former, we get

$$\begin{aligned}&\int_{A_{2^{-i},1-2^{-i}}(E)} |R|^2 \\ &\leq c \cdot 2^{4i} \cdot \mathrm{Vol}(A_{0,1}(E)) \cdot \left(1 + \left(\frac{1}{\mathrm{Vol}(A_{0,1}(E))}\int_{A_{2^{-(i+1)},1-2^{-(i+1)}}(E)} |R|^2\right)^{\frac{3}{4}}\right).\end{aligned} \tag{5.10}$$

From (5.9), (5.10) and Lemma 5.1, we obtain (1.28), which concludes the proof of Theorem 1.26. □

Remark 5.11. By using the observation in Remark 1.4, together with Remark 4.8, it follows easily that Theorem 1.26 can be extended to the case in which the Einstein condition is dropped and (0.2) with the assumption $|\lambda| \leq 3$ is replaced by (4.9) with the assumption $|\mathrm{Ric}_{M^4}| \leq 3$. Indeed, the effect of this change is just to add a definite constant to the right-hand sides of (5.9), (5.10).

Remark 5.12. We are grateful to Fang-Hua Lin for pointing out to us the formal similarity between our iteration argument and the one used in proving Theorem 2.1 in [LiSch], which provides a mean value inequality for subharmonic functions.

6. Collapse implies L_2 concentration of curvature

In this Section we prove Theorem 0.1, one of our main results on collapsing.

Proof of Theorem 0.1. Let C be as in (0.3). From the discussion of Sections 2–4, and a standard covering argument, we get under the assumptions of Theorem 0.1, that there exists $v > 0$, $\beta > 0$, $p_1, \ldots, p_N$, such that $N \leq \beta C$, and such that if the metric is rescaled, $g \to s^{-2}g$, then the hypothesis of Theorem 1.26 holds with $E = M^4 \setminus \bigcup_i B_s(p_i)$.

Theorem 0.1 follows immediately from Theorem 1.26 (the key estimate); see in particular (1.28). □

Remark 6.1. There are many examples of collapsing sequences of Ricci flat Kähler metrics on $K3$ surfaces with fixed lower diameter bound; see e.g. [GsWi]. The behavior of any such collapsing sequence is governed by Theorem 0.1 and Theorem 9.1 (which relies on Theorem 0.8).

7. Negative Einstein constant; noncollapse and exponential decay

We begin this section with the proof of Theorem 0.14 and some related remarks. Next we recall some standard facts which, in combination with Theorem 0.1, yield Theorem 7.8, our noncollapsing result in the Kähler case.

We also prove a result on exponential decay of the volume and of the L_2-norm of the curvature, for Einstein 4-manifolds with $\lambda = -3$.

Proof of Theorem 0.14; noncollapse.

Proof. It suffices to assume that the hypothesis of Theorem 0.1, say for the case $s = 1$, is satisfied. By (1.1), if $\lambda = \pm 3$, then the left-hand side of (0.6) is bounded below by

$$\frac{1}{6}\left(\mathrm{Vol}(M^4) - \sum_i \mathrm{Vol}(B_1(p_i))\right).$$

Now the claim follows from (0.6). □

Remark 7.1. Recall that Theorem 0.14 is a partial replacement in dimension 4 for the Heintze-Margulis theorem, whose proof, while employing the relation between nilpotency and collapse, rests otherwise on considerations which are entirely different from ours.

Remark 7.2. In any dimension, if the Ricci curvature has a positive lower bound, then (absent any further assumptions) a bound on the diameter is provided by Myers' theorem. If $\mathrm{Ric}_{M^n} \geq n-1$, then a relative volume comparison, [GvLP], gives an inequality in which the constant does not depend on C, namely,

$$\mathrm{Vol}(B_1(p)) \geq \frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_\pi(\underline{p}))} \cdot \mathrm{Vol}(M^n) \qquad (\text{for all } p \in M^n).$$

Remark 7.3. We do not know whether in the absence of an a priori lower bound on $\mathrm{Vol}(M^4)$, collapse with Einstein constant ± 3 can occur.

Noncollapse in the Kähler case. In the Kähler case, the volume can be expressed in terms of the Kähler class. Namely,

$$\mathrm{Vol}(M^n) = \frac{1}{(n/2)!}[\omega]^{\frac{n}{2}}(M^n)\,. \tag{7.4}$$

However, for a sequence of Kähler metrics for which the Kähler class degenerates, the volume can go to zero.

If, in addition, M^n is Einstein and the Einstein constant does not vanish, we make the normalization,

$$\mathrm{Ric}_{M^n} = \pm(n-1)g\,. \tag{7.5}$$

Then the first Chern form, $c_1(R)$, satisfies

$$\begin{aligned} c_1(R) &= \frac{1}{2\pi}\mathrm{Ric}_{M^n}(J,\,\cdot) \\ &= \pm\frac{(n-1)}{2\pi}\omega \end{aligned} \tag{7.6}$$

and

$$\mathrm{Vol}(M^n) = \left(\frac{2\pi}{n-1}\right)^{\frac{n}{2}} |c_1^{\frac{n}{2}}(M^n)|\,. \tag{7.7}$$

Thus we obtain:

Theorem 7.8 (Noncollapsing for $c_1 \neq 0$)**.** *Let M^4 denote a Kähler-Einstein manifold with $c_1 \neq 0$, satisfying* (0.15)*. Then for w as in* (0.16)*, M^4 is not ϑ-collapsed, where*

$$\vartheta = \frac{w}{4}\cdot\frac{c_1^2(M^4)}{\chi(M^4)}\,. \tag{7.9}$$

Proof. Theorem 7.8 follows immediately from Theorem 0.14, together with (7.7). □

For more general lower volume bounds which follow from Seiberg-Witten theory, see [LeBru1], [LeBru2], [Tau], [Wi].

Exponential decay of volume. If $\lambda = -3$ and (0.3) holds, then (1.1) implies $\mathrm{Vol}(M^4) < \infty$.

Theorem 7.10 (Exponential decay of volume)**.** *There exist β, $\gamma > 0$, c, such that if M^4 denotes a complete Einstein* 4*-manifold satisfying* (0.3)*,* (0.15)*, then there exist $p_1, \ldots, p_N$, with*

$$N \leq \beta\cdot C\,, \tag{7.11}$$

such that for $r \geq 5$,

$$\mathrm{Vol}(M^4 \setminus \bigcup_i B_r(p_i)) \leq c\cdot C\cdot e^{-\gamma r}\,. \tag{7.12}$$

Proof. Consider the collection of balls, $B_1(p)$, such that

$$\int_{B_1(p)} |R|^2 \geq \frac{1}{6}\cdot\theta\cdot t\cdot\tau\,,$$

where the notation is as in (1.18). By a standard covering argument there exists a disjoint subcollection, $\{B_1(p_i)\}$, such that the original collection is contained in

$\bigcup_i \overline{B_5(p_i)}$. Since the collection, $\{B_1(p_i)\}$, is disjoint and (0.3) holds, there are at most N such balls, where N satisfies (7.11) for suitable β.

Because we assume (0.15), i.e., $\lambda = \pm 3$, it follows from (1.18) that the set, $M^4 \setminus \bigcup_i \overline{B_5(p_i)}$, is t-collapsed with locally bounded curvature.

Let c be as in (1.28) of Theorem 1.26. Let t denote the smallest integer such that $t > 2c$. For $i = 1, \ldots, t$, apply Theorem 1.26 to each of the sets, $M^4 \setminus \bigcup_i \overline{B_{r+i}(p_i)}$, add the inequalities corresponding to (1.28), and use

$$\fint_{M^4 \setminus \bigcup_i \overline{B_{r+i}(p_i)}} |R|^2 \geq 24\,.$$

The theorem follows. $\square$

Remark 7.13. As mentioned to us by M. Gromov, by the theorem of J. Lohkamp asserting the C^0-density of metrics with negative Ricci curvature in the space of all Riemannian metrics, a negative upper bound on the Ricci curvature is not in general sufficient to guarantee exponential decay of the volume.

Remark 7.14. In view of Theorem 1.26, it follows from Theorem 7.10 that the square of the L_2-norm of the curvature decays exponentially as well.

Sufficiently pinched Ricci curvature.

Remark 7.15. Since the results of this section are essentially formal consequences of Theorem 1.26, they extend to the case in which the Einstein condition is dropped and the assumption, $|\lambda| = 3$, is replaced by the assumption that the Ricci tensor is sufficiently pinched: $0 < a \leq |\mathrm{Ric}_{M^4}| \leq 3$. The particular constants in the conclusions depend on the pinching.

8. ϵ-REGULARITY

In this section, we prove Theorem 0.8, the improved ϵ-regularity theorem.

Proof of Theorem 0.8. In order to apply the results of [ChCoTi2] directly in proving Proposition 8.2 below, it will be convenient to make the following reduction.

Let $0 < \eta < \frac{1}{2}$. If $q \in B_{\frac{1}{2}}r(p)$, then $B_{\eta r}(q) \subset B_r(p)$. Clearly, it suffices to prove the theorem for all such $B_{\eta r}(q)$. As a consequence, we may assume that $r \leq \eta$, for some fixed $\eta < \frac{1}{2}$ (and continue to consider $B_r(p)$). An appropriate value of η will be determined in Proposition 8.2.

The argument has two main steps.

Step 1. The first step is the reduction in (8.1) below, which can be viewed as the analog in our context of a critical initial step in Gromov's proof of his celebrated theorem on almost flat manifolds — a short loop with holonomy which is not too big actually has holonomy comparable to its length.

Under the assumptions of Theorem 0.8, either the hypothesis of Theorem 4.4 of [An3] holds or we can assume that after rescaling, $g \to (2r)^{-2} g$, the assumptions of Theorem 1.26 are satisfied, with $E = B_1(p)$, $T_1(E) = B_2(p)$. In the latter case, by Theorem 1.26, there exists a definite constant, c, as in (1.28), such that

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2 \leq c\,. \tag{8.1}$$

Step 2. We will show that if one passes to a smaller concentric ball, whose radius can be estimated from below, then the hypothesis, (1.12), of Theorem 4.4 of [An3] is satisfied. An application of that theorem will complete the proof of Theorem 0.8. (Step 2 does not rely on (4.4); compare however (8.9).

Our claim is a direct consequence of the following proposition, in which a crucial point is the absence from the hypothesis of a lower volume bound. The proposition asserts that if on some interval, certain conditions are verified, then as r decreases, the quantity appearing on the left-hand side of (1.12) (and (8.5) below) decays at a definite rate. (It is this quantity which the hypothesis of Theorem 4.4 of [An3] requires to be $\leq \tau$.)

Proposition 8.2. *For all $C_1 > 0$, there exists $\eta = \eta(C_1) > 0$, such that if*

$$0 < r \leq \eta\,, \tag{8.3}$$

$$\int_{B_r(p)} |R|^2 \leq 4\pi^2\,, \tag{8.4}$$

$$\frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2 \leq C_1\,, \tag{8.5}$$

$$\frac{\mathrm{Vol}(B_{\eta r}(p))}{\mathrm{Vol}(B_{\eta r}(\underline{p}))} \leq \frac{1}{4}\,, \tag{8.6}$$

(8.7)

then

$$\frac{\mathrm{Vol}(B_{\eta r}(\underline{p}))}{\mathrm{Vol}(B_{\eta r}(p))} \int_{B_{\epsilon r}(p)} |R|^2 \leq (1-\eta) \frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} |R|^2\,. \tag{8.8}$$

Proof. By scaling, we can suppose $r = 1$, $\mathrm{Ric}_{M^4} \geq -3\eta^2$, $\underline{p} \in \underline{M}^4_{-\eta^2}$.

Under the assumption that for some sufficiently small $\epsilon > 0$, (8.4)–(8.6) hold but (8.8) fails, we will construct a closed submanifold with boundary, $U \subset B_1(p)$, for which the boundary term, $TP_\chi(\partial U)$, in the Chern-Gauss-Bonnet formula satisfies

$$0 < \int_{\partial U} TP_\chi < \frac{1}{2}\,. \tag{8.9}$$

Since, by (1.1), (8.4), we also have

$$0 < \int_U P_\chi < \frac{1}{2}\,, \tag{8.10}$$

the Chern-Gauss-Bonnet formula gives $0 < \chi(U) < 1$, a contradiction.

Assume that for some $\eta > 0$, (8.4)–(8.6) hold but (8.8) fails. Since each of the factors in the quantity on the left-hand side of (8.5) is a nondecreasing function of r, it follows that

$$\frac{\mathrm{Vol}(B_{\frac{1}{4}}(\underline{p}))}{\mathrm{Vol}(B_{\frac{1}{4}}(p))} \geq (1-\eta) \frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_1(p))}\,, \tag{8.11}$$

$$⨍_{B_1(p)\setminus B_{\frac{1}{4}}(p)} |R|^2 \leq c \cdot \eta\,, \tag{8.12}$$

for some absolute constant c (arising from a relative volume comparison in dimension 4).

We can assume $c \cdot \eta^{\frac{1}{2}} < c \cdot \tau$, where τ is the constant in (1.12) and c is chosen so small that the hypothesis of Theorem 4.4 of [An3] is valid for balls $B_r(q) \subset (B_1(p) \setminus B_{\frac{1}{4}}(p))$. Thus, for some absolute constant, c_1, we have the pointwise curvature bound

$$|R| \leq c_1 \cdot \eta^{\frac{1}{2}} \qquad (\text{on } B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{2}}(p))\,. \tag{8.13}$$

By (8.11), the set, $A_{\frac{1}{4},1}(p)$, is an almost volume annulus. Hence we can make use of the constructions underlying the proof of the "almost volume annulus implies almost metric annulus" theorem of [ChCo0], as well as subsequent related constructions of [ChCoTi2]. (The reduction, $r \leq \eta$, and hence the assumption (8.3) enables us to quote directly from [ChCoTi2]. As in Sections 2, 3 of [ChCoTi2], this normalization leads to the function, $\mathbf{r}$, appearing in (8.14)–(8.17). By the same token, after our rescaling, $g \to r^{-2}g$, we have $\mathrm{Ric}_{M^4} \geq -3\eta^2$ and the above-mentioned metric annulus lies in a metric cone.)

Let $\Psi = \Psi(\eta) > 0$ denote some definite function (independent of M^4) such that $\Psi \to 0$ as $\eta \to 0$.

According to Section 4 of [ChCo0] and Sections 2, 3 of [ChCoTi2], there exists Ψ and a function, $\mathbf{r} : B_{\frac{3}{4}}(p) \setminus B_{\frac{1}{2}}(p) \to [0,1]$, such that

$$\Delta \mathbf{r}^2 = 8\,, \tag{8.14}$$

$$|\mathbf{r} - r| < \Psi\,, \tag{8.15}$$

$$⨍_{\mathbf{r}^{-1}(a)} |\nabla \mathbf{r} - \nabla r|^2 < \Psi\,, \tag{8.16}$$

$$|\nabla \mathbf{r}| < c\,. \tag{8.17}$$

In addition, the following holds. For a a regular value of $\mathbf{r}$, denote by $g_{\mathbf{r}^{-1}(a)}$, the induced metric at points of $\mathbf{r}^{-1}(a)$, and by $II_{\mathbf{r}^{-1}(a)}$, the second fundamental form. Then for some subset, $A \subset [\frac{1}{2}, \frac{3}{4}]$, of regular values of $\mathbf{r}$, if $a \in A$, we have

$$\left|1 - \frac{\mathrm{Vol}(\mathbf{r}^{-1}(a))}{\mathrm{Vol}(\partial B_a(p))}\right| \leq \Psi\,, \tag{8.18}$$

$$⨍_{\mathbf{r}^{-1}(a)} |II_{\mathbf{r}^{-1}(a)} - \frac{1}{\mathbf{r}} g_{\mathbf{r}^{-1}(a)} \otimes \nabla \mathbf{r}|^2 \leq \Psi\,. \tag{8.19}$$

It is important to note that the integral in (8.19) is normalized by volume. To see (8.18), (8.19) observe that the key assumption of [ChCo0], assumption (4.10) of that paper, concerns a *ratio* of volumes and that the quantities appearing in all subsequent estimates are normalized by volume, as are all estimates of Sections 2, 3 of [ChCoTi2] (which depend on the estimates of Section 4 of [ChCo0]). Note in particular that this holds for (4.84) of [ChCo0], which is the L_2 estimate on the Hessian of the function $\mathbf{r}^2$. Estimate (8.18) is just a restatement of (3.11), which is part of the conclusion of Theorem 3.7 of [ChCoTi2].

Since the boundary term, $TP_\chi(\mathbf{r}^{-1}(a))$, in the Chern-Gauss-Bonnet formula contains terms that are of degree 3 in the second fundamental form, $II_{\mathbf{r}^{-1}(a)}$, from (8.13), (8.18), (8.19), it does *not* follow immediately that

$$\left|\int_{\mathbf{r}^{-1}(a)} TP_\chi - \frac{\mathrm{Vol}(\mathbf{r}^{-1}(a))}{\mathrm{Vol}(\partial B_a(\underline{p}))}\right| \leq \Psi \cdot \mathrm{Vol}(\partial B_a(\underline{p}))\,. \tag{8.20}$$

However, since the curvature on the annulus, $B_{\frac{3}{4}}(p)\setminus\overline{B_{\frac{1}{2}}(p)}$, is uniformly bounded, this annulus has local bounded covering geometry in the sense of [ChFuGv]. Thus, there exists $\underline{s}>0$ (independent of M^4) such that for all $q\in B_{\frac{3}{4}}(p)\setminus\overline{B_{\frac{1}{2}}(p)}$, the universal covering space, $\widetilde{B_{\underline{s}}(q)}$, of $B_{\underline{s}}(q)$ has C^∞ bounded covering geometry with respect to the pull-back metric. In particular, the injectivity radius on $\widetilde{B_{\underline{s}}(q)}$ has a definite positive lower bound. By pulling back the function, $\mathbf{r}$, to $\widetilde{B_{\underline{s}}(q)}$, we reduce to the noncollapsed case.

Now, just as in Section 3 of [ChCoTi2], we can argue by contradiction. After passing to a suitable subsequence, a sequence of counterexamples (in manifolds M_i^4) would converge in the C^∞-topology to a portion of an annulus in a flat cone, and the corresponding functions, $\mathbf{r}_i$, would converge in the C^∞-topology to the distance function from the vertex of this flat limit cone. For all $a\in[\frac{1}{2},\frac{3}{4}]$, this implies convergence of second fundamental forms for sequences of level surfaces, $\mathbf{r}_i^{-1}(a)\to\mathbf{r}^{-1}(a)$, a contradiction.

Thus, taking $U=\mathbf{r}^{-1}((0,a])$, we get (8.20), and hence, (8.9). □

By combining (8.1) with Proposition 8.2, the proof of Theorem 0.8 can be reduced to an application of Theorem 4.4 of [An3]. □

Remark 8.21. Already in deriving (8.13), we made use of Theorem 4.4 of [An3] in a situation in which no a priori lower bound on volume is assumed; compare Remark 1.31. This was made possible by the initial reduction given in (8.1).

Bounded Ricci curvature.

Remark 8.22. Theorem 0.8 can be extended to the case in which the Einstein condition is dropped, the assumption $|\lambda|\le 3$ is replaced by $|\mathrm{Ric}_{M^4}|\le a$, provided in the conclusion, and the condition, $|R|\le c$, is replaced by $C^{1,\alpha}$ bounded covering geometry, $\alpha<1$, or $L_{2,p}$-bounded covering geometry, $p<\infty$. Additionally, one can deduce a definite bound on the L_p-norm of curvature for all $p<\infty$.

To see this, note that in view of Remark 5.11, the only point in the argument which requires modification is (8.10) in the proof of Proposition 8.2. Although P_χ need not be a positive multiple of $|R|^2\cdot\mathrm{Vol}(\,\cdot\,)$, this continues to hold up to an error that is bounded by a definite multiple of $|\mathrm{Ric}_{M^4}|^2$. The neighborhood, U, is close in the Gromov-Hausdorff sense to a ball with center the vertex of a flat cone (which might be very collapsed). Thus, the error term is bounded by a definite multiple of the volume of a small neighborhood of the center of U. Thus, the error term is not only small, but small with respect to the area of the boundary. Once again, we get $0<\chi(U)<1$, a contradiction.

9. Consequences of ϵ-regularity

From Theorem 0.8 and a standard covering argument, we get:

Theorem 9.1 (Bound on the number of blowup points). *There exist $c>0$, $\beta>0$, such that if M^4 denotes a complete Einstein 4-manifold satisfying* (0.2), (0.9), *then there exist $p_1,\dots,p_N$, with*

$$N\le\beta\cdot C\,,\tag{9.2}$$

such that for all q,

$$|R(q)| \leq c \cdot \sup_{p_\alpha} \max\left((\overline{q, p_\alpha})^{-2}, 1 \right) . \tag{9.3}$$

Moreover, if $\lambda = 0$, *then*

$$|R(q)| \leq c \cdot \sup_{p_\alpha} (\overline{q, p_\alpha})^{-2} . \tag{9.4}$$

Remark 9.5. For $n > 4$, analogs of the above results are conjectured; compare Section 11. But there remains the possibility that if M^n is sufficiently collapsed relative to the size of its diameter, the pointwise norm of the curvature might be arbitrarily large for all $p \in M^n$. Similarly, for $n > 4$, the Gromov-Hausdorff limit of a collapsing sequence with a uniform bound on diameter might conceivably have no points with locally Euclidean neighborhoods.

Remark 9.6. One may ask whether for complete noncompact Ricci flat manifolds satisfying (0.3), the curvature estimate, (9.4), can be improved, i.e., if decay is actually faster than quadratic. This is known to hold if, in addition, the volume growth is Euclidean; see e.g. [BaKaNa], [ChTi1].

Remark 9.7. By employing the theory of collapse with bounded curvature, it can be shown that for M^4 a complete noncompact Ricci flat 4-manifold satisfying (0.3), with sub-Euclidean volume growth, every tangent cone, with the base point deleted, can be written locally as the quotient of a noncollapsed Ricci flat 4-manifold by a group of isometries whose identity component is nilpotent.

Similarly, in the case of negative Einstein constant, $\lambda = -3$, it follows that if M^4 satisfies (0.3), (0.15), then for any sequence, $p_i \to \infty$, there is a subsequence, $\{p_{i_j}\}$, such that (M^4, p_{i_j}) converges in the Gromov-Hausdorff sense to space Y, which is locally the quotient of a smooth Einstein 4-manifold with bounded curvature by a group of isometries, whose identity component is nilpotent.

Note that in certain important special cases, Einstein metrics with Killing fields on 4-manifolds are known to be given by a local ansatz; see e.g. [GibHaw], [CaPe]. We intend to discuss these matters at greater length elsewhere, including issues which are more global in nature.

10. Moduli spaces

In this section, we discuss the implications of our main theorems for compactifications of the moduli spaces of Einstein metrics on a given compact 4-manifold M^4. Much of our discussion also applies to complete Einstein metrics with finite L_2-norm of curvature; compare Remarks 9.6, 9.7. Our approach uses Gromov-Hausdorff convergence of sequences, (M^4, g_k), or in case there is no a priori bound on the diameter, $\underline{N}$-pointed Gromov-Hausdorff convergence.

Remark 10.1. Our results have an obvious extension to Gromov-Hausdorff limits (respectively $\underline{N}$-pointed Gromov-Hausdorff limits) of sequences, (M_k^4, g_k), in which the underlying manifold is not fixed. All that is actually required is a bound, $\chi(M_k^4) \leq C$, and for noncollapsing theorems, a lower bound on volume: $\mathrm{Vol}(M_k^4) \geq v > 0$.

The completion of the moduli space of Einstein metrics on a *fixed* 4-manifold is studied in [An3] using the extrinsic L_2 metric on the moduli space. By way of comparison with the present approach based on $\underline{N}$-pointed Gromov-Haudorff convergence, we note that for (M^4, g_i) a sequence with $\mathrm{diam}(M^4, g_i) \to \infty$, the assumption that (M^4, g_i) converges with respect to the extrinsic L_2 metric provides a strong additional constraint on the sequence; for further discussion see Remark 10.10.

Among other important theorems, [An3] contains the first written results on the collapsing case, notably, collapse with locally bounded curvature and hence, by [ChGv2], the presence of an F-structure away from a definite number of points. One key consequence of our main results is that "locally bounded curvature" can be replaced by "bounded curvature".

In [An3], a qualitative analogy with the case of constant curvature metrics on Riemann surfaces is proposed. The analogy is, of course, not complete, since in dimension 4, the curvature can concentrate and the moduli space can have positive dimension for $\lambda > 0$.

Gromov-Hausdorff compactifications. Let M^4 denote a smooth compact connected 4-manifold. For fixed $\lambda > 0$, let $\mathcal{M}(M^4, \lambda)$ denote the moduli space of isometry classes of Einstein metrics on M^4, with Einstein constant λ. For $\lambda \leq 0$, there is no a priori bound on the diameter, so we consider the moduli space of $\underline{N}$-pointed isometry classes $\mathcal{M}(M, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, \lambda)$. Two $\underline{N}$-pointed Einstein manifolds, $(M, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, g)$, $(M, \underline{m}'_1, \ldots, \underline{m}'_{\underline{N}}, g')$, are $\underline{N}$-pointed isometric if there exists an isometry between (M, g), (M, g'), carrying $\underline{m}_j$ to $\underline{m}'_j$, for all j.

Below, for $\lambda \neq 0$, we make the normalization $|\lambda| = 3$, which can be achieved by scaling. Due to scale invariance of the condition, $\lambda = 0$, in that case, we impose the additional normalization $\mathrm{Vol}((M^4, g)) = 1$.

For $\lambda = 3$, the space $\mathcal{M}(M^4, \lambda)$ carries certain natural metrics. Here, we use the weakest of these, the Gromov-Hausdorff metric. Since 2) below does in fact hold (i.e., at regular points, weak convergence implies strong convergence) this yields the strongest possible results. With respect to the Gromov-Hausdorff metric, bounded subsets of $\mathcal{M}(M^4, \lambda)$ are typically incomplete. For $\lambda \leq 0$, there is no a priori bound on the diameter of an Einstein manifold, (M^4, g), and the diameter of $\mathcal{M}(M^4, \lambda)$ with respect to the Gromov-Hausdorff metric is typically infinite. In this case, we employ the topology of $\underline{N}$-pointed Gromov-Hausdorff convergence. We write $(M_k, g_k, \underline{m}_{k,1}, \ldots, \underline{m}_{k,\underline{N}}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$ whenever $(M_k, g_k, \underline{m}_{k,j}) \stackrel{d_{GH}}{\to} (Y_j, \underline{y}_j)$, for all $1 \leq j \leq \underline{N}$, i.e., whenever $(M_k, g_k, \underline{m}_{k,j})$ converges to $(Y_j, \underline{y}_j)$ in the pointed Gromov-Hausdorff sense. Since we allow $\underline{N} > 1$ and $\overline{m_{k,j_1}, m_{k,j_2}} \to \infty$, for all $j_1 \neq j_2$, an understanding of the possible limiting collections, $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$, provides a global picture of the degenerating sequence.

By using Gromov's compactness theorem, bounded subsets of $\mathcal{M}(M^4, \lambda)$ can be completed by adding suitable compact connected length spaces Y. The completion of such a bounded subset is compact.

For $\lambda \leq 0$, using the pointed version of Gromov's compactness theorem, the completion of $\mathcal{M}(M^4, \underline{m}_1, \ldots, \underline{m}_{\underline{N}}, \lambda)$ can be compactified by adding certain collections of noncompact connected pointed length spaces $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$. Namely, we add such a collection whenever there exists a sequence, $(M^4, g_k, \underline{m}_{k,j})$, such that $(M^4, g_k, \underline{m}_{k,1}, \ldots, \underline{m}_{k,\underline{N}}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$.

In the above formulation the two most basic issues are the following; see below for further amplification.

1) Describe explicitly the geometric structure of the collections of pointed spaces, $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$, which must be added in the compactification.

2) Show that near regular points, $y \in Y$, the convergence is actually in the strongest possible sense.

In the Kähler case, additional important issues involving the complex structure arise, but will not be discussed here.

In what follows, the notion of *concentration point* of the curvature plays an important role. Let $(M^4, g_k, \underline{m}_{k,1}, \ldots, \underline{m}_{k,\underline{N}}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$ as above. We say $y_j \in Y_j$ is *not* a concentration point of curvature, if for some subsequence, $(M^4, g_s, \underline{m}_{s,j}) \stackrel{d_{GH}}{\to} (Y_j, \underline{y}_j)$, the point, y_j, is *not* the point limit of a sequence of points, $p_{s,\alpha}$, with $p_{s,\alpha}$ a blowup point of the curvature as in (9.3) of Theorem 9.1. Thus, by Therem 9.1, if y_j is not a concentration point, then near y_j, the space, Y_j, is the Gromov-Hausdorff limit of a sequence of spaces with *bounded curvature.*

Let $\{y_{j,\alpha_j}\}$ denote the set of concentration points in Y_j. Relation (9.2) of Theorem 9.1 implies that the cardinality, N, of the set, $\bigcup_j \{y_{j,\alpha_j}\}$, satisfies $N \leq \beta \cdot 8\pi^2 \cdot \chi(M^4)$, for some absolute constant, β, independent of M^4.

Noncollapsed limit spaces. Let Y^4 (respectively, $\{(Y_1, y_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$ denote some noncollapsed Gromov-Hausdorff limit (respectively, $\underline{N}$-pointed Gromov-Hausdorff limit) of a sequence of compact Einstein metrics on a fixed compact manifold M^4. Then Y^4 (respectively, Y_j^4) is known to be a connected smooth Einstein manifold away from the set of concentration points of the curvature, at which the singularities are of orbifold type.

Away from the concentration points, for all k, the convergence takes place in the C^k topology on compact subsets. In particular, 2) above has a positive answer in these instances; for the above, see [An1], [An3], [Na], [Ti]; compare also [AnCh].

Next, we describe those cases in which noncollapsed limit spaces are known to arise.

For $\lambda = 3$, by Myers' theorem, there is an a priori bound on the diameter and the completion of $\mathcal{M}(M^4, 3)$ is itself compact. Thus, those Y which arise as limit points are compact as well. In the Kähler case, $\mathrm{Vol}((M^4, g))$ has an a priori lower bound, which implies that $Y = Y^4$ is noncollapsed. Seiberg-Witten theory provides lower volume bounds under more general assumptions, e.g. if M^4 admits a symplectic structure; see [LeBru1], [LeBru2], [Tau], [Wi]. In the general case, the question of whether Y can be collapsed remains open.

Let $\lambda = 0$. Recall that in this case, we impose the additional constraint, $\mathrm{Vol}((M^4, g)) = 1$. Since complete noncompact manifolds with nonnegative Ricci curvature have infinite volume, it follows from relative volume comparison that if $(Y, \underline{y})$ is a pointed limit space, then Y is noncompact if and only if it is everywhere collapsed. Moreover, given a sequence of 2-pointed Einstein metrics, $(M^4, g_{k,j}, \underline{m}_{k,j}) \stackrel{d_{GH}}{\to} \{(Y_1, \underline{y}_1), (Y_2, \underline{y}_2)\}$, where possibly, $\overline{m_{i,1}, m_{i,2}} \to \infty$, it follows that Y_1 is collapsed if and only if it is noncompact, and this holds if and only if Y_2 is collapsed and noncompact. If $Y_1 = Y_2 = Y^4$ is compact, and hence noncollapsed, then the above discussion applies. The noncompact collapsed case will be discussed below.

As a particular example, recall that in the case of $K3$ surfaces, both collapsing and noncollapsing behavior can occur.

In case $\lambda = -3$, we have the following basic results.

Theorem 10.2. *Let $\lambda = -3$ and let $\{(M^4, g_i)\}$ satisfy*

$$\operatorname{diam}(M^4, g_i) \to \infty\,. \tag{10.3}$$

If for some $v > 0$,

$$\operatorname{Vol}((M^4, g_i)) \geq v\,, \tag{10.4}$$

then there exists a pointed subsequence, $(M^4, g_k, \underline{m}_k) \overset{d_{GH}}{\to} (Y^4, \underline{y})$, where Y^4 is a noncollapsed complete noncompact orbifold. Outside of a compact subset, the curvature of Y^4 is bounded and (9.4) of Theorem 9.1 holds. If the (M^4, g_i) are Kähler, then (10.4) is satisfied.

Proof. This follows immediately from Theorems 0.14 and 9.1. □

Theorem 10.5. *Let $\lambda = -3$. There exist $\underline{\beta}, c > 0$ with the following properties. Let $(M^4, g_k, \underline{m}_{k,1}, \dots, \underline{m}_{k,\underline{N}}) \overset{d_{GH}}{\to} \{(Y_1^4, \underline{y}_1), \dots, (Y_{\underline{N}}^4, \underline{y}_{\underline{N}})\}$, where $\overline{m_{i,j_1}, m_{i,j_2}} \to \infty$, for all $j_1 \neq j_2$, and Y_j^4 is noncollapsed for all j. Then*

$$\underline{N} \leq \underline{\beta} \cdot \chi(M^4)\,. \tag{10.6}$$

If Y_j^4 is smooth for some j, then

$$\int_{Y_j^4} |R|^2 \geq c\,. \tag{10.7}$$

If $\underline{N}$ is chosen as large as possible such that the above hypotheses are satisfied, then

$$\lim_{k\to\infty} \operatorname{Vol}(M^4, g_k) = \operatorname{Vol}(Y_1^4) + \cdots + \operatorname{Vol}(Y_{\underline{N}}^4)\,. \tag{10.8}$$

Proof. As above, each Y_j^4 is a complete Einstein orbifold with finite volume and the convergence is smooth away from at most a definite number of points at which the curvature concentrates. It follows that

$$\lim_{k\to\infty} \operatorname{Vol}(M^4, g_k) \geq \operatorname{Vol}(Y_1^4) + \cdots + \operatorname{Vol}(Y_{\underline{N}}^4)\,.$$

Relation (10.7) follows from Theorem 4.5.

These statements imply (10.6). In fact, if we assume

$$\underline{N} \geq \frac{16\pi^2}{c} \cdot \chi(M^4) + \beta \cdot 8\pi^2 \cdot \chi(M^4) + 1\,,$$

where $8\pi^2 \cdot \beta \cdot \chi(M^4)$ bounds the number of concentration points, then at least $\frac{16\pi^2}{c} \cdot \chi(M^4)$ of the spaces, Y_j^4, are smooth. Since for such Y_j^4, the convergence $(M^4, g_k, \underline{m}_j) \to (Y_j^4, \underline{y}_j)$ is also smooth on compact subsets, this gives for k sufficiently large,

$$\int_{(M^4, g_k)} |R|^2 > 8\pi^2 \cdot \chi(M^4)\,,$$

which contradicts (1.3).

Now suppose that $\underline{N}$ is chosen as large as possible subject to the conditions $\overline{m_{i,j_1}, m_{i,j_2}} \to \infty$, for all $j_1 \neq j_2$, and Y_j^4 is noncollapsed for all j.

We claim that if $r >> 1$ and k is sufficiently large, then in the L_2-sense, almost all of the curvature on $M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})$ is concentrated on a finite subset of blowup points of cardinality $\leq \beta \cdot \chi(M^4)$. If we grant this momentarily, and note that since $\lambda = -3$, we have the pointwise relation, $|R|^2 > 1$, it follows that

$$\lim_{k\to\infty, r\to\infty} \mathrm{Vol}(M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})) = 0\,,$$

from which (10.8) follows.

In actuality, the claim is a slight generalization of Theorem 0.1. The maximality of $\underline{N}$ implies that $M^4 \setminus \bigcup_j B_r(\underline{m}_{k,j})$ collapses as $k \to \infty, r \to \infty$. Also, each noncollapsed Y_j^4 has finite volume and bounded curvature outside a compact subset. Thus, the claim follows by applying the chopping theorem of [ChGv3] near $\bigcup_j \partial B_r(\underline{m}_{k,j})$ and then repeating the proof of Theorem 0.1. □

Remark 10.9. As mentioned at the end of Section 0, the present paper arose from an initial attempt in 1987 to prove Theorem 10.2. What we lacked at that time was the chopping theorem for local bounded curvature, Theorem 3.13 (or alternatively, the transgression form, $\mathcal{T}P_\chi$) and the specific estimate (1.11).

