MAT561: Physics for Mathematicians II (Spring 2008)

Instructor

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Coordinates:

MAT561 meets Mondays and Wednesdays at 15:50-17:10 in the Light Engineering Lab 154.

Outline

This is the second part of an ambitious two-part course in theoretical physics aimed at the graduate mathematics student. It is a fundamental part of the RTG Program in Geometry and Physics with the purpose of introducing many of the basic concepts, theories, and principles which form the basis of our current understanding of the Universe. The topics covered in the MAT 560 included the classical (non-)relativistic dynamics of particles and fields.

Prerequisite

The prerequisite for MAT 561 is MAT 560 or permission from the instructor.

In the second semester we will attempt to cover as much as possible from the following outline of topics:

- Classical field theory (continued):
  - Classical spinor fields
  - Sigma models
  - Supersymmetry
- Classical Mechanics:
  - Hamiltonian formalism
- Quantum Mechanics:
  - Heisenberg mechanics
  - Schrödinger mechanics

- Dirac notation
  - Spinors
  - Feynmann Path Integral
- Quantum Fields I:
  - Canonical quantization
  - Path integral quantization
- Quantum Fields II:
  - Review of gauge systems
  - Gauge fixing and ghosts: Fadeev-Popov method
  - BRST method
  - BV method
- String theory I:
  - Bosonic string
  - Fermionic strings
  - Green-Schwarz string
- String theory II:
  - Covariant quantization
  - Backgrounds

As was the case in the first semester, the course will be taught from a physical perspective (mathematical rigor is not emphasized). The aim is to gain familiarity with and intuition for many of the concepts usually taught to a student of theoretical physics over the course of his/her undergraduate and graduate training.

Textbooks

The unorthodox nature of this program makes it impossible to follow a textbook. Nevertheless, I have assigned Theodore Frankel's "The Geometry of Physics: An Introduction" as a text. This book, as many others, is written to teach physicists mathematics and not the other way around. We will try to reverse-engineer and follow various sections of this book throughout the semester.

Due to the vast amount of material and the lack of centralized supporting literature, a lot of work will be required on the students' part to keep up. Besides the completion of regularly assigned homework sets, the student should read up on the topics being presented in the class.

Announcements

- The MAT 560 page has been updated to include all the old homework assignments.

Assignments

1. If you did not take MAT 560 you might want to look over the homework assignments from last semester.

2. Homework 1 due Wednesday March 5 in class: Do *all* the problems in this (PDF) set.
3. Homework 2 due Wednesday March 26 in class: Do *all* the problems in this (PDF) set.

Resources

This is a list of resources/references from MAT 560 to which we will add this semester:

- In class I recommended taking a look at V.I. Arnold's "Mathematical Methods of Classical Mechanics. I will try to remember to put it on reserve in the library.
- Matt Young reminded me of the notes on Classical Field Theory and Supersymmetry by Daniel S. Freed in the 1996-1997 IAS program in Quantum Field Theory. As Matt points out, there is quite a bit of overlap with what we are/will be doing in class from a pure math perspective.
- Thanks go out again to Matt Young for pointing out this article on how to derive the Euler-Lagrange equation in a coordinate invariant manner.
- Some of the material on special relativity I am presenting in class is based on Bernard Schutz' book "A first course in general relativity" [Spires]
- Matt Young has sent me a copy of the "Toronto Lectures on Physics" by Shlomo Sternberg on various topics covered in MAT 560. He recommends the first 5 chapters. The rest of the lectures also have some nice stuff which we will see in the coming weeks.

MAT561 Homework 1

Due Wednesday, March 5^{th}.

Abstract

As usual, you may not skip any exercises and your solutions must show that you have understood the solution to the problem. The last 2 problems use concepts from last semester's course. It is important for you to understand them so if you are having difficulty, please come see me.

1 Reading

Chapters 19 in Frankel [1].