Remark 10.10. In Theorem III of [An3], under the strong additional assumption that the sequence, (M^4, g_k), converges with respect to the extrinsic L_2 metric, it is shown that the limit consists of a finite number of complete orbifolds of finite volume. Moreover, it is asserted that the curvature of each of these is bounded outside of a compact subset. The argument indicated in [An3] for this assertion is not valid. To establish this fact (even assuming convergence with respect to the L_2 metric) Theorem 9.1, one of the main results of the present paper, is required.

Again with the strong additional hypothesis of extrinsic L_2 convergence, Theorem III of [An3] contains statements corresponding to (10.6), (10.7), (10.8). The argument given in [An3] for (10.6) is not correct as stated, since it rests on the presumed applicability of the good chopping theorem of [ChGv3], whereas the curvature of Y_j^4 is not shown to be bounded outsided a compact subset; compare also Remark 4.7. However, Anderson has pointed out that assuming extrinsic L_2 convergence and granted Theorem 5.3 and Remark 5.4 of [An3], a proof can be given.

The question of whether for $\lambda < 0$, *every* sequence, (M^4, g_i), has a subsequence which converges in the extrinsic L_2 metric is raised in ii) of p. 33 of [An3]. It remains open.

Collapsed limit spaces. Let Y (respectively, $\{(Y_1, \underline{y}_1), \ldots, (Y_{\underline{N}}, \underline{y}_{\underline{N}})\}$) denote a collapsed Gromov-Hausdorff limit space (respectively, an $\underline{N}$-pointed Gromov-Hausdorff limit space). As noted above, the cardinality of the set, $\bigcup_j \{y_{j,\alpha_j}\}$, of concentration points, is bounded by $\beta \cdot \chi(M^4)$.

By Theorem 5.1 of [An3], away from the set, $\bigcup_j \{y_{j,\alpha_j}\}$, the approximating spaces, (M^4, g_k), are collapsed with locally bounded curvature and hence, by [ChGv2], admit an F-structure of positive rank. Equivalently, $Y_j \setminus (\bigcup_{\alpha_j} y_{j,\alpha_j})$ is a limit space with locally bounded curvature.

It follows directly from Theorem 9.1 that the following strengthening holds.

Theorem 10.11. *A compact subspace of* $Y_j \setminus \bigcup_j \{y_{j,\alpha_j}\}$ *is a collapsed limit space with bounded curvature.*

Apart from the qualitative statement in Theorem 10.11, we have quantitative estimates corresponding to (9.3), (9.4), of Theorem 9.1. The theory of collapse with bounded curvature, including the existence of standard N-structures, describes the geometry of Y_j as well as the convergence on a fixed scale. In particular, for k sufficiently large, regions near regular points, $y_j \in Y_j$, regions of (M^4, g_k) fibre over regions of Y_j with nilpotent fibres. These fibrations are almost Riemannian submersions. The existence of these fibrations can be viewed as an appropriate version of 2) above (weak convergence implies stronger convergence) in the collapsing case. For further information on collapse with bounded curvature, see [ChFuGv].

For additional information concerning the collapsing structure at infinity of noncompact limits, see Remark 9.7.

As mentioned above, for $\lambda = 3$, absent a lower volume bound, it is not known whether collapsed limit spaces can occur. For $\lambda \leq 0$, collapsed $\underline{N}$-pointed limit spaces occur whenever $\mathrm{diam}((M^4, g_i)) \to \infty$ (which can happen).

11. Further directions

In this section, we speculate on some possible extensions of our main results.

Anti-self-dual metrics. A metric g on an oriented Riemannian 4-manifold is called anti-self-dual if its self-dual Weyl tensor $W_+(g)$ vanishes. Partial progress on the study of such metrics has been made in [An4], [TiVia1], [TiVia2].

We conjecture that the curvature estimate in Theorem 0.8 still holds for anti-self-dual metrics with constant scalar curvature (and Kähler metrics with constant scalar curvature). Moreover, we believe that there should be versions of Theorems 0.1 and 0.14 for anti-self-dual metrics with constant scalar curvature. One problem here is to establish a local volume estimate in terms of local Sobolev constant; for the global version, see [TiVia1], [TiVia2].

Kähler metrics of constant scalar curvature. We conjecture that the curvature estimate in Theorem 0.8 still holds for Kähler metrics with constant scalar curvature; compare [TiVia1], [TiVia2].

Ricci flow. The normalized Ricci flow on $[0, T) \times M$ is a solution to the equation,

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) - \frac{r}{n} g)\,, \tag{11.1}$$

where $g(t)$ is a family of metrics on M, $n = \dim M$ and r is the average of the scalar curvatures of $g(t)$. Assume that $n = 4$ and $g(t)$ is an entire solution of (11.1), i.e., $T = \infty$.

In view of recent work for the Yang-Mills flow by Hong and the second author, [HoTi], we conjecture that the curvature and injectivity radius estimates in Theorems 0.1, 0.8 and 0.14 still hold for $g(t)$ as t tends to infinity. Of particular interest is the case of shrinking Ricci solitons.

Higher dimensions. Finally, we consider the higher-dimensional case. We conjecture that, for n arbitrary, if the L_2 bound on curvature is replaced by an $L_{\frac{n}{2}}$ bound, then Theorems 0.1, 0.8 continue to hold. Additionally, we conjecture that for n arbitrary, given an L_2 bound on curvature, Theorem 0.14 holds and the conclusions of Theorems 0.1 and 0.8 are valid off suitable subsets of finite $(n-4)$-dimensional Hausdorff measure. (Analogous statements can be conjectured in higher dimensions for entire solutions of the normalized Ricci flow.) Note that in the particular case of special holonomy, the anti-self duality of the curvature tensor implies that $|R|^2 \cdot \mathrm{Vol}(\,\cdot\,)$ is a definite multiple of a certain characteristic form $C(M^n)$; see e.g. [ChTi2]. Although this relation is analogous to (1.1), in fact $C(M^n) \neq P_\chi$. This circumstance makes it unclear how to extend our present approach to the higher-dimensional case of special holonomy.

12. Appendix; The transgression form $\mathcal{T}P_\chi$

Proof based on $\mathcal{T}P_\chi$. The notation in this appendix is as in Section 1.

On a $(\theta \cdot t)$-collapsed manifold with locally bounded curvature, there exists an *essentially canonical* form, $\mathcal{T}P_\chi$, satisfying

$$d\,\mathcal{T}P_\chi = P_\chi, \tag{12.1}$$

$$|\mathcal{T}P_\chi(p)| \le c(n) \cdot (r_{|R|}(p))^{-(n-1)}\,. \tag{12.2}$$

The bound, (12.2), on $|\mathcal{T}P_\chi|$ can be transformed into one in which the scale, $r_{|R|}$, is absent by means of (1.24).

A proof of the key estimate, Theorem 1.26, based on the form, $\mathcal{T}P_\chi$, proceeds along the same lines as the one based on equivariant good chopping, modulo the following proviso. In deriving the counterpart of (1.25), an argument employing a cutoff function and Stokes' theorem replaces our previous argument.

Construction of $\mathcal{T}P_\chi$. The detailed construction of the form, $\mathcal{T}P_\chi$, is more technical than the proofs of the chopping theorems, Theorems 3.1, 3.13. In addition to properties ii)-iv) of standard N-structures, it relies on property i) as well; compare Remark 2.1. The construction goes roughly as follows.

Consider first the case of a sufficiently collapsed manifold, M^n, with bounded curvature. Suppose also, that for some standard N-structure, and for the saturation of some ball, $B_{r(n)}(p)$, that all orbits, $\mathcal{O}$, have fixed dimension, k, and in addition, the second fundamental forms of these orbits are bounded, $|II_{\mathcal{O}}| \le c(n)$. Consider the Whitney sum of the connections obtained by orthogonally projecting the Riemannian connection of an invariant metric, $\overline{g}$, onto the sub-bundles which are tangent to and orthogonal to the orbits. Using the fact that the orbits are nilmanifolds (whose Euler characteristic vanishes) and the fact that the connection on the complementary bundle is locally the pull-back of a connection on an $(n-k)$-dimensional quotient, it follows that the Chern-Gauss-Bonnet form of this connection vanishes identically.

More generally, if the N-structure has an atlas consisting of charts of the above type, then the corresponding connections can be glued together to produce a connection, $\widehat{\nabla}$, for which the Chern-Gauss-Bonnet form vanishes identically, $P_\chi^{\widehat{\nabla}} \equiv 0$.

In the general bounded curvature case, the orbits in a given chart have positive dimension, which need not be constant. However, by using the bound on the second fundamental form in property i) of standard N-structures, a connection, $\widehat{\nabla}$, with vanishing Chern-Gauss-Bonnet form can still be constructed. Recall that property i) states that any point lies in a tubular neighborhood, $T_{3r(n)}(\mathcal{O}_q)$, where the second fundamental form, $II_{\mathcal{O}_q}$, satisfies $|II_{\mathcal{O}_q}| \le c(n)$; see Section 2.

The invariant metric, $\widetilde{g}$, lies at a definite distance from the given metric, g, and the connection, $\widehat{\nabla}$, is constructed from that of $\widetilde{g}$ by an essentially canonical local procedure. Thus, Chern-Weil theory give rise to a standard form, $\mathcal{T}P_\chi$, with

$$d\mathcal{T}P_\chi = P_\chi - P_\chi^{\widehat{\nabla}} = P_\chi\,.$$

Moreover, the above-mentioned bounds imply $|\mathcal{T}P_\chi| \le c(n)$.

Since, in the case of collapse with bounded curvature, the connection, $\widehat{\nabla}$, is constructed by an essentially canonical local procedure, the construction has a direct extension to the case of sufficient collapse with locally bounded curvature. In this case, by scaling, we get the bound (12.2), for the norm of the $(n-1)$-form $\mathcal{T}P_\chi$.

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Lecture 1 - Examples of Collapsing Manifolds

January 26, 2010

Note: These lectures are entirely expository; the results and methods are due to others. Appropriate credit is indicated throughout the notes.

1 Collapsing

We are mainly interested in the phenomenon of sequences of manifolds with injectivity radii limiting to zero, while sectional curvatures remain bounded.

Any compact Riemannian manifold can be said to converge to a point by multiplying its metric by a constant δ^2 and letting $\delta \to 0$. What is meant by "converge to a point" will be made precise later, but note that such a manifold's volume and diameter do converge to zero. This kind of process in rather trivial. Note that only in the flat case does this produce a family with bounded curvature.

However, many Riemannian manifolds admit some collapsing process that leaves curvature bounded. This is "geometric collapse," or more properly *collapse with bounded curvature.*

2 Example: $\mathbb{S}^3$

The first example (historically) is Berger's collapsing 3-sphere. We first describe the classical Hopf fibration. The unit 3-sphere can be defined as the set of points $(z, w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. The 1-sphere $\mathbb{S}^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$ acts on $\mathbb{S}^3$ by multiplication: if $\phi = e^{i\theta} \in \mathbb{S}^1$ we define

$$\phi.(z, w) \;=\; \left(e^{i\theta} z,\, e^{i\theta} W\right).$$

The orbit of a point under this $\mathbb{S}^1$ action are called Hopf circles. This generates a foliation of $\mathbb{S}^3$ by $\mathbb{S}^1$, that is actually a fiber bundle (actually a principle bundle).

There is also a simple map $\mathbb{S}^3 \to \mathbb{C}^* \approx \mathbb{S}^2$, called the Hopf map, given by

$$(z,\, w) \mapsto\; zw^{-1}.$$

It is easy to show that if (z, w) and $(\tilde{z}, \tilde{w})$ map to the same point in $\mathbb{S}^2$ then $(z, w) = c(\tilde{z}, \tilde{w})$ for some constant c, with, necessarily, $|c| = 1$. Therefore $c \in \mathbb{S}^1$, and we see that the fibers of this submersion are precisely the Hopf circles. The O'Niell formulas show that $\mathbb{S}^2$ is a half-radius sphere, of constant sectional curvature 4.

It is possible to see this map in a more "active" setting. Topologically $\mathbb{S}^3$ is just the Lie group $SU(2)$. Let X, Y, Z be a basis of its Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ with brackets

$$[X,\, Y] \;=\; 2Z \quad [Y,\, Z] \;=\; 2X \quad [Z,\, X] \;=\; 2Y,$$

and dual basis $\eta, \mu, \zeta \in \mathfrak{g}^*$. Let $g = \eta \otimes \eta \,+\, \mu \otimes \mu \,+\, \zeta \otimes \zeta$ be the standard bi-invariant metric on $SU(2)$; in fact this is the round metric on $\mathbb{S}^3$:

$$\begin{aligned} R(X,Y)Z \;&=\; -\frac{1}{4}[[X,Y],Z] \;=\; 0 \\ R(X,Y)Y \;&=\; -\frac{1}{4}[[X,Y],Y] \;=\; X. \end{aligned}$$

Now let $g_\delta \;=\; \delta^2 \eta \otimes \eta \,+\, \mu \otimes \mu \,+\, \zeta \otimes \zeta$ be another metric; g_δ is left-invariant but not bi-invariant. One verifies that sectional curvatures are bounded, but that the injectivity radius at each point is $|\delta|$, as given by the geodesic with tangent vector X. The limiting object is $\mathbb{S}^2$ with sectional curvature 4.

3 Example: Free effective torus actions

The 3-sphere example can be generalized. Berger's collapse is just the scaling of the metric along the orbits of the circle-action while leaving the metric unchanged on the perpendicular distribution. Now suppose a torus T^k acts freely (isotropy groups are trivial) and isometrically on a Riemannian manifold M^{n+k}. Now M supports an integrable tangential distribution (of dimension k) and a perpendicular distribution (of dimension n). The metric can be likewise decomposed: letting $T_p \subset T_pM$ and $P_p \subset T_pM$ indicate the tangential and perpendicular distributions, respectively, we can write $g \;=\; g_T \,+\, g_P$.

Not let $g_\delta \;=\; \delta^2 g_T + g_P$. Pick a point $p \in M$; we will estimate the sectional curvatures at p. Let $N^n \subset M^{n+k}$ be a transverse submanifold, defined in a neighborhood of p, that contains p. Let $y^1, \ldots, y^n$ be coordinates on N with $p = (0, \ldots, 0)$. Let $\tilde{x}^1, \ldots, \tilde{x}^k$ be coordinates on T^k with the identity e having coordinates $x^i = 0$. The freeness of the actions allows these coordinate functions to push forward to functions $x^1, \ldots, x^k$ locally near p, where $x^1 = \cdots = x^k = 0$ on the transversal N. Finally extend the functions $y^1, \ldots, y^n$ to a neighborhood of p by projection along the fibers onto N. This gives a coordinate system $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ in a neighborhood of p.

The coordinate fields $\frac{\partial}{\partial x^i}$ are tangent to the fibers, although the fields $\frac{\partial}{\partial y^i}$ are not. Write $\frac{\partial}{\partial y^i} = X_i + V_i$ where the fields X_i are parallel to the fibers and the fields V_i are perpendicular to the fibers.

The original metric has the form

$$g = \begin{pmatrix} A & B \\ B & C+D \end{pmatrix},$$

where $A_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$, $B_{ij} = \left\langle \frac{\partial}{\partial x^i}, X_j \right\rangle$, $C_{ij} = \langle X_i, X_j \rangle$, and $D_{ij} = \langle V_i, V_j \rangle$. Note that these matrices are functions of the coordinates y^i only, since the torus action is isometric. The new metrics have the form

$$g_\delta(x,y) = \begin{pmatrix} \delta^2 A & \delta^2 B \\ \delta^2 B & \delta^2 C + D \end{pmatrix}.$$

This metric is singular and it is not clear that sectional curvature remains bounded. Now make a change of coordinates: $u^i = \delta x^i$, and let $\tilde{A}_{ij} = \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle$ and $\tilde{B}_{ij} = \left\langle \frac{\partial}{\partial u^i}, X_i \right\rangle$. Then in the new coordinates we have

$$\begin{aligned} g_\delta(u,y) &= \begin{pmatrix} \delta^2 \tilde{A} & \delta^2 \tilde{B} \\ \delta^2 \tilde{B} & \delta^2 C + D \end{pmatrix} = \begin{pmatrix} A & \delta B \\ \delta B & \delta^2 C + D \end{pmatrix} \\ \lim_{\delta \to 0} g_\delta &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \end{aligned}$$

a generalized warped product metric. It is clear now that the g_δ have bounded curvature. It is also possible to prove that injectivity radii converge to 0, so we indeed have a prototype for collapse with bounded curvature.

Theorem of Cheeger-Gromov: This example is in essence the only kind of collapse with bounded curvature, at least as observed on the scale of the injectivity radius.

4 Example: Nilmanifolds

Nilmanifolds provide the prototype for collapse with bounded curvature. Note that tori are Lie groups with Abelian algebras.

Let G be a finite dimensional Lie group with Lie algebra $(\mathfrak{g}, [,])$. Its *descending central series* is defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$. If $\mathfrak{g}^k = 0$ for some k, then $\mathfrak{g}$ is called a *nilpotent Lie algebra* and g a *nilpotent Lie group*. Note the possible conflict with the use of this term in the group-theoretic setting. In fact there is no conflict. The Lie algebra operation $[,]$ and the group's commutator operation (also denoted $[,]$) are related by the formula

$$[X_1, X_2] = \frac{1}{2} \frac{d^2}{dt^2} [\exp(tX_1), \exp(tX_2)].$$

This can be exploited easily to show that a Lie group that is nilpotent in the group-theoretic sense is nilpotent in the Lie algebra sense. The converse is slightly more difficult to see, but also true.

Any (finite-dimensional) nilpotent Lie group N is isomorphic to a group of $n \times n$ upper-triangular matrices with 1's along the diagonal; it's Lie algebra is a Lie algebra of strictly upper triangular matrices (ie. with 0's along the diagonal). Given $q > 0$ let g_q be the norm

$$|A|^2 \;=\; \sum_{ij} (a_{ij})^2 \, q^{2(i-j)}.$$

This gives rise to a left-invariant metric that is bi-invariant only when $q = 1$. Alternatively we can let $\mathfrak{g}_k$ be the vector space of matrices a_{ij} where $a_{ij} = 0$ unless $i - j = k$, and let g_q be the metric so that $g_q|_{\mathfrak{g}^k} = q^{2k} g$. Because $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$ (a simple consequence of the Jacobi identity), we get the estimate

$$|[X,Y]|_q^2 \;\le\; |XY - YX|_q^2 \;\le\; 2(n-2)|X|_q^2 |Y|_q^2,$$

which we use to derive the estimates

$$\begin{aligned}
|\nabla_X Y|_q &\;\le\; 3\sqrt{2(n-2)}|X|_q^2 |Y|_q^2 \\
|\operatorname{Rm}(X,Y)Z|_q &\;\le\; 42(n-2)|X|_q |Y|_q |Z|_q
\end{aligned}$$

(hint for (1): use the Koszul formula). This implies that sectional curvatures are bounded independently of q.

Let N denote the Lie group of upper triangular matrices under consideration, and let $\Gamma \subset N$ be a cocompact subgroup; for instance the group of upper triangular matrices with integral entries and 1's along the diagonal. The metric is invariant under left translation by Γ so the metric descends to the quotient $M = \Gamma$
N. As $q \to 0$ the diameter of M goes to zero while its sectional curvature remains bounded. Thus M is an example of an "almost-flat manifold", one which supports a sequence of metrics with $\operatorname{diam}(M)^2 \cdot \max|\sec(M)| \searrow 0$. We shall prove later that there is no flat metric on M.

Gromov's theorem states that this is in fact the only way for a manifold to collapse to a point with bounded curvature.

Consider the Heisenberg group, the group of upper triangular 4×4 matrices with 1's along the diagonal. It is 3-dimensional, its natural bi-invariant metric has both positive and negative curvatures, and its Riemann tensor is not parallel. The left quotient by the integer subgroup gives a twisted circle-bundle over the torus. This is the (compact) prototype for Thurston's nilgeometry.

5 Example: Solvmanifolds

This previous example can be extended to the solvegeometry. Let G be a solvable Lie group, given by the upper triangular $n \times n$ matrices, with Lie algebra $\mathfrak{g}$. Define a sequence of normal subgroups by $G^0 = G$, $G^1 = [G, G]$ and $G^k = [G^1, G^{k-1}]$, and put on a metric g_q so that $v \in G^i$ and v is perpendicular to G^{i-1} gives $|v|^2 = q^{2i}$.

Let $\Gamma \subset G$ be a cocompact discrete subgroup, for instance the integer subgroup. Now sending $q \to 0$ we get collapse to a manifold $\Gamma \backslash (G/G^1)$, which is isometric to the torus T^n. The "collapsed" directions constitute a fibration by nilmanifolds, each isomorphic to $(G^1 \cap \Gamma) \backslash G$. Indeed this produces a fiber bundle

$$(G_1 \cap \Gamma) \backslash G_1 \to \Gamma \backslash G \to \Gamma \backslash G / G_1.$$

This is an example of Fukaya's theorem, than any collapse to a lower dimensional Riemannian manifold produces a fiber bundle, where the fibers are nilmanifolds, along which the collapse occurs.

Lecture 2 - Topology and Convergence in the Space of Metric Spaces

January 28, 2009

1 The Hausdorff distance

1.1 Basic Properties

Given a bounded metric space X, the set of closed sets of X supports a metric, the Hausdorff metric. Whether X is bounded or not, there is a compact, locally compact topology on the space of closed sets. If $A, B \subset X$ are closed sets, define their *Hausdorff distance* $d_H(A, B)$ to be the number

$$\inf\left\{\, r \mid B\,\mathrm{is\,in\,the}\,r-\mathrm{neighborhood\,of}\,A\,\mathrm{and}\,A\,\mathrm{is\,in\,the}\,r-\mathrm{neighborhood\,of}\,B\,\right\}.$$

We can say this more precisely as follows. We say B is r-close to A (or B is in the r-neighborhood of A) if

$$B \;\subset\; \bigcup_{x\in A} B(x, r).$$

Then the Hausdorff distance is the infimum of all r such that B is r-close to A and A is r-close to B. There is still another equivalent definition. Given a point $p \in X$ and a closed set $A \subset X$, define

$$d(p, A) \;=\; \inf_{y\in A} \mathrm{dist}(p, y).$$

Then the Hausdorff distance is

$$d_H(A, B) \;=\; \max\left\{ \sup_{x\in A} d(x, B),\; \sup_{y\in B} d(y, A) \right\}$$

That is, $d_H(A, B)$ is the farthest distance any point of B is from the set A, or the farthest any point of A is from B, whichever is greater.

Theorem 1.1 *If X is a bounded metric space, the set of closed sets of X is itself a metric space with the Hausdorff metric.*

<u>*Pf*</u> We verify the metric space axioms. First, the symmetry of d_H is clear by definition. Second, d_H satisfies the triangle inequality because if C is in the r-neighborhood of B and B is in the s-neighborhood of A, then C is in the $(r+s)$-neighborhood of A. Likewise A is in the $(r+s)$-neighborhood of C. Thus $d(A,C) \leq d(A,B) + d(B,C)$. Finally $d_H(A,B) = 0$ implies $A \subset \overline{B} = B$, because if B is in every r-neighborhood of A then every point of A is a limit point of B. Likewise $B \subset \overline{A} = A$. $\square$

If X is not bounded, the metric space axioms continue to hold, but $d_H(A,B)$ could well be infinity.

1.2 Compactness

Denote the closed subset of X by $\mathfrak{C}(X)$ (or just $\mathfrak{C}$ for short). Given a closed set A and a number r, let $\mathfrak{B}(A,r)$ be the set of all $D \in \mathfrak{C}$ with $d_H(B,A) < r$. Since d_H is a metric on $\mathfrak{C}$, we know that the balls $\mathfrak{B}(A,r)$ are open, and form a neighborhood base.

Obviously the balls with rational radius also form a base, so the induces topology on $\mathfrak{C}$ is *first countable*. All metric spaces are Hausdorff, so $(\mathfrak{C}, d_H)$ is Hausdorff. One can state this directly: since distinct closed sets are separated by a finite distance, say ϵ, so the balls of radius, say, $\epsilon/4$ around each is disjoint.

If X is noncompact, then the topology associated to the Hausdorff distance is neither compact nor even locally compact. To see the local noncompactness, simply pick a sequence $x_i \in X$ that has no convergent subsequence, and define the closed sets X_i to be $X_i = \{x_j\}_{j=1}^{i}$. Given any neighborhood $\mathfrak{N}$ of $X_\infty = \{x_j\}_{j=1}^{\infty}$, each $X_i \in \mathfrak{N}$.

If X is noncompact, $(\mathfrak{C}(X), d_H)$ is not even locally compact. For instance if the base space X is nondiscrete (it has the property that, given any point $x \in X$ and any number $\epsilon > 0$, there is a point $y \in X$ with $d(x,y) < \epsilon$), then it is not locally compact. As an example, we will will show that $\mathbb{R}$ is not locally compact. Let $A = [0, \infty)$ be the half-line, and consider its r-neighborhood $B(A,r)$ (wlog assume $r < \frac{1}{2}$). Define the A_i inductively by setting $A_0 = A$ and $A_i = A_{i-1} - (i, i + r/2)$. We have $d_H(A_i, A_j) = r/2$ for any $i \neq j$, so there are no Cauchy subsequences, and therefore no convergent subsequences.

In fact, the metric topology on $(\mathfrak{C}(\mathbb{R}), d_H)$ is not even locally paracompact. There exist closed sets A such that every neighborhood of A contains an uncountable discrete subset.

In sharp contrast we have the following theorem.

Theorem 1.2 *If X is compact, then $(\mathfrak{C}(X), d_H)$ is compact.*

<u>*Pf*</u>

Let A_i be a sequence of open sets. Each A_i, has a $\frac{1}{j}$-net consisting of $< N_j \in \mathbb{N}$ elements (an ϵ-net is a maximal discrete ϵ-separated subset; the compactness of X guarantees the existence of the number N_j). Let $A_i^k \subset A_i$ be the union of the $\frac{1}{j}$-nets in A_i for $1 \leq j \leq k$; note that the cardinality of A_i^k is at most $N_1 + \cdots + N_j$.

Fixing k, some subsequence $A_{i_k}^k$ converges in the Hausdorff topology, to a some discrete set A^k. Since $A_{i_k}^k$ is $\frac{1}{k}$-close to A_{i_k}, we have that, for large i_k, A_{i_k} is $\frac{3}{k}$-close to A^k. We can require that $A_{i_k}^k$ is a subsequence of $A_{i_{k+1}}^{k+1}$, which means $A^k \subset A^{k+1}$. Since A^k is ϵ-close to $A_{i_k}^k$ for large i_k, and $A_{i_{k+1}}^k$ is $\frac{1}{k}$-close to $A_{i_{k+j}}^{k+j}$ which is ϵ-close to A^{k+j}, we have that A^k is $(\frac{1}{k} + 2\epsilon)$-close to A^{k_j}, any $\epsilon > 0$ so that A^k is $\frac{1}{k}$-close to A^{k+j}.

The diagonal subsequence $A_{k_k}^k$ converges to some set A^∞, in which each A^k is $\frac{1}{k}$-dense. Consider the sequence A_{k_k}. Since $A_{k_k}^k$ is $\frac{1}{k}$-dense in A_{k_k} and is also $\frac{1}{k}$-close to A^∞, this means A_{k_k} is $\frac{1}{k}$-close to A^∞. This implies A_{k_k} converges to $\overline{A^\infty}$. □

A topology does exist on $\mathfrak{C}(X)$ that is both locally compact and compact, regardless of the compactness of X. Let a base for this topology be set of the form $N_{K,\epsilon}(A)$, where $K \subset X$ is compact, $A \subset X$ is closed, and $\epsilon > 0$, where we define

$$N_{K,\epsilon}(A) \;=\; \{\, B \in \mathfrak{C}(X) \mid d_H(A \cap K, B \cap K) \;<\; \epsilon \,\}.$$

This topology on $\mathfrak{C}(X)$ is called the *pointed Hausdorff topology.* If X is compact, it is the metric topology. If X is noncompact, this topology is not induced by any metric.

2 The Gromov-Hausdorff distance

The Gromov-Hausdorff distance was invented by Gromov for the purpose of making precise the notions of "closeness" and "convergence." Recall that his "Almost Flat Manifold" theorem states that a compact bounded-curvature manifold that is "close" to being a point has a finite normal cover that is "close" to being a nilmanifold. The idea behind the Gromov-Hausdorff distance is not difficult; here is what Gromov himself has to say:

- "Either you have no inkling of an idea or, once you have understood it, the very idea appears so embarrassingly obvious that you feel reluctant to say it aloud... "
- "I knew [of] it [the Gromov-Hausdorff metric] for a long time, but it just seemed too trivial to write. Sometimes you just have to say it."[1]

The Gromov-Hausdorff distance significantly extends the idea of the Hausdorff distance (and is not equivalent to it). Given two closed subsets A and B of any metric space (not necessarily subsets of the same space), we define

$$d_{GH}(A, B) \;=\; \inf_{f,g} d_H(f_{A \to X}(A),\, g_{B \to X}(B))$$

[1]Taken from Cheeger's lecture 'Mikhail Gromov: How Does He Do It?'.

where the notation $f_{A\to X}$ (resp. $g_{B\to X}$) denotes an isometric embedding of A into some metric space X (resp. isometric embeddings of B into X) and the infimum is taken over all possible such embeddings.

In general the topology associated to the Gromov-Hausdorff distance is neither locally compact nor locally paracompact. To redress this we define the *pointed Gromov-Hausdorff topology*. This is a topology on the set of pointed sets (defined to be pairs (A,p) where A is a closed subset of a metric space and $p \in A$). A local base for this topology are the sets of the form $N_{K,\epsilon}(A)$ (where A is closed, $K \subset A$ is compact and $p \in K$, and $\epsilon > 0$); we define $N_{K,\epsilon}(A)$ to be the set of pointed closed sets (B,q) so that there exists a compact subset $J \subset B$, $q \in J$, and so that there are isometric embeddings $f : A \cap K \to X$ and $g : B \cap J \to X$ into some space X so that $f(p) = g(q)$ and the Hausdorff distance satisfies $d_H(f(A\cap K),\, g(B \cap J)) < \epsilon$.

This topology is locally compact and compact. If the Gromov-Hausdorff topology is restricted to compact closed sets, the Gromov-Hausdorff topology and the pointed Gromov-Hausdorff topology coincide.

3 The Lipschitz, $C^{k,\alpha}$, and $L^{p,k}$ topologies

The Gromov-Hausdorff topology is not suitable for questions of differentiability or even topology, since Gromov-Hausorff limits can jump differentiable structures, topologies, and even dimensions. For example a sequence of tori can converge to a round sphere, or to a circle or to a point.

Thus the Gromov-Hausdorff topology is completely inadequate when studying Riemannian structures (curvature, etc), and we have to find something sharper. Let $f : M \to N$ be a map between metric spaces. Define the dilation of f to be

$$dil(f) \;=\; \sup_{p,q\in M} \left\{ \frac{\mathrm{dist}_N(f(p), f(q))}{\mathrm{dist}_M(p,q)} \right\}.$$

We allow $dil(f)$ to take values in $[0,\infty]$. We define the *Lipschitz distance* between compact homeomorphic metric spaces M, N by

$$Lip(M,N) \;=\; \inf_{\substack{f:M\to N,\\ f\ homeo}} |\log(dil(f))| \;+\; |\log(dil(f^{-1}))|.$$

One easily verifies that this is a metric (up to equivalence of isometric metric spaces). If M is compact then the induced topology is locally compact. If M is noncompact, one can define a "local Lipschitz topology," meaning convergence occurs iff it occurs when restricted to compact subsets of the original metric spaces M, N. The convergence is essentially of Lipshitz type: for instance the graphs of $\frac{1}{n}\sin(n\pi t)$ over the unit interval for $n \in \mathbb{Z}$ converge to the unit interval. If one includes Riemannian metrics of type $C^{0,1}$, then the space of Riemannian metrics on a compact manifold M is locally compact and complete in the Lipschitz topology; this can be seen by examining the sequence of metrics on a chart in M diffeomorphic to a Euclidean ball and applying the Arzela-Ascoli theorem.

It is possible to further refine the Lipschitz topology in the category of Riemannian manifolds. Given a sequence of Riemannian manifolds (M_i, g_i), one says that they converge to (M, g) in the $C^{k,\alpha}$- or $L^{k,p}$-topology if there are homeomorphisms $f : M \to M_i$ such that the following holds: Given any coordinate chart $U \subset M$ with coordinates $\{x^1, \ldots, x^n\}$, with pullback metrics $g_{i,jk} dx^j \otimes dx^k$, the functions $g_{i,jk}$ converge to g_{jk} in the $C^{k,\alpha}$- or $L^{p,k}$-sense.

Lecture 3 - The Gromov-Hausdorff Topology

February 4, 2010

Gromov-Hausdorff distance and the Gromov-Hausdorff topology are central to these lectures.

1 Equivalent formulations of the Gromov-Hausdorff distance

Proposition 1.1 *The Gromov-Hausdorff distance $d_{GH}(X,Y)$ is the infimum of the Hausdorff distances between X and Y taken among all metrics on $X\coprod Y$ that restrict to the given metric on X and on Y.*

<u>*Pf*</u>

Define $\tilde{d}_{GH} \;=\; \inf\{d_H(X,Y)\}$, where the infimum is taken over metrics on $Z = X\coprod Y$ that restrict to the given metrics on X and Y. Since d_{GH} is an infimum taken over a larger set,

$$d_{GH} \;\le\; \tilde{d}_{GH}.$$

Now consider a metric on some ambient space Z that restricts to the given metrics on X and Y. Put $\alpha = d_H(X,Y)$. Define a function $\bar{d}: X\coprod Y \times X\coprod Y \to \mathbb{R}^{\ge 0}$ as follows. If $x_1, x_2 \in X$ and $y_1, y_e \in Y$, set $\bar{d}(x_1,x_1) = d(x_1,x_2)$, $\bar{d}(y_1,y_2) = d(y_1,y_2)$, $\tilde{d}(x_1,y_1) = d(x_1,y_1)$ if $d(x_1,y_1) \ge \alpha/2$, and $\bar{d}(x_1,y_1) = \alpha/2$ if $d(x_1,y_1) < \alpha/2$. We check that the triangle inequality holds. By the symmetry of the distance function we only have to check that

$$\tilde{d}(x,y) \;\le\; \tilde{d}(x,x') + \tilde{d}(x',y).$$

There are four cases. First if $d(x,y) \ge \alpha/2$ and $d(x',y) \ge \alpha/2$ then

$$\tilde{d}(x,y) \;=\; d(x,y) \;\le\; d(x,x') + d(x',y) \;=\; \tilde{d}(x,x') + \tilde{d}(x',y).$$

Second if $d(x,y) \ge \alpha/2$ and $d(x',y) < \alpha/2$ then

$$\tilde{d}(x,y) \;=\; d(x,y) \;\le\; d(x,x') + d(x',y) \;<\; \tilde{d}(x,x') + \alpha/2 \;<\; \tilde{d}(x,x') + \tilde{d}(x',y).$$

Third if $d(x,y) < \alpha/2$ and $d(x',y) \geq \alpha/2$ then

$$\tilde{d}(x,y) \;=\; \alpha/2 \;\leq\; d(x',y) \;<\; d(x,x') \,+\, d(x',y) \;=\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Finally if $d(x,y) < \alpha/2$ and $d(x',y) < \alpha/2$ then

$$\tilde{d}(x,y) \;=\; \alpha/2 \;=\; \tilde{d}(x',y) \;<\; \tilde{d}(x,x') \,+\, \tilde{d}(x',y).$$

Therefore from isometric embeddings $X \hookrightarrow Z$, $y \hookrightarrow Z$, we have found a metric on $X \coprod Y$, that restricts to the given metrics on X and Y, and so that the Hausdorff distance from X to Y is preserved. This proves that

$$\bar{d}_{GH}(X,\,Y) \;\leq\; d_{GH}.$$

$\square$

A map $f : X \to Y$ (not necessarily continuous) between metric spaces is called an ϵ-GHA (for "Gromov-Hausdorff approximation") if $|d_Y(f(x_1), f(x_2)) \,-\, d_X(x_1,x_2)| < \epsilon$ for all $x_1, x_2 \in X$, and Y is in the ϵ-neighborhood of $f(X)$. We can define a new distance function between metric spaces, called $\widehat{d_{GH}}$, by setting

$$\widehat{d_{GH}}(X,Y) \;=\; \inf\{\, \epsilon > 0 \mid \text{there are}\, \epsilon\text{−GHA's}\, f : X \to Y \,\text{and}\, g : Y \to X \,\}.$$

It is a simple exercise to prove that this is a metric: if there is an ϵ_1-GHA $f : X \to Y$ and an ϵ_2-GHA $g : Y \to Z$, then the composition satisfies

$$\begin{aligned}
&|d_Z(gf(x_1), gf(x_2)) \,-\, d_X(x_1,x_2)| \\
&\quad\leq\; |d_Z(gf(x_1), gf(x_2)) \,-\, d_Y(f(x_1), f(x_2))| \,+\, |d_Y(f(x_1), f(x_2)) \,-\, d_X(x_1,x_2)| \\
&\quad\leq\; \epsilon_1 \,+\, \epsilon_2
\end{aligned}$$

and it is also easy to show that the $(\epsilon_1 + \epsilon_2)$-neighborhood of $fg(X)$ is Z. Taking infima, we have that $\widehat{d_{GH}}(X,Z) \;\leq\; \widehat{d_{GH}}(X,Y) + \widehat{d_{GH}}(Y,Z)$.

Proposition 1.2 *The metrics $\widehat{d_{GH}}$ and d_{GH} are equivalent (though they are not the same).*

<u>*Pf*</u>

We prove that any sequence that converges in one metric converges in the other. If $X_i \to X$ in the d_{GH} sense, we can easily construct 2ϵ-approximations $f_i : X_i \to X$. To do this, note that for all big enough i, since $X_i \coprod X$ has a metric in which X is in the ϵ-neighborhood of X_i (and vice-versa), we can pick any map that sends a point $p \in X_i$ to some point $f(p) \in X$ a distance at most 2ϵ away from p. This can be done, for instance, by choosing a denumerated, finite set of points $X_F \subset X$ that is ϵ-dense (by the compactness of X), and sending any point $p \in X_i$ to the nearest point of X_F. If two or more points are equally close to p, then send p to the point of X_F that is lower in the denumeration. Then for $p, q \in X_i$ we have $|d(p,q) - d(f(p), f(q))| < 2\epsilon$, and X is clearly in the 2ϵ-neighborhood of $f(X_i)$.

Conversely, if $X_i \to X$ in the $\widehat{d_{GH}}$-topology, we can construct metrics on $X_i \coprod X$, for large enough i in which X is in the 2ϵ-neighborhood of X_i. Construct a distance function d so that if $f_i : X_i \to X$ is a 2ϵ-approximation put $d(x_i, f(x_i)) = \epsilon$, and given any other $x_i \in X_i$ and $x \in X$,

$$d(x_i, x) \;=\; \epsilon + \inf_{x_i' \in X_i} \left(d(x_i, x_i') \,+\, d(f(x_i'), x) \right).$$

We verify the triangle inequality. First assume $x_i, x_i' \in X_i$. The only case to verify is $d(x_i, x_i') \le d(x_i, x) + d(x, x_i')$, where $x \in X$. We have

$$\begin{aligned} d(x_i, x) + d(x, x_i') &= 2\epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i'') \,+\, d(f(x_i''), x)) \,+\, \inf_{x_i'' \in X_i} (d(x_i', x_i'') \,+\, d(f(x_i''), x)) \\ &\ge 2\epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i'') \,+\, d(f(x_i''), x) \,+\, d(x_i', x_i'') \,+\, d(f(x_i''), x)) \\ &\ge 2\epsilon \,+\, d(x_i, x_i') \;\ge\; d(x_i, x_i'). \end{aligned}$$

Next assume $x_i \in X_i$ and $x \in X$. If $x' \in X$ then

$$\begin{aligned} d(x_i, x) &= \epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x)) \\ &\le \epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x') \,+\, d(x', x)) \\ &= d(x_i, x') \,+\, d(x', x), \end{aligned}$$

and if $x_i' \in X_i$ then

$$\begin{aligned} d(x_i, x) &= \epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x)) \\ &\le \epsilon + \inf_{x_i'' \in X_i} (d(x_i, x_i') + d(x_i', x_i'') + d(f(x_i'), x)) \\ &= d(x_i, x_i') \,+\, d(x_i', x). \end{aligned}$$

Finally assume $x, x' \in X$. Given any $x_i \in X_i$ then

$$\begin{aligned} d(x, x') &< 2\epsilon \,+\, inf_{x_i'' \in X_i} (d(f(x_i''), x) + d(f(x_i''), x')) \\ &\le 2\epsilon \,+\, inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x) + d(x_i, x_i'') + d(f(x_i''), x')) \\ &\le 2\epsilon \,+\, inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x)) \,+\, inf_{x_i'' \in X_i} (d(x_i, x_i'') + d(f(x_i''), x')) \\ &= d(x, x_i) \,+\, d(x_i, x') \end{aligned}$$

□

2 Properties of the Gromov-Hausdorff metric

Proposition 2.1 *d_{GH} is a metric on the set of compact metric spaces, modulo isometry.*

Pf

If X and Y are isometric then clearly $d_{GH}(X,Y)=0$.

Conversely assume $d_{GH}(X,Y)=0$. Then there is a sequence of distance functions d_i on $X\coprod Y$ with $d_i|_X=d_X$ and $d_i|_Y=d_Y$ so that $d_{i,H}(X,Y)\to 0$. Let $\epsilon_j>0$ be a sequence that converges to 0. For each j construct finite sets of points $\mathcal{X}_j=\{x_k\}$ and $\mathcal{Y}_j=\{y_k\}$ with the following properties: $\mathcal{X}_j$ is ϵ_j-dense in X, $\mathcal{Y}_j$ is ϵ_j-dense in Y, and for large enough i the sets $\mathcal{X}_j$ and $\mathcal{Y}_j$ are ϵ_j-close in the Hausdorff metric. We also require that $\mathcal{X}_j\subset\mathcal{X}_{j+1}$ and $\mathcal{Y}_j\subset\mathcal{Y}_{j+1}$, so that $\mathcal{X}=\bigcup_j\mathcal{X}_j$ is dense in X and $\mathcal{Y}=\bigcup_j\mathcal{Y}_j$ is dense in Y.

Now consider the distance functions $\{d_i\}$ restricted to $\mathcal{X}_j\cup\mathcal{X}_j$. Because this set is finite, a subsequence d_{i_j} converges to a limiting pseudometric $\overline{d}_j$. Passing to ever more refined subsequences of d_i as j increases and taking a diagonal subsequence (which we also call d_i), we get convergence to a pseudometric d on $\mathcal{X}\cup\mathcal{Y}$, a dense subset of $X\coprod Y$, and therefore convergence on $X\coprod Y$.

Given any ϵ_j, a given point $x\in X$ is ϵ_j-close to a point $x_j\in\mathcal{X}$, which is ϵ_j-close to a point of $y_j\in\mathfrak{Y}$. Taking a limit $y=\lim_j y_j$ we have that $d(x,y)=0$. Similarly given an arbitrary point $y\in Y$ we can find a point $x\in X$ with $d(x,y)=0$.

Identify points $a,b\in X\coprod Y$ with $d(a,b)=0$ and call the moduli space M. Since $d|_X=d_X$ and $d|_Y=d_Y$ and the triangle inequality holds, a point in X is identified with a unique point in Y, and vice-versa. We now have natural isometric equivalences $X\to M$ and $Y\to M$ so that X and Y are isometric. □

Proposition 2.2 *The Gromov-Hausdorff topology on the set of compact metric spaces is second countable.*

Pf

If a topology is Hausdorff and separable it is second countable, or better, a separable metric space is second countable. Consider the set $\tilde{\mathcal{X}}$ of finite metric spaces where all distances are rational. There are countably many such spaces. To see that these spaces are dense, consider a compact metric space X. We can construct a sequence X_i of such finite spaces that converge to X by letting $X_i\subset X$ be a 2^{-i}-dense set of points. The metric space X_i has a distance of less than 2^{-i} from some finite metric space $\tilde{X}_i$ with rational distance, so that $\tilde{X}_i\to X$. This proves that $\tilde{\mathcal{X}}$ is dense. □

As it happens, the pointed Gromov-Hausdorff distance is not second countable. We have already constructed an uncountable collection of subsets of $\mathbb{R}^1$ that are Hausdorff distance 1 from each other.

Lemma 2.3 (Gromov's Precompactness Lemma) *Let $N:\mathbb{N}\to\mathbb{N}$ be monotonic. Assume $\mathfrak{M}$ is a collection of metric spaces so that each $M\in\mathfrak{M}$ has a $\frac{1}{j}$-dense discrete subset of cardinality $\leq N(j)$. Then $\mathfrak{M}$ is precompact.*

Pf

Let $\{M_i\} \subset \mathfrak{M}$, and let $\tilde{M}_{i,j} \subset M_i$ be a $\frac{1}{j}$-dense subset of cardinality $\leq N(j)$. By replacing $N(j)$ with $\sum_{i=1}^{j} N(i)$ we can assume that $\tilde{M}_{i,j} \subset \tilde{M}_{i,j+l}$. Fixing j and letting $i \to \infty$ we get convergence of $\tilde{M}_{i,j}$ along a subsequence to a space $\tilde{M}_j$. Passing to further refinements of the subsequence and taking a diagonal sequence, we get a sequence of distance functions d_k that converge on each $\tilde{M}_j$, and therefore on $\tilde{M} = \overline{\bigcup_j \tilde{M}_j}$. Now given $\epsilon > 0$ there is an i so that $\tilde{M}_i$ is ϵ-close to $\tilde{M}$, and there is a j so that $M_{i,j}$ is ϵ-close to both $\tilde{M}_i$ and to M_i. Thus M_i converges to $\tilde{M}$. $\square$

Lecture 4 - Convergence Theorems

February 9, 2010

1 Volume comparison and the Heintze-Karcher theorem

Given a complete embedded submanifold $N^k \subset M^n$, we can parametrize M locally near N by some neighborhood of N in $N^k \times \mathbb{R}^{n-k}$, via the normal exponential map. This requires identifying $N^k \times \mathbb{R}^{n-k}$ with the normal tangent bundle $T^\perp N$, which can be done locally. Namely fixing a point $p \in N$, one identifies $T_q^\perp N$ with $T_p^\perp N$ by parallel transport in the normal bundle, whenever there is a unique geodesic from p to q. Putting coordinates $\{x^1, \dots, x^k\}$ on N near p and coordinates $\{x^{k+1}, \dots, x^n\}$ on $\mathbb{R}^{n-k}$, we can express the metric tensor g_{ij} and the volume form $dVol = \sqrt{\det g_{ij}}dx^1 \wedge \cdots \wedge dx^n$ in components.

Lemma 1.1 (Heintze-Karcher (1978)) *Assume $N^k \subset M^n$ is an embedded submanifold. Let ξ be a geodesic perpendicular to N. If all sectional curvatures along ξ are $\geq \kappa$ and the mean curvature vector of N^k at $\xi(0)$ is H, then*

$$\sqrt{\det g_{ij}(t)} \;\leq\; f_{n,k,\kappa,H}(t),$$

where t measures the Riemannian distance to N, and $f_{n,k,\kappa,H}$ is a function uniquely determined by n, k, κ, and H. This inequality holds until when $t \geq 0$ and until $\det \mathcal{A}(t, \xi)$ *has a zero.*

<u>*Pf*</u>

Let $\{x^1, \dots, x^n\}$ be coordinates near $p \in N$, as above. The identification of nearby normal tangent spaces implies

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} \;=\; -B_{\partial/\partial x^j}\left(\frac{\partial}{\partial x^i}\right)$$

where $1 \leq i \leq k$ and $k+1 \leq j \leq n$. Now assume $x^{k+1}, \dots, x^n$ are spherical coordinates with $x^n = t$ being the radial coordinate.

Letting $\xi(t)$ be a geodesic with initial direction perpendicular to N at p, we can choose frames $\{E_1, \ldots, E_n\}$ along ξ so that $E_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq k$ and so that $\nabla_{\partial/\partial t} \frac{\partial}{\partial x^j} = E_j$ for $k+1 \leq j \leq n-1$. Let $A_{ij} = g\left(\frac{\partial}{\partial x^i}, E_j\right)$, $1 \leq i, j \leq n-1$. It is easy to prove that $\det A_{ij} = \sqrt{\det g_{ij}}$.

Letting $\mathcal{R}(t)$ be the linear operator

$$(\mathcal{R}A)_{ij} \;=\; g\left(\mathrm{Rm}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t},\, E_j\right),$$

we have that $\frac{d^2}{dt^2}A_{ij} + (\mathcal{R}A)_{ij} = 0$. This system of differential equations has initial conditions

$$\begin{aligned} A(0) &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\ A'(0) &= \begin{pmatrix} -B_{\partial/\partial t} & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

Now consider the solution $B(t)$ of $B'' + \kappa B = 0$ with initial conditions $B(0) = A(0)$, $B'(0) = A'(0)$. The Index theorem implies (for instance) that the trace of $A(t)$ is less than or equal to the trace of $B(t)$. Then $\det(A)^{1/n-1} \leq \frac{1}{n-1}Tr(A) \leq \frac{1}{n-1}Tr(B) = \det(B)^{1/(n-1)}$. $\square$

Corollary 1.2 (Cheeger's lemma (1970)) *Assume M is a compact Riemannian manifold. Then inj_M is bounded from below by a constant determined by $\mathrm{Vol}(M)$, $\mathrm{Diam}(M)$, and $\kappa = \min_{p \in M} \sec(M)$.*

<u>*Pf*</u>

Given a small geodesic lasso, there is a geodesic loop of shorter (or equal) length in its free homotopy class. Such a loop ξ has mean curvature zero, so integrating the Heinze-Karcher inequality gives

$$\mathrm{Vol}(M) \leq C(n, \kappa, \mathrm{Diam}(M))\, l(\xi).$$

$\square$

Theorem 1.3 (Bishop-Gromov volume comparison) *If $\mathrm{Ric} \geq (n-1)H$ for some real number H, then $\mathrm{Vol}\, B_p(r) \leq \mathrm{Vol}\, B^H(r)$, where $B^H(r)$ indicates the ball of radius r in the spaceform of constant sectional curvature H.*

<u>*Pf*</u>

The Weitzenböck formula

$$\frac{1}{2}\triangle|\nabla f|^2 \;=\; |\nabla^2 f|^2 \,+\, \langle \nabla f, \nabla \triangle f\rangle \,+\, \mathrm{Ric}(\nabla f, \nabla f) \qquad (1)$$

gives, after plugging in a distance function r,

$$0 \;=\; |\nabla^2 r|^2 \;+\; \frac{\partial}{\partial r}\triangle r \;+\; \mathrm{Ric}(\nabla r, \nabla r). \tag{2}$$

Now use spherical coordinates $\{r, x^2, \ldots, x^n\} = \{r, x\}$ and set

$$A = \sqrt{\det(g_{ij})}.$$

Then

$$\frac{\partial^2}{\partial r^2}\log A \;=\; \frac{\partial}{\partial r}\triangle r \;=\; -|\nabla^2 r|^2 \;-\; \mathrm{Ric}(\nabla r, \nabla r) \tag{3}$$

So with $J^{n-1} = A$, we get

$$\frac{\partial}{\partial r^2}\log J \;=\; -\frac{1}{n-1}|\nabla^2 r|^2 \;-\; \frac{1}{n-1}\,\mathrm{Ric}(\nabla r, \nabla r) \tag{4}$$

$$\frac{J''}{J} - \left(\frac{J'}{J}\right)^2 \;=\; -\frac{1}{n-1}|\nabla^2 r|^2 \;-\; \frac{1}{n-1}\,\mathrm{Ric}(\nabla r, \nabla r) \tag{5}$$

and with

$$\frac{\partial}{\partial r}\log J \;=\; \frac{1}{n-1}\frac{\partial}{\partial r}\log A \;=\; \frac{\triangle r}{n-1}, \tag{6}$$

we get another "Heintze-Karcher inequality"

$$(n-1)\,\frac{J''}{J} \;=\; \frac{(\triangle r)^2}{n-1} \;-\; |\nabla^2 r|^2 \;-\; \mathrm{Ric}(\nabla r, \nabla r) \;\leq\; -\mathrm{Ric}(\nabla r, \nabla r) \;\leq\; -(n-1)H. \tag{7}$$

Conversely, on the space form of sectional curvature H we get exactly the equation $J'' + HJ = 0$. Comparing these differential equations and integrating gives the desired inequality. $\square$

2 Convergence Theorems for Noncollapsed manifolds

In 1967 Cheeger proved the following finiteness theorem

Theorem 2.1 (Cheeger's Diffeofiniteness) *Let $\mathcal{M}(n, \Lambda, \delta, \nu)$ be the set of Riemannian manifolds of dimension n, sectional curvature $|\sec| < \Lambda$, diameter $\mathrm{diam} < \delta$, and volume $\mathrm{Vol} > \nu$. Then there are only finitely many diffeomorphism types of manifolds in $\mathcal{M}(n, \Lambda, \delta, \nu)$.*

<u>*Idea of Pf*</u> First a lower bound on the injectivity radius is established. Given a manifold $M \in \mathcal{M}$ one establishes that finitely many Euclidean charts of definite size can cover the

manifold, where each chart is given “Riemann normal coordinates”. Finally one shows that the transitions between different “normal coordinate” regimes is controlled by the curvature. With control over the number of charts and the transition functions between them, the finiteness of diffeomorphism classes follows. $\square$

Theorem 2.2 (Gromov's Precompactness Theorem) *Let $\mathcal{M}(n, \lambda, \nu, D)$ be the set of compact manifolds of dimension n of volume greater than ν, diameter less than D, and Ricci curvature greater than λ. Then $\mathcal{M}(n, \nu, \lambda)$ is precompact in the pointed Gromov-Hausdorff topology.*

<u>*Pf*</u>

Let $\{(M_i, p_i)\} \subset \mathcal{M}(n, \lambda)$ be a sequence of such manifolds. Choose a maximal $\frac{1}{j}$-separated set of points in M (in particular, this set is $\frac{1}{j}$-dense). Consider the (disjoint) balls of radius $\frac{1}{2j}$ around each point. Each has volume

$$\begin{aligned}
\operatorname{Vol} B(p, 1/2j) &\geq \frac{\operatorname{Vol} B^{\Lambda}(1/2j)}{\operatorname{Vol} B^{\Lambda}(2D)} \operatorname{Vol} B(p, 2D) \\
&\geq \frac{\operatorname{Vol} B^{\Lambda}(1/2j)}{\operatorname{Vol} B^{\Lambda}(2D)} \operatorname{Vol} B(p_i, D) \\
&= C(j, R, \Lambda) \operatorname{Vol} B(p_i, D).
\end{aligned}$$

Thus there are fewer than $N(j, D, \Lambda) = 1/C(j, D, \Lambda)$ points in our maximal $\frac{1}{j}$-separated set. Now Gromov's Precompactness Lemma states that the balls $B(p_i, D) \subset M_i$ converge along a subsequence to some metric space set $\tilde{M}$. $\square$

In 1981 Gromov extended this result to prove precompactness in the Lipschitz topology. Specifically

Theorem 2.3 (Gromov's $C^{1,1}$-precompactness) *The space $\mathcal{M}(n, \Lambda, \delta, \nu)$ is precompact in the Lipschitz topology. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, \delta, \nu)$ converges along a subsequence to a differentiable manifold with a C^0 metric, and a $C^{1,1}$ distance function.*

In 1987 this theorem was improved (independently) by S. Peters and Greene-Wu. We have

Theorem 2.4 ($C^{1,\alpha}$-precompactness) *The space $\mathcal{M}(n, \Lambda, \delta, \nu)$ is precompact in the Lipschitz topology. Any sequence of manifolds $\{M_i\} \in \mathcal{M}(n, \Lambda, \delta, \nu)$ converges along in the $C^{1,\alpha}$ topology to a differentiable manifold with a $C^{1,\alpha}$ metric.*

This theorem holds locally in appropriate settings; for instance the proof goes through almost unchanged on star-shaped domains. Better control over the curvature tensor yields improved convergence: if the Riemann tensors are bounded in the C^k sense, then convergence is in the $C^{k+1,\alpha}$ topology. In a sense this theorem is a kind of global (though slightly weakened) Arzela-Ascoli or Rellich theorem.

We have already discussed spaces with Ricci curvature bounded from below in the context of Gromov's precompactness theorem. In some cases Gromov's theorem can be sharpened.

Theorem 2.5 (Anderson-Cheeger) *The space of compact n-dimensional manifolds with (possibly negative) lower bounds on Ricci curvature, (positive) lower bounds on injectivity radius and upper bounds on volume is precompact in the Gromov-Hausdorff topology. Sequences of such manifolds subconverge in the Gromov-Hausdorff distance to manifolds in the $C^{1,\alpha}$-differentiability class, and metrically converge in the $C^{0,\alpha}$-topology.*

Theorem 2.6 (Cheeger-Colding) *The space of compact n-dimensional manifolds with definite lower bounds on Ricci curvature, (positive) lower bounds on the volume of unit balls and upper bounds on volume is precompact in the Gromov-Hausdorff topology. Off a singular set of codimension 2 or greater, convergence is in the $C^{0,\alpha}$-topology to a connected differentiable manifold.*

Lecture 5 - The Bieberbach Theorem I

February 16, 2010

1 Compact flat manifolds: Bieberbach's theorem on crystallographic groups

A discrete subgroup G of the group of Euclidean motions $\subset \mathcal{O}(n) \ltimes \mathbb{R}^n$ is called a crystallographic group if it acts freely and has compact fundamental domain. A crystallographic group determines a compact flat manifold, and a compact flat manifold is determined by (at least one) crystallographic group. Certain fundamental questions arise. Given n, are there finitely many flat manifolds of dimension n? Is any flat manifold covered by a torus? Are these coverings normal? This can be rephrased: given a crystallographic group G, does it have a normal Abelian subgroup of finite index and maximal rank? Bieberbach answered this in the affirmative.

Given a motion $\alpha \in \mathcal{O}(n) \ltimes \mathbb{R}^n$ we can project to its rotational part $A = A(\alpha)$ and its translational part $t = t(\alpha)$. If A is a transformation we can write $\mathbb{R}^n = E_0 \oplus \cdots \oplus E_k$ where A acts as a rotation of angle θ_i on E_i, and possibly acts as a reflection on E_k (this is to account for the nonorientable case). The θ_i are called the *principle rotational angles*.

Theorem 1.1 (Bieberbach, 1911) *Assume $G \subset \mathcal{O}(n) \ltimes \mathbb{R}^n$ acts freely on $\mathbb{R}^n$. If $\alpha \in G$ then all principle rotational angles of $A(\alpha)$ are rational. If the translational parts of G span some subspace $S \subset \mathbb{R}^n$, then the pure translations of G span E.*

Since the translational parts of elements of a crystallographic group spans $\mathbb{R}^n$, it follows that a flat manifold is covered by a torus. Bieberbach was also able to use this with a theorem on group extensions to prove that there are only finitely many quotients.

As a special case of his almost-flat manifold theorem, Gromov proved a stronger form of Bieberbach's theorem.

Theorem 1.2 *Let G be a crystallographic group. Then* *i) There is a translational, normal subgroup* $\Gamma \triangleleft G$ *of finite index (indeed* $ind(\Gamma : G) < 2(4\pi)^{\frac{1}{2}n(n-1)}$*).*

ii) If $\alpha \in G$ *then either* α *is a translation or the smallest nonzero principle angle* $A(\alpha)$ *is greater than* $\frac{1}{2}$*.*

iii) Further, if $\alpha \in G$ *and* $0 < \theta_1 < \cdots < \theta_k$ *are the nonzero principle rotational angles of* $A = A(\alpha)$*, then*

$$\theta_l \geq \frac{1}{2}(4\pi)^{l-k}$$

Via (i), this formulation directly shows that there are finitely many flat manifolds of a given dimension.

2 Statement of Gromov's theorem on almost flat manifolds

Gromov's far-reaching extension of Bieberbach's theorem states that almost-flat manifolds have a finite normal cover isomorphic to a nilmanifold. Specifically he proved

Theorem 2.1 (Gromov 1978) *Let* M^n *be a compact Riemannian manifold and set* $K = max|\sec(M)|$ *and* $d = \text{diam}(M)$*. If* $d^2K < exp(-exp(n^2))$*, then* M^n *is covered by a nilmanifold. More specifically,*

- $\pi_1(M)$ *contains a torsion-free nilpotent normal subgroup* Γ *or rank* n*,*
- *The quotient* $G = \Lambda\backslash\pi_1(M)$ *has order* $\leq 2(6\pi)^{\frac{1}{2}n(n-1)}$ *and is isomorphic to a subgroup of* $\mathcal{O}(n)$*,*
- *The finite covering of* M *with fundamental group* Γ *and deckgroup* G *is diffeomorphic to a nilmanifold* $\Gamma\backslash N^n$*, and*
- *The simply connected Lie group* N *is uniquely determined by* $\pi_1(M)$*.*

Gromov claims his proof is a generalization of Bieberbach's proof, although such a view is hard to support considering his introduction of several radical new techniques. Ruh (1982) proved that collapsed manifolds are actually *infranil*. A compact manifold M is called infranil if it is a quotient of a nilpotent Lie group N by affine transformations such that the image of the holonomy action of the canonical flat affine connection is finite.

3 Finsler geometry on $\mathcal{O}(n)$

Proposition 3.1 *Given an operator $A \in \mathcal{O}(n)$, there is a decomposition*

$$\mathbb{R}^n \;=\; E_0 \oplus E_1 \oplus \ldots \oplus E_k$$

where A acts as a simple rotation through $\pm\theta_i$ on E_i, and we arrange

$$0 \;=\; \theta_0 \;<\; \theta_1 \;<\; \ldots \;<\; \theta_k$$

(possibly $E_0 = \{0\}$). If A is orientation-reversing then E_k is 1-dimensional and we put $\theta_k = \pi$.

<u>*Pf*</u> The E_i are the eigenspaces of A. The $\pm\theta_i$ are the corresponding eigenvalues. □

We call the θ_i the *principle angles* of $A \in SO(n)$. Define $|A|$ by

$$|A| \;=\; \theta_k \;=\; \max_{x \in \mathbb{R}_n} \measuredangle(x, Ax).$$

A norm on $\mathfrak{so}(n)$ (the operator norm) can be defined by stating that for $X \in \mathfrak{so}(n)$, we have

$$|X| \quad = \quad \max\{\, |Xv| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\}.$$

Of course this is the magnitude of the largest eigenvalue of a. Note that this norm is Ad-invariant:

$$\begin{aligned}
|Ad_A X| \quad &= \quad \max\{\, |AXA^{-1}v| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\} \\
&= \quad \max\{\, |XA^{-1}v| \mid v \in \mathbb{R}^n \text{ and } |v| = 1 \,\} \\
&= \quad \max\{\, |Xw| \mid w \in \mathbb{R}^n \text{ and } |Aw| = 1 \,\} \\
&= \quad \max\{\, |Xw| \mid w \in \mathbb{R}^n \text{ and } |w| = 1 \,\} \;=\; |a|.
\end{aligned}$$

Proposition 3.2 *Left-translating the $|\cdot|$ norm on $\mathfrak{so}(n)$ to each tangent space on $SO(n)$ gives a Finsler metric, with (right) invariant distance function*

$$d(A, B) \;=\; |AB^{-1}|.$$

If $X \in \mathfrak{so}(n)$ has $|X| \leq \operatorname{diam}(SO(n)$ and $A = \exp(X)$, then

$$|X| \;=\; |A|.$$

<u>*Pf*</u>

The Finsler metric is obtained by left-translation, so therefore the distance function will be *right* invariant. Writing $d(A, B) = d(AB^{-1}, Id)$, we only have to verify that $d(C, Id) =$

$|C|$. Let C be any element of $SO(n)$, with principle rotational angles $\theta_k, \theta_{k-1}, \ldots$. Let $C(t)$, $t \in [0,1]$, be the path in $SO(n)$ consisting of matrices with the eigenspace decomposition of C, but with the primary rotational angles given by $t\theta_k$. Then $|\dot{C}(t)| = \theta_k$ and so clearly $L(C(t)) = \int_0^1 |\dot{C}(t)|dt = \theta_k = |C|$. Therefore $\mathrm{dist}(C, Id) \leq |C|$. We can show that $C(t)$ is a distance-minimizing path. Let $\mathcal{C}(t)$ be another path with $\mathcal{C}(0) = Id$, $\mathcal{C}(1) = C$, and principle rotation angles $\theta_k(t)$. Given any such path we can create a new path $\tilde{\mathcal{C}}(t)$ with the same principle rotational angles, but with the same eigenspace decomposition of C. Since $|\dot{\tilde{\mathcal{C}}}(t)| = |\dot{\mathcal{C}}(t)| = |\dot{\theta}_k(t)|$, these paths have the same pathlengths. With $L(\mathcal{C}(t)) = \int_0^1 |\dot{\theta}_k(t)|dt$, we see that this is minimized when θ_k is linear.

To prove the last statement, note that $\gamma(t) = \exp(tX)$ realizes the minimum distance from Id to A. Since $\dot{\gamma}(t) = dL_{\gamma(t)}X$ we have $|\dot{\gamma}(t)| = |X|$ so that $|A| = \int_0^1 |\dot{\gamma}(t)|dt = |X|$. $\square$

In this Finsler metric there are is no point with a unique minimizing path joining it to the origin. To see this note that since pathlength depends only on the largest principle rotational angle, the other rotational angles can be modified in any way, and as long as they remain less than the largest angle, pathlength will be unaffected.

Given $A, B \in SO(n)$ let $K_A : SO(n) \to SO(n)$ act on B by conjugation: $K_A(B) = ABA^{-1}$. Since $K_A(Id) = Id$ we can regard $dK_A : \mathfrak{so}(n) \to \mathfrak{so}(n)$. We can also define $\mathrm{Ad}_A : \mathfrak{so}(n) \to \mathfrak{so}(n)$ by $\mathrm{Ad}_A X = AXA^{-1}$.

Lemma 3.3 *Given $A \in SO(n)$ and $X \in \mathfrak{so}(n)$, we have*

$$\begin{aligned} dK_A(X) &= \mathrm{Ad}_A(X) & (1)\\ \exp(\mathrm{Ad}_A X) &= K_A(\exp(X)) & (2)\\ \mathrm{Ad}_{\exp(tY)}X &= \mathrm{Exp}(t\,\mathrm{ad}_Y)X \triangleq \sum_{i=0}^{\infty} \frac{1}{i!}\left(t\,\mathrm{ad}_Y\right)^i X. & (3)\end{aligned}$$

<u>*Pf*</u>

Equation (1) follows from taking the derivative

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_A \exp(tX) = AXA^{-1}.$$

Keeping in mind that $SO(n)$ is a matrix group, expression (2) is just $e^{AXA^{-1}} = Ae^X A^{-1}$. One way to prove (3) is to find a Taylor series expression for $\mathrm{Ad}_{\exp(tY)}$, and to prove that the radius of convergence is infinite. We have

$$\begin{aligned} \left.\frac{d^k}{dt^k}\right|_{t=0} \mathrm{Ad}_{\exp(tY)}X &= \left.\frac{d^{k-1}}{dt^{k-1}}\right|_{t=0} (Y\exp(tY)X\exp(-tY) - \exp(tY)X\exp(-tY)Y)\\ &= \left.\frac{d^{k-1}}{dt^{k-1}}\right|_{t=0} \mathrm{Ad}_{\exp(tY)}\mathrm{ad}_Y X\\ &= \left(\mathrm{ad}_Y\right)^k X.\end{aligned}$$

The radius of convergence is infinite because the matrix $(\mathrm{ad}_Y)^k X$ is a polynomial expression of order k in the entries of Y and of order 1 in the entries of X. $\square$

Proposition 3.4 *The exponential map* $\exp : \mathfrak{so}(n) \to SO(n)$ *is length-nonincreasing, as measured in the Finsler norm.*

Pf

In addition to the Finsler norm is the bi-invariant metric g on $SO(n)$, the geodesics of which are precisely the left- or right-translates of paths of the form $\exp(tY)$, $Y \in \mathfrak{so}(n)$. If ∇^L is the canonical left-invariant connection (zero on any left-invariant vector field), the Riemannian connection is

$$\nabla_X Y \;=\; \frac{1}{2}\left(\nabla^L_X Y \,+\, \nabla^L_Y X\right) \,+\, \frac{1}{2}[X,Y].$$

Let $\gamma(t)$ be a geodesic with initial direction X, meaning $\gamma(t) = \exp(tX)$. Let $J_Y(t)$ be the Jacobi field along γ with initial conditions $J_Y(0) = 0$ and $\nabla_{\dot\gamma} J_Y(0) = Y$. The map $d\exp_X : \mathfrak{so}(n) \to T_{\exp(X)}SO(n) \approx \mathfrak{so}(n)$ is just $J_Y(1)$. Also, $d\exp_{tX} Y \;=\; \frac{1}{t}J_Y(t)$.

Now consider the Jacobi equation $0 = \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y + R(Y,\dot\gamma)\dot\gamma = \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - \frac{1}{4}[\dot\gamma,[\dot\gamma,Y]]$. Since $\dot\gamma$ is left-invariant, for any vector field Y along γ one easily checks

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y \,+\, R(Y,\dot\gamma)\dot\gamma \;=\; \frac{1}{4}\nabla^L_{\dot\gamma}\nabla^L_{\dot\gamma}Y \,+\, \frac{1}{2}\left[\dot\gamma, \nabla^L_{\dot\gamma}Y\right],$$

so the Jacobi equation can be written $\nabla^L_{\dot\gamma}\nabla^L_{\dot\gamma}Y + 2\left[\dot\gamma,\nabla^L_{\dot\gamma}Y\right] = 0$. Putting $K(t) = dL_{\gamma(t)^{-1}}J(t) \in \mathfrak{so}(n)$, the Jacobi equation reads

$$\ddot K \,+\, 2[X,\dot K] \;=\; 0.$$

Thus $\dot K(t) \;=\; \mathrm{Exp}(2t\,\mathrm{ad}_X)\cdot \dot K(0) \;=\; \mathrm{Ad}_{\exp(2tX)}\,Y$, and since $|\dot K(t)| \;=\; |\dot K(0)| \;=\; |Y|$ and $|\dot J_Y(t)| = |\dot K(t)|$, we have

$$|J_Y(t)| = |K(t)| \;\le\; \int_0^t |\dot K(t)|dt \;=\; t|\dot K(0)| \;=\; t|Y|.$$

Therefore $|d\exp_{tX} Y| \;=\; \frac{1}{t}|J_Y(t)| \;\le\; |Y|$, and so the exponential map is length-nonincreasing. $\square$

Proposition 3.5 *Given* $A, B \in \mathcal{SO}(n)$*, we have* $d([A,B],\,Id) \;\le\; 2d(A,\,Id)d(B,\,Id)$.