2 Spinors in various dimensions

This problem is taken from the appendix of [2]. Consider the Clifford algebra relation

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}\mathbf{1} \tag{1}$$

where $a, b = 1, \ldots, d$ are d-dimensional spinor indices, η is the Minkowski metric in the "mostly plus" convention and the Γs and $\mathbf{1}$ are matrices in $GL(n, \mathbb{C})$ for some n. Suppose $d = 2k + 2$ is even[1] and group the matrices into $k + 1$ matrices

$$\gamma^{0\pm} = \frac{1}{2}\left(\pm\gamma^0 + \gamma^1\right), \gamma^{i\pm} = \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right), i = 1, \ldots, k. \tag{2}$$

[1] If d is odd, we proceed with the construction as if we were in $d - 1$ dimensions. The missing Dirac matrix is then taken to be the chirality matrix (c.f. equation 6).

Compute their anti-commutators. Conclude that $(\gamma^{i-})^2 = 0$ for all $i = 0, \ldots, k$. This implies that in the representation space $\mathbb{C}^n$ there is a vector ζ which is annihilated by all these "lowering operators"

$$\gamma^{i-}\zeta = 0 \quad \forall i = 0, \ldots, k. \tag{3}$$

From this vector we may construct others $\zeta^{(\mathbf{s})}$, $\mathbf{s} = (s_0, \ldots, s_k)$ by acting in all ways with the "raising operators" γ^{i+}. Argue from the anti-commutation relations that we get an $n = 2^{k+1}$-dimensional representation this way. We could label these representations by 0s and 1s but let us instead shift this by $-\frac{1}{2}$ resulting in a labeling $\mathbf{s} = (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ and

$$\zeta^{(\mathbf{s})} = (\gamma^{k+})^{s_k+\frac{1}{2}} \ldots (\gamma^{0+})^{s_0+\frac{1}{2}}\zeta. \tag{4}$$

Starting in $k = 1$ and using $\zeta^{(\mathbf{s})}$ as a basis, derive the explicit form of the Dirac matrices Γ^0 and Γ^1. Do the same for $k = 2$.[2] Give an inductive prescription to generate an explicit set of Dirac matrices for general k.

We know that the Lorentz generators $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ satisfy the $\mathfrak{so}(d-1, 1)$ relations. Use this to show that the operators[3]

$$S^i \equiv i^{\delta_{i,0}-1}\Sigma^{2i,2i+1} \tag{5}$$

commute with each other. Show that the eigenvalues of these operators are the $\{s^i\}$.

Define the chirality matrix

$$\gamma \equiv i^{-k}\gamma^0\gamma^1 \ldots \gamma^{d-1} \tag{6}$$

and show that it anti-commutes with the Dirac matrices, commutes with the Lorentz generators, and squares to 1. By writing γ in terms of the S^i operators, show that $\gamma = +\mathbf{1}$ on spinors $\zeta^{(\mathbf{s})}$ with even numbers of $s^i = -\frac{1}{2}$ and $\gamma = -\mathbf{1}$ on those with an odd number of $s^i = -\frac{1}{2}$.

[2] Due to the definition of $\Gamma^{a\pm}$ the basis you get will not agree with the basis used in class. The various explicit representations are equivalent (they are related by similarity transformations which preserve the Clifford algebra relations) but some have special properties. The *Weyl basis* used in class gave a nice block-diagonal form for the Lorentz generators. There is also the *Majorana basis* in which the Dirac matrices are all real. In the *Dirac basis* the representation of the physical degrees of freedom in a spinor is nice. In this problem Polchinski is using a basis (I don't know of a name but we can call it a Fock basis) which the spinor is built up by acting on the ground state by raising operators in a nice way.

[3] The annoying factor of $i^{\delta_{i,0}}$ is just there to remove the factor of $-i$ in S^0.

Argue from the construction of the Dirac matrices above that the irreducible spinor representations are unique up to a change of basis. Since the complex conjugate matrices $\pm(\gamma^a)^*$ satisfy the same Clifford algebra relations, they are related to the γs by a similarity transformation. Noting that in Polchinski's basis $\zeta^{(\mathbf{s})}$ the matrices $\gamma^{a\pm}$ are real, γ^a is imaginary for $a = 3, 5, 7, \ldots$. Define the matrices

$$B_1 = \gamma^3\gamma^5 \ldots \gamma^{d-1} \quad \text{and} \quad B_2 = \gamma B_1 \tag{7}$$

and show that

$$B_1\gamma^a B_1^{-1} = (-1)^k\gamma^{a*} \quad \text{and} \quad B_2\gamma^a B_2^{-1} = (-1)^{k+1}\gamma^{a*} \tag{8}$$

What does a similarity transformation of the Lorentz generators by these matrices do to the generators? Use this to construct from ζ^* and the Bs a spinor which transforms like ζ, that is, construct the charge conjugate of ζ. Square the charge conjugation operation to determine in which dimensions the Majorana condition can be imposed.