Pf

Putting $A = \exp(X)$ and $B = \exp(Y)$, we can connect A with BAB^{-1} with the curve

$$\gamma(t) \;=\; \exp\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right) \qquad t \in [0,1].$$

Now

$$d([A,B],\,Id) \;=\; d(A,\,BAB^{-1}) \;\leq\; \int_0^1 |\dot{\gamma}(t)|\,dt.$$

Estimating $|\dot{\gamma}(t)|$ we have

$$\begin{aligned}
|\dot{\gamma}(t)| \;&=\; \left|\frac{d}{dt}\exp\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \;=\; \left|d\exp_{\mathrm{Exp}(t\,\mathrm{ad}_Y)X}\left(\frac{d}{dt}\,\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \\
&\leq\; \left|\frac{d}{dt}\left(\mathrm{Exp}(t\,\mathrm{ad}_Y)X\right)\right| \;=\; \left|\,\mathrm{Exp}\left(t\,\mathrm{ad}_Y\right)\,[Y,X]\,\right| \\
&=\; \big|[Y,X]\big| \;\leq\; 2|X||Y| \;=\; 2|A||B|.
\end{aligned}$$

$\square$

Corollary 3.6 *A discrete subgroup of $SO(n)$ generated by elements of norm less than $1/2$ is nilpotent.*

<u>*Pf*</u>

On generators, $|[A,B]| < \min\{|A|,|B|\}$. Chains of commutators therefore converge on the identity, and since the subgroup is assumed discrete, any such chain must eventually terminate with the identity element. $\square$

Now define a norm on the group of Euclidean symmetries $\mathcal{O}(n) \ltimes \mathbb{R}^n$ by

$$|\alpha| \;\triangleq\; \max\{\,|r(\alpha)|,\, c\cdot|t(\alpha)|\,\}. \tag{4}$$

Proposition 3.7 *Any set of Euclidean motions $\{\alpha_1,\ldots,\alpha_k\}$ with $d(\alpha_i,\alpha_j) \geq \max\{|\alpha_i|,|\alpha_j|\}$ has $k \leq 3^{n+\frac{1}{2}n(n-1)}$.*

<u>*Pf*</u> We will work in the tangent space $\mathfrak{so}(n)\times\mathbb{R}^n$ at the identity. Let $A_i = r(\alpha_i)$ and $a_i = t(\alpha_i)$, and let $S_i \in \mathfrak{so}(n)$ be such that $\exp\,S_i = A_i$. Define a norm on $\mathfrak{so}(n)\times\mathbb{R}^n$ by $|(S,a)| = \max\{|S|, c|a|\}$, where c is any constant. Put $w_i = (S_i, a_i)$. Then $\tilde{w}_i = \frac{w_i}{|w_i|}$ lie on the unit sphere, and

$$\begin{aligned}
|\tilde{w}_i \,-\, \tilde{w}_j| \;&\geq\; \left|\frac{w_i}{|w_j|} \,-\, \tilde{w}_j\right| \,-\, \left|\frac{w_i}{|w_j|} \,-\, \tilde{w}_i\right| \\
&=\; \frac{1}{|w_j|}\left(|w_i \,-\, w_j| \,-\, \big||w_i| \,-\, |w_j|\big|\right).
\end{aligned}$$

By using $|d\exp_X Y| \;\leq\; |Y|$ from above, it is a simple matter to show $d(\alpha_i,\alpha_j) \leq |w_i - w_j|$. Also, $|w_i| = |\alpha_i|$, so therefore

$$|\tilde{w}_i \,-\, \tilde{w}_j| \;\;\geq\;\; \frac{1}{|\alpha_j|}\left(\max\{|\alpha_i|,|\alpha_j|\} \,-\, \big||\alpha_i| \,-\, |\alpha_j|\big|\right) \;=\; 1.$$

Thus we have found points on the unit sphere in $\mathfrak{so}(n)\times\mathbb{R}^n$ with unit mutual separation. There is a uniform upper bound on how many such points there can be in any such collection.
$\square$

Lecture 6 - The Bieberbach Theorem II

February 18, 2010

Let G be a crystallographic group. Given $A \in G \subset \mathcal{O}(n) \ltimes \mathbb{R}^n$, recall the norm

$$|A| \;=\; \max\{|r(\alpha)|,\, c|t(\alpha)|\} \tag{1}$$

where the norm $|\cdot|$ on $\mathcal{O}(n)$ was defined previously, and the norm $|\cdot|$ on $\mathbb{R}^n$ is the distance to the origin.

1 Two lemmas

Put $G_\rho = \{\alpha \in G \,|\, |t(\alpha)| < \rho\}$, and $G_\rho^\epsilon = \{\alpha \in G \,|\, |t(\alpha)| < \rho_i,\, |r(\alpha)| < \epsilon\}$. We will use the notation $B(s)$ to indicate the ball of radius s centered at the origin in $\mathbb{R}^n$. We will use "d" to indicate the Finsler metric on $\mathcal{O}(n)$ and "dist" to indicate the Euclidean distance function on $\mathbb{R}^n$.

The most important consequence of the following lemma is that, regardless of how small ϵ is, the pseudogroup G_ρ^ϵ has plenty of elements if ρ is big enough.

Lemma 1.1 *Given $R > 0$ and $\epsilon \in \left(0, \frac{1}{2}\right)$, there is some $\rho > R$ such that the set translation parts $t(\alpha)$ of elements $\alpha \in G_\rho^\epsilon$ is $\rho/4$-dense in $B(3\rho/4)$.*

<u>Pf</u>

Let r be the radius of the fundamental domain, and put $\rho_i = (R + r) \cdot 10^i$. If the lemma is false, there is an $x_i \in B(\rho_i)$ such that $\mathrm{dist}(x_i, t(G_{\rho_i}^\epsilon)) > \rho_i/4$. However the set of translational parts in G_{ρ_i} is r-dense in $B(3\rho_i/4)$, so there is some $\alpha_i \in G_{\rho_i}$ so that $|t(\alpha_i) - x_i| \le r$.

If the lemma is false we prove that the rotational parts $r(\alpha_1), r(\alpha_2), \ldots$ are all ϵ-separated from each other. Using $A_i = r(\alpha_i)$ and $a_i = t(\alpha_i)$, we compute $t(\alpha_i\alpha_j^{-1}) = -A_iA_j^{-1}a_j + a_i$. Then, with $i > j$,

$$|t(\alpha_i\alpha_j^{-1})| \;\le\; |A_iA_j^{-1}a_j| + |a_i - x_i| + |x_i| \;\le\; \rho_j + r + \frac{3}{4}\rho_i \;<\; \rho_i,$$

so that $\alpha_i \alpha_j^{-1} \in G_{\rho_i}$. Also

$$|t(\alpha_i \alpha^{-1}) - x_i| \leq |A_i A_j^{-1} a_j| + |a_i - x_i| < \rho_{i-1} + r < \frac{\rho_i}{4},$$

so that $\alpha_i \alpha_j^{-1} \notin G^\epsilon_{\rho_i}$. Therefore $|r(\alpha_i \alpha_j^{-1})| \geq \epsilon$. Thus we have a sequence $r(\alpha_1), r(\alpha_2), \ldots$ of ϵ-separated elements of $\mathcal{O}(n)$, an impossibility. $\square$

Lemma 1.2 *If $\epsilon < \frac{1}{2}$, then the group $\langle G^\epsilon_\rho \rangle$ generated by G^ϵ_ρ is d-nilpotent with $d = d(n)$.*

Pf

Fix the constant in (1) to be ϵ/ρ, so that $|\alpha| < \epsilon$ iff $\alpha \in G^\epsilon_\rho$. Define a *short basis* $\{\alpha_1, \ldots, \alpha_d\}$ inductively by selecting $\alpha_1 \in G^\epsilon_\rho$ so that α_1 has minimal norm, and selecting an element $\alpha_i \in G^\epsilon_\rho - \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ of minimal norm among all elements of $G^\epsilon_\rho - \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$.

We first prove that $d(\alpha_i, \alpha_j) \geq \max\{|\alpha_i|, |\alpha_j|\}$; a lemma of Finsler geometry now directly gives that $d \leq 3^{n^2}$. Wlog put $i > j$. Arguing for a contradiction, assume $|\alpha_i \alpha_j^{-1}| = d(\alpha_i, \alpha_j) < |\alpha_i| \leq \epsilon$. It follows from the definitions that $\alpha_i \alpha_j^{-1} \in G^\epsilon_\rho$, but since α_i was chosen minimally in G^ϵ_ρ it follows that $\alpha_i \alpha_j^{-1} \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$. But then $\alpha_i = (\alpha_i \alpha_j^{-1}) \alpha_j \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$, a contraction.

We can also prove that $|[\alpha, \beta]| < \min\{|\alpha|, |\beta|\}$. Note first that the rotational parts satisfy $|r[\alpha, \beta]| \leq 2|r(\alpha)||r(\beta)| \leq 2|\alpha||\beta|$. Before considering the translational parts, note that given any $A \in SO(n)$ and $x \in \mathbb{R}^n$ we have $|(Id - A)x| \leq 2\sin(|A|/2) \cdot |x| \leq |x|$. The commutator therefore satisfies

$$\begin{aligned}
t[\alpha, \beta] &= -ABA^{-1}B^{-1}b - ABA^{-1}a + Ab + a \\
&= A(I - B)A^{-1}a + AB(I - A^{-1})B^{-1}b \\
|t[\alpha, \beta]| &\leq 2|a| \sin\left(\frac{1}{2}|B|\right) + 2|b| \sin\left(\frac{1}{2}|A^{-1}|\right) \\
&\leq |r(\beta)||t(\alpha)| + |r(\alpha)||t(\beta)| \leq 2\frac{\rho}{\epsilon}|\alpha||\beta|.
\end{aligned}$$

Therefore $[\alpha_i, \alpha_j] \in \langle \alpha_1, \ldots, \alpha_{\min\{i,j\}-1} \rangle$, so that the length of commutators amongst generators of $\langle G^\epsilon_\rho \rangle$ is bounded by d. Induction on the formula $[\alpha\beta, \gamma] = [\beta, \gamma] \cdot [[\gamma, \beta], \alpha] \cdot [\alpha, \gamma]$ proves that the length of any commutator chain of elements in $\langle G^\epsilon_\rho \rangle$ is bounded by d. $\square$

2 Proof of (ii)

The lemmas suffice to show that G^ϵ for $\epsilon < 1/2$ is actually a translation group. Assume there is some $\gamma \in G$ with $|r(\gamma)| < \frac{1}{2}$. Let $\mathbb{R}^n = E \oplus E^\perp$ be the orthogonal decomposition where E is the subspace of maximal rotational angle of $r(\gamma)$. Given some $x \in \mathbb{R}^n$ let $x = x^E + x^\perp$ denote the corresponding orthogonal vector decomposition.

Let $\delta \in (0, \frac{1}{2})$ (to be chosen later, and will depend on γ alone) and put $\rho = 2|t(\gamma)|$. Pick some $x \in E$ with $|x| = \frac{3}{4}\rho$. The first lemma guarantees some $\alpha \in G^{\delta}_{\rho}$ with $|t(\alpha) - x| \leq \frac{1}{4}\rho$. Consequently $|t(\gamma)| \leq |t(\alpha)|$ and $|t(\alpha)| < 2|t(\alpha)^E|$. Let $\alpha_0 = \alpha$ and

$$\alpha_k \;=\; [\alpha_{k-1}, \gamma]$$

be the k-fold iterated commutator. For convenience put $A_k = r(\alpha_k)$, $a_k = t(\alpha_k)$ and $C = r(\gamma)$, $c = t(\gamma)$. From Finsler geometry we have

$$|A_{k+1}| \;=\; |[A_k, C]| \;\leq\; 2|C||A_k| \;<\; |A_k|$$

so that $|A_i| < |A| \leq \delta$. Consider the decomposition

$$\begin{aligned} t(\alpha_{k+1}) \;\triangleq\; a_{k+1} \;&=\; -A_k C A_k^{-1} C^{-1} c \,-\, A_k C A_k^{-1} a_k \,+\, A_k c \,+\, a_k \\ &=\; (Id \,-\, C)\, a_k \,+\, (Id \,-\, [A_k, C])\, C a_k \,+\, A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c \\ &=\; (Id \,-\, C)\, a_k \,+\, (Id \,-\, A_{k+1})\, C a_k \,+\, A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c. \end{aligned}$$

It is easy to prove that as long as $|B| = \theta \leq \pi$, then $|(Id - B)f| \leq 2\sin(\theta/2)|f| \leq |B||f|$. As a first application we get

$$\begin{aligned} |a_{k+1}| \;&\leq\; |C||a_k| \,+\, |A_{k+1}||a_k| \,+\, |A_k||c| \\ &\leq\; \left(\frac{1}{2} \,+\, \delta\right) |a_k| \,+\, \delta|a|. \end{aligned}$$

Since commutators are at most $d = d(n)$ long, by iterating we get (after choosing δ) that $|a_{k+1}| < |a|$. From this, we can also estimate $|a_{k+1}|$ from below. Using

$$a^E_{k+1} \;=\; (Id \,-\, C)\, a^E_k \,+\, [(Id \,-\, A_{k+1})\, C a_k]^E \,+\, \left[A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c\right]^E$$

and another triangle inequality,

$$\begin{aligned} |a^E_{k+1}| \;&\geq\; |\,(Id \,-\, C)\, a^E_k| \,-\, |\,[(Id \,-\, A_{k+1})\, C a_k]^E\,| \,-\, |\,\left[A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c\right]^E\,| \\ &\geq\; |\,(Id \,-\, C)\, a^E_k| \,-\, |\,(Id \,-\, A_{k+1})\, a_k| \,-\, |A_k C \left(Id \,-\, A_k^{-1}\right) C^{-1} c| \\ &\geq\; 2|a^E_{d-1}| \sin\left(\frac{\theta}{2}\right) \,-\, |A_d||a_{d-1}| \,-\, |A_{d-1}||c| \\ &\geq\; 2|a^E_{d-1}| \sin\left(\frac{\theta}{2}\right) \,-\, 2\delta|a| \end{aligned}$$

Choosing δ small enough and iterating this inequality d times, recalling that $|a^E_0| > \frac{1}{2}|a_0| > \frac{1}{2}|c|$, we get that $|a^E_d| > 0$. This contradicts the fact that d-fold commutators vanish. $\square$

3 Proof of (i)

To prove that the translation subgroup Γ is normal, let $\alpha \in G$ and $\beta \in \Gamma$ be the transformations $\alpha(x) = Ax + a$ and $\beta(x) = x + b$. Then

$$\begin{aligned} r(\alpha\beta\alpha^{-1}) \;&=\; I \\ t(\alpha\beta\alpha^{-1}) \;&=\; Ab. \end{aligned}$$

so conjugation fixes Γ. (This also follows from the fact that $\mathbb{R}^n \lhd SO(n) \ltimes \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n$.)

Finally note that the rotational parts of any two elements in G/Γ are separated by at least $\frac{1}{2}$. For if $\alpha, \beta \in G$ have $|r(\alpha)r(\beta)^{-1}| < \frac{1}{2}$ then $|AB^{-1}| = 0$ is translational, so in particular $r(\alpha) = r(\beta)$, and $\alpha \equiv \beta \,(mod\, \Gamma)$. Thus elements of G/Γ can be mapped injectively onto a set of $\frac{1}{2}$-separated points on $SO(n)$. Therefore G/Γ is uniformly bounded in terms of n.

Lecture 7 - Gromov's almost flat manifold theorem I

February 23, 2010

Let $D = D(M)$ is the diameter of M and $K = \max_{p\in M} |sec_p|$ is the largest sectional curvature that appears on M. Gromov proves the following:

Theorem 0.1 (Gromov's almost flat manifold theorem) *There is an $\epsilon > 0$ so that if M is a compact Riemannian manifold and $D^2 K < \epsilon$, then $\pi_1(M)$ has a nilpotent subgroup Γ of finite index, and M is a finite quotient of a nilmanifold.*

1 Short loops, short relations, and the Gromov product

The condition $D^2K \leq \epsilon$ is scale-invariant, so we work in the scale $D(M) = 1$. Making ϵ smaller enforces tighter bounds on K. The maximal rank radius r_{max} of the exponential map, which satisfies $r_{max} \geq \pi/\sqrt{K} \geq \pi\epsilon^{-1/2}D$, can be made arbitrarily large compared to the diameter. We can, somewhat informally, regard exp as a kind of large covering map for M, and most of our work will be done in a large ball in this space.

A *short loop* will be a geodesic from the basepoint to itself, of length shorter than the rank radius. A *short homotopy* is a homotopy through curves that are shorter than the rank radius. Given two short loops a, b, the usual path-product can be deformed, through a homotopy that keeps the endpoints fixed, to a geodesic loop. If this deformation can be done through a short homotopoy, then the final path, denoted $\beta * \alpha$ is called the *product between short loops a and b* (also called "Gromov's product"), and is unique, by Klingenberg's lemma. If $|\alpha| + |\beta| < r_{max}$ then $\beta * \alpha$ is defined. If the deformation cannot be done through a short homotopy, then $\beta * \alpha$ is not defined.

Denote by Γ the set of short loops. Clearly $\alpha \in \Gamma$ implies $\alpha^{-1} \in \Gamma$. Associativity holds when the sum of lengths of the paths is less than the rank radius: $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$ if $|\gamma| + |\beta| + |\alpha| < r_{max}$. Thus Γ is (essentially) a pseudogroup.

It is fairly easy to show that Γ possesses all generators and relations that exist in $\pi_1(M)$. To see that generators exist in Γ, note that any element of $\pi_1(M)$ can be expressed as a product of elements of length $\leq 2D(M)$ (this can be seen, for instance, by working in the universal covering: if a representative path there crosses fundamental domain boundaries, is can be deformed into a series of paths, the interior of each of which lies in a single fundamental domain).

It is slightly more difficult to see that the relations are all present in Γ as well (as long as say $\epsilon^{1/2} < \pi/5$).

2 Homotopy errors

One would like the Gromov product of short loops to commute with composition of holonomy actions. But the Gromov product involves a homotopy, so this cannot be true in general. However there is a homotopy approximation theorem. Let $\gamma, \tilde{\gamma} : [0,1] \to M$ be homotopic paths from p to q, and let $\triangle$ denote the homotopy between them. Let $|\triangle|$ the area of the homotopy, and let $K(\triangle)$ denote the largest sectional curvature at any point of $\triangle$, and let L denote the length of the longest path in the homotopy. Now let $X, \tilde{X}$ solve $\nabla_{\dot{\gamma}} X = 0$, $\nabla_{\dot{\tilde{\gamma}}} \tilde{X} = 0$ with $X(0) = \tilde{X}(0)$, and let $Y, \tilde{Y}$ solve $\nabla_{\dot{\gamma}} Y = 0$, $\nabla_{\dot{\tilde{\gamma}}} \tilde{Y} = 0$ with $Y(0) = \tilde{Y}(0) = 0$. Then

$$\begin{aligned} |X(1) - \tilde{X}(1)| &\leq \frac{4}{3} \cdot K(\triangle) \cdot |\triangle| \\ |Y(1) - \tilde{Y}(1)| &\leq \frac{4}{3} \cdot K(\triangle) \cdot L \cdot |\triangle|. \end{aligned}$$

This allows an estimation of the difference between the rotational and translational parts of the Gromov product of two loops and the standard loop product. Considering the triangle $\triangle$ (in T_pM) formed by α, β, and $\beta * \alpha$, we get (say) $|\triangle| \leq \frac{3}{2} \frac{|t\alpha||t\beta|}{2}$. Then

$$d\left(r(\beta * \alpha),\, r(\beta) r(\alpha)\right) \leq K |t(\alpha)||t(\beta)|.$$

If L denotes the length of the largest path in the homotopy then $L \leq |t\alpha| + |t\beta|$, so

$$\big|t(m(\beta)m(\alpha)) - t(\beta * \alpha)\big| = \big|r(\beta)t(\alpha) + t(\beta) - t(\beta * \alpha)\big| \leq K\left(|t(\alpha)| + |t(\beta)|\right)|t(\alpha)||t(\beta)|.$$

For commutators the situation is perhaps better than expected:

- $d\left(r[\beta,\alpha],\, [r\beta, r\alpha]\right) \leq \frac{4}{3} K \left(|t\alpha||t\beta| + \frac{1}{2}|t[\beta,\alpha]|\left(|t\alpha| + |t\beta|\right)\right)$
- $|t[\beta,\alpha] - t[m\alpha, m\beta]| \leq \frac{4}{3} K \left(|t\beta| + |t\alpha|\right)\left(|t\alpha||t\beta| + \frac{1}{2}|t[\beta,\alpha]|\left(|t\alpha| + |t\beta|\right)\right)$
- $\sqrt{K}|t[\beta,\alpha]| \leq |r\alpha| \sinh\left(\sqrt{K}|t\beta|\right) + |r\beta| \sinh\left(\sqrt{K}|t\alpha|\right) + \frac{2}{3} K |t\alpha||t\beta| \sinh\left(\sqrt{K}\left(|t\alpha| + |t\beta|\right)\right)$

Recall that for rotations $A, B \in \mathcal{O}(n)$ we have $|[A,B]| \leq 2|A||B|$. For Euclidean motions α, use $|\alpha| = \max\{|r(\alpha)|,\, 3\sqrt{K}|t(\alpha)|\}$.

Thus if $|\alpha|, |\beta| \le \frac{1}{3}$, then

$$\begin{aligned} |t(\alpha)|, |t(\beta)| &< \frac{1}{9K^{\frac{1}{2}}} \\ |m[\beta,\alpha]| &\le 2.4|m\alpha||m\beta| \le 0.8\min\{|m\alpha|,|m\beta|\}. \end{aligned}$$

3 Commutator length

Gromov's striking application is that the commutator length of the subgroup of $\pi_1(M)$ generated by elements of rotation $< \frac{1}{3}$ is bounded in terms of the dimension n only.

Proposition 3.1 *Given $\delta < \frac{1}{3}$, let $\widetilde{\Gamma}^\delta$ be some collection of short loops α with $r\alpha < \delta$. Then the group $\left\langle \widetilde{\Gamma}^\delta \right\rangle \subset \pi_1(M)$ is nilpotent with degree of nilpotency bounded by a dimensional constant $d = d(n)$.*

Pf Let $\widetilde{\Gamma}^\delta$ be any set of short loops α with $r\alpha < \delta$. We choose a short basis: pick $\alpha_1 \in \widetilde{\Gamma}^\delta$ with minimal $|\alpha_1|$. Pick $\alpha_{i+1} \in \widetilde{\Gamma}^\delta$ with $|m\alpha_{i+1}|$ minimal in $\widetilde{\Gamma}^\delta - \langle \alpha_1, \ldots, \alpha_i \rangle$. It is not necessary to prove that this process stops.

We can prove that $|m(\alpha_i\alpha_j^{-1})| \ge \max\{|m\alpha_i|, |m\alpha_j|\}$. For if $|m(\alpha_i\alpha_j^{-1})| < \max\{|\alpha_i|, |\alpha_j|\}$ (wlog $|\alpha_i| < |\alpha_j|$) then $\alpha_i\alpha_j^{-1} \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$ and so too $\alpha_i = \left(\alpha_i\alpha_j^{-1}\right)\alpha_j \in \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$, a contradiction, so therefore the short basis has $|m(\alpha_i, \alpha_j^{-1})| \ge \max\{|m\alpha_i|, |m\alpha_j|\}$. After accounting for the homotopy error, we get, say,

$$|m(\alpha_i)m(\alpha_j^{-1})| \;\ge\; \max\left(|m(\alpha_i)| - \frac{1}{27}|m(\alpha_j)|,\, |m(\alpha_j)| - \frac{1}{27}|m(\alpha_i)|\right).$$

Our lemma from the Finsler geometry of $\mathcal{SO}(n)$, still goes through, and there is a uniform bound on d. We have already proved that $|m[\alpha,\beta]| < \min\{|m\alpha|, |m\beta|\}$, so for elements $\alpha, \beta \in \widetilde{\Gamma}^\delta$, it follows that $[\alpha_i, \alpha_j] \in \left\langle \alpha_1, \ldots, \alpha_{\min\{i,j\}=1} \right\rangle$ so that commutators of basis elements have length $\le d$. An induction on $[\alpha\beta, \gamma] = [\beta, \gamma][[\gamma, \beta], \alpha][\alpha, \gamma]$ proves that any commutator in $\langle \alpha_1, \ldots, \alpha_d \rangle$ has length $\le d$. □

4 Density of subgroups of small rotational parts

Let Γ^δ be the set of short loops α with $r\alpha < \delta$, let Γ_ρ be the set of short loops α with $t\alpha < \rho$, and put $\Gamma_\rho^\delta = \Gamma^\delta \cap \Gamma_\rho$.

We have found pseudogroups Γ^δ that generate nilpotent subgroups of $\pi_1(M)$. But so what? It is not even clear that such pseudogroups are nontrivial, let alone generators of finite-index subgroups of $\pi_1(M)$. Gromov has a Dirichlet-principle type argument that says that $\langle \Gamma_\rho^\delta \rangle \subset \pi_1(M)$ has 'lots of elements'. Enough, in fact, to generate finite-index subgroups of $\pi_1(M)$.

Lemma 4.1 *Given $\delta < \frac{1}{3}$ and $R < \infty$, there is an $\epsilon > 0$ and a $\rho > R$ so that if $DK^2 < \epsilon$, then the set of translational parts of elements in Γ_ρ^δ is $\rho/4$-dense in the ball of radius $3\rho/4$.*

<u>*Pf*</u>

Pick $\rho_i = 20^i(D(M) + R)$. Assuming the lemma is false, pick $x_i \in B_{3\rho_i/4}$ so that $\mathrm{dist}(x_i, \Gamma^\delta) \geq \rho_i/4$. Since the translation parts of $\pi_1(M)$ is $D(M)$-dense in ρ_i, we can find $\alpha_i \in \pi_1(M)$ with $|t\alpha_i - x_i| < \rho_i/4$, though by assumption $r\alpha_i \geq \delta$.

Fixing i we can show that the rotation parts $r\alpha_i$ are all δ-separated. For convenience put $A_i = r\alpha_i$ and $a_i = t\alpha_i$, so that $m\alpha_i : T_pM \to T_pM$ is given by $X \mapsto A_iX + a_i$. Note that

$$\begin{aligned}
|t(\alpha_i * \alpha_j^{-1})| &\leq |-A_iA_j^{-1}a_j| + |a_i - x_i| + |x_i| + \mathit{error} \\
&\leq \rho_j + D + \frac{\rho_i}{4} + \mathit{error} < \rho_i \\
|t(\alpha_i * \alpha_j^{-1}) - x_i| &= |-A_iA_j^{-1}a_j + a_i - x_i| + \mathit{error} \\
&\leq |a_j| + |a_i - x_i| \leq \rho_j + d + \mathit{error} < \rho_i/4.
\end{aligned}$$

The 'error' is controlled by $\frac{1}{2}\min\{|a_i|, |a_j|\}$. The second inequality implies $\alpha_i * \alpha_j^{-1} \notin \Gamma^\delta$, but the first inequality has $|\alpha_i * \alpha_j^{-1}| < \rho_i$, meaning $r(\alpha_i * \alpha_j^{-1}) \geq \delta$, so $r\alpha_i$ is δ-separated from each of the $r\alpha_1, \ldots, r\alpha_{i-1}$.

For each δ, there is a uniformly finite number of elements of $\mathcal{SO}(n)$ that can be δ-separated, so, as long as ρ_d is within the maximal rank radius, the length of the list $\alpha_1, \ldots, \alpha_d$ is bounded in terms of n. $\square$

Lecture 8 - Gromov's almost flat manifold theorem II

February 25, 2010

1 Small rotation implies almost-translation

Lemma 1.1 *Given any small number $\eta > 0$, there are numbers $\epsilon = \epsilon(n, \delta, \eta, R)$, $\rho = \rho(\delta, \eta) > R$ so that $\gamma \in \Gamma_\rho^{1/3}$ implies $\gamma \in \Gamma_\rho^\eta$, provided M is ϵ-flat.*

<u>*Pf*</u>

Assume there is a motion γ with $r\gamma < \frac{1}{3}$ and $t\gamma < \rho$, where ρ will be chosen momentarily. Put $C = r\gamma$ and $c = t\gamma$, so the affine holonomy action $m\gamma : T_pM \to T_pM$ is $X \mapsto CX + c$. Write $\mathbb{R}^n = E + E^\perp$ where E is the plane of maximum rotation of C. Pick $x \in E$ with $|x| = 2|c|$. There is some $\rho > 2|c|$ so that $t\left(\Gamma_\rho^\delta\right)$ is $\rho/4$-dense in $B(3\rho/4)$, so there is a short loop α with $|t\alpha - x| < \rho/4$ and $r\alpha < \delta$. Note that $|t\gamma| < |t\alpha|$. Put $\alpha_0 = \alpha$ and $\alpha_k = [\alpha_{k-1}, \gamma]$. For convenience write $A_k = r\alpha_k$ and $a_k = t\alpha_k$. Note that

$$\begin{aligned}
|A_{k+1}| &= |r[\alpha_k, \gamma]| \\
&\leq K\left(2|t\alpha_k||t\gamma| + |t[\alpha_k, \gamma]|\left(|t\alpha_k| + |t\gamma|\right)\right) + |[A_k, C]| \\
&\leq K\left(2|t\alpha_k||t\gamma| + |t[\alpha_k, \gamma]|\left(|t\alpha_k| + |t\gamma|\right)\right) + 2|C||A_k|
\end{aligned}$$

Since $k + 1 < d$ (the constant $d = d(n)$ is the nilpotency degree from the previous lemma), we can choose ϵ (therefore K) so small, that after iterating at most d times, we get $|A_k| < |A| < \delta$. Now we can estimate the translation parts of α_k from above

$$\begin{aligned}
a_{k+1} &= -A_kCA_k^{-1}C^{-1}c - A_kCA_k^{-1}a_k + A_kc + a_k + error \\
&= (Id - C)\,a_k + (Id - A_{k+1})\,Ca_k + A_kC\left(Id - A_k^{-1}\right)C^{-1}c + error.
\end{aligned}$$

Thus

$$\begin{aligned}
|a_{k+1}| &\leq |C||a_k| + |A_{k+1}||a_k| + |A_k||c| + error \\
&= \left(\frac{1}{3} + \delta\right)|a_k| + \delta|c| + error \\
&< \left(\frac{1}{3} + \delta\right)|a_k| + \delta|a| + error.
\end{aligned}$$

Iterating, we get $|a_{k+1}| < |a|$. From this we can estimate $|a^E_{k+1}|$ from below.

$$\begin{aligned}|a^E_{k+1}| &\geq |C||a^E_k| - |A_{k+1}||a_k| - |A_k||c| - error\\ &\geq |C||a^E_k| - 2\delta|a| - error.\end{aligned}$$

If $|C|$ is large enough compared to δ and ϵ (which controls the error), then $|a^E_d| > 0$ after iterating $d = d(n)$ times, an impossibility. Therefore $|C|^d$ is small compared to ϵ. $\square$

Note that, for an appropriately large ρ and small ϵ, the set $\Gamma^{1/3}_\rho$ is a pseudogroup under the Gromov product. This is because rotational parts cannot build up under the product: as soon as they always remain smaller than η. Note also that because at least $\rho/|t(\alpha)|$ iterations of α still lie in Γ^η_δ, we have

$$\alpha \in \Gamma^\delta_\eta \quad\Longrightarrow\quad |r(\alpha)| \leq \eta\frac{|t(\alpha)|}{\rho}.$$

Finally note that translational errors for products in $\Gamma^{1/3}_\rho$ are extremely tiny:

$$\begin{aligned}&|t(\beta * \alpha) - t(\alpha) - t(\beta)| \leq \eta\frac{|t(\alpha)||t(\beta)|}{\rho}(1+\epsilon)\\ &|t([\beta,\alpha])| \leq 2\eta\frac{|t(\alpha)||t(\beta)|}{\rho}(1+\epsilon).\end{aligned}$$

2 Gromov's Normal Basis

We have found a normal subgroup of finite index in $\pi_1(M)$, but we have not found that the corresponding cover is a Lie group. The first step in finding a Lie group is constructing a "Gromov normal basis".

Let $\delta_1 \in \Gamma^\eta_\rho$ be a minimal element. By the commutator estimates, $[\delta_1, \Gamma^\delta_\rho] = 0$. Given any $\alpha \in \Gamma^\eta_\rho$ let $\alpha_i = \delta^i_1 * \alpha$ whenever the product is in Γ_ρ. Since Gromov products among elements in Γ^η_ρ are almost-translations, we can find a unique $\alpha_j \in \{\alpha_i\}$ with

$$\begin{aligned}\langle t(\alpha_j), \delta_1\rangle &\geq 0\\ \langle t(\tilde\alpha_{j-1}), \delta_1\rangle &< 0.\end{aligned}$$

Put $\tilde\alpha = \alpha_j$. Finally let α' be the projection of $\tilde\alpha$ onto the subspace orthogonal to $t(\delta_1)$ in $\mathbb{R}^n = T_p(M)$. Define the product $\beta' * \alpha'$ to be the projection of the product $\tilde\beta * \tilde\alpha$ onto the compliment. The collection of the α' with this product form a pseudogroup Γ'.

Using that $|t(\delta_1)|$ is smaller than $|t(\alpha)|$ we can prove $|t(\alpha')| \leq |t(\tilde\alpha)| \leq 1.5|t(\alpha')|$, and therefore the translation estimates from above still apply to products in Γ'.

Perform the process again: there is some $\delta'_2 \in \Gamma'$ (projection of some $\tilde\delta_2$) with shortest translation part among elements of Γ', etc.

Due to the denseness of Γ_ρ^η in $\mathbb{R}^n$, this process will terminate with exactly n elements $\delta_1, \tilde{\delta}_2, \tilde{\tilde{\delta}}_2, \ldots$. The subgroup of Γ_ρ^η generated by these n elements is actually Γ_ρ^η itself.

This is because any element of Γ_ρ^η is first translated by δ_1 so it is 'almost' in the complimentary plane. Then is is translated by δ_2 so it is 'almost' in the plane complimentary to δ_1, δ_2, etc, until it is translated until it is 'almost' at the origin. But this contradicts the minimality of the n^{th} element δ_n.

Let G_k indicate the subpseudogroup generated by $\delta_1, \ldots, \delta_k$. The commutator estimates indicate that $[\delta_i, \delta_j]$ is mush smaller than δ_i or δ_j. Its representative $\widetilde{[\delta_i, \delta_j]}$ will also be much smaller. But since δ_i, δ_j were chosen minimally, this implies $[\delta_i, \delta_j] \subset G_{\min\{i,j\}-1}$.

Therefore each element in Γ_ρ^η has a unique expression in the form $\delta_1^{k_1} * \cdots * \delta_n^{k_n}$.

3 Construction of the Lie group and the covering map

According to a theorem of Malcev, the product of two elements of Γ^η is

$$\delta_1^{k_1} * \cdots * \delta_n^{k_n} \; * \; \delta_1^{l_1} * \cdots * \delta_n^{l_n} \; = \; \delta_1^{P_1} * \cdots * \delta_n^{P_n},$$

where the P_i are polynomials of degree $\leq n+1-i$ in the $k_1, \ldots, k_n, l_1, \ldots, l_n$, called the Malcev polnomials.

Now we can identify $p = \delta_1^{k_1} * \cdots * \delta_n^{k_n}$ with the lattice point $\sum k_i \delta_i$, and use the Malcev polynomials to determine a product on this lattice. Replacing the integers k_i with real numbers, and still using the polynomials to determine products, we now have a nilpotent group structure on $\mathbb{R}^n$.

Through the exponential map, large balls around the origin in $\mathbb{R}^n$ can be identified with large balls around a basepoint in the universal cover $\tilde{M}$ of M. Now Γ^η acts by left translation on $\mathbb{R}^n$ and also by deck transformation on $\tilde{M}$. These actions are almost compatible, so after an appropriate center-of-mass averaging, we get a Γ^η-equivariant map from the Lie group to $\tilde{M}$.

Lecture 9 - Fukaya's Theorem

March 2, 2010

1 Statement

Theorem 1.1 (Fukaya) *Given n, $\mu > 0$, there is a number ϵ so that whenever N^n, M are Riemannian manifolds with $|sec| \leq 1$, $\mathrm{inj}(N) > \mu$, and $d_{GH}(N, M) < \epsilon$, then there is a (differentiable?) submersion $f : M \to N$ so that (M, N, f) is a fiber bundle, the fibers are quotients of nilmanifolds, and $e^{-\tau(\epsilon)} < |df(\xi)|/|\xi| < e^{\tau(\epsilon)}$.*

We use τ to indicate a function of ϵ with $\lim_{\epsilon\to 0} \tau(\epsilon) = 0$. We set up some notation that will be used throughout.

$$\begin{aligned} R &= \min\{\mu, 1\}/2 \\ \sigma &= \text{a small number}, \quad 0 < \epsilon << \sigma << 1 \\ r &= \sigma R \end{aligned}$$

2 Embedding into an l^2 space

Let (Z, d) be a discrete metric space, with ϵ-almost isometries into M and N, $j_M : Z \to M$ and $j_N : Z \to N$. Since $d_{GH}(M, N) < \epsilon$, we can choose (Z, d) and j_M, j_N so that $Z = (z_1, \dots)$ is a countable set, M (resp. N) is in the ϵ-neighborhood of $j_N(Z)$ (resp. $j_M(z)$), and so that $j_M(Z)$ (resp. $J_N(z)$) is ϵ-dense and $\epsilon/4$-separated in M (resp. N).