Compute the effect on the chirality matrix of a similarity transformation by the Bs. Use this to determine in which dimensions the Majorana condition can be imposed on a Weyl spinor. Such spinors are simply called Majorana-Weyl.

3 Solutions to the Dirac equation

This problem is from §3 of [3]. Suppose a Dirac spinor Ψ solves the massive Dirac equation. Recall that the components of Ψ satisfy the Klein-Gordon equation and argue that any such Ψ can be written as a linear combination of plane waves[4]

$$u(k)\mathrm{e}^{ik\cdot x} \tag{9}$$

with wave number k_a. What is the dispersion relation $k^2(m)$? Plug this *anzats* into the Dirac equation to get an algebraic equation. Now perform a standard trick: Argue physically or mathematically that if $k^0 > 0$, you can always go to a frame in which the wave number is $(k_a) = (-m, 0, 0, 0)$. Solve the Dirac equation for Ψ in this "rest frame" in terms of a 2-component spinor ξ.

Now boost this solution to a frame with rapidity η in the 3-direction: $(k_a) = (-m\coth\eta, 0, 0, m\sinh\eta)$. In this frame, a good

[4]This is Fourier's decomposition.

basis for the ξ is the σ^3 eigen-basis ξ^s with $s = 1, 2$ with

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{10}$$

Supposing a the boost is highly relativistic, what form does the Dirac spinor take for $s = 1$ and $s = 2$? Define the "helicity" operator

$$h = \frac{1}{2}\hat{k}^i \ell^i \ . \tag{11}$$

where $\hat{k}$ is the unit wave number indicating the direction of motion of the plane wave and the $\ell^i = \frac{1}{2}\epsilon^{ijk}\Sigma_{jk}$ are the spacial rotation generators. The helicity is the projection of the spinor's "angular momentum" or "spin" onto it's direction of motion. Argue that the helicity of a massless spinor is well-defined meaning that it does not depend on the reference frame.

The choices about reference frames made above may look *ad hoc* but, as usual by Lorentz covariance, any other frame is equivalent to this one. In particular we have found that any spinor with $k^0 > 0$ satisfying the Dirac equation describes 2 real degrees of freedom which, in the case of $m \neq 0$ we may think of as "spin up" and "spin down" along some spacial axis or or in the case $m = 0$ helicity $\pm\frac{1}{2}$.

4 Coupling to gauge fields

The Lagrangian of a free Dirac spinor is invariant under a global $U(1)$ action $\Psi \mapsto \mathrm{e}^{i\alpha}\Psi$. Gauge this global symmetry thereby constructing the coupling of Dirac spinors to gauge fields. What is the conserved current?

References

[1] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*

[2] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," SPIRES entry *Cambridge, UK: Univ. Pr. (1998) 531 p*

[3] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," SPIRES entry *Reading, USA: Addison-Wesley (1995) 842 p*

MAT561 Homework 2

Due Wednesday, March 26$^{\text{th}}$.

Abstract

As usual, you may not skip any exercises and your solutions must show that you have understood the solution to the problem.

1 Superspace conventions

In class I tried to keep Frankel's [2] conventions and really screwed the whole thing up. In this exercise we will work through a consistent set of conventions I lifted from the very clear and careful book [1] by Buchbinder and Kuzenko. Their choice is very clever: They use "mostly-plus" conventions for the Minkowski metric $\eta = \mathrm{diag}(-1, +1, +1, +1)$ which is the popular choice for geometers and relativists but they define the Clifford algebra relation with an extra sign, that is, they take $\{\gamma^a, \gamma^b\} = -2\eta^{ab}\mathbf{1}$. The result is that in the discussion of spinors the conventions closely resemble the "mostly-minus" choice favored by practitioners of quantum field theory and pretty much every book ever written on the subject and the only cost is an extra factor of $i = \sqrt{-1}$ in front of all your Dirac matrices (and of course all consequences thereof).

A left-handed Weyl spinor representation (also known as the $\mathbf{2}$)will be taken to have its $SL(2,\mathbb{C})$ index down ψ_α. The "metric" tensor $\varepsilon^{\alpha\beta}$ is normalized by $\varepsilon^{12} = +1$ and its inverse is defined by $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$. An index on a left-handed Weyl spinor is raised according to the convention $\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta$.[1] Show that this implies that $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$. The complex conjugate representation $\bar{\mathbf{2}}$ has index structure given by $\chi_{\dot\alpha}$. Indices are raised and lowered with the conjugate $\varepsilon_{\dot\alpha\dot\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$ with

[1]Note that there is no factor of i here contrary to the conventions in class.

the same conventions as above. Define the pairing of co- and contravariant spinors ψ_α and χ^α and their conjugates by $\langle\chi\psi\rangle \equiv \chi^\alpha\psi_\alpha$ and $[\bar{\chi}\bar{\psi}] \equiv \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$. Compute $\langle\psi\chi\rangle$ and $[\bar{\psi}\bar{\chi}]$ in terms of these. We usually drop the bracket notation when there can be no confusion. Show that $\psi_\alpha\chi_\beta = \psi_\beta\chi_\alpha + \varepsilon_{\alpha\beta}\psi\chi$ and $\bar{\psi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} = \bar{\psi}_{\dot{\beta}}\bar{\chi}_{\dot{\alpha}} - \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\chi}$.

The Pauli matrices are extended to a 4-covector (note the placement of the spacetime index!) of matrices by $(\sigma_a) = (\mathbf{1}, \vec{\sigma})$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and their index structure is defined to be $(\sigma_a)_{\alpha\dot{\alpha}}$. Define the associated matrices $(\tilde{\sigma}_a)^{\dot{\alpha}\alpha} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_a)_{\beta\dot{\beta}}$ and show that $(\tilde{\sigma}_a) = (\mathbf{1}, -\vec{\sigma})$. These index structures define a natural matrix multiplication *exempli gratia* $(\sigma^a\tilde{\sigma}^b)_\alpha{}^\beta = (\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}^b)^{\dot{\alpha}\beta}$. Verify the formulæ

$$\begin{aligned} (\sigma^a\tilde{\sigma}^b + \sigma^b\tilde{\sigma}^a)_\alpha{}^\beta &= -2\eta^{ab}\delta^\beta_\alpha \\ (\tilde{\sigma}^a\sigma^b + \tilde{\sigma}^b\sigma^a)^{\dot{\alpha}}{}_{\dot{\beta}} &= -2\eta^{ab}\delta^{\dot{\alpha}}_{\dot{\beta}} \\ (\sigma^a)_{\alpha\dot{\alpha}}(\tilde{\sigma}_a)^{\dot{\beta}\beta} &= -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \end{aligned} \tag{2}$$

Define the combinations $(\sigma^{ab})_\alpha{}^\beta \equiv -\frac{1}{4}(\sigma^a\tilde{\sigma}^b - \sigma^b\tilde{\sigma}^a)_\alpha{}^\beta$ and $(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv -\frac{1}{4}(\tilde{\sigma}^a\sigma^b - \tilde{\sigma}^b\sigma^a)^{\dot{\alpha}}{}_{\dot{\beta}}$. Show that $(\sigma^{ab})_{\alpha\beta} = (\sigma^{ab})_{\beta\alpha}$ and similarly for $\tilde{\sigma}^{ab}$. Furthermore, verify that as 2-forms, σ_{ab} is self-dual and $\tilde{\sigma}_{ab}$ is anti-self-dual. Show that this, together with the equations above, means that a symmetric spin-tensor $S_{\alpha\beta} = S_{\beta\alpha}$ is equivalent to a self-dual 2-form and analogously for $S'_{\dot{\alpha}\dot{\beta}}$. Take the results you have derived so far to conclude that in all we have that an anti-symmetric spin-tensor $A_{\alpha\beta}$ is a scalar, a symmetric one $(S'_{\dot{\alpha}\dot{\beta}})$ $S_{\alpha\beta}$ is a(n anti-)self-dual 2-form, and one with mixed indices $V_{\alpha\dot{\alpha}}$ is a vector. In terms of irreducible representations of the 4-dimensional Lorentz group these statements are condensed into the decomposition rules