Consider the space $\mathbb{R}^Z = l^2(Z)$, the Hilbert space on Z. If ϵ is small compared to μ then we can define $f_N : N \to \mathbb{R}^Z$ by setting

$$p \mapsto (\mathrm{dist}_N(p, z_1), \dots) .$$

This map is 1-1, but not differentiable since $\mathrm{dist}_N(z_i, \cdot)$ is Lipschitz and not C^1 (also it is not a map into $l^2(Z)$ unless $\#\{Z\} < \infty$). However we can compose this with a C^∞

cutoff function $h : \mathbb{R} \to \mathbb{R}$ that is constant at 0, and equals zero outside a definite radius. Specifically,

$$\begin{aligned} h(t) &= 1 \quad \text{if } t \le 0 \\ h(t) &= 0 \quad \text{if } t \ge r \\ h'(t) &\in [-\kappa/r, 0) \quad \text{if } t \in (0, r/8] \cup [7r/8, r) \\ h'(t) &\in [-\kappa/r, -2/r] \quad \text{if } t \in (r/8, 7r/8). \end{aligned}$$

Now define

$$f_N(p) = (h(\mathrm{dist}_N(p, z_1)), \dots) .$$

Let

$$K = \sup_{x \in N} \# \left(B_r(x) \cap j_N(Z)\right) .$$

The following hold, for appropriate constants C, C_1, C_2:

- f_N is an embedding
- $\exp^\perp : T^\perp N :\to \mathbb{R}^Z$ is a diffeomorphism out to radius $C\sqrt{K}$.
- (quasi-isometry) we have $|df_N(\xi)|/|\xi| \in (C_1\sqrt{K}, C_2\sqrt{K})$
- If $d_N(x, y)$ is small enough compared to ϵ, σ, and μ, then

$$d(x, y) \le CK^{-1/2} \operatorname*{dist}_{\mathbb{R}^Z}(f_N(x), f_N(y)).$$

For a proof see A. Katsuda, *Gromov's convergence theorem and its applications* (1984). We would like to say something about a similar map $M \to \mathbb{R}^Z$, but we cannot expect the distance functions $\mathrm{dist}_M(z_i, \cdot)$ can themselves ever be made differentiable. Yet we can smooth them. For $p \in M$ set

$$d_z(p) = \fint_{B_\epsilon(z)} \mathrm{dist}_M(p, y)\, dy.$$

Then d_z is C^1 (but not C^2), for if $\xi \in T_pM$ then

$$\xi(d_z)(p) = \fint_{B_\epsilon(z)} \xi(\mathrm{dist}_M(p, y))\, dy,$$

and $\xi(\mathrm{dist}_M(p, y))$ is defined almost everywhere.

Proposition 2.1 *The maps $j_N : N \to \mathbb{R}^Z$, $j_M : M \to \mathbb{R}^Z$ are embeddings, and $j_M(M)$ is in the $6\epsilon\sqrt{K}$-tubular neighborhood of $j_N(N)$.*

Pf

We prove the last statement. Since M and N are ϵ-close in the Gromov-Hausdorff sense, we can find a distance function d on $M \coprod N$ that restricts to the Riemannian distance on M and N respectively, and so that M is in the ϵ-neighborhood of N and vice-versa, and with $\text{dist}(j_M(z_i), j_N(z_i)) < \epsilon$. Let p be any point of M and let $p' \in N$ be a point with $d(p, p') < \epsilon$. Then

$$\begin{aligned} d(p, j_M(z_i)) &\le d(p', j_N(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i)) \\ d(p', j_N(z_i)) &\le d(p, j_M(z_i)) + d(p, p') + d(j_N(z_i), j_M(z_i)) \end{aligned}$$

so that

$$|\text{dist}_M(p, j_M(z_i)) \,-\, \text{dist}_N(p', j_N(z_i))| \;\le\; 2\epsilon.$$

Since $|h'(t)| \le 2$ we have

$$|h(\text{dist}_M(p, j_M(z_i))) \,-\, h(\text{dist}_N(p', j_N(z_i)))| \;\le\; 4\epsilon.$$

Then

$$\begin{aligned} |f_M(p) \,-\, f_N(p')|^2 \;&=\; \sum_i \left(h(\text{dist}_M(p, j_M(z_i))) \,-\, h(\text{dist}_N(p', j_N(z_i)))\right)^2 \\ &\le\; 16K\epsilon^2. \end{aligned}$$

Using the averaged quantity $d_{z_i}(p)$ in place of $\text{dist}_N(z_i, p)$ changes the estimates by at most 2ϵ, so we get the result. $\square$

Now we have a map $f : M \to N$ given by

$$f \;=\; f_N^{-1} \circ \pi \circ \exp^{\perp -1} \circ f_M$$

where π indicates the projection from the normal bundle of $f_N(N)$ onto N.

3 $f : M \to N$ is a fiber bundle

We have to prove that $f_M(M)$ is transverse to the fibers of the normal bundle of $f_N(N)$ in $\mathbb{R}^Z$. This follows directly from the following proposition.

Proposition 3.1 *Given any $\nu > 0$, one can choose ϵ, σ so that the following holds. If $p \in M$ and $p' = f(p)$, then given any $\xi' \in T_{p'}N$ there exists a $\xi \in T_pM$ such that*

$$\frac{|df_M(\xi) \,-\, df_N(\xi)|}{|df_N(\xi)|} \;\le\; \nu.$$

Pf

Let $l' : [0, t']$ be a unit-speed geodesic in N with $l'(0) = p'$ and $\frac{Dl'}{dt} = \xi'$. Let $l : [0, t]$ be a geodesic in M with $l(0) = p$ and $\mathrm{dist}_{\mathbb{R}^Z}(l(t), l'(t')) < \epsilon$. Now let l'_i be a geodesic from $j_N(z_i)$ to p' and let l_i be a geodesic from a point $y \in B_\epsilon(j_M(z_i))$ to p. Let θ_i be the angle between l and l_i, and let θ'_i be the angle l' and l'_i. We prove that

$$\left| \frac{d}{dt}\Big|_{t=0} h(\mathrm{dist}_N(y, p)) \; - \; \frac{d}{dt}\Big|_{t=0} h(\mathrm{dist}_N(j_N(z_i), p')) \right| \; \leq \; \nu.$$

We break the proof into two parts; when $\mathrm{dist}(j_N(z_i), p) < r/8 - \epsilon$ or $\mathrm{dist}(j_N(z_i), p) > 7r/8 + \epsilon$, and when $\mathrm{dist}(j_N(z_i), p) \in [r/8 - \epsilon, 7r/8 + \epsilon]$.

In the first case,

$$\begin{aligned} \left| \frac{d}{dt} h(\mathrm{dist}_M(y, p)) \right| \; &= \; h' \cdot \mathrm{dist} \; \leq \; \kappa r/8 \\ \left| \frac{d}{dt} h(\mathrm{dist}_N(j_N(z_i), p')) \right| \; &= \; h' \cdot \mathrm{dist} \; \leq \; \kappa r/8 \end{aligned}$$

so that

$$\left| \frac{d}{dt} h(\mathrm{dist}_N(y, p)) \; - \; \frac{d}{dt} h(\mathrm{dist}_N(j_N(z_i), p')) \right| \; \leq \; 2\kappa r/8 < \kappa\sigma/8$$

Now we consider the second case. By the first variation formula, we have to prove that $|\theta_i - \theta'_i|$ is small. By Toponogov's comparison theorem, we have to prove the following.

Lemma 3.2 *Given $\delta > 0$, $\mu > 0$, there is a ν with the following properties. Given $\delta R < t_1, t_2 < R$, assume $l_1 : [0, t_1] \to M$, $l_2 : [0, t_2] \to M$ are geodesics with $l_1(0) = l_2(0) = p$, and $l'_1 : [0, t'_1] \to N$, $l'_2 : [0, t'_2] \to N$ are minimal geodesics with $l'_1(0) = l'_2(0) = p'$ with $d(l'_1(t'_1), l_1(t_1)) < \nu$, $d(l'_2(t'_2), l_2(t_2)) < \nu$. If θ and θ' are the angles formed by $l_1(0)$, $l_2(0)$ and $l'_1(0)$, $l'_2(0)$ respectively, then $|\theta - \theta'| < \mu$.*

Pf

□

□

Lecture 10 - F-structures I

March 9, 2010

1 Partial Actions

Def A *partial action*, A, of a topological group G on a Hausdorff space X is given by

i. The *domain* of the action: a neighborhood $\mathcal{D} \subset G \times X$ of $\{e\} \times X$.

ii. A continuous map $A : \mathcal{D} \to X$, also written $(g, x) \to gx$, such that $(g_1g_2)x = g_1(g_2x)$ whenever (g_1g_2, x) and (g_1, g_2x) lie in $\mathcal{D}$.

To emphasize the domain, a partial action A can be written $(A, \mathcal{D})$.

We can form an equivalence relation on the set of partial actions. Two partial actions $(A_1, \mathcal{D}_1)$ and $(A_2, \mathcal{D}_2)$ are equivalent if for any subset $\mathcal{D} \subset \mathcal{D}_1 \cap \mathcal{D}_2$, we have $A_1|_{\mathcal{D}} = A_2|_{\mathcal{D}}$; we will denote an equivalence class by $[A]$. Any global action defines a local action; an equivalence class which has such a member will be called *complete*. Notice that if G is connected, any two global actions in the same equivalence class are identical. An equivalence class of partial actions is called a *local action*.

In the smooth category, the class of local actions of a Lie group G is just the class of homomorphisms from the Lie algebra of G to the Lie algebra of vector fields on X. The completeness of an action is the same as the *global* integrability of the individual vector fields.

A subset $X_0 \subset X$ is called $[A]$-invariant if whenever $x_0 \in X_0$ and $(g, x_0) \in \mathcal{D}$ (for some $\mathcal{D}$ associated to a partial action $A \in [A]$) then $gx_0 \in X_0$. The intersection of $[A]$-invariant sets is $[A]$-invariant, so any point x lies in a minimal $[A]$-invariant set, called the orbit of x, denoted $\mathcal{O}_x$ or just $\mathcal{O}$. The orbits partition the space X.

A local action $[A]$ on X can be restricted to any subset $U \subset X$ by restricting the domain $\mathcal{D}$ of any representative of $[A]$ to any open subset $\mathcal{D}'$ that contains $\{e\} \times U$ and which obeys (*ii*) from above. If $(A_1, \mathcal{D}_1)$ represents a local action on U_1 and $(A_2, \mathcal{D}_2)$ represents a local action on U_2 and if $A_1|_{\mathcal{D}_1 \cap \mathcal{D}_2} = A_2|_{\mathcal{D}_1 \cap \mathcal{D}_2}$, a partial action on $U_1 \cup U_2$ is can be constructed with domain $\mathcal{D}_1 \cup \mathcal{D}_2$. This of course defined a local action on $U_1 \cup U_2$. Unlike an action, a local action $[A]$ on X pulls back along any local homeomorphism $f : Y \to X$ to a local action $f^*[A]$ on Y.

2 $\tilde{\mathfrak{g}}$-structures and F-structures

Def A *sheaf*, $\mathfrak{F}$, on a topological space X is an association between open sets $U \subset X$ and groups that satisfies the following three axioms.

a) $\mathfrak{F}(U)$ is a group whenever U is an open subset of X

b) If $V \subseteq U$ is an inclusion of open sets, there is a homomorphism (the restriction homomorphism) $\rho_{VU} : \mathfrak{F}(U) \to \mathfrak{F}(V)$ subject to the restrictions that *(i)* $\mathfrak{F}(\varnothing) = \{0\}$, *(ii)* $\rho_{UU} = \mathrm{Id}$, and *(iii)* $W \subseteq V \subseteq U$ implies $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$.

c) If $\{V_\alpha\}$ is an open covering of U and $s_\alpha \in \mathfrak{F}(V_\alpha)$ satisfies $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there exists a unique element $s \in \mathfrak{F}(U)$ so that $s|_{U_\alpha} = s_\alpha$.

If $\mathfrak{F}$ only satisfies (*a*) and (*b*) it is called a *presheaf*. The salient feature of sheafs is the *stalk* that exists over each point, and the nature of their global connectedness. Let $\mathfrak{F}$ be a sheaf over M, and let $p \in M$. Let $\{U\alpha\}_{\alpha \in \mathcal{A}}$ be the family of open sets containing p; in fact the $\mathfrak{F}(U_\alpha)$ constitute a directed family of groups. The direct limit is called the stalk at p. A topology can be put on the space of stalks: a neighborhood base is given by the images of the "sections" $\mathfrak{F}(U)$ in the space of stalks. Stalks can be defined if just a presheaf structure exists, and then sections of the space of stalks constitute a sheaf (the *sheafification* of the presheaf).

Let $\mathfrak{F}$ be a sheaf of connected topological groups (note there is some question about topology here; we just accept that there are two topologies, the sheaf topology, and a topology that makes the stalks into Lie groups— for differential geometric applications, usually the sheaf topology is ignored). An *action* of $\mathfrak{F}$ on X is given by a local action of $\mathfrak{F}(U)$ for each open U such that the local actions agree with the sheaf restriction maps. To be explicit, when $x \in V \subset U$ and $g \in \mathfrak{F}(U)$, we have $gx = \rho_{VU}(g)x$ wherever gx and $\rho_{VU}(g)x$ are defined.

A set $S \subset X$ is called invariant if $S \cap U$ is invariant under $\mathfrak{g}(U)$ for all open subsets $U \subset X$. A minimal invariant set is called an orbit. The orbits partition X, and a set that is the disjoint union of orbits is called *saturated*.

We denote the stalk at x by $\mathfrak{F}_x$. If $f : X \to Y$ is a local homeomorphism, we denote by $f^* \mathfrak{F}$ the pullback sheaf.

Def An action of a sheaf $\mathfrak{F}$ is called a *complete local action* if whenever $x \in X$ there exists a neighborhood $V(x)$ of x and a local homeomorphism $\pi : \tilde{V}(x) \to V(x)$ so that $\tilde{V}(x)$

is Hausdorff and

i. If $\tilde{x} \in \pi^{-1}(x)$, then for any open neighborhood $W \subset \tilde{V}(x)$ of x, the structure homomorphism $\mathfrak{F}(W) \to \mathfrak{g}_{\tilde{x}}$ is an isomorphism

ii. The local action of $\pi^* \mathfrak{F}(\tilde{V}(x))$ on $\tilde{V}(x)$ is complete.

If $\pi : \tilde{V}(x) \to V(x)$ is a covering space, the deck transformation group Γ induces a natural action on $\pi^* \mathfrak{F}$, called the *holonomy action*. For $\gamma \in \Gamma$ it is easy to verify $\gamma(gx) = \gamma(g)\gamma(x)$. Specifically, if $g \in \mathfrak{F}(U)$ then $\gamma(g)$ acts on elements $x \in \gamma(U)$ via $\gamma(g) = \gamma \circ g \circ \gamma^{-1}$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ on X is a sheaf, $\mathcal{G}$, of connected topological groups and a complete local action of $\mathcal{G}$ on X such that the sets $V(x)$ and $\tilde{V}(x)$ can be chosen so that

i. $\pi : \tilde{V}(x) \to V(x)$ is a normal covering map

ii. For all x, $V(x)$ is saturated

iii. For all $\mathcal{O}$, if $x, y \in \overline{\mathcal{O}}$, then $V(x) = V(y)$.

Condition (*iii*) actually implies that $\mathcal{G}$ is a locally constant sheaf on $\overline{\mathcal{O}}$, though not necessarily on neighborhoods of $\overline{\mathcal{O}}$. A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called *pure* if it is a locally constant sheaf on each $V(x)$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called an F-structure if each stalk $\mathcal{G}_x$ is isomorphic to a torus, and the sets $\tilde{V}(x)$ can be chosen to be finite coverings. If one can choose $\tilde{V}(x) = V(x)$, then $\mathcal{G}$ is called a *T-structure*. If $\tilde{V}(x)$ can be chosen independently of x then $\mathcal{G}$ is called an *elementary F-structure*.

If $\mathcal{G}$ is a $\tilde{\mathfrak{g}}$-structure with sheaf $\mathcal{G}$ and $\mathcal{G}' \subset \mathcal{G}$ is a subsheaf, then, since the action of $\mathcal{G}$ descends to $\mathcal{G}'$, the sheaf $\mathcal{G}'$ comes with a complete local action. This defines a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}'$ called a *substructure*.

Proposition 2.1 *If X is a compact manifold that carries an F-structure of positive rank, then $\chi(X) = 0$.*

Pf

On each $\tilde{V}(x)$ a torus acts with no common fixed points, so almost all of its elements have a fixed-point free action. Given such an element with no fixed points, one finds a one-parameter subgroup that acts on $\tilde{V}(x)$, and so $\chi(\tilde{V}(x)) = 0$, so $\chi(V(x)) = 0$. Essentially the same argument shows that $\chi(V(x) \cap V(y)) = 0$. Covering X with finitely many $V(x)$, we get the result. $\square$

Lecture 11 - F-structures II

March 11, 2010

1 Atlases and Polarizations

If X is a manifold and for all U, $g(U)$ is a connected topological group and its local action on U is effective, then the restriction maps will be injective. For this kind of $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$, on each $V(x)$ there is a unique pure substructure $\mathcal{G}_\alpha \subset \mathcal{G}\,|_{V(x)}$ with stalk $\mathcal{G}_{\alpha,x} = \mathcal{G}_x$. If every $V(x)$ can be chosen this way, we call $\mathcal{G}$ an effective $\tilde{\mathfrak{g}}$-structure.

We define the *rank* of a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ at x to be $\dim \mathcal{O}_x$, and we say $\mathcal{G}$ has positive rank if $\dim \mathcal{O}_x > 0$ for all x.

Def If $\mathcal{G}$ is an effective $\tilde{\mathfrak{g}}$-structure, a collection $\{(U_\alpha, \mathcal{G}_\alpha)\}$ is called an *atlas* for $\mathcal{G}$ if

- the U_α are connected, saturated (w.r.t. $\mathcal{G}$), and open, and form a locally finite covering of X
- each $\mathcal{G}_\alpha \subset \mathcal{G}\,|_{U_\alpha}$ is pure
- given any x, there is an α with $\mathcal{G}_{\alpha,x} = \mathcal{G}_x$.

A *subatlas* $\mathcal{A}' \subset \mathcal{A}$ is an atlas $\{(U'_\alpha, \mathcal{G}'_\alpha)\}$ so that $U'_\alpha \subset U_\alpha$ and $\mathcal{G}'_\alpha = \mathcal{G}_\alpha\,|_{U'_\alpha}$.

A substructure $\mathcal{P} \subseteq \mathcal{G}$ is called a *polarization* for $\mathcal{G}$ if $\mathcal{P}$ has an atlas so that the rank of $\mathcal{P}_\alpha$ is positive and constant on U_α (the rank of $\mathcal{P}$ may vary with α). A polarization $\mathcal{P}$ is called *pure* if $\mathcal{P}$ is a pure $\tilde{g}$-structure.

Proposition 1.1 (regular atlases) *If the F-structure $\mathcal{G}$ on the manifold X (possibly open) has an atlas $\{(U_\alpha, \mathcal{G}_\alpha)\}$, then it has an atlas $\{(\underline{U}_\alpha, \mathcal{G}_\alpha)\}$ for $\mathcal{G}$ with the following properties:*

(1) The sets $\underline{U}_\alpha$ have compact closure

(2) If $x \in \underline{U}_{\alpha_1} \cap \cdots \cap \underline{U}_{\alpha_k}$, then (for some ordering) $\mathcal{G}_{\alpha_1,x} \subseteq \cdots \subseteq \mathcal{G}_{\alpha_k,x}$

(3) *Given any $x \in \underline{U}_\alpha$, there is at most one $\underline{U}_\beta$ with $\mathcal{G}_{\alpha,x} = \mathcal{G}_{\beta,x}$. If the manifold is compact or if (1) is dropped, we can assume strict inclusion in (2).*

Pf

(1) is clear.

(2) We argue inductively. Assume $x \in U_\beta \cap U_\gamma$ but $\mathcal{G}_{\beta,x} \nsubseteq \mathcal{G}_{\gamma,x}$ and $\mathcal{G}_{\gamma,x} \nsubseteq \mathcal{G}_{\beta,x}$. Since $\mathcal{G}_y \neq \mathcal{G}_{\beta,y} \neq \mathcal{G}_{\gamma,y}$ for any $y \in U_\beta \cap U_\gamma$, so that $U_\beta \cap U_\gamma$ is covered by other domains in the atlas. Thus we can replace U_β by $U_\beta - \overline{U_\gamma}$ and U_γ by $U_\gamma - \overline{U_\beta}$, and still retain $X = \bigcup U_\alpha$.

(3) First assume (1) an be dropped or that the manifold is compact. Let $U_1, \ldots, U_k$ be a maximal subcollection so that $\bigcup U_i$ is connected and whenever $x \in U_i \cap U_j$, then $\mathcal{G}_{i,x} = \mathcal{G}_{j,x}$. Set $\underline{U}_1 = U_1$ and let $\underline{U}_2, \ldots, \underline{U}_l$ be the connected components of $\bigcup_i U_i$ where the union is over the U_i the have nonzero intersection with U_1. Now consider the U_i that do not intersect U_1, and repeat this process.

Doing this for all such subcollections, the result follows. □

Proposition 1.2 (invariant metrics) *Assume X is a manifold, and let $\mathcal{A} = \{(U_\alpha, \mathcal{G}_\alpha)\}$ be a regular atlas for $\mathcal{G}$. If $\mathcal{G}$ has the property that each $\tilde{V}(x) \to V(x)$ is a finite normal covering, then X has a $\mathcal{G}$-invariant metric.*

Pf

Let $\mathcal{A}' \subset \mathcal{A}$. With a partial ordering of the U_α coming from (2) of Proposition 1.1, we can choose U_α to be maximal. Cover U'_α by sets $V(x_1), \ldots, V(x_k)$ with $\overline{V(x_i)} \subset U_\alpha$. Put some metric on $V(x_1)$, lift it to $\tilde{V}(x_1)$, and average it over the action of $\mathcal{G}$ and over the deck action. Project back to $V(x_1)$. Put a metric on $V(x_2)$ that agrees with the invariant metric on $V(x_1)$ on the overlap, and perform the same averaging. Eventually this gives an invariant metric on U'_α. This same procedure can be done on some U'_β, only the starting metrics on the $V(x_i)$ must now agree with the metric on U_α where the intersection is nonempty. □

2 Examples

Reason for passing to coverings

Let K be the Klein bottle. The torus T acts on the orientable 2-cover of K (which is again the torus). The action of T on the cover gives rise to a local action, which passes back to K. Let $\mathcal{G}$ be the F-structure defined here, with a locally constant sheaf $\mathfrak{g}$. Clearly no local action of $\mathfrak{g}$ on K is complete, but passing to the cover gives a complete action.

Canonical action of a sheaf on its total space

Let $\mathfrak{g}$ be a locally constant sheaf of topological groups over a topological space X, with projection π. Let $\mathfrak{g}^* = \pi^*(\mathfrak{g})$ denote the pullback sheaf. There is a canonical local action of

$\mathfrak{g}^*$ on the total space of the sheaf $\mathfrak{g}$. This action is pure, the orbits are just the fibers, and $(\pi^{-1}(X), \mathfrak{g}^*)$ is a pure polarization.

A non-polarized F-structure with a polarization

Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. The Clifford torus, just the set of points $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{C}^2$, acts on S^3 via multiplication. Let $\mathcal{G}$ be the F-structure obtained from this action. The structure is pure, though not of constant rank. The natural atlas is just $\{(U_\alpha, \mathcal{G})\}$. It is not a polarization however, since the rank is nonconstant. In fact, there is no polarized atlas for this structure; one must pass to a substructure. Any one-parameter subgroup of the torus besides either of the factors themselves, yields a pure polarized T-structure.

An F-structure with no pure polarization

We consider $\mathbb{S}^3$ as above. If $U \in \mathbb{S}^3$ intersects $(z_1, 0)$ but not $(0, z_2)$, then $\mathfrak{g}(U)$ is the circle acting by $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, z_2)$. Similarly if U meets $(z_2, 0)$ but not $(z_1, 0)$. If U meets neither circle, then $\mathfrak{g}(U)$ is the torus. If U meets both circles, then $\mathfrak{g}(U) = \{\mathrm{Id}\}$.

The Solvgeometry Let A be a matrix $A \in SL(2, \mathbb{Z})$, so A can be considered a map $A : T^2 \to T^2$. Let M^3 be its mapping torus. If A is nilpotent, M^3 is a nilmanifold, and it supports an F-structure of rank 1. If A has distinct real eigenvalues, it is a solvmanifold. In this case there is a pure F-structure of rank 2, with exactly two substructures of rank 1, each corresponding to an eigenvalue of A.

A pure F-structure with no polarization

We construct this space is two steps. First let $\mathcal{E}_\theta$ be the flat space $[0,1] \times \mathbb{C} / \sim$, where $\{0\} \times \mathbb{C}$ is identified to $\{1\} \times \mathbb{C}$ via $(0, v) \mapsto (1, e^{2\pi i\theta} v)$. The torus naturally acts on this space, so we get a pure T-structure of nonconstant rank. Any closed subgroup of the torus produces a pure polarized T-structure. Note that $\mathcal{E}_\theta$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{C}$, though has a different metric structure.

Now consider the space diffeomorphic to $[0,1] \times \mathbb{S}^1 \times \mathbb{C}$, but give each $\theta \times \mathbb{S}^1 \times \mathbb{C}$ the metric structure of $\mathcal{E}_\theta$. Since $\mathcal{E}_0$ is isometric to $\mathcal{E}_1$, we can identify $\{0\} \times \mathbb{S}^1 \times \mathbb{R}^2$ and $\{1\} \times \mathbb{S}^1 \times \mathbb{R}^2$; we will call this space simply $\mathcal{E}$. The torus acts on each slice, but if we follow the action around, the holonomy on the action group is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which has a single eigenvalue which corresponds to the rotation that fixes the the base circle in each $\mathcal{E}_\theta$. Let $\mathcal{G}$ denote the corresponding T-structure. Clearly no polarization exists for $\mathcal{G}$, since the stalk at any point of any base circle is has dimension higher than the dimension of the orbit there. Any substructure must have stalk a subgroup of T with the same holonomy, but the only eigenvalue of the matrix above corresponds to the action that fixed the base circles of the $\mathcal{E}_\theta$. There fore the unique substructure of $\mathcal{G}$ has orbits of rank zero, so is therefore nonpolarized.

Lecture 12 - F-structures III

March 16, 2010

1 Pure polarized collapse

Assume the (possibly noncompact) manifold X admits a pure polarized F-structure. This means $\mathfrak{g}$ is a locally constant sheaf whose orbits all have the same dimension. On such a manifold we can split the metric into two parts $g = g' + h$ where h vanishes on vectors tangent to the orbits and g' vanishes on vectors perpendicular to orbits. Set

$$g_\delta = \delta^2 g' + h. \tag{1}$$

Theorem 1.1 (Pure Polarized Collapse) *As $\delta \to 0$ the metric g_δ collapses everywhere. Also $\mathrm{dist}_{g_\delta}(p, q)$ decreases with δ, and the sectional curvature is bounded on any compact set.*

Pf

We examine the curvature at the point p by constructing special coordinates near p. Let k denote the dimension of the orbits Let N^{n-k} be any submanifold through p transverse to the orbits. Given coordinates $y^1, \dots, y^{n-k}$ on N we can extend these coordinate functions to a neighborhood of p by projecting along the orbits. Finally a k-torus acts locally on the orbits themselves; the push-forward of a basis of its Lie algebra is an independent Abelian set of Killing fields parallel to the orbits, and which span the distribution defined by the orbits. The Frobenius theorem says we can integrate these to get the remaining coordinate functions $x^1, \dots, x^k$ with coordinate fields $\frac{d}{dx^i}$ equal to the original killing fields. We can choose the origin on any orbit to be its point of intersection with N.

The coordinates field $\frac{d}{dy^i} = X_i + V_i$ can be decomposed into a part parallel to the orbits X_i and a part perpendicular to the orbits V_i. Now make the change of coordinates $u^i = \delta x^i$. Then

$$g_\delta = \begin{pmatrix} \left\langle \frac{d}{dx^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta \left\langle \frac{d}{dx^i}, \frac{d}{dy^j} \right\rangle_{g_1} \\ \delta \left\langle \frac{d}{dy^i}, \frac{d}{dx^j} \right\rangle_{g_1} & \delta^2 \left\langle X_i, X_j \right\rangle_{g_1} + \left\langle V_i, V_j \right\rangle_{g_1} \end{pmatrix}.$$

As $\delta \to 0$ the metric converges to a warped product metric.

2 Polarized collapse

If the polarization is not pure, it means that the various U_α in the atlas are such that the corresponding pure substructures $\mathcal{G}_\alpha$ possibly have different ranks (though the rank is constant on each U_α). We have to modify the metric on each U_α separately, and at the same time push the various U_α away from each other.

Theorem 2.1 *If $\mathcal{G}$ is a polarized F-structure on the compact manifold X, then X admits a sequence of metrics g_δ so that*

(1) The manifold (X, g_δ) collapses

(2) $\mathrm{diam}_{g_\delta}(X) < \mathrm{diam}_{g_1}(X)|\log \delta|$

(3) $\mathrm{Vol}_{g_\delta}(X) < \mathrm{Vol}_{g_1}(X)\delta^k|\log\delta|^n$, *some* $k \geq 1$

(4) Sectional curvature $|K|$ is uniformly bounded.

Pf

Let $\{(U_i, \mathcal{G}_i)\}_{i=1}^N$ be an atlas. Let $f_\alpha : U_\alpha \to [1, 2]$ be a collection of functions, constant on the orbits of $\mathcal{G}$, so that $f_i = 1$ in a neighborhood of ∂U_i, and so that $\bigcup_i f_i^{-1}(2) = X$. Put

$$\rho_i \;=\; \delta^{\log_2 f_i}.$$

We start with the metric $g_0 = \log_2(\delta)\, g$. On U_i we can write

$$g_0 \;=\; g_1' \,+\, h_1,$$

where g_1' is tangent to the orbits of $\mathcal{G}_1$ and h_1 is perpendicular. Then define g_1 by

$$g_1 \quad = \quad \begin{cases} \rho_1^2 g_1' + h_1 & \text{on } U_1 \\ g_0 & \text{on } X \,-\, U_1 \end{cases}$$

Proceed inductively. Once g_{i-1} has been chosen, set $g_{i-1} = g_i' + h_i$ on U_i where g_i' is parallel to the orbits of $\mathcal{G}_i$ and h_i is perpendicular, and put

$$g_i \quad = \quad \begin{cases} \rho_i^2 g_i' + h_i & \text{on } U_i \\ g_{i-1} & \text{on } X \,-\, U_i \end{cases}$$

Now *(1)*, *(2)*, and *(3)* are obvious, where $k = \min \mathrm{rank}\, \mathcal{G}_i$.

We check that sectional curvature is bounded. Let $p \in X$; let $l = \dim \mathcal{O}_p$. Let U_i indicate the atlas charts in which p lies. If we work on a normal atlas, we can arrange that $\{\mathcal{G}_j\}$ etc, where the structures have rank $l_1 > \cdots > l_s \geq k$. The metric near p is changed

s times, and we will keep track of the changes in curvature as the metric is changed each time.

Let N^{n-l_j} be a submanifold transverse to the orbits of $\mathcal{G}_j$, and choose coordinates $(\underline{x}^1, \ldots, \underline{x}^l, \underline{y}^1, \ldots, \underline{y}^{n-l})$ as before with $p = (0, \ldots, 0)$, where the coordinate fields $\frac{d}{d\underline{x}^i}$ are just the action fields of $\mathcal{G}_\alpha$ and the $\underline{y}^1, \ldots, \underline{y}^{n-l}$ are constant on the orbits of $\mathcal{G}_\alpha$.

First we scale the coordinates

$$x^i \;=\; \log\delta \cdot \underline{x}^i \qquad\qquad y^i \;=\; \log\delta \cdot \underline{y}^i.$$

In the new coordinates, we still have that $\frac{d\rho_j}{dx^i} = 0$, but also that

$$\begin{aligned}
\frac{d\rho_j}{dy^i} &= \frac{1}{\log 2}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j}\delta^{\log_2 f_j} \\
\frac{d^2\rho_j}{dy^k dy^i} &= \frac{1}{\log\delta}\frac{1}{\log 2}\frac{d^2 f_j}{d\underline{y}^k d\underline{y}^i}\frac{1}{f_j}\delta^{\log_2 f_j} - \frac{1}{\log\delta}\frac{1}{\log 2}\frac{df_j}{d\underline{y}^k}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j^2}\delta^{\log_2 f_j} \\
&\quad + \left(\frac{1}{\log 2}\right)^2 \frac{df_j}{d\underline{y}^k}\frac{df_j}{d\underline{y}^i}\frac{1}{f_j^2}\delta^{\log_2 f_j}.
\end{aligned}$$

Therefore in these coordinates, the functions ρ_j'/ρ_j and ρ_j''/ρ_j are bounded as $\delta \to 0$. Since by the induction assumption the previous metric g_{j-1} has bounded curvature, so does the new metric.

$\square$

3 Nonpolarized Collapse

Let $\mathcal{G}$ be an F-structure on the manifold M. We construct what is called a 'slice polarization.'

3.1 Pure structure

Let Σ_i be the union of orbits of $\mathcal{G}$ of dimension i. Let Σ_{ϵ_i} denote the set of points of Σ_i a distance of ϵ_i or greater from $\partial\Sigma_i$ (this is a "thickening" of Σ_i). If N is any submanifold let $\nu(N)$ denote the normal bundle. Let S_{ϵ_i,r_i} denote the set $\{v \in \nu(\Sigma_{\epsilon_i})\, s.t.\, \|v\| < r_i\}$, and let Σ_{ϵ_i,r_i} denote the image of S_{ϵ_i,r_i} under the exponential map. If r_i is chosen small enough, the exponential map is a diffeomorphism.

Lemma 3.1 *There is an invariant metric g and numbers ϵ_i, r_i so that*

(1) $\bigcup \Sigma_{\epsilon_i,r_i} = M$

(2) If $i < j$, then $\pi_i = \pi_i \circ \pi_j$ on $\Sigma_{\epsilon_i,r_i} \cap \Sigma_{\epsilon_j,r_j}$.

$\square$

Now set $U_i = \Sigma_{\epsilon_i,r_i}$. If $q \in U_i$, then parallel translation from q to $\pi_i(q)$ along a geodesic induces an injection $\mathcal{G}_q \to \mathcal{G}_{\pi_i(q)}$.

Lemma 3.2 *There exists an inner product $\langle,\rangle_p$ on g_p, the Lie algebra of stalks $\mathcal{G}_p$, that is invariant under the action of $\mathcal{G}_p$ and under the projections π_i whenever $\pi_i(q)$ is defined.*

$\square$

For $p \in S_{\epsilon_i,r_i}$ let K_p^i be the (not necessarily closed) subgroup of $\mathcal{G}_p$ whose lie algebra is the orthogonal complement of the isotropy group of p. Set $K_p^i = \pi_i^{-1}(K_{\pi_i(p)})$. It follows from the previous lemmas that the assignment $p \to K_p^i$ is invariant under the local action of $\mathcal{G}_p$.