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \ , \ \bar{\mathbf{2}} \otimes \bar{\mathbf{2}} = \mathbf{1} \oplus \bar{\mathbf{3}} \ , \text{ and } \mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{4} \tag{3}$$

where $\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{4}$ are, respectively, the scalar, spinor, conjugate spinor, self-dual tensor, anti-self-dual tensor, and vector (a.k.a. defining) representations labelled by their dimensions. Note that the adjoint $\mathbf{6} = \mathbf{3} \oplus \bar{\mathbf{3}}$ is irreducible as a real representation.

Show that the matrices

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \tilde{\sigma}^a & 0 \end{pmatrix} \tag{4}$$

satisfy the Clifford algebra relations (with the extra sign). Explain why this implies that a Dirac spinor must have the form

$$(\Psi_{\hat{\alpha}}) = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} . \tag{5}$$

The rotation generators are defined by $\Sigma^{ab} = -\frac{1}{4}[\gamma^a, \gamma^b]$ and a Dirac spinor is defined to transform as $\delta\Psi = \frac{1}{2}\omega_{ab}\Sigma^{ab}\Psi$.

Moving on to superspace, we take $\mathbb{R}^{4|4}$ to have coordinates $(x^A) = (x^a, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$. Differentiation proceeds as defined in class. Conjugation, however, is taken to reverse the order of monomials. For example $(\theta^\alpha\theta^\beta)^* = \bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}}$. Go back and look at the definition of the contraction of dotted spinors. Show that $\langle\psi\chi\rangle^* = [\bar{\chi}\bar{\psi}]$. In particular note that $(\theta^2)^* = \bar{\theta}^2$. Carefully analyzing what this implies for derivatives, one finds that for a general superfunction f with parity $|f|$ the conjugate of the derivative is $(\partial_\alpha f)^* = -(-)^{|f|}\bar{\partial}_{\dot{\alpha}}\bar{f}$.

The global supersymmetry transformation with constant spinor parameter ϵ is defined to act on the coordinates as

$$\delta\theta^\alpha = \epsilon^\alpha \ , \ \delta\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}} \ , \ \delta x^a = i(\theta\sigma^a\bar{\epsilon} - \epsilon\sigma^a\bar{\theta}) \ . \tag{6}$$

We can write this as $\delta x^A = -i(\epsilon Q + \bar{\epsilon}\bar{Q})x^A$ with the supercharges defined by

$$Q_\alpha = i\partial_\alpha + (\sigma^a\bar{\theta})_\alpha\partial_a \ , \ \bar{Q}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}} - (\theta\sigma^a)_{\dot{\alpha}}\partial_a \tag{7}$$

Show that the only non-vanishing bracket (ignoring rotations) is

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}}P_a \tag{8}$$

where $P_a = -i\partial_a$. By taking into consideration that a scalar superfield $V(x,\theta,\bar{\theta})$ is invariant under supersymmetry $V'(x',\theta',\bar{\theta}') = V(x,\theta,\bar{\theta})$ (tensor transformation law) show that $\delta V(x,\theta,\bar{\theta}) = V'(x,\theta,\bar{\theta}) - V(x,\theta,\bar{\theta})$ where by definition we keep only the linear part, is given by

$$\delta V(x,\theta,\bar{\theta}) = i(\epsilon Q + \bar{\epsilon}\bar{Q})V(x,\theta,\bar{\theta}) \ . \tag{9}$$

Introduce the covariant derivatives

$$D_\alpha = \partial_\alpha + i(\sigma^a\bar{\theta})_\alpha\partial_a \ , \ \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i(\theta\sigma^a)_{\dot{\alpha}}\partial_a \tag{10}$$

and show that they commute with the supercharges. Show that the only non-vanishing commutator of the Ds is