We can now describe the collapsing procedure. Let f_i, ρ_i be as before. Fix q and let $U_{i_1}, \ldots, U_{i_j}$, $i_1 < \cdots < i_j$ be the U_i with $q \in U_i$. Let $Z_{i_1} \subseteq \cdots \subseteq Z_{i_j}$ denote the subspaces of T_qM tangent to the orbits of $K_q^{i_1}, \ldots, K_q^{i_j}$. Let $W_{i_j} \subseteq \cdots \subseteq W_{i_1}$ denote the subspaces $W_{i_1} = \pi_{i_1}^{-1}(\mathcal{O}_{\pi_{i_1}(q)}), \ldots, W_{i_j} = \pi_{i_j}^{-1}(\mathcal{O}_{\pi_{i_j}(q)})$. Note that also $Z_{i_j} \subseteq W_{i_j}$.

Now let g be the invariant metric from Lemma 3.1. Set $g_0 = \log^2 \delta \cdot g$, and write a decomposition for g_0

$$g_0 \;=\; g_1' + h_1 + k_1,$$

corresponding to $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Put

$$g_1 \quad = \quad \begin{cases} \rho^2 g_1' + h_1 + \rho^{-2} k_1 & p \in U_1 \\ g_0 & \text{otherwise} \end{cases}$$

Proceed by induction, letting $g_{l-1} = g_l' + h_l + k_l$ be the decomposition according to Z_{i_l}, $Z_{i_l}^{\perp} \cap W_{i_l}$, $W_{i_l}^{\perp}$, and putting

$$g_l \quad = \quad \begin{cases} \rho^2 g_l' + h_l + \rho^{-2} k_l & p \in U_l \\ g_{l-1} & \text{otherwise} \end{cases}$$

First we claim that curvature is bounded as $\delta \to 0$. We establish a coordinate system. Let

$$m_i \;=\; \dim \Sigma_i \;-\; i \;=\; \dim \Sigma_i \;-\; \mathrm{rank}_{\mathbb{F}} \Sigma_i,$$

and let $s^1, \ldots, s^{m_{i_1}}$ be coordinates on Σ_{i_1} constant on the orbits. Extend these to U_{i_1} via π_{i_1}. Let $s^{m_{i_1}+1}, \ldots, s^{m_{i_2}}$, be coordinates on U_{i_2}, constant on the orbits. Extend these to

$U_{i_1} \cap U_{i_2}$. Proceed in this way, finally getting coordinates $s^1, \ldots, s^{m_{i_j}}$ on $U_{i_1} \cap \cdots \cap U_{i_j}$. Now compliment these coordinates with additional coordinates $t^1, \ldots, t^{n-i_j-m_{i_j}}$ that are constant on the orbits of $K_q^{i_j}$ and so that $s^1, \ldots, s^{i_j}, t^1, \ldots, t^{n-i_j-m_{i_j}}$ is a complete system that is transverse to the orbits of $\mathbb{F}$. Finally let $x^1, \ldots x^{i_j}$ be coordinates so that $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_k}}$ are fields generated by the action of $K_q^{i_k}$.

Now we compute the curvature. First consider the change of metric $g_0 \mapsto g_1$. Relabel the coordinates

$$\begin{aligned}
z^1 &= s^1 \\
&\vdots \\
z^{m_{i_1}} &= s^{m_{i_1}} \\
y^1 &= s^{m_{i_1}+1} \\
&\vdots \\
y^{m_{i_j}-m_{i_1}} &= s^{m_{i_j}} \\
y^{m_{i_j}-m_{i_1}+1} &= t^1 \\
&\vdots \\
y^{n-i_j-m_{i_1}} &= t^{n-i_j-m_{i_j}} \\
x^1 & \\
&\vdots \\
x^{i_j}. &
\end{aligned}$$

The orthogonal decomposition of the tangent space given by $Z_{i_1}, Z_{i_1}^\perp \cap W_{i_1}, W_{i_1}^\perp$ roughly corresponds to the selection of the x, y, z coordinates. Working in the Σ_{i_1} stratum, $x^1, \ldots, x^{i_1}$ are coordinates on the rank i_1 orbits themselves; this roughly corresponds to Z_{i_1}. The subspace $Z_{i_1}^\perp \cap W_{i_1}$ is the subspace directly perpendicular to the stratum; this essentially parametrizes the orbits of $\mathbb{F}$ not in Σ_{i_1}, that is, captures the y coordinates, and also captures the remaining x^k. Finally $W_{i_1}^\perp$ parametrizes the orbits of Σ_{i_1}; in fact the coordinate functions $\frac{d}{dx^1}, \ldots, \frac{d}{dx^{i_j}}$ project to zero in this space, or else the action of some of the other strata $\Sigma_{i_2}, \ldots, \Sigma_{i_j}$ would act on the Σ_1 stratum, which is impossible, and also the action fields are tangent to the orbits and $W_{i_1}^\perp$ is perpendicular to all obits. Since the y are coordinates on strata and the strata are perpendicular to $W_{i_1}^\perp$, we get that $\frac{d}{dy^k}$ is perpendicular to $W_{i_1}^\perp$ as well.

Thus we decompose the vectors

$$\begin{aligned}
\frac{d}{dx} &= b_x^1 v_{x,1} + b_x^2 v_{x,2} \\
\frac{d}{dy} &= b_y^1 v_{y,1} + b_y^2 v_{y,2} \\
\frac{d}{dz} &= b_z^1 v_{z,1} + b_z^2 v_{z,2} + b_z^3 v_{z,3}
\end{aligned}$$

according to the decomposition $Z_{i_1}, Z_{i_1}^{\perp} \cap W_{i_1}, W_{i_1}^{\perp}$. Multiplying the coordinate functions by $\log\delta$, we have again that $|\rho''_{i_1}/\rho_{i_1}|$ and $|\rho'_{i_1}/\rho_{i_1}|$ are bounded. We get the following matrix for g.

$$(\log\delta)^2\, g \;=\; \begin{pmatrix} (b_x^1)^2+(b_x^2)^2 & b_x^1 b_y^1 + b_x^2 b_y^2 & b_x^1 b_z^1 + b_x^2 b_z^2 \\ b_x^1 b_y^1 + b_x^2 b_y^2 & (b_y^1)^2+(b_y^2)^2 & b_y^1 b_z^1 + b_y^2 b_z^2 \\ b_x^1 b_z^1 + b_x^2 b_z^2 & b_y^1 b_z^1 + b_y^2 b_z^2 & (b_z^1)^2+(b_z^2)^2+(b_z^3)^2 \end{pmatrix}$$

therefore

$$g_1 \;=\; \begin{pmatrix} \rho^2(b_x^1)^2+(b_x^2)^2 & \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho^2 b_x^1 b_y^1 + b_x^2 b_y^2 & \rho^2(b_y^1)^2+(b_y^2)^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^2 b_y^1 b_z^1 + b_y^2 b_z^2 & \rho^2(b_z^1)^2+(b_z^2)^2+\rho^{-2}(b_z^3)^2 \end{pmatrix}$$

We make the change of coordinates $x \mapsto \rho_{i_1} x$, $z \mapsto \rho_{i_1}^{-1} z$. In the new coordinates the matrix reads

$$g_1 \;=\; \begin{pmatrix} (b_x^1)^2+\rho^{-2}(b_x^2)^2 & \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 \\ \rho b_x^1 b_y^1 + \rho^{-1} b_x^2 b_y^2 & \rho^2(b_y^1)^2+(b_y^2)^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 \\ \rho^2 b_x^1 b_z^1 + b_x^2 b_z^2 & \rho^3 b_y^1 b_z^1 + \rho b_y^2 b_z^2 & \rho^4(b_z^1)^2+\rho^2(b_z^2)^2+(b_z^3)^2 \end{pmatrix}.$$

We must deal with the $\rho^{-1} b_x^2$ term somehow. As we choose δ differently, the b_x^2 (and the other b_K^i for $K = x, y, z$, $i = 1, 2, 3$) will be different. Let $b_{x,\delta}^2$ denote b_x^2 in he metric g_δ. Since the coordinate fields d/dx^k, $1 \le k \le i_1$, are inside of Z_{i_1} to first order, we get

$$\begin{aligned} \lim \frac{b_x^2(q)}{\rho} &= \lim_{\delta\to 0} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\rho - 0} \\ &= \lim_{\delta\to 0} \frac{1}{\log\delta} \frac{b_{x,\delta}^2(q) - b_{x,0}^2(q)}{\delta/\log\delta - 0} \\ &= 0 \end{aligned}$$

Letting $\delta \to 0$, the limiting matrix is just

$$g_1 \;=\; \begin{pmatrix} (b_x^1)^2 & 0 & 0 \\ 0 & (b_y^2)^2 & 0 \\ 0 & 0 & (b_z^3)^2 \end{pmatrix}.$$

To continue, we now focus attention on U_{i_2}. We readjusts choice of coordinates, so now

$$\begin{aligned}
z^1 &= s^1\\
&\vdots\\
z^{m_{i_2}} &= s^{m_{i_2}}\\
y^1 &= s^{m_{i_2}+1}\\
&\vdots\\
y^{m_{i_j}-m_{i_2}} &= s^{m_{i_j}}\\
y^{m_{i_j}-m_{i_2}+1} &= t^1\\
&\vdots\\
y^{n-i_j-m_{i_2}} &= t^{n-i_j-m_{i_j}}\\
x^1 &\\
&\vdots\\
x^{i_j}.&
\end{aligned}$$

One considers the splitting of the tangent space via $Z_{i_2}, Z_{i_2}^{\perp} \cap W_{i_2}, W_{i_2}^{\perp}$, and repeats the computation of the curvature matrix as above.

To see collapse, the idea is that the orbits are almost totally geodesic as the collapsing proceeds. To be specific, let $q \in M$ and let $\mathcal{O}_q$ be its orbit. Choose r so that the exponential map on vectors perpendicular to the orbits is a diffeomorphism on vectors of length $< r$. There is a number c so that $\mathrm{dist}(q, \partial T_{r/2}(\mathcal{O}_q)) > c$. However there is a closed loop that is noncontractible in $T_{r/2}(\mathcal{O}_q)$ and has length $< c'\delta$. For $\delta < c/c'$ this implies there is a noncontractible geodesic in $T_{r/2}(\mathcal{O}_q)$ of length $< c'\delta$; hence the injectivity radius converges to 0 at q.

3.2 Nonpure collapse

If the structure is not pure, then we work on a regular atlas $U_1, \ldots, U_A$. Over U_α we have a pure substructure G_α, and we can carry out the procedure above. If we order the atlases so $\mathcal{G}_{1,p} \subset \cdots \subset \mathcal{G}_{A,p}$, then the orbit stratification near p for higher U_α refines that for lower U_α. We must also modify the cutoff functions ρ_i^α to be equal to 1 in some neighborhood of ∂U_α; this way the charts in the atlas are pushed away from each other as well as the strata inside each chart.

Lecture 13 - F-structures IV - Collapsing implies existence of an F-structure

March 18, 2010

Theorem 0.1 *Given a manifold M^n, there is a decomposition $M^n = K^n \cup H^n$ where H admits an F-structure of positive rank and if $p \in K$, then there is a $c = c(n) < \infty$ such that*

$$\sup_{y \in B_{ci_p}(p)} |\mathrm{Rm}_y|^{1/2} i_p \;\geq\; c^{-1}.$$

We first try to explain the idea behind the proof. The constant $c(n)$ is chosen so that $\sup_{y \in B_{ci_p}(p)} |\,\mathrm{Rm}_y\,|^{1/2} i_p < c^{-1}$ implies $B_{ci_p}(p)$ is *almost flat* in the sense that there exists a quasi-isometry from some flat manifold into some large subset (compared to the injectivity radius) of $B_{ci_p}(p)$. We construct (elementary) F-structures on flat manifolds, which then pass to these almost-flat balls. A technical argument remains on how to "glue" the F-structures together on overlaps. This is achieved by showing that the F-structures' local actions are "almost" the same, in the C^1-sense. Then a stability theorem is used: if a Lie group has two actions that are "close enough" in the C^1-sense, the actions can be perturbed so as to coincide.

Essentially the orbits of the F-structure correspond to the "most collapsed directions."

1 F-structures on complete flat manifolds

The "soul theorem" states that a complete manifold M^n of nonnegative curvature is isometric to the total space of the normal bundle of a compact, totally geodesic flat submanifold, called the *soul*, S^k, of M^n.

Let $\pi_1(S^k)$ be the fundamental group (of course $\pi_1(S^k) \approx \pi_1(M^n)$ as $S^k \hookrightarrow M^n$) is a homotopy equivalence). The Bieberbach theorem states that there is an Abelian normal subgroup $A \lhd \pi_1(S^k)$ of finite rank $\leq \lambda(k)$, corresponding to which is a finite cover (of $\leq \lambda(k)$ sheets) of S^k by a torus T^k.

Considered as an Abelian Lie group, T^k acts on itself, although this does not necessarily give rise to a T-structure on the total space of the normal bundle. But Cheeger-Gromov give a method for defining an F-structure can be defined on the normal bundle, based on the existence of short loops. The idea is as follows. An Abelian group $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$ of discrete isometries of the covering space gives rise to a continuous group of commuting isomorphisms of the covering space. Assume Γ is invariant under conjugation with π_1. Let $\triangle \subset \pi_1$ be the discrete Abelian normal subgroup of finite index guaranteed by the Bieberbach theorem, and assume it commutes with Γ. Then Γ gives rise to a torus T_k which acts (possibly noneffectively) on $\mathbb{R}^n/\triangle$. There is an induced action of π_1 on $Aut(T^k)$, so we get an F-structure on $\mathbb{R}^n/pi_1 = M$.

2 Locally collapsed regions

Given $y \in M$ and $R > 0$ define the quantity $v(y, R)$ by

$$v(y,R) \triangleq \sup_{x \in B_{Ri_y}(y)} |\mathrm{Rm}_x|^{\frac{1}{2}} i_y.$$

By an *h-quasi-isometry* (for $h \in [1, \infty)$) between Riemannian manifolds U and V will mean a homeomorphism $f : U \to V$ differentiable of degree at least $C^{k,\alpha}$, so that $\frac{1}{h} g_U \le f^* g_V \le h g_U$. Of course a 1-quasi-isometry is an isometry.

Lemma 2.1 *Given $h > 0$, $k < \infty$, there is a $\delta = \delta(h, k, n)$ and an $R = R(h, k, n)$ so that if $v(y, \delta^{-1}) < \delta$ then there is a flat manifold F with soul S so that*

- *i) an h-quasi-isometry $f : U \to U_F$ from some subset $y \in U \subset B_{ki_y}(y)$ a neighborhood U_F in F, where also U contains $B_{\frac{1}{4}ki_y}(y)$,*
- *ii) $\mathrm{dist}(f(y), S) \le R$,*
- *iii) $\mathrm{Diam}(S) \le R$.*

<u>*Pf*</u>

Assume (i) is false. Put $\delta_i = i^{-1}$. By scale invariance we can assume that $i_y = 1$ and $|\,\mathrm{Rm}\,| < 1/i$ on $B_i(y)$, but there is no h-quasi-isometry from any neighborhood of y to any tubular neighborhood $B_{i \cdot i_y}(S)$ of any soul in any flat manifold.

But by Cheeger-Gromov convergence, as $i \to \infty$ the sets $B_i(y)$ converge in the $C^{1,\alpha}$-topology to a complete flat manifold with unit injectivity radius at a point.

Thus for large enough i, there is indeed an h-quasi-isometry from $B_i(y)$ to a subset of this flat manifold.

If (ii) or (iii) is false, we can repeat the argument. However, in the limiting flat manifold the soul is a finite distance away, so it is clear that we can chose a subset $U_i \subset B_i(y)$ with $y \in U_i$ that maps onto some tubular neighborhood. $\square$

The h-quasi-isometry is actually too weak a notion. It is important that holonomies converge, not just distances. However since the convergence above occurs in the $C^{1,\alpha}$-topology (in particular, in the C^1 topology), holonomies around geodesic loops based at y converge to the respective holonomies in the flat case.

3 Joining of locally defined F-structures

In this section I will describe how F-structures are defined locally, and how they are joined together. Pick $h > 0$. Let $p \in M$ and suppose curvature satisfies $|\,\mathrm{Rm}\,| < \delta\, i_p^{-2}$ inside $B_{i_p\delta^{-1}}(p)$. Then there is some flat manifold, Y_p, and an h-quasi-isometry between a some large subset of $U_p \subset B_{i_p\delta^{-1}}(p)$ and a large subset of Y_p.

There is an F-structure on Y_p, however we do not want the entire F-structure. We will consider a loop at p to be a "short loop" if it is a geomdesic lasso and its length is a definite multiple of the injectivity radius. Corresponding to short geodesic loops at p are short almost-geodesic loops in Y_p, which can be homotoped to (nontrivial!) short geodesic loops. If the loops at p have small holonomy, then (by Bieberbach's theorem) the corresponding loops in Y_p have zero holonomy and therefore correspond to geodesic loops in the covering torus, so correspond to an orbit of the F-structure. Let $\gamma_1, \dots, \gamma_k$ be the loops at p with small holonomy (say, maximal rotation angle $< 1/4$); a simple argument shows this list is nontrivial. Corresponding to these are loops $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ in Y_p, corresponding to which is an F-structure of constant rank k. It is this F-structure which passes down to a neighborhood near p.

Now consider two nearby points p, q with overlapping neighborhoods U_p, U_q. Let $\gamma_1^p, \dots, \gamma_k^p$ and $\gamma_1^q, \dots, \gamma_l^q$ be the short loops at p, q, respectively, with maximal holonomy angle $< \frac{1}{4}$; these lead to possibly different F-structures on $U_p \cap U_q$, although $U_p \cap U_q$ is saturated for either structure.

We claim is that a third structure exists on a neighborhood of $U_p \cap U_q$, which contains both previous structures. One can "slide" the loops $\gamma_1^p, \dots, \gamma_k^p$ and $\gamma_1^q, \dots, \gamma_l^q$ to a point $p' \in U_p \cap U_q$. At p' these loops still have small holonomy and short length, so define an F-structure on a neighborhood of p'.

Now we can replace U_p with $U_p - \overline{U_q}$ and the same with U_p. Repeating this process, we get at least one F-structure defined in a neighborhood of each point, so that if two such structures overlap, then one contains the other.

If $|\,\mathrm{Rm}\,|^{1/2} i_x$ is small enough, the orbits of the F-structures will converge in the C^1 sense. A stability theorem (Grove-Karcher (1973)) says that if two Lie groups produce actions that are close enough in the C^1-sense, the actions can be perturbed so as to coincide.

Lecture 14 - Singularities of F-structures I - Classification of singularities in dimension 4

March 23, 2010

1 Three singularity models

1.1 Rong's Structure I

In this case we let $U_{1,k}$ be the solid torus bundle over $\mathbb{S}^1$, constructed as follows. Let $N = D^2 \times \mathbb{S}^1 \times [0,1]$, and identify $\{(r,\theta_1,\theta_2,0)\}$ with $\{(r,\theta_1,\theta_2,1)\}$ by the map that fixes r and maps the θ's by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let $\mathcal{F}_{1,k}$ denote the natural F-structure on $U_{1,k}$, which is given on each fiber by the torus action. We easily see that $(U_{1,k}, \mathcal{F}_{1,k})$ is polarizable iff $k = 0$.

1.2 Rong's Structure II

Let Y_1, Y_2, and Y_3 be copies of $D^2 \times [0,1] \times \mathbb{S}^1$. The F-structure on Y_1 and Y_3 will be rotation in the $\mathbb{S}^1$ factor, and the F-structure on the Y_2 factor will be the torus acting on the $\mathbb{S}^1$ factor. Join Y_1 to Y_2 by gluing $D^2 \times 1 \times \mathbb{S}^1 \subset Y_1$ to $D^2 \times 0 \times \mathbb{S}^1 \subset Y_2$ via the identity map, and join Y_2 to Y_3 by gluing $D^2 \times 1 \times \mathbb{S}^1 \subset Y_2$ to $D^2 \times 0 \times \mathbb{S}^1 \subset Y_3$, via some automorphism of the torus. This automorphism can be represented by a matrix in $\mathrm{SL}(2,\mathbb{Z})$. If this matrix has two distinct eigenvectors, then there are two distinct polarized substructures. If the matrix in nilpotent, then up to change of basis this matrix is (1), and the structure is not polarizable. We call this structure $\mathcal{F}_{2,k}$. It is polarizable iff $k = 0$.

1.3 Rong's Structure III

This is simply $D^2 \times \mathbb{S}^1 \times J$ where J is the line or the half-line. The F-structure is the obvious pure T^2 structure. The singular locus is an open or half-open cylinder, and obviously this structure contains a polarized substructure.

2 Singularity classification

We consider pure F-structures. Given an F-structure $\mathcal{F}$ let $\mathcal{Z}(\mathcal{F})$ be the singular locus. We'll use $\mathcal{Z}_0$ to indicate a connected component of $\mathcal{Z}(\mathcal{F})$. Let $\mathcal{E}(\mathcal{F})$ be the set of exceptional $\mathbb{S}^1$ orbits that have nontrivial intersection with $\mathcal{Z}(\mathcal{F})$. Put $\mathcal{W}(\mathcal{F}) = \mathcal{Z}(\mathcal{F}) \cup \mathcal{E}(\mathcal{F})$.

2.1 Rank 3 structures

We prove that any rank 3 structure has a polarizable substructure. We will work with an invariant metric. Let W_0 be a singular component. We can lift locally to a finite normal cover, on which a 3-torus acts without fixed points. Let $\mathcal{O}_x$ be a singular orbit; we have $\mathrm{Dim}\,\overline{\mathcal{O}}_x$ equals 1 or 2.

If $\mathrm{Dim}\,\overline{\mathcal{O}}_x = 1$, the isotropy group at x is a 2-torus, which acts effectively (and isometrically) on $T_xM^\perp \approx \mathbb{R}^3$. However this is impossible, for $\mathfrak{so}(3)$ has no 2-dimensional abelian subalgebra.

If $\mathrm{Dim}\,\overline{\mathcal{O}}_x = 2$, the isotropy group is a circle, which acts effectively and isometrically on $T_xM^\perp \approx \mathbb{R}^2$. This action has a single fixed point (the origin), so the orbit $\mathcal{O}_x$ is isolated. A disk $D^2 \subset \mathbb{R}^2$ can be identified with a 2-disk in M via the (normal) exponential map. The images of this disk under the action of the various elements of $\mathcal{F}$ give a tubular neighborhood of $\mathcal{O}_x$ the structure of a 2-disk bundle over a 2-torus.

It is possible to find a polarization $\mathcal{P} \subset \mathcal{F}$. Namely, if an open set $U \in \mathcal{O}_x \times D^2$ intersects the exceptional orbit, assign it the group T^2, and if not assign it the group T^3.

2.2 Rank 2 structures

If $\mathcal{O}_x$ is an exceptional orbit then $\mathrm{Dim}\,\mathcal{O}_x = 1$. Consider the action of the isotropy group at x on $T_xM^\perp \approx \mathbb{R}^3$. This gives an embedding $\mathbb{S}^1 \hookrightarrow \mathrm{SO}(3)$, so there is a fixed $\mathbb{R}^1$. This means the orbit is not isolated, and the singular locus is $\mathbb{S} \times I$, $\mathbb{R} \times I$, $\mathbb{S} \times \mathbb{R}$, $\mathbb{S} \times \mathbb{R}^+$, the torus, the Klein bottle, or the 2-plane, However we can rule out $\mathbb{R} \times I$ and $\mathbb{R}^2$, because these cannot be foliated by $\mathbb{S}^1$-orbits.

Now we must distinguish between pure and mixed structures. If the F-structure is a pure structure of rank 2, then the finite cylinder is no longer a possibility. If a connected singular locus $\mathcal{Z}_0$ is a torus, then an ϵ-neighborhood $T_\epsilon(\mathcal{Z}_0)$ is a disk bundle over a torus. There is an $\mathbb{S}^1$ action on the singular locus however, which trivializes another direction. This gives $T_\epsilon(\mathcal{Z}_0)$ the structure of a $D^2 \times \mathbb{S}^1$-bundle (solid torus bundle) over $\mathbb{S}^1$. The structure over this neighborhood admits a polarization iff the structure group is solvable.

If the structure is pure but the manifold is not compact, a singular component $\mathcal{Z}_0$ can be an infinite cylinder. In this case the neighborhood $T_\epsilon(\mathcal{Z}_0)$ is a D^2-bundle over the cylinder, and is necessarily trivial. Such a structure is always admits polarization.

If the structure is mixed, then the singular components that are contained in rank 2 neighborhoods can be finite cylinders that abut rank 1 charts. Consider the boundary of $T_\epsilon(Z_0)$ in this case, which can be described as $\mathbb{S}^1 \times D^2 \cup_{f_1} I \times T^2 \cup_{f_2} \mathbb{S}^1 \times D^1$, where the gluing maps are f_1, f_2 are automorphisms of the torus. Up to homotopy this is a lens space $\mathbb{S}^1 \times D^2 \cup_{f_1 f_2^{-1}} \mathbb{S}^1 \times D^2$. The F-structure restricted to this subset is polarizable iff the the $\mathbb{S}^1$ actions induced on the interior by the boundary maps are multiples of each other. This is the case if $f_1 f_2^{-1} \in \mathrm{SL}(2, \mathbb{Z})$ has two distinct eigenvectors, in which case there are two distinct polarized substructures.

Lecture 15 - Singularities of F-structures II - Removability of Singularities

April 6, 2010

1 Characteristic Forms and Transgressions

Let G be Lie group with algebra $\mathfrak{g}$. Let

$$\mathcal{P} : \mathfrak{g}^{\otimes k} \to \mathbb{R}$$

be a *symmetric invariant polynomial*, which is to say, a map that is

i) Symmetric: $\mathcal{P}(\eta_1, \ldots, \eta_i, \ldots, \eta_j, \ldots, \eta_k) = \mathcal{P}(\eta_1, \ldots, \eta_j, \ldots, \eta_i, \ldots, \eta_k)$

ii) Invariant: $\mathcal{P}(\mathrm{Ad}_g\, \eta_1, \ldots, \mathrm{Ad}_g\, \eta_2) = \mathcal{P}(\eta_1, \ldots, \eta_k)$, and

iii) Polynomial: a sum of elementary multilinear maps on $\mathfrak{g}$ of degree k,

where $\eta_1, \ldots, \eta_k \in \mathfrak{g}$ and $g \in G$. The derivative of Ad is ad, so letting $g(t) = \exp(t\theta)$ and putting this into (*ii*) and taking a derivative gives

ii') $\sum \mathcal{P}(\eta_1, \ldots, [\theta, \eta_i], \ldots, \eta_k) = 0.$

Now let M^n be a manifold with structure group G (normally G is $O(n)$, $SO(n)$, or $U(n)$), and let Ω_i be a $\mathfrak{g}$-valued l_i-form for $i \in \{1, \ldots, k\}$. We can define a $\sum l_i$-form

$$\mathcal{P}(\Omega_1, \ldots, \Omega_k)$$

in the obvious way (inserting forms to evaluate the Ω_i to $\mathfrak{g}$, then taking the polynomial). One easily proves that

$$d\,\mathcal{P}(\Omega_i, \ldots, \Omega_k) = \sum (-1)^{l_1 + \cdots + l_{i-1}}\, \mathcal{P}(\Omega_1, \ldots, d\Omega_i, \ldots, \Omega_k),$$

which is the usual rule for wedge products. If θ is a $\mathfrak{g}$-valued 1-form (eg. a connection 1-form), then (*ii'*) is *ii")* $\sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\,\ldots,\,[\theta,\Omega_i],\,\ldots,\,\Omega_k) \;=\; 0.$

Adding (using the multilinearity), we get

$$d\,\mathcal{P}(\Omega_i,\ldots,\Omega_k) \;=\; \sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\ldots,d\Omega_i+[\theta,\Omega_i],\ldots,\Omega_k).$$

If θ is indeed a connection 1-form then $D = d + [\theta,\cdot]$, so we get

$$d\,\mathcal{P}(\Omega_i,\ldots,\Omega_k) \;=\; \sum(-1)^{l_1+\cdots+l_{i-1}}\,\mathcal{P}(\Omega_1,\ldots,D\Omega_i,\ldots,\Omega_k).$$

Therefore $\mathcal{P}(\Omega_1,\ldots,\Omega_k)$ is a closed $(l_1+\cdots+l_k)$-form whenever the Ω_i are covariant-constant (ie. $D\Omega_i = 0$). If $\Omega_i = \Omega = d\theta + \frac{1}{2}[\theta,\theta]$ is the curvature 2-form, then $D\Omega = 0$. Therefore, assigned to each connection is a curvature 2-form and so a deRham class in $H^{2k}(M)$. Given a connection θ let $\mathcal{P}_\theta$ denote the representative $2k$-form.

The question is whether this class is unique. To answer this, let θ_0, θ_1 be two connection 1-forms, and let $\theta_t = t\theta_1 + (1-t)\theta_0$ be the interpolation between them. Corresponding to the connection θ_t is the curvature tensor $\Omega_t = d\theta_t + \frac{1}{2}[\theta_t,\theta_t]$. Since Ω is a form of even degree, we compute

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{P}(\Omega_t,\ldots,\Omega_t) &= k\,\mathcal{P}(d\Omega_t/dt,\Omega_t,\ldots,\Omega_t)\\
&= k\,\mathcal{P}(d(\theta_1-\theta_0)+[\theta_t,\theta_1-\theta_0],\,\Omega_t,\ldots,\Omega_t)\\
&= k\,\mathcal{P}(D_t(\theta_1-\theta_0),\,\Omega_t,\ldots,\Omega_t)\\
&= kd\,\mathcal{P}(\theta_1-\theta_0,\Omega_t,\ldots,\Omega_t)
\end{aligned}$$

where $D_t\alpha = d\alpha + [\theta_t,\alpha]$. Therefore

$$\mathcal{P}_{\theta_1} - \mathcal{P}_{\theta_0} \;=\; k\,d\int_0^1 \mathcal{P}(\theta_1-\theta_0,\Omega_t,\ldots,\Omega_t)\,dt,$$

and so $\mathcal{P}(\Omega_1,\ldots,\Omega_1)$ and $\mathcal{P}(\Omega_0,\ldots,\Omega_0)$ define the same cohomology class. The $(2k-1)$-form $k\int_0^1 \mathcal{P}(\theta_0-\theta_1,\Omega_t,\ldots,\Omega_t)\,dt$ is often called a *transgression form*, and denoted $\mathcal{TP} = \mathcal{TP}(\theta_1,\theta_0)$. We have

$$\underset{\theta_1}{\mathcal{P}} - \underset{\theta_0}{\mathcal{P}} + d\,\mathcal{TP}(\theta_1,\theta_0) \;=\; 0$$

2 Characteristic numbers

If n is even and $G = SO(n)$, let $\mathcal{P}(\eta_1,\ldots,\eta_{n/2})$ be the Pfaffian. If a manifold M^n has structure group $SO(n)$ on its frame bundle then this defines a characteristic class, the Euler class. Put $\mathcal{P}_\chi = \mathcal{P}(\Omega,\ldots,\Omega)$ for the Levi-Civita curvature 2-form Ω; this defines the Euler class. (Of course an Euler class can be defined on any even-dimensional $SO(k)$ principle bundle, but we are only concerned with the frame bundle.) Now let X be a vector field on M

with isolated zeros. Replace X with $X/|X|$, so X is defined and C^∞ outside a finite number of singular points. At these singular points the index of $X/|X|$ is defined, and the Euler number of M is the sum of the indices of these singular points. Away from the singularities we have a splitting of the tangent bundle into a parallel and orthogonal distribution. If

$$\theta \;=\; \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is the corresponding block decomposition of the the Levi-Civita connection θ, then define a new connection

$$\theta' \;=\; \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$$

Needless to say, $A = 0$, since it is an $\mathfrak{o}(1)$-valued 1-form. Since

$$\Omega' \;=\; \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & dD + \frac{1}{2}[D,D] \end{array}\right)$$

we have that $\mathcal{P}(\Omega', \ldots, \Omega') = 0$ for any symmetric invariant polynomial $\mathcal{P}$ of degree $\frac{n}{2}$. By the previous section, we have "transgressed" $\mathcal{P}_\chi$ outside the zeros of X:

$$\mathcal{P}_\chi \,+\, d\mathcal{T}\mathcal{P}_\chi \;=\; 0.$$

Letting p_i be the zeros of X and putting $B(i,\epsilon) = B_{p_i}(\epsilon)$, we have

$$\int_{M - \bigcup_i B(i,\epsilon)} \mathcal{P}_\chi \;=\; \sum_i \int_{\partial B(i,\epsilon)} \mathcal{T}\mathcal{P}_\chi.$$

A classical theorem of Weyl gives that

$$\lim_{\epsilon\to 0} \int_{\partial B(i,\epsilon)} \mathcal{T}\mathcal{P}_\chi \;=\; -C\, Ind_{p_i}(X/|X|)$$

where $C = C(n)$ is a constant. Therefore

$$\chi(M) \;=\; \frac{1}{C}\int_M \mathcal{P}_\chi.$$

In dimension 4 it turns out that $C = 8\pi^2$, and $\mathcal{P}_\chi$ is a quadratic functional of the Riemann tensor:

$$\chi\left(M^4\right) \;=\; \frac{1}{8\pi^2}\int_M \frac{1}{24}R^2 \,-\, \frac{1}{2}|\overset{\circ}{\mathrm{Ric}}|^2 \,+\, |W|^2$$

On the other hand, let $\mathcal{P}_\tau = Tr(\Omega\wedge\Omega)$. It can be proven that $\mathcal{P}_\tau = |W^+|^2 - |W^-|^2$ and

$$\tau \;=\; \frac{1}{3}p_1 \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 \,-\, |W^-|^2$$

where τ is the signature of the manifold.

3 Characteristic numbers of manifolds with boundary

Assume the boundary of M is C^∞. If X is perpendicular to the boundary, it is easy to modify the Weyl formula to get

$$\chi(M) \;=\; \frac{1}{C}\int_M \mathcal{P}_\chi \,+\, \frac{1}{C}\int_{\partial M} \mathcal{T}\mathcal{P}_\chi.$$

On the other hand another term is introduced to the signature formula

$$\tau(M^4) \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 \,+\, \frac{1}{12\pi^2}\int_{\partial M} \mathcal{T}\mathcal{P}_\tau \,+\, \eta_{\partial M}.$$

The functional $\eta_{\partial M}$ is called the η-invariant. This invariant is defined for any 3-manifold and depends only on the Riemannian structure of ∂M (not how it is embedded as the boundary of M). It is additive over disjoint unions.

Using the Hirzebruch L-polynomial a formula for the signature, in terms of the Riemann tensor, can be obtained for any manifold of dimension $4k$. The corresponding signature formula for $4k$-manifolds with boundary has the boundary corrections coming from both a transgression form and an eta-invariant for the boundary $(4k-1)$-manifolds. See the papers of Atiyah-Patodi-Singer for more information.

4 Signatures of the structures $\mathcal{F}_{1,k}$

Let DT^n indicate the solid torus with boundary T^{n-1}. Recall that Rong's non-polarizable structure $\mathcal{F}_{1,k}$ can be considered to be a disk bundle over a 2-torus, or as a solid torus bundle over a circle. As a solid torus bundle, it is

$$\mathcal{F}_{1,k} \;=\; [0,1]\times DT^3/\sim$$

where $\sim$ identifies $\{0\}\times DT^3$ with $\{1\}\times DT^3$ with the matrix

$$\left(\begin{array}{cc} 1 & k \\ 0 & 1 \end{array}\right).$$

To see that is is a disk bundle over a torus, consider the projection on each solid torus that takes DT^3 to its central circle. Since the central circle is mapped to itself isomorphically, this is well-defined globally.

We claim that the signature of this oriented manifold-with-boundary is precisely k. To see this, note that there are two homology classes in $H_2(M,\partial M;\mathbb{Z})$, one of which is carried by the central 2-torus, denoted T, and the other is fiber, denoted D. We claim the intersection form is

$$\left(\begin{array}{cc} \pm k & 1 \\ 1 & 0 \end{array}\right).$$

That $D \cdot D = 0$ and $D \cdot T = 1$ are obvious. To see that $T \cdot T = k$ we shall perturb T to another 2-dimensional submanifold T' and show that T and T' intersect transversely in k places, and that the orientations of the intersections are consistent.

Now let S be the meridian circle on the boundary $T^2 \approx \partial(\{1\} \times DT^3)$. This is identified to the circle $S' \subset \partial\partial(\{0\} \times DT^3) \approx T^2$ that wraps around the meridian once and the longitude k times. Let $S(t)$ be a circle in $\{t\} \times DT^3$ with the following property. If $\pi_t : [0,1] \times DT^3 \to DT^3$ is the projection onto the second factor, then the image $\pi_t(S(t))$ is a smooth homotopy from the circle $\pi_0(S(0))$ (that wraps around the boundary $1-k$ times) and the circle $\pi_1(S(1))$ (that wraps around the boundary $1-0$ times), and so that halfway through the homotopy $\pi_t(S(t))$, the circle intersects the boundary circle in precisely k points. Now consider the 2-surface-with-boundary T' that $S(t)$ defines in $[0,1] \times DT^3$. This surface intersects the central cylinder precisely k times. Also, the boundary circle $S(1)$ is identified to the boundary circle $S(0)$ under $\sim$. After identification, we have therefore have $T \cdot T' = \pm k$. It is also clear that T' is smoothly homotopic to T.

5 Embedding of $\mathcal{F}_{1,k}$ into a collapsed manifold

Let M be a collapsed manifold with a pure F-structure. All singular irremovable singular orbits are (quotients of) 2-tori, denoted say T, with an exponential tubular neighborhood isomorphic to one of the $\mathcal{F}_{1,k}$. We can prove that there is some $\rho > 0$ so that the injectivity radius for the exponential map off of T is at least ρ.

By the Cheeger-Gromov-Fukaya work on N-structures, there is a critical radius ϵ, so that if this exponential map has injectivity radius $< \epsilon$ then this direction is part of an orbit of a larger N-structure. However, Rong proves that on a definite neighborhood of a singular orbit, the N-structure is in fact just the original F-structure. One way to see this is to note that the singular orbit will remain singular. If there is another collapsed direction, then the N-structure must have a 3-dimensional stalk.

Any singular fiber is therefore 2-dimensional and the isotropy killing fields are therefore 1-dimensional. This implies that a singular fiber is *isolated*. However this is impossible, since the singular fibers of the F-structure are not isolated.

This implies that the normal injectivity radius from the singular locus of the F-structure is at least ϵ.

6 Volume bounds

Let Z be a connected component of the singular locus. Then $T_\rho(Z)$ (or a double cover) is diffeomorphic to the structure $\mathcal{F}_{1,k}$. By hypothesis, $T_\rho(Z)$ has very small volume, $|sec| \leq 1$, and boundary diffeomorphic to a nilmanifold. Note that the second fundamental form of the boundary is controlled.

It is possible to extend $T_\rho(Z)$ is a complete manifold of small volume and controlled curvature. Near infinity we can give $T_\rho(Z)$ the structure of an almost-flat manifold crossed with a half-open interval. Using the Atiyah-Patodi-Singer formula

$$\tau(T_\rho(Z)) \;=\; \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 \;+\; \frac{1}{12\pi^2}\int_{\partial M} \mathcal{TP}_\tau \;+\; \eta_{\partial M}.$$

we get that η_{N^3}, where $N^3 = \partial T_\rho(Z)$ is very small, where $\mathcal{TP}_\tau = 0$ because the second fundamental form vanishes, and where $\int |W^+|^2 - |W^-|^2$ is controlled by the (very small) volume and bounded sectional curvature. Therefore $|\tau(M^4)|$ is very small and therefore zero, contradicting that $\tau\left(T_\rho(Z)\right) = k$ (unless $k = 0$ and $T_\rho(Z)$ is the trivial disk bundle over the 2-torus).

Lecture 16 - Boundary of the space of manifolds of bounded sectional curvature

April 8, 2010

1 Boundary consists generically of objects that are locally quotients of manifolds

Let $\mathcal{M}(n, D)$ be the set of Riemannian manifolds with diameter $\leq D$, dimension n, and all sectional curvatures in $[-1, 1]$. By Gromov's precompactness theorem $\mathcal{M}(n, D)$ is precompact in the Gromov-Hausdorff topology, with limiting objects being length spaces. The question is, what is the structure of the length spaces on the boundary?

Let $\tilde{M}_i \in \mathcal{M}(n, D)$ be a sequence that is Cauchy with respect to the Gromov-Hausdorff distance. There is another sequence, M_i, that is ϵ-close in the Lipschitz sense, and has uniformly controlled derivatives of curvature (depending on ϵ). In fact, we have the following theorem

Theorem 1.1 *Given any $\epsilon > 0$, if $\tilde{M}^n$ is a Riemannian manifold with $|K| \leq 1$ and $Diam(M) \leq D$, then there is a Riemannian manifold M^n with $Lip(\tilde{M}, M) < \epsilon$ and*

$$|\nabla^k \operatorname{Rm}| \ \leq \ C(n, k, \epsilon). \tag{1}$$

<u>*Pf*</u> (somewhat heuristic) Locally lift the metric tensor to a Euclidean ball, smooth via convolution with some C_c^∞ function that is C^∞-close to a δ-function, and then pass back down after averaging. This gives a metric that is very close to the original metric, but which has derivatives that depend on the C_c^∞ function that was used. Do this on patches that cover the manifold, being careful to glue the patches together smoothly using a partition of unity argument. The gluing process will not perturb the metric too much, because control can be gained over the multiplicity of the covering. $\square$

This means that given a sequence $\tilde{M}_i$ there is a sequence M_i with $d_{GH}(M_i, \tilde{M}_i) < \epsilon$, but where M_i has uniform C^∞ control on the metric. Then also $d_{GH}(M_\infty, \tilde{M}_\infty) \leq \epsilon$ where $M_\infty = \lim_i M$, $\tilde{M}_\infty = \lim_i \tilde{M}_i$.

Now consider points $p_i \in M_i$ with the p_i converging to some $p \in M_\infty$. Let $B_i \subset T_{p_i}M_i$ be unit balls in the respective tangent spaces, with the pullback metrics, and projections $\pi_i : B_i \to M_i$. Passing to a subsequence if necessary, the metrics on these balls converges in the C^∞ sense (this can be seen using, for instance, Deturck-Kazdan's harmonic coordinate trick). We investigate what happens on the pushdown back to the base space.

Given $p_i \in M_i$ we define G_{p_i}, the *local group at* p_i as follows. A differentiable map $\gamma : U \to B_i$ is *admissible* if $o \in U \subset \frac{1}{2}B_i$ and $\pi_i \circ \gamma = \pi_i$ (in particular, γ is a local isometry). Two admissible maps $\gamma_1 : U_1 \to B_i$ and $\gamma_2 : U_2 \to B_i$ are equivalent if they agree on $U_1 \cap U_2$. An element of G_i is an equivalence class of admissible maps. Any equivalence class is represented by a maximal element. Further, it is possible to define the product of equivalence classes, making G_i into a pseudogroup. The local group G_i partitions B_i, and clearly $\pi_i(B_i) = B_i/G_i$.

Now we can take a limit of the G_i in the following sense. Each element of G_i is represented by a differentiable map $\frac{1}{2}B_i \to B_i$ with uniformly bounded derivatives. Thus we can embed the discrete space G_i into the space $L = C^\infty(\frac{1}{2}B_i, B_i)$. The Arzela-Ascoli theorem indicates that L is compact. Gromov's convergence theorem says that the space of closed subsets of a compact space is compact in the Gromov-Hausdorff topology, so the G_i converge, after possibly passing to another subsequence, so we can assume $G_i \to G$.

It is easily seen that G is a pseudogroup, and it can be proved that G has a differentiable structure. In fact we can prove that G is nilpotent. To do this, we find a point p near $o \in B_\infty$ with trivial isotropy, so that a neighborhood of $\pi_\infty(p)$ is Riemannian. Then the Fukaya-Ruh fibration theorem states that, near p, there is a map $M_i \to M_\infty$ with fibers isomorphic to infranil manifolds, for large enough i. Since G_i embeds into the fundamental group of these infranils, so that k-fold commutators vanish (for k depending only on n, by the proof of Gromov's almost-flat manifold theorem). Since eventually the G_i embed in G and $\bigcap \bigcup G_i$ is dense in G, this implies that k-fold commutators of G also vanish.

We can extend the local group G to a simply connected nilpotent group, and extend the ball B_∞ to an unbounded subset of $\mathbb{R}^n$. We get that locally near p the manifold B_∞ is isomorphic to the quotient of a neighborhood of the origin in $\mathbb{R}^n$ by a simply connected nilpotent Lie group.

2 Structure of limits of smooth sequences

More can be said about the structure of limits of smooth manifolds. If the isotropy of G at $p \in B_\infty$ is finite, then a neighborhood of p has a Riemannian structure. If isotropy is not finite, then a neighborhood of p is the quotient of some $\mathbb{R}^k$ by a Lie group whose identity component is a torus in $SO(k)$.

To see this, assume $\xi \in \mathfrak{g}$ satisfies $\xi(p) = p$. Then ξ integrates out to a subgroup of $SO(n)$, which is necessarily compact. Consider the representation of the Lie algebra $\mathbb{C}\,\xi$ on

$\mathfrak{g}$ via the adjoint representation.

For some reason, this must be a semisimple representation.

But then there is some eigenvector η, with $\operatorname{ad}_\xi \eta = \alpha\eta$. If $\alpha \neq 0$ then the subalgebra $\mathbb{C}\,\xi \oplus \mathbb{C}\,\eta \subset \mathfrak{g}$ is not nilpotent, a contradiction.

3 Fibrations over the frame bundle

Let $M_i \to M$ be a smoothed sequence of manifolds converging to the boundary. Over each point $p_i \in M_i$ we have a ball B_i and projections $\pi_i : B_i \to M_i$. On the ball we have pseudogroups G_i with $B_i/G_i \approx \pi_i(B_i)$, and with the G_i converging to a local Lie group G, and the metrics on the balls B_i converging to a metric on B. Then B/G is isometric to a neighborhood of $\lim_i p_i \in M$

Now consider the frame bundles FM_i over the M_i, with a Riemannian metric that comes from the metric on the base space, and a fixed metric on the fibers (which are isomorphic to $O(n)$ or $SO(n)$). Now consider the balls B_i, and consider the pullback bundle FB_i. The pseudogroups G_i act isometrically on B_i, so they therefore act isometrically on FB_i. Furthermore this action is free, since an isometry that fixes a point of FB_i fixes B_i and therefore fixes FB_i as well.

Taking a limit we get that the groups G act freely and isometrically on the limiting bundle FB_∞. This proves that a neighborhood of a point in the limit of the FB_i is a manifold. Furthermore, since we have $FM_i/O(n) \approx M_i$, we also have $FM_\infty/O(n) \approx M_\infty$.

Lecture 18 - Einstein Manifolds I

April 15, 2010

1 Isoperimetric and Sobolev constants

If Ω is an n-dimensional domain with a Riemannian metric and $\nu > 0$, we define the ν-isoperimetric constant of Ω to be

$$I_\nu(\Omega) \;=\; \inf_{\Omega' \subset\subset \Omega} \frac{\mathrm{Area}(\partial\Omega')}{\mathrm{Vol}(\Omega')^{\frac{\nu-1}{\nu}}}$$

where Area indicates Hausdorff $(n-1)$-measure. If Ω is a closed Riemannian manifold, we take the infimum over domains Ω' with $\mathrm{Vol}\,\Omega' \leq \frac{1}{2}\mathrm{Vol}\,\Omega$; if some such restriction is not made then of course the infimum is zero. Note that if $\nu < n$ then $I_\nu(\Omega) = 0$.

On the other hand we define the ν-Sobolev constant of Ω by

$$S_\nu(\Omega) \;=\; \inf_{f \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla f|}{\left(\int_\Omega |f|^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}}.$$

If Ω is a closed Riemannian manifold, we take the infimum over functions with $\mathrm{Vol}(\mathrm{supp}\, f) < \frac{1}{2}\mathrm{Vol}(\Omega)$; if some such restriction is not made then of course the infimum is zero.

Theorem 1.1 (Federer-Fleming)

$$I_\nu(\Omega) \;=\; S_\nu(\Omega).$$

Pf

Pf that $S_\nu(\Omega) \leq I_\nu(\Omega)$.

With

$$\int |\nabla f| \;\geq\; S_\nu(\Omega)\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}},$$

we can let $f \equiv 1$ on Ω', $f \equiv 0$ outside $\Omega'^{(\epsilon)}$ (the ϵ-thickening of Ω'), and $f(p) = 1 - \epsilon^{-1}\operatorname{dist}(\Omega', p)$ on $\Omega'^{(\epsilon)} - \Omega'$. As $\epsilon \searrow 0$ we have

$$\begin{aligned} \lim_{\epsilon\searrow 0}\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} &= \operatorname{Vol}(\Omega')^{\frac{\nu-1}{\nu}} \\ \lim_{\epsilon\searrow 0}\int |\nabla f| &= \lim_{\epsilon\searrow 0}\frac{\operatorname{Vol}(\Omega'^{(\epsilon)} - \Omega')}{\epsilon} = \operatorname{Area}(\partial\Omega'). \end{aligned}$$

Therefore

$$\operatorname{Area}(\partial\Omega') = \lim_{\epsilon\searrow 0}\int |\nabla f| \geq \lim_{\epsilon\searrow 0} S_\nu(\Omega)\left(\int f^{\frac{\nu-1}{\nu}}\right)^{\frac{\nu}{\nu-1}} = S_\nu(\Omega)\operatorname{Vol}(\Omega')^{\frac{\nu}{\nu-1}}.$$

Pf that $I_\nu(\Omega) \leq S_\nu(\Omega)$.

Given a nonnegative C_c^∞ function $f : \Omega \to \mathbb{R}$ and given a number t, let $A_t = f^{-1}(t)$ and let $\Omega_t = f^{-1}([t,\infty])$. Locally (near a regular point of f) we can parametrize Ω' by letting f be one coordinate, and putting some coordinates on A_t. We can split the cotangent bundle by letting $df/|df|$ be one covector in an orthonormal coframe. Then if $d\sigma_t$ indicates the wedge product of the remaining vectors, we have Then $dV = \frac{1}{|\nabla f|}df \wedge d\sigma_t$. Therefore

$$\begin{aligned} \int_M |\nabla f|\, dV &= \int_{min(f)}^{max(f)}\int_{A_t} d\sigma_t\, df = \int_0^\infty \operatorname{Area}(A_t)\, dt \\ &\geq I_\nu(\Omega)\int_0^\infty \operatorname{Vol}(\Omega_t)^{\frac{\nu-1}{\nu}}\, dt \end{aligned}$$

The equality $\int_M |\nabla f|\, dV = \int_0^\infty \operatorname{Area}(A_t)\, dt$ is called the *coarea formula*. Changing the order of integration, *á la* calculus III, gives

$$\begin{aligned} \int f^{\frac{\nu}{\nu-1}} &= \frac{\nu}{\nu-1}\int_\Omega\int_0^{f(p)} t^{\frac{1}{\nu-1}}\, dt\, dVol(p) \\ &= \frac{\nu}{\nu-1}\int_0^\infty\int_{\Omega_t} t^{\frac{1}{\nu-1}}\, dV\, dt = \frac{\nu}{\nu-1}\int_0^\infty t^{\frac{1}{\nu-1}}\operatorname{Vol}(\Omega_t)\, dt \end{aligned}$$

The result follows from the following lemma.

Lemma 1.2 *If $g(t)$ is a nonnegative decreasing function and $s \geq 1$, then*

$$\left(s\int_0^\infty t^{s-1}g(t)\, dt\right)^{\frac{1}{s}} \leq \int_0^\infty g(t)^{\frac{1}{s}}\, dt$$

Pf

We have

$$\frac{d}{dT}\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}} \;=\; T^{s-1}g(T)\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}-1}$$
$$\leq\; T^{s-1}g(T)^{\frac{1}{s}}\left(s\int_0^T t^{s-1}\,dt\right)^{\frac{1}{s}-1} \;=\; g(T)^{\frac{1}{s}}.$$

Since $\frac{d}{dT}\int_0^T g(t)^{\frac{1}{s}}\,dt \;=\; g(T)^{\frac{1}{s}}$, we have

$$\left(s\int_0^T t^{s-1}g(t)\,dt\right)^{\frac{1}{s}} \;\leq\; \int_0^T g(t)^{\frac{1}{s}}\,dt$$

for all T. $\square$

2 Sobolev embedding

As long as $1 \leq p < \nu$ we have

$$\begin{aligned}
\left(\int_\Omega |\nabla f|^p\right)^{\frac{1}{k}} \;&\geq\; |\Omega|^{\frac{1}{p}-1}\int_\Omega |\nabla f| \\
&\geq\; S_\nu|\Omega|^{\frac{1}{p}-1}\left(\int_\Omega f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \;=\; S_\nu|\Omega|^{\frac{1}{p}-\frac{1}{\nu}}\left(\frac{1}{|\Omega|}\int_\Omega f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \\
&\geq\; S_\nu|\Omega|^{\frac{1}{p}-\frac{1}{\nu}}\left(\frac{1}{|\Omega|}\int_\Omega f^{\frac{p\nu}{\nu-p}}\right)^{\frac{\nu-p}{p\nu}} \;=\; S_\nu\left(\int_\Omega f^{\frac{p\nu}{\nu-p}}\right)^{\frac{\nu-p}{p\nu}}.
\end{aligned}$$

This gives the Sobolev embedding

$$W^{1,p} \hookrightarrow L^{\frac{p\nu}{\nu-p}}.$$

Likewise we have $W^{2,p} \hookrightarrow W^{1,\frac{p\nu}{\nu-p}} \hookrightarrow L^{\frac{p\nu}{\nu-2p}}$ and so forth, giving

$$W^{k,p} \hookrightarrow L^{\frac{p\nu}{\nu-kp}}.$$

Thus we see this holds on any Riemannian manifold, as long as the ν-isoperimetric constant (where $\nu \geq 0$) is controlled.

3 The elliptic equation for Einstein metrics

On any Riemannian manifold,

$$\begin{aligned}
(\triangle \mathrm{Rm})_{ijkl} &= \mathrm{Rm}_{ijkl,ss} \;=\; \mathrm{Rm}_{ijsl,ks} + \mathrm{Rm}_{ijks,ls} \\
&= \mathrm{Rm}_{ijsl,sk} + \mathrm{Rm}_{ijks,sl} \\
&\quad + \mathrm{Rm}_{skip}\mathrm{Rm}_{pjsl} + \mathrm{Rm}_{skjp}\mathrm{Rm}_{ipsl} + \mathrm{Rm}_{sksp}\mathrm{Rm}_{ijpl} + \mathrm{Rm}_{sklp}\mathrm{Rm}_{ijsp} \\
&\quad + \mathrm{Rm}_{slip}\mathrm{Rm}_{pjks} + \mathrm{Rm}_{sljp}\mathrm{Rm}_{ipks} + \mathrm{Rm}_{slkp}\mathrm{Rm}_{ijps} + \mathrm{Rm}_{slsp}\mathrm{Rm}_{ijkp} \\
&= \mathrm{Ric}_{li,jk} - \mathrm{Ric}_{lj,ik} + \mathrm{Ric}_{kj,il} + \mathrm{Ric}_{kj,jl} + \\
&\quad + \mathrm{Rm}_{skip}\mathrm{Rm}_{pjsl} + \mathrm{Rm}_{skjp}\mathrm{Rm}_{ipsl} + \mathrm{Rm}_{sksp}\mathrm{Rm}_{ijpl} + \mathrm{Rm}_{sklp}\mathrm{Rm}_{ijsp} \\
&\quad + \mathrm{Rm}_{slip}\mathrm{Rm}_{pjks} + \mathrm{Rm}_{sljp}\mathrm{Rm}_{ipks} + \mathrm{Rm}_{slkp}\mathrm{Rm}_{ijps} + \mathrm{Rm}_{slsp}\mathrm{Rm}_{ijkp}
\end{aligned}$$

Schematically we can write

$$\triangle \mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \mathrm{Ric}\,.$$

In the Einstein case $\mathrm{Ric} = const$, so $\triangle \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}$.

If T is a tensor on any Riemannian manifold we have

$$\begin{aligned}
|T|\triangle|T| &= \langle T,\, \triangle T\rangle + |\nabla T|^2 - |\nabla |T||^2 \\
&\geq \langle T,\, \triangle T\rangle \\
&\geq -|T||\triangle T|.
\end{aligned}$$

Putting $f = c(n)|\,\mathrm{Rm}\,|$ we therefore have $\triangle f \geq -|f|^2$.

4 The L^p theory on Einstein manifolds

For the time being we assume that $|\,\mathrm{Rm}\,| \in L^{\frac{n}{2}}$. Later we shall discuss justifications for this assumption. Let ϕ be a C_c^∞ function with $\mathrm{Vol}(\mathrm{supp}\,\phi) \leq \frac{1}{2}\mathrm{Vol}(M)$. The Sobolev inequality gives

$$\begin{aligned}
\left(\int \phi^{2\gamma}|\,\mathrm{Rm}\,|^{p\gamma}\right)^{\frac{1}{\gamma}} &= S_n^{-2}\int |\nabla(\phi|\,\mathrm{Rm}\,|^{\frac{p}{2}})|^2 \\
&\leq 2S_n^{-2}\int |\nabla\phi|^2|\,\mathrm{Rm}\,|^p + \frac{p^2}{2}S_n^{-2}\int \phi^2|\,\mathrm{Rm}\,|^{p-2}|\nabla|\,\mathrm{Rm}\,||^2 \qquad (1)
\end{aligned}$$

where we use the abbreviation $\gamma = \frac{n}{n-2}$. If f is a positive function it is easy to compute

$$\begin{aligned}
(p-1)\int \phi^2 f^{p-2}|\nabla f|^2 &= -2\int \phi f^{p-1}\langle\nabla\phi, \nabla f\rangle - \int \phi^2 f^{p-1}\triangle f \\
&\leq \frac{p-1}{2}\int \phi^2 f^{p-2}|\nabla f|^2 + \frac{2}{p-1}\int |\nabla\phi|^2 f^p - \int \phi^2 f^{p-1}\triangle f \\
\int \phi^2 f^{p-1}|\nabla f|^2 &\leq \left(\frac{2}{p-1}\right)^2\int |\nabla\phi|^2 f^p - \frac{2}{p-1}\int \phi^2 f^{p-1}\triangle f.
\end{aligned}$$

With $f = |\operatorname{Rm}|$ and $\triangle|\operatorname{Rm}| \geq -C|\operatorname{Rm}|^2$ we therefore have

$$\int \phi^2|\operatorname{Rm}|^{p-1}|\nabla f|^2 \quad \leq \quad \left(\frac{2}{p-1}\right)^2 \int |\nabla\phi|^2|\operatorname{Rm}|^p + \frac{2}{p-1}\int \phi^2|\operatorname{Rm}|^{p+1}.$$

Putting back into (1) we get

$$\frac{S_n^2}{2}\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p + \frac{2p^2}{p-1}\int \phi^2|\operatorname{Rm}|^{p+1}. \tag{2}$$

The first step is to put $|\operatorname{Rm}|$ in a slightly higher L^p space.

Lemma 4.1 *Assume that Ω is a domain with Sobolev constant $S_n = S_n(\Omega)$, and with*

$$\left(\int_\Omega |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma}S_n^2.$$

Then if $\phi \in C_c^\infty(\Omega)$ we have

$$\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \quad \leq \quad 4S_n^{-2}\left(1+(2\gamma)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n}{2}}$$

<u>*Pf*</u>

Since $\frac{1}{\gamma}+\frac{2}{n}=1$ Hölder's inequality gives

$$\begin{aligned}\frac{1}{2}S_n^2\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} &\leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p \\ &\quad + \frac{2p^2}{p-1}\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\end{aligned}$$

$$\left(\frac{1}{2}S_n^2 - \frac{2p^2}{p-1}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right)\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^p.$$

Therefore we require the $L^{n/2}$-norm of $|\operatorname{Rm}|$ to be small compared to p and S_n. If we let $p = n/2$ we get

$$\left(\frac{1}{2}S_n^2 - \frac{n^2}{n-2}\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}}\right)\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2p}{p-1}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n}{2}}.$$

If we require $\left(\int_{\operatorname{supp}\phi}|\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} < \frac{1}{4}\frac{1}{n}\frac{1}{\gamma}S_n^2$, then

$$\frac{1}{4}S_n^2\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \leq \left(1+\left(\frac{2n}{n-2}\right)^2\right)\int |\nabla\phi|^2|\operatorname{Rm}|^{\frac{n}{2}}. \tag{3}$$

$\square$

Lemma 4.2 *There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and*

$$\left(\int_\Omega |\operatorname{Rm}|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \frac{1}{4n\gamma} S_n^2,$$

then

$$\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \leq C\, S_n^{-2}\, p^{\frac{n}{2}} \sup |\nabla\phi|^2 \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p$$

<u>*Pf*</u>

We start from (2). Since $\frac{1}{\gamma^2} + \frac{2}{n} + \frac{2}{n}\frac{1}{\gamma} = 1$, Hölder's inequality gives

$$\begin{aligned}
\int \phi^2 |\operatorname{Rm}|^{p+1} &= \int \phi^{\frac{2}{\gamma}} |\operatorname{Rm}|^{\frac{p}{\gamma}} |\operatorname{Rm}|^{\frac{2p}{n}} \phi^{\frac{4}{n}} |\operatorname{Rm}| \\
&\leq \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma^2}} \left(\int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p\right)^{\frac{2}{n}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{2}{n}\frac{1}{\gamma}}
\end{aligned}$$

Using (2) and the Schwartz inequality,

$$\begin{aligned}
&\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\
&\quad\leq \left(1 + \left(\frac{2p}{p-1}\right)^2\right) \int |\nabla\phi|^2 |\operatorname{Rm}|^p + \frac{2p^2}{p-1} \int \phi^2 |\operatorname{Rm}|^{p+1} \\
&\quad\leq \left(1 + \left(\frac{2p}{p-1}\right)^2\right) \int |\nabla\phi|^2 |\operatorname{Rm}|^p + \frac{1}{\gamma}\frac{S_n^2}{2} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\
&\qquad + \frac{2}{n}\left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}\frac{1}{\gamma}} \left(\frac{2p^2}{p-1}\right)^{\frac{n}{2}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p.
\end{aligned}$$

At this point we note that, if we restrict ourselves to $p \geq \frac{n}{2} \geq$, then $\frac{p}{p-1} \leq 2$. Therefore

$$\begin{aligned}
&\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\
&\quad\leq \frac{5n}{2} \int |\nabla\phi|^2 |\operatorname{Rm}|^p + 2^n p^{\frac{n}{2}} \left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}\frac{1}{\gamma}} \left(\int \phi^{2\gamma} |\operatorname{Rm}|^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p.
\end{aligned}$$

Now using (3) we get

$$\begin{aligned}
&\frac{S_n^2}{2}\left(\int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \\
&\quad\leq \frac{5n}{2} \int |\nabla\phi|^2 |\operatorname{Rm}|^p + (1 + (2\gamma)^2)\, 2^{n+1} p^{\frac{n}{2}} \left(\frac{S_n^2}{2}\right)^{-\frac{n}{2}} \int |\nabla\phi|^2 |\operatorname{Rm}|^{\frac{n}{2}} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p \\
&\quad\leq \frac{5n}{2} \int |\nabla\phi|^2 |\operatorname{Rm}|^p + (1 + (2\gamma)^2)\, 2^{n+1} (2n\gamma)^{-\frac{n}{2}} p^{\frac{n}{2}} \sup |\nabla\phi|^2 \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p.
\end{aligned}$$

If we let

$$C(n) \ = \ 10n(1+(2\gamma)^2)2^{n+1}(2n\gamma)^{-\frac{n}{2}}$$

then we can write

$$S_n^2 \left(\int \phi^{2\gamma} |\,\mathrm{Rm}\,|^{p\gamma} \right)^{\frac{1}{\gamma}} \ \leq \ C\, p^{\frac{n}{2}} \sup |\nabla \phi|^2 \int_{\mathrm{supp}\,\phi} |\,\mathrm{Rm}\,|^p.$$

$\square$

Lecture 19 - Einstein Manifolds II

April 20, 2010

1 Moser Iteration

With S_n being the Sobolev (=isoperimetric) constant, recall the lemma from last time:

Lemma 1.1 *There exists a $C = C(n)$ so that if $p \geq \frac{n}{2}$ and*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \;\leq\; \frac{1}{4n\gamma},$$

then

$$\left(S_n^{-n} \int \phi^{2\gamma} |\operatorname{Rm}|^{p\gamma} \right)^{\frac{1}{p\gamma}} \;\leq\; C^{\frac{1}{p}}\, p^{\frac{n}{2p}}\, \sup |\nabla \phi|^{\frac{2}{p}} \left(S_n^{-n} \int_{\operatorname{supp}\phi} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

We can apply this iteratively to obtain a local C^∞ bound.

Theorem 1.2 *There exists a constant $C = C(n)$ so that if*

$$\left(S_n^{-n} \int_\Omega |\operatorname{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \;\leq\; \frac{1}{4n\gamma},$$

then

$$\sup_{B_q(r/2)} |\operatorname{Rm}| \;\leq\; C(n)\, r^{-\frac{n}{p}} \left(S_n^{-n} \int_{B_q(r)} |\operatorname{Rm}|^p \right)^{\frac{1}{p}}$$

<u>*Pf*</u>

Given r, let $r_i = \frac{r}{2}\left(1 + \frac{1}{2^i}\right)$. Let ϕ_i be a function with $\phi_i \equiv 1$ inside $B_q(r_i)$, $\phi_i \equiv 0$ outside $B_q(r_{i-1})$, and $|\nabla \phi_i| \leq 2(r_{i-1} - r_i)^{-1} = r^{-1} 2^{i+2}$. Putting

$$\Phi_i \;=\; \left(S_n^{-n} \int_{B_q(r_i)} |\operatorname{Rm}|^{p\gamma^i} \right)^{\frac{1}{p\gamma^i}},$$

we have from lemma (1.1) that

$$\begin{aligned}\Phi_{i+1} &\leq C^{p^{-1}\gamma^{-i}} \left(p\gamma^i\right)^{\frac{n}{2}p^{-1}\gamma^{-i}} (4r^{-1}2^i)^{2p^{-1}\gamma^{-i}} \Phi_i \\ &= (C\,r^{-2}\,p)^{p^{-1}\gamma^{-i}} (4\gamma)^{p^{-1}i\gamma^{-i}} \Phi_i\end{aligned}$$

Iterating, we get

$$\begin{aligned}\Phi_{i+1} &\leq (C\,r^{-2}\,p)^{\frac{1}{p}\sum_{j=0}^{i}\gamma^{-j}} (4\gamma)^{\frac{1}{p}\sum_{j=0}^{i}j\gamma^{-j}} \Phi_0 \\ &\leq (C\,r^{-2}\,p)^{\frac{1}{p}\sum_{j=0}^{\infty}\gamma^{-j}} (4\gamma)^{\frac{1}{p}\sum_{j=0}^{\infty}j\gamma^{-j}} \Phi_0\end{aligned}$$

We have

$$\begin{aligned}\sum_{j=0}^{\infty}\gamma^{-j} &= \frac{1}{1-\gamma^{-1}} = \frac{n}{2} \\ \sum_{j=0}^{\infty}j\gamma^{-j} &= \frac{\gamma}{(\gamma-1)^2} = \left(\frac{n}{2}\right)^2\frac{1}{\gamma}.\end{aligned}$$

so that

$$\begin{aligned}\Phi_{i+1} &\leq (C\,r^{-2}\,p)^{\frac{1}{p}\frac{n}{2}} (4\gamma)^{\frac{4}{p}\gamma^{-1}n^{-2}} \Phi_0 \\ &= C(n,p)\,r^{-\frac{n}{p}}\,\Phi_0.\end{aligned}$$

We therefore have

$$\lim_{i\to\infty}\Phi_i = \lim_{i\to\infty}\left(S_n^{-n}\int_{B_q(r_i)}|\,\mathrm{Rm}\,|^{r\gamma^i}\right)^{\frac{1}{p}\frac{2}{n}} = \sup_{B_q(r/2)}|\,\mathrm{Rm}\,|.$$

$\square$

We have proven the standard ϵ-regularity lemma:

Theorem 1.3 *There exist constants* $\epsilon_0 = \epsilon_0(n, S_n)$ *and* $C = C(n, S_n)$ *so that*

$$\int_{B_q(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}} \leq \epsilon_0$$

implies

$$\sup_{B_q(r/2)}|\,\mathrm{Rm}\,| \leq C\,r^{-2}\left(\int_{B_q(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

2 Kähler geometry

An *almost complex structure* on a manifold is a tensor $J : T_pM \to T_pM$ such that $J^2 = -1$ (namely, $J(J(X)) = -X$ for all $X \in T_pM$); clearly this resembles multiplication by i in $\mathbb{C}^n$. A manifold is a *complex manifold* if is has domains $U_\alpha \subset M$ and maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ with transition functions $\phi_{\beta\alpha} = \phi_\beta \phi_\alpha^{-1}$ being *holomorphic*.

A complex manifold automatically carries an almost complex structure: in a coordinate chart $(x^1, y^1, \ldots, x^n, y^n)$ we just define $J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$ and $J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$. If the transition functions are holomorphic, then this definition is consistent; the preservation of this definition of J is known as the Cauchy-Riemann condition. But on the other hand, when does the existence of an almost complex structure imply that there are charts with holomorphic transition functions? When this is the case, the almost complex structure is said to be a *complex structure*, or we say that J is *integrable*. The Newlander-Nirenberg theorem provides the answer. Let

$$N(X,Y) \;=\; [JX, JY] \,-\, J[JX, Y] \,-\, J[X, JY] \,-\, [X,Y]$$

be the *Nijenhuis tensor*. On a complex manifold it is automatic that $N \equiv 0$.

Theorem 2.1 (Newlander-Nirenberg) *An almost complex manifold is a complex manifold iff $N(X,Y) = 0$ for all smooth vector fields X, Y.*

A metric g on an almost complex manifold is called *Hermitian*, *J-Hermitian*, or *compatible with the almost complex structure* if $g(X,Y) = g(JX, JY)$. In that case we can create the *Kähler form* ω by setting

$$\omega(X,Y) \;=\; g(JX, Y).$$

It is easy to see that the symmetry of g implies the antisymmetry of ω, making it a 2-form. We say that a manifold (M, J, g) is a *Kähler manifold* if J is integrable and if ω is a closed 2-form: $d\omega = 0$.

Note that ω is a *real* 2-form, meaning $\omega(X,Y) \in \mathbb{R}$ whenever X, Y are real sections of $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$. If η is another real form in the same DeRham cohomology class, then of course $\eta - \omega = d\phi$ for some 1-form ϕ. However the so-called $\partial\bar{\partial}$-lemma provides more:

$$\eta \,-\, \omega \;=\; \sqrt{-1}\partial\bar{\partial}\phi$$

for some *function* ϕ. If a fixed Kähler form ω is given, then ϕ is often called the *Kähler potential* for the Kähler form η. Note that for a given potential ϕ the form $\eta = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is not necessarily the Kähler form of a Riemannian metric, because the associated metric $g_\eta(X,Y) = -\eta(JX, Y)$ (though symmetric) might not be everywhere positive definite.

In a sense, Kähler geometry is the intersection of Riemannian and symplectic geometry. To be more precise, recall that a metric is Kähler if its holonomy is in $U(n)$, Riemannian if its holonomy is in $SO(2n)$, and symplectic if its holonomy is in $Sp(n)$. Note that

$$U(n) \ = \ SO(n) \cap Sp(n),$$

so that a metric with holonomy in both $SO(2n)$ and $Sp(n)$ is Kähler.