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma^a)_{\alpha\dot{\alpha}}\partial_a \ . \tag{11}$$

Define the squares $D^2 = \langle DD \rangle$ and $\bar{D} = [\bar{D}\bar{D}]$ and show that[2]

$$[D^2, \bar{D}_{\dot{\alpha}}] = -4i\partial_{\alpha\dot{\alpha}} D^{\alpha} \text{ and } [\bar{D}^2, D_{\alpha}] = +4i\partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \ . \tag{12}$$

Now show that

$$D^{\alpha}\bar{D}^2 D_{\alpha} = \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \ . \tag{13}$$

Using these equations show that[3]

$$D^2\bar{D}^2 + \bar{D}^2 D^2 - 2D^{\alpha}\bar{D}^2 D_{\alpha} = 16\Box \tag{14}$$

and

$$[D^2, \bar{D}^2] = -4i\partial_{\alpha\dot{\alpha}}[D^{\alpha}, \bar{D}^{\dot{\alpha}}] \ . \tag{15}$$

2 Non-linear σ-model

If you do not know what a Kähler manifold is, find out. Consider the action of n chiral fields Φ^i and their anti-chiral conjugates $\bar{\Phi}^{\bar{\imath}}$

$$S[\Phi^i, \bar{\Phi}^{\bar{\imath}}] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K[\Phi^i, \bar{\Phi}^{\bar{\imath}}] \tag{16}$$

for K a general real function of Φ and $\bar{\Phi}$. Show that this action has as a symmetry any (holomorphic) field transformation

$$\Phi \mapsto f(\Phi) \tag{17}$$

which takes

$$K[\Phi, \bar{\Phi}] \mapsto K[\Phi, \bar{\Phi}] + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \ . \tag{18}$$

Note that ($\bar{\Lambda}$) Λ is (anti-)chiral. Define the components

$$\varphi^i = \Phi^i| \ , \ \psi^i_{\alpha} = D_{\alpha}\Phi^i| \ , \ F^i = -\frac{1}{4}D^2\Phi^i| \tag{19}$$

and similarly for the conjugates. Find the component form of the action. In particular, put the kinetic term for the scalars in the the form $-\int \mathrm{d}^4x\, g_{i\bar{\jmath}}(\varphi, \bar{\varphi})\, \partial^a\bar{\varphi}^{\bar{\jmath}}\partial_a\varphi^i$. A theory with such a scalar kinetic term is called a non-linear σ-model. Why is it appropriate to refer to K as the Kähler potential?

[2]Hint: Start with $\bar{D}_{\dot{\alpha}}D^2$ and move the $\bar{D}$ to the right using the basic rule (11). Do the same for the other case.

[3]You shouldn't have to do much at this point but remember $D_{\alpha}D_{\beta}D_{\gamma} = 0$ identically.

Study the transformation of the component fields (19) under the holomorphic transformation (17). Observe that F is not transforming covariantly, that is, if we interpret φ^i as a coordinate and the transformation as a reparameterization, ψ^i is transforming as a tensor but F^i is not. Fix this by introducing the Christoffel symbols

$$\Gamma^I_{JK} = \frac{1}{2} g^{IL} \left(\partial_J g_{KL} + \partial_K g_{JL} - \partial_L g_{JK} \right) \tag{20}$$

and defining the combinations

$$\begin{aligned} \mathcal{F}^i &= F^i - \frac{1}{4} \Gamma^i_{jk} \langle \psi^j \psi^k \rangle \\ \bar{\mathcal{F}}^{\bar{\imath}} &= \bar{F}^{\bar{\imath}} - \frac{1}{4} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} [\bar{\psi}^{\bar{\jmath}} \bar{\psi}^{\bar{k}}] \,. \end{aligned} \tag{21}$$