3 Elliptic systems on canonical manifolds

3.1 Extremal Kähler manifolds

On any Kähler manifold, we have

$$\begin{aligned}
\mathrm{Ric}_{i\bar{\jmath},s\bar{s}} &= \mathrm{Ric}_{s\bar{\jmath},i\bar{s}} \\
&= \mathrm{Ric}_{s\bar{\jmath},\bar{s}i} + \mathrm{Rm}_{i\bar{s}s\bar{t}}\mathrm{Ric}_{t\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}t\bar{\jmath}}\mathrm{Ric}_{s\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},\bar{\jmath}i} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}t\bar{\jmath}}\mathrm{Ric}_{s\bar{t}} \\
&= R_{,i\bar{\jmath}} + \mathrm{Ric}_{i\bar{t}}\mathrm{Ric}_{t\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}t\bar{\jmath}}\mathrm{Ric}_{s\bar{t}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}_{i\bar{\jmath},\bar{s}s} &= \mathrm{Ric}_{i\bar{s},\bar{\jmath}s} \\
&= \mathrm{Ric}_{i\bar{s},s\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= \mathrm{Ric}_{s\bar{s},i\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}} \\
&= R_{,i\bar{\jmath}} - \mathrm{Rm}_{i\bar{s}i\bar{t}}\mathrm{Ric}_{t\bar{s}} + \mathrm{Rm}_{i\bar{s}t\bar{s}}\mathrm{Ric}_{i\bar{t}}
\end{aligned}$$

Schematically

$$\triangle\,\mathrm{Ric} = \mathrm{Rm} * \mathrm{Ric} + \nabla^2 R.$$

Therefore if the metric is, for instance, CSC Kähler, then we have an elliptic system

$$\begin{aligned}
\triangle\,\mathrm{Rm} \ &= \ \mathrm{Rm} * \mathrm{Rm} + \nabla^2\,\mathrm{Ric} \qquad (1) \\
\triangle\,\mathrm{Ric} \ &= \ \mathrm{Rm} * \mathrm{Ric}\,. \qquad (2)
\end{aligned}$$

It is known that many Kähler manifolds do not admit CSC (much less Kähler-Einstein) Kähler metrics. A generalization of the CSC condition was proposed by Calabi, who proposed minimizing the functional

$$\mathcal{C}(\omega) \ = \ \int R^2 \omega^n$$

over metrics is a fixed class.

If the Kähler metric is extremal in the sense of Calabi, then we do not necessarily have constant scalar curvature, but in fact we have $\triangle X = -Ric(X)$ where $X = R_{,i}$. In this case we have the elliptic system

$$\begin{aligned}
\triangle \mathrm{Rm} &= \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \mathrm{Ric} \\
\triangle \mathrm{Ric} &= \mathrm{Rm} * \mathrm{Ric} + \nabla X + \overline{\nabla} X \\
\triangle X &= \mathrm{Ric} * X.
\end{aligned}$$

3.2 Other cases

There are two other cases of metrics with elliptic systems. The first is the case of metrics with so-called harmonic curvature, namely $\mathrm{Rm}_{ijkl,i} = 0$; this is equivalent to the metric being CSC and $W_{ijkl,i} = 0$.

The other is case of 4-dimensional CSC Bach flat metrics, which includes for isntance the CSC half-conformally flat metrics. There are higher dimensional generalizations of the Bach tensor, but I don't know if making them zero yields an elliptic system.

Lecture 20 - Einstein Manifolds III - Compactness under diameter and volume constraints

April 22, 2010

1 Convergence

Assume for the moment that the Sobolev constant is globally controlled. We have

Theorem 1.1 *Let M_i be a sequence of n-manifolds with Einstein metrics, with*

i) The Einstein constants λ_i are controlled: $\underline{\lambda} \;\leq\; \lambda_i \;\leq\; \overline{\lambda}$.

ii) Energy is controlled: $\int_{M_i} |\,\mathrm{Rm}_i\,|^{\frac{n}{2}} \leq \Lambda$,

iii) Diameters are bounded from above: $\mathrm{Diam}(M_i) \leq D$

iv) Volumes are bounded from below: $\mathrm{Vol}\, M_i \geq \nu$

v) The Sobolev constant is controlled: $S_n(M_i) \;\geq\; C_S$

Then some subsequence of M_i converges to a length space M_∞. There exists a number $N = N(n, D, \nu, C_S, \Lambda)$ so that away from at most N many point-like singularities, the space M_∞ has the structure of an Einstein manifold, and (i)-(v) continue to hold.

<u>Pf</u> Let r be some small number. We divide M_i into a "good" set $G_i(r)$, and a "bad" set $B_i(r)$ as follows. Given $p \in M_i$, we let $p \in G_i(r)$ if $\int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\leq\; \epsilon = \epsilon(n, S_n)$, and let $B_i(r) = M_i - G_i(r)$ otherwise. That is,

$$B_i(r) \;=\; \left\{ p \in M_i \;\Big|\; \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;>\; \epsilon_0 \right\}.$$

Now cover $B_i(r)$ with balls $\{B_{2r}(p_{i,k})\}_k$ of radius $2r$ in such a way that the half-radius balls $\{B_r(p_{i,k})\}_k$ are disjoint. The procedure for doing this is as follows. Let $p_{i,1} \in M_i$ be any point in $B_i(r)$. If $p_{i,1}, \ldots, p_{i,l}$ are points in $B_i(r)$ with pairwise separation $> 2r$, then let $p_{i,l+1} \in B_i(r)$ be any point that has a distance $> 2r$ away from each $p_{i,1}, \ldots, p_{i,l}$ assuming any such point exists. Clearly this process must end after Λ/ϵ_0 points have been chosen.

Since each of the balls ball $B_(p_{i,k})(2r)$ has volume bounded from above by $C(n)r^n$ (Bishop volume comparison), most of the manifold's volume lies in $G_{i,r}$. Also, $|\,\mathrm{Rm}\,| < \alpha\, r^{-2}$ on $G_{i,r}$, where α can be made as small as desired by adjusting ϵ_0.

Fixing r, a subsequence of the $G_i(r)$ converges in the Gromov-Hausdorff and the $C^{1,\alpha}$-sense to some manifold $G_\infty(r)$. Now let r be smaller, and repeat the process, starting with the subsequence already found. Continuing this with countable many values of r that decline toward 0, a diagonal subsequence will converge to a manifold whose closure is a manifold-with-singularities. The singularities are point-like. Let $M_\infty = \overline{\bigcup_r G_\infty(r)}$ denote the completion of the limit.

The convergence for each choice of r is in the $C^{1,\alpha}$ topology, due to the fact that curvature is bounded. It can be shown that the convergence is actually in the C^∞ sense, using the following argument. Note that curvature is bounded on the interior of each $G_i(r)$. This means that on a ball of definite radius one can pass to a ball in the tangent space with the pullback metric. There we have harmonic coordinates, and the equation

$$\begin{aligned} \triangle(g_{ij}) &= -2\mathrm{Ric}_{ij} + Q(g, \partial g) \\ &= -2\lambda\, g_{ij} + Q(g, \partial g) \end{aligned}$$

(following DeTurck-Kazdan, 1981). Since the g_{ij} (and therefore the coefficients on the Laplacian) are controlled in the $C^{1,\alpha}$-sense, Schauder theory gives uniform bounds on $C^{2,\alpha}(g_{ij})$. Bootstrapping this fashion gives uniform $C^{k,\alpha}$ bounds on the functions g_{ij}. □

Note that, ostensibly, curvature grows like $o(r^{-2})$ near the singularities of M_∞.

2 The nature of the singularities

With $|\,\mathrm{Rm}\,| = o(r^{-2})$ near the singularities, it is possible to prove the existence of flat tangent cones at the identity. Since we are working in dimension bigger than 2, any such cone must be a standard cone over a quotient of $\mathbb{S}^{n-1}$. However it is *not* possible to prove (in the general case of $|\,\mathrm{Rm}\,| = o(r^{-2})$) that a neighborhood of the singular point is homeomorphic to (a neighborhood of) such a cone. In particular the tangent cone need not be unique. However if curvature grows *strictly* slower than r^{-2}, namely $|\,\mathrm{Rm}\,| = O(r^{-2+\epsilon})$, or, even better, $|\,\mathrm{Rm}\,| < C$, then the Grove-Shiohama theory of critical points allows us to determine that tangent cones are indeed unique.

To improve the growth of $|\,\mathrm{Rm}\,|$ we would like to implement the Moser iteration process despite the presence of the point-like singularities on our Einstein manifolds. However this would appear impossible, as we do not know that our elliptic inequality $\triangle u \geq -u^2$ holds weakly across the singularity. Specifically, in the Moser iteration argument, the first stage is the Sobolev inequality, which provides

$$\left(\int \phi^{2\gamma}|\operatorname{Rm}|^{p\gamma}\right)^{\frac{1}{\gamma}} \;\le\; C\int |\nabla\phi|^2|\operatorname{Rm}|^p \;+\; Cp\int \phi^2\left|\nabla|\operatorname{Rm}|^{\frac{p}{2}}\right|^2.$$

Then it is required that an integration-by-parts be performed on the right-most term, to obtain a Laplacian term. Although the Sobolev inequality is easily seen to hold despite the singularity, integration-by-parts does not. However we have access to the following lemma, first proved in the context of singularities of Yang-Mills instantons.

Theorem 2.1 (Sibner's lemma) *Assume 2-sided volume growth bounds, Sobolev constant bounds, and $\triangle u \ge -fu$ where $f \in L^{n/2}(B-\{o\})$ ($B = B_o(r)$ is any ball) and $u \ge 0$. There exists an $\epsilon_0 > 0$ so that if $\operatorname{supp}\eta \subset B$, then $\int_{\operatorname{supp}\eta}|f|^{n/2} < \epsilon_0$ implies*

$$\int \eta^2|\nabla u^k|^2 \;\le\; C\int|\nabla\eta|^2|u^k|^2 \tag{1}$$

whenever $k > \frac{1}{2}\frac{n}{n-2}$.

$\square$

If the conclusion of Sibner's lemma holds, then clearly the integration-by-parts argument can proceed, and $|\operatorname{Rm}|$ can be bootstrapped into a higher L^p space. Note that $\frac{n}{2} > \frac{n}{n-2}$ when $n > 4$.

In dimension 4 equality holds and Sibner's lemma just fails, so we have to look to the geometry of Einstein manifolds to provide the additional rigidity that can allow improved in regularity. One way this can be found is in an *improved Kato inequality.* The classical (and quite trivial) Kato inequality reads $|\nabla|T||^2 \le |\nabla T|^2$ for any tensor T. In the case of Einstein manifolds it is possible to improve this inequality:

$$|\nabla\operatorname{Rm}|^2 \;\ge\; (1+\eta)|\nabla|\operatorname{Rm}||^2$$

where $\eta = \eta(n) > 0$ (in fact, $\eta = \frac{1}{3}$ in the 4-dimensional case). The reference is Bando-Kasue-Nakajima, 1989.

To use this information, first note that $|T|\triangle|T| + |\nabla|T||^2 = \langle\triangle T, T\rangle + |\nabla T|^2$ and

$$\begin{aligned}\frac{1}{1-\delta}|T|\triangle|T|^{1-\delta} &= -\delta|\nabla|T||^2 \,+\, |T|^{1-\delta}\triangle|T| \\ &= |\nabla T|^2 \,-\, (1+\delta)|\nabla|T||^2 \,+\, |T|^{-\delta}\langle T, \triangle T\rangle\,.\end{aligned}$$

Letting $T = \operatorname{Rm}$ we have

$$\begin{aligned}\triangle|T|^{1-\eta} &\ge (1-\eta)|T|^{-1-\eta}\langle T, \triangle T\rangle \\ &\ge -C(n)|T|\cdot|T|^{1-\eta}.\end{aligned}$$

This is again of the form $\triangle u \ge -fu$ where $f \in L^{\frac{n}{2}}$, but now $u \in L^{\frac{1}{1-\eta}\frac{n}{2}}$, and improvement. Therefore Sibner's lemma goes through, even in dimension 4, despite the presence of singularities.

At this point Moser iteration proceeds nearly unchanged and we get $|\mathrm{Rm}| \in L^\infty_{loc}$ despite the presence of singularities. This implies that the point-like singularities are in fact orbifold points.

If $B_o(r)$ is a ball around a singular point o, one can pass to an orbifold cover. This is a Euclidean ball $\tilde{B}(r)$ around the origin that has a discrete group $\Gamma \subset SO(n)$ and a C^∞ map $\pi : (\tilde{B}(r) - \{pt\}/\Gamma \to B_o(r) - \{o\}$. Let $\tilde{g}_{ij} = \pi^*(g)_{ij}$ be the pullback metric. With $|\widetilde{\mathrm{Rm}}|$ bounded on $\tilde{B}(r)$ we can construct coordinates so that the metric components are $C^{1,1}$ functions. Therefore harmonic coordinates can be constructed (again by the results of Kazdan-DeTurck), in which we have the equation

$$\triangle(g_{ij}) \;=\; -2\mathrm{Ric}_{ij} \,+\, Q(g, \partial g).$$

Combined with $g_{ij} = \lambda\,\mathrm{Ric}_{ij}$ a bootstrapping argument commences, which gives $g_{ij} \in C^\infty$.

3 Various statements of the compactness theorem, and the naturality of the hypotheses

When $|\mathrm{Ric}|$ is controlled, say $|\mathrm{Ric}| < \overline{\lambda}$, then the Sobolev constant is controlled in terms of $\overline{\lambda}$, D, and ν; therefore the hypothesis on Sobolev constants is superfluous. The reference for this is Croke, 1980. The defining equation for Einstein metrics is

$$\mathrm{Ric}_{ij} \;=\; \lambda g_{ij}$$

Since Ric is scale invariant, if $\lambda \neq 0$ we can scale λ by scaling the metric. The numbers ν and D can also be modified by adjusting the scale, so we can fix just one of the numbers λ, ν, D. In the case that $\lambda \neq 0$ we choose to set $\lambda = \pm 1$, and in the case $\lambda = 0$ we fix $\mathrm{Vol} = 1$. Note that in the case of positive Einstein constant, we have $D \leq \pi/\sqrt{\lambda}$ by Myers' Theorem.

Finally we comment on the energy controls. It is rare that we can control $\int |\mathrm{Rm}|^{\frac{n}{2}}$ on a manifold when $n > 4$. However in the 4-dimensional case, we have the Chern-Gauss-Bonnet integral formula $8\pi^2\chi(M) = \int \frac{1}{24}R^2 \,-\, \frac{1}{2}|\overset{\circ}{\mathrm{Ric}}|^2 \,+\, |W|^2$. In the Einstein case, this is

$$\chi(M) \;\;=\;\; \frac{1}{8\pi^2}\int \frac{1}{24}R^2 \,+\, |W|^2.$$

It is standard that $|\mathrm{Rm}|^2 \,=\, \frac{1}{6}R^2 + 2|\overset{\circ}{\mathrm{Ric}}|^2 + |W|^2$ in dimension 4, but by adjusting the constants slightly,

$$\chi(M) \;\;=\;\; \int |\mathrm{Rm}|^2.$$

Thus the L^2-norm of the Riemannian curvature is controlled by a topological quantity. We can therefore restate our proposition

Theorem 3.1 *Let* $\mathcal{M}^n_{-1} = \mathcal{M}(\nu, D, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* n*-dimensional Einstein manifold with* $\lambda = -1$
- $\mathrm{Vol}(M) \geq \nu$
- $\mathrm{Diam}(M) \leq D$
- $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}} \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Theorem 3.2 *Let* $\mathcal{M}^n_{1} = \mathcal{M}(\nu, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* n*-dimensional Einstein manifold with* $\lambda = 1$
- $\mathrm{Vol}(M) \geq \nu$
- $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}} \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Theorem 3.3 *Let* $\mathcal{M}^n_{0} = \mathcal{M}(D, \nu, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* n*-dimensional Einstein manifold with* $\lambda = 0$
- $\mathrm{Vol}(M) \geq \nu$
- $\mathrm{Diam}(M) \leq D$
- $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}} \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Theorem 3.4 *Let* $\mathcal{M}^4_{-1} = \mathcal{M}(\nu, D, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* 4*-dimensional Einstein manifold with* $\lambda = -1$
- $\mathrm{Vol}(M) \geq \nu$
- $\mathrm{Diam}(M) \leq D$
- $\chi(M) \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Theorem 3.5 *Let* $\mathcal{M}_1^4 = \mathcal{M}(\nu, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* 4*-dimensional Einstein manifold with* $\lambda = 1$
- $\mathrm{Vol}(M) \geq \nu$
- $\chi(M) \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Theorem 3.6 *Let* $\mathcal{M}_0^4 = \mathcal{M}(D, \Lambda)$ *be the set of manifolds* M *such that*

- M *is an* 4*-dimensional Einstein manifold with* $\lambda = 0$ *and* $\mathrm{Vol}(M) = 1$
- $\mathrm{Diam}(M) \leq D$
- $\chi(M) \ \leq \ \Lambda.$

Then the conclusions of the Proposition 1.1 hold.

Lecture 21 - Einstein Manifolds IV

April 27, 2010

1 Sobolev constants on Einstein manifolds

We use the notation VR $B_p(r) \;=\; r^{-n}\,\mathrm{Vol}\;B_p(r)$.

Lemma 1.1 *Let M be an Einstein manifold. There are constants $\epsilon_1, C > 0$ that depend only on n so that $\sup_{B_p(r)} |\,\mathrm{Rm}\,| < \epsilon_1\, r^{-2}$ implies $S_n(B_p(r)) > C \cdot (\mathrm{VR}\;B_p(r))^{\frac{1}{n}}$.*

<u>*Pf*</u>

Assume $\pi : \widetilde{M} \to M$ is a k-to-1 covering space where M is a manifold, possibly with boundary. If $\Omega \subset M$ is a domain with boundary $\partial\Omega$, and if $\widetilde{\Omega} = \pi^{-1}(\Omega)$ is its lift, then also $\widetilde{\partial\Omega} = \partial\widetilde{\Omega}$, and we have

$$\begin{aligned} |\partial\Omega| &= k^{-1}|\partial\widetilde{\Omega}| \\ &\geq k^{-1}|\widetilde{\Omega}|^{\frac{n-1}{n}} \\ &= k^{-1}k^{\frac{n-1}{n}}|\Omega|^{\frac{n-1}{n}} \;=\; k^{-\frac{1}{n}}|\Omega|^{\frac{n-1}{n}} \end{aligned}$$

So that $S_n(M) \;\geq\; k^{-\frac{1}{n}}\,S_n(\widetilde{M})$.

The hypotheses of the lemma are scale-invariant so we can assume $|\,\mathrm{Rm}\,| \leq \epsilon_1$ on $B = B_p(1)$. If the lemma is false, there are examples of such balls B_i with $|\,\mathrm{Rm}_i\,| < \epsilon_i \searrow 0$ $S_n(B_i) < C_i\,(\mathrm{Vol}\;B_i)^{\frac{1}{n}}$ and $C_i \searrow 0$. If a subsequence of the numbers $\{\mathrm{Vol}\,B_i\}_i$ remains bounded away from 0, the Cheeger lemma implies that injectivity radii are bounded, and we can take a limit. The Sobolev constant is continuous under taking $C^{0,1}$ limits of Riemannian manifolds, and so on this limiting manifold-with-boundary B_∞, we have $S_n(B_\infty) = 0$, an impossibility.

Therefore $\mathrm{Vol}\,B_i \searrow 0$, and we are in the collapsing situation. With $|\,\mathrm{Rm}_i\,| \leq \epsilon_i$ we know that B_i is almost-flat, and possesses an N-structure. Passing to the frame bundle FB_i over B_i, this N-structure has the following structure. There is a normal cover $\widetilde{FB_i} \to FB_i$

so that $\widetilde{FB}_i$ is the total space of an $\mathbb{R}^k$-bundle over a manifold with controlled injectivity radius.

Let $f_i : B_i \to \mathbb{R}^{\geq 0}$ be a $W_0^{1,1}$ function with $\int |\nabla f_i| \leq C_i \, (\mathrm{Vol}\, B_i)^{\frac{1}{n}} \left(\int f_i{}^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$. By normalizing f we can assume $\int f_i{}^{\frac{n-1}{n}} = 1$. We can lift f_i to an $SO(n)$-invariant function on FB_i. Let $\tilde{f}_i$ be its lift to $\widetilde{FB}_i$. On $\widetilde{FB}_i$ we can restrict to a submanifold $\widetilde{FB}_i^C$ where the fibers are reduced from $\mathbb{R}^k$ to a cube of fixed size. Then $\pi : \widetilde{FB}_i^C \to FB_i$ is (generically) a k-to-1 local homeomorphism, where $k = (\mathrm{Vol}\ SO(n)\ \cdot\ \mathrm{Vol}\ B_i)^{-1}$. Therefore

$$\int |\nabla \widetilde{f}_i| \ \leq\ C_i \left(\int \widetilde{f}_i^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}$$

This is impossible. □

Theorem 1.2 *Assume $B_p(r)$ be a ball in a Riemannian manifold, on which the standard ϵ-regularity theorem holds, namely that there are constants $C = C(n) < \infty$ and $\epsilon_0 = \epsilon_0(n) > 0$ so that*

$$S_n(B_p(r)) \int |\, \mathrm{Rm}\, |^{\frac{n}{2}} \ \leq\ \epsilon_0$$

implies

$$\sup_{B_p(r/2)} |\, \mathrm{Rm}\, | \ \leq\ C\, r^{-2} \left(S_n(B_p(r)) \int_{B_p(r)} |\, \mathrm{Rm}\, |^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

If we set

$$H = H(p,r) \quad = \quad \sup_{B_q(s) \subset\subset B_p(r)} \frac{1}{\mathrm{VR}\ B_q(s)} \int_{B_q(s)} |\, \mathrm{Rm}\, |^{\frac{n}{2}},$$

then there exist numbers $C_1 = C_1(n) < \infty$ and $\epsilon_1 = \epsilon_1(n) > 0$ so that

$$H < \epsilon \qquad \text{implies} \qquad \sup_{B_p(r/2)} |\, \mathrm{Rm}\, | \ \leq\ C_1 r^{-2} H^{\frac{2}{n}}.$$

Pf

We can choose ϵ_1 small enough so that if $|\, \mathrm{Rm}\, | \leq C_1 (r/2)^{-2} \epsilon_1^{\frac{2}{n}}$ on each half-radius ball, then the hypotheses of Lemma 1.1 are met, and then standard ϵ-regularity gives the conclusion.

If not, then there is a point p_1 so that on the ball $B_{p_1}(r2^{-1})$, the conclusion of the theorem is false. Now consider the half-radius subballs of $B_{p_1}(r2^{-1})$. Let p_2 be a point with $B_{p_2}(r2^{-2}) \subset B_{p_1}(r2^{-1})$ on which the conclusion of the theorem is false. Continuing in this manner we have a sequence of points p_i and balls $B_{p_i}(r2^{-i})$ for which the conclusion is false, meaning $|\,\mathrm{Rm}\,| \geq C_1 2^{2i} r^{-2} \epsilon_1^{\frac{2}{n}}$. However this cannot continue indefinitely, because $|\,\mathrm{Rm}\,|$ is not infinity at any point. Therefore there is a point p_i so that the conclusion is false on $B_{p_i}(r2^{-i})$ but so that the conclusion is true on every subball of half-radius. Now we are in the situation of the previous paragraph, and we have a contradiction. $\square$

2 Notations and notions of collapsing

We define $\mathrm{VR}\, B_p(r) = r^{-n}\,\mathrm{Vol}\, B_p(r)$. Define $r_{|R|}(p)$, called the *local curvature radius*, by

$$r_{|R|}(p) \;=\; \sup\; \left\{\, r > 0 \;\middle|\; r^{-2}|\,\mathrm{Rm}\,| \;\leq\; 1 \text{ on } B_p(r) \,\right\}.$$

This is the largest number μ so that scaling the metric by μ^2 produces a ball $B_p(1)$ with $|\,\mathrm{Rm}\,| \leq 1$. Note that $r_{|R|}(p) = \infty$ iff the manifold is flat. A slightly different notion is the *s-local curvature radius*,

$$r^s_{|R|}(p) \;=\; \sup\; \left\{\, 0 < r < s \;\middle|\; r^{-2}|\,\mathrm{Rm}\,| \;\leq\; 1 \text{ on } B_p(r) \,\right\}.$$

This is used in case where we intentionally want to restrict the scale. We can also define the *local energy radius* $\rho(p)$ by

$$\rho(p) \;=\; \sup\; \left\{\, r > 0 \;\middle|\; \frac{1}{\mathrm{VR}\, B_p(r)} \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;<\; \epsilon_0 \right\}$$

and the *s-local energy radius* $\rho^s(p)$ by

$$\rho^s(p) \;=\; \sup\; \left\{\, 0 < r < s \;\middle|\; \frac{1}{\mathrm{VR}\, B_p(r)} \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;<\; \epsilon_0 \right\}.$$

This is the largest ball (of radius $\leq s$) on which the standard ϵ-regularity theorems are guaranteed to hold.

A set $E \subset M^n$ is said to be *v-collapsed on the scale r* if

$$p \in E,\; s \leq r \quad\Longrightarrow\quad \mathrm{VR}\, B_p(s) \;\leq\; v.$$

If no scale is mentioned, it is understood that the scale is 1. The set E is said to be *v-collapsed with locally bounded curvature* if

$$p \in E,\; s \leq r_{|R|}(p) \quad\Longrightarrow\quad \mathrm{VR}\, B_p(s) \;\leq\; v$$

and E is said to be *(v, σ)-collapsed with locally bounded curvature* if

$$p \in E,\; s \leq r^{\sigma}_{|R|}(p) \quad\Longrightarrow\quad \mathrm{VR}\, B_p(s) \;\leq\; v.$$

3 Statement of the Cheeger-Tian results

Throughout we assume M^4 is a compact Einstein manifold, normalized to have Einstein constant $\lambda \in \{0, 3, -3\}$. In the case $\lambda = 0$ we normalize so $\mathrm{Vol}(M^4) = 1$.

Theorem 3.1 (ϵ-regularity) *There exists numbers ϵ, c so that when $p \in M$ and $r < 1$,*

$$\int_{B_r(p)} |\operatorname{Rm}|^2 \;\le\; \epsilon,$$

implies

$$\sup_{B_{r/2}(p)} |\operatorname{Rm}| \;\le\; c\, r^{-2}.$$

Theorem 3.2 (Collapse implies concentration of curvature) *There are constants $v > 0$, $\beta < \infty$, $c < \infty$ so that*

$$s^{-4}\ \mathrm{Vol}\ B_s(p) \;\le\; v$$

for all $p \in M$, $s < 1$ implies there are points $p_1, \dots, p_N \in M$ with

$$N \;\le\; \beta \int_M |\operatorname{Rm}|^2$$

such that

$$\int_{M - \bigcup B_s(p_i)} |\operatorname{Rm}|^2 \;\le\; c \sum_{i=1}^{N} s^{-4}\ \mathrm{Vol}\ B_s(p_i).$$

Theorem 3.3 (Noncollapsing) *There exists a constant $w > 0$ so that $|\lambda| = 3$ implies there is some point p with*

$$\mathrm{Vol}\ B_1(p) \;\ge\; w \cdot \frac{\mathrm{Vol}\ M}{\int_M |\operatorname{Rm}|^2}$$

Lemma 3.4 (The 'Key Estimate') *There is a $c < \infty$, $\delta > 0$, $t > 0$ so that, whenever $E \subset M$ is a bounded open subset, $T_r(E)$ is t-collapsed on the scale r, and*

$$\int_{B_r(p)} |\operatorname{Rm}|^2 \;\le\; \delta$$

for all $p \in E$, then

$$\int_E |\operatorname{Rm}|^2 \;\le\; c\, r^{-4}\ \mathrm{Vol}\left(A_r(E)\right).$$

We use $T_s(E)$ indicates the s-tube around E (the set of points of distance $< s$ from E), and we use $A_r(E)$ to denote the "annulus" of radius r around E: $A_r(E) = T_r(E) - \overline{E}$. A set E is said to be t-collapsed on the scale a if $p \in E$ implies $a^{-4}\,\mathrm{Vol}\,B_a(p) \le t$. We say E is t-collapsed if it is t-collapsed on the scale 1.

Lecture 24 - Einstein Manifolds VII - Epsilon regularity

May 6, 2010

1 Energy ratio improvement

We have called scale-invariant quantity $\frac{1}{\mathrm{VR}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}$ the "energy ratio." It is convenient to modify this, and consider the quantity

$$\frac{\mathrm{Vol}^\lambda\, B(r)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}.$$

Since we have $\mathrm{Ric}\geq\lambda g$, this quantity is more useful in the use of relative volume comparison. If we restrict ourselves to $a\leq 1$ then these quantities are equivalent.

Lemma 1.1 *Assume M^n is an Einstein manifold, $r\leq 1$, and $\int_{B_p(r)}|\,\mathrm{Rm}\,|^{\frac{n}{2}}<\delta$. Then there exist numbers $C<\infty$, $\eta>0$ so that either*

$$\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r/2)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2\ \leq\ (1\,-\,\eta)\,\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^2 \tag{1}$$

or else the annulus $B_p(5r/8)-B_p(3r/8)$ has

$$\begin{aligned}|\,\mathrm{Rm}\,| &< Cr^{-2}\sqrt{\eta}\\ \frac{\mathrm{Vol}\ B_p(r)}{\mathrm{Vol}\ B_p(r/2)} &\geq (1-\eta)\frac{\mathrm{Vol}^\lambda\, B(r)}{\mathrm{Vol}^\lambda\, B(r/2)}.\end{aligned}$$

<u>*Pf*</u>

If (1) does not hold, then

$$\begin{aligned}\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r/2)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2 &\geq (1\,-\,\eta)\,\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r)}|\,\mathrm{Rm}\,|^2\\ &\geq (1\,-\,\eta)\,\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}\int_{B_p(r/2)}|\,\mathrm{Rm}\,|^2,\end{aligned}$$

and we have $\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r/2)} \geq (1-\eta)\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\ B_p(r)}$. On the other hand

$$\int_{B_p(r)-B_p(r/2)} |\,\mathrm{Rm}\,|^2 \ \leq\ \left(\frac{1}{1-\eta}\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}^\lambda\, B(r)}\frac{\mathrm{Vol}\ B_p(r)}{\mathrm{Vol}\ B_p(r/2)}\ -\ 1\right)\int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2.$$

Bishop-Gromov volume comparison gives $\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}^\lambda\, B(r)}\frac{\mathrm{Vol}\ B_p(r)}{\mathrm{Vol}\ B_p(r/2)} \leq 1$, so therefore

$$\int_{B_p(r)-B_p(r/2)} |\,\mathrm{Rm}\,|^2 \ \leq\ \frac{\eta}{1-\eta}\int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2.$$

The Key Estimate now gives

$$\int_{B_p(r)-B_p(r/2)} |\,\mathrm{Rm}\,|^2 \ \leq\ \frac{\eta}{1-\eta}Cr^{-4}\left(\mathrm{Vol}\ B_p(r)\ -\ \mathrm{Vol}\ B_p(r/2)\right).$$

Now let $q \in B_p(5r/8) - B_p(3r/8)$, so that $B_q(r/8) \subset B_p(r) - B_p(r/2)$. We have

$$\begin{aligned}
\int_{B_q(r/8)} |\,\mathrm{Rm}\,|^2 &\leq \int_{B_p(r)-B_p(r/2)} |\,\mathrm{Rm}\,|^2 \\
&\leq \frac{\eta}{1-\eta}Cr^{-4}\left(\mathrm{Vol}\ B_p(r)\ -\ \mathrm{Vol}\ B_p(r/2)\right) \\
&\leq \frac{\eta}{1-\eta}Cr^{-4}\ \mathrm{Vol}\ B_q(2r) \\
&\leq \frac{\eta}{1-\eta}Cr^{-4}\ \mathrm{Vol}\ B_q(r/8)\frac{\mathrm{Vol}^\lambda\, B_q(2r)}{\mathrm{Vol}^\lambda\, B_q(r/8)}
\end{aligned}$$

so that

$$\frac{\mathrm{Vol}^\lambda\, B(r/8)}{\mathrm{Vol}\ B_q(r/8)}\int_{B_q(r/8)} |\,\mathrm{Rm}\,|^2 \quad\leq\quad \frac{\eta}{1-\eta}Cr^{-4}\mathrm{Vol}^\lambda B_q(2r).$$

With $r \leq 1$ we have that there exists a C so that

$$\frac{\mathrm{Vol}^\lambda\, B(r/8)}{\mathrm{Vol}\ B_q(r/8)}\int_{B_q(r/8)} |\,\mathrm{Rm}\,|^2 \quad\leq\quad \frac{\eta}{1-\eta}C.$$

Now if η is chosen small enough that $C\eta/(1-\eta) < \epsilon_0$, then ϵ-regularity holds and we get

$$|\mathrm{Rm}_q| \ \leq\ C\, r^{-2}\sqrt{\eta}$$

□

Lemma 1.2 *The second alternative in Lemma 1.1 does not hold.*

Pf

The small curvature and almost-volume annulus together imply the existence of a Cheeger-Colding function $\hat{r}$ that has the following properties

$$\begin{aligned}
&\triangle \hat{r}^2 \;=\; 8 \\
&|\hat{r} \,-\, r| \;\leq\; \Phi \\
&\frac{1}{|\hat{r}^{-1}(a)|}\int_{\hat{r}^{-1}(a)} |\nabla \hat{r} \,-\, \nabla r|^2 \;\leq\; \Phi \\
&|\nabla \hat{r}| \;\leq\; C \\
&\left| 1 \,-\, \frac{|\hat{r}^{-1}(a)|}{|\partial B_p(a)|} \right| \;\leq\; \Phi \\
&\frac{1}{|\hat{r}^{-1}(a)|}\int_{\hat{r}^{-1}(a)} \left| II_{\hat{r}^{-1}(a)} \,-\, \frac{1}{\hat{r}} g_{\hat{r}^{-1}(a)} \otimes \nabla \hat{r} \right| \;\leq\; \Phi.
\end{aligned}$$

for some $\Phi = \Phi(\eta)$ where $\lim_{\eta \to 0} \Phi = 0$. We can pass to the universal covering space, where the injectivity radius is bounded and we can take a limit. On this space the injectivity radius is bounded, so it is possible to take a limit as $\eta \to 0$. On the limit space the function $\hat{r}$ has $\nabla^2 r = \frac{1}{\hat{r}} g$, so the limit is a warped product with level sets of $\hat{r}$ being space forms. This gives $C^{1,\alpha}$-convergence of $\hat{r}$.

Therefore $\hat{r}^{-1}$ converges in the *pointwise* sense to a space form. Thus the annulus has (almost) the metric structure of an annulus in a Euclidean cone.

Now consider again the Chern-Gauss-Bonnet theorem

$$\chi(B_p(3r/4)) \;=\; \int |\,\mathrm{Rm}\,|^2 \;+\; \int_{\partial B_p(3r/4)} \mathcal{TP}_\chi.$$

The boundary term converges to the Euclidean boundary term, which is *positive*. Since the left-hand side is negative due to the F-structure, we have a contradiction. $\square$

Theorem 1.3 (ϵ-regularity) *If $r \leq 1$ and $\int_{B_p(r)} |\,\mathrm{Rm}\,|^2 \;\leq\; \delta$, then for some $\mu > 0$*

$$\sup_{B_p(\mu r)} |\,\mathrm{Rm}\,|^2 \;\leq\; C r^{-2}.$$

Pf

The Key Estimate gives

$$\begin{aligned}
&\int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2 \;\leq\; C r^{-4}\,|\mathrm{Vol}\; B_p(r) - \mathrm{Vol}\; B_p(r/2)| \\
&\frac{\mathrm{Vol}^\lambda\, B(r/2)}{\mathrm{Vol}\; B_p(r/2)} \int_{B_p(r/2)} |\,\mathrm{Rm}\,|^2 \;\leq\; C
\end{aligned}$$

for some C. Lemmas 1.1 and 1.2 give

$$\frac{\mathrm{Vol}^{\lambda}\, B(r2^{-k-1})}{\mathrm{Vol}\, B_p(r2^{-k-1})} \int_{B_p(r/2)} |\operatorname{Rm}|^2 \;\leq\; C\eta^k.$$

Choosing $k > \frac{\log(\epsilon_0/C)}{\log(\eta)}$ gives

$$\frac{\mathrm{Vol}^{\lambda}\, B(r2^{-k-1})}{\mathrm{Vol}\, B_p(r2^{-k-1})} \int_{B_p(r/2)} |\operatorname{Rm}|^2 \;\leq\; \epsilon_0,$$

whereupon standard ϵ-regularity goes through. $\square$