Check that the Christoffel symbols not used here all vanish. Show that the component action can be put into the form

$$\begin{aligned} S &= - \int \mathrm{d}^4 x \, g_{i\bar{\jmath}} \left(\partial^a \bar{\Phi}^{\bar{\jmath}} \partial_a \Phi^i - \bar{\mathcal{F}}^{\bar{\jmath}} \mathcal{F}^i + \frac{i}{4} \psi^i \sigma^a \overleftrightarrow{\nabla}_a \bar{\psi}^{\bar{\jmath}} \right) \\ &\qquad - \frac{1}{16} \int \mathrm{d}^4 x \, \mathcal{R}_{i\bar{\jmath}k\bar{l}} \, \psi^i \bar{\psi}^{\bar{\jmath}} \psi^k \bar{\psi}^{\bar{l}}, \end{aligned} \tag{22}$$

where we have introduced the covariant derivatives

$$\begin{aligned} \nabla_a \psi^i_\alpha &= \partial_a \psi^i_\alpha + \Gamma^i_{jl} (\partial_a \varphi^j) \psi^l_\alpha \\ \nabla_a \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} &= \partial_a \bar{\psi}^{\bar{\imath}}_{\dot{\alpha}} + \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{l}} (\partial_a \bar{\varphi}^{\bar{\jmath}}) \bar{\psi}^{\bar{l}}_{\dot{\alpha}} \end{aligned} \tag{23}$$

and $\mathcal{R}_{i\bar{\jmath}k\bar{l}} = K_{ik\bar{\jmath}\bar{l}} - g^{m\bar{n}} K_{ik\bar{n}} K_{m\bar{\jmath}\bar{l}}$ in terms of derivatives of the Kähler potential. Show that this is the $IJKL = i\bar{\jmath}k\bar{l}$ component of the Riemann tensor

$$\mathcal{R}^I{}_{JKL} = \partial_K \Gamma^I_{LJ} - \partial_L \Gamma^I_{KJ} + \Gamma^I_{KE} \Gamma^E_{LJ} - \Gamma^I_{LE} \Gamma^E_{KJ} \tag{24}$$

with the I index lowered.

References

[1] I. L. Buchbinder and S. M. Kuzenko, "Ideas and methods of supersymmetry and supergravity: Or a walk through superspace," SPIRES entry *Bristol, UK: IOP (1998) 656 p*

[2] T. Frankel, "The geometry of physics: An introduction," SPIRES entry *Cambridge, UK: Univ. Pr. (1997) 654 p*

Classical Field Theory and Supersymmetry

Daniel S. Freed

IAS/Park City Mathematics Series
Volume 11, 2001

Classical Field Theory and Supersymmetry

Daniel S. Freed

Introduction

These notes are expanded versions of seven lectures given at the IAS/Park City Mathematics Institute. I had the impossible task of beginning with some basic formal structure of mechanics and ending with sufficient background about supersymmetric field theory for Ronan Plesser's lectures on mirror symmetry. As a result the details may be hard to follow, but hopefully some useful pictures emerge nonetheless. My goal is to explain some parts of field theory which only require fairly standard differential geometry and representation theory. (I also rely on basics of supermanifolds to treat fermionic fields; see the lectures by John Morgan.)

The formal aspects of lagrangian mechanics and field theory, including symmetries, are treated in Lectures 1 and 2; fermionic fields and supersymmetries are introduced in Lecture 4. The key examples and some basic concepts are discussed in Lecture 3. In some sense the heart of the material occurs in Lecture 5. It includes the relation between classical free fields and quantum particles, and also a discussion of free approximations to nonfree lagrangian theories. Physicists use these ideas without comment, so they are important for the mathematician to master. The final two lectures introduce some supersymmetric field theories in low dimensions. Each lecture concludes with exercises for the industrious reader.

I have written about this material elsewhere, most completely in joint articles with Pierre Deligne in *Quantum Fields and Strings: A Course for Mathematicians, Volume 1*, American Mathematics Society, 1999. My briefer *Five Lectures on Supersymmetry*, also published by the American Mathematics Society in 1999, provides a lighter introduction to some of the topics covered here. I have shamelessly lifted text from these other writings. At the same time I have given shamefully few references. The volumes *Quantum Fields and Strings: A Course for Mathematicians* have plenty of material for the mathematician who would like to pursue this subject further. Standard physics texts on quantum field theory and supersymmetry cover much of this material, but from a somewhat different point of view.

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LECTURE 1
Classical Mechanics

Particle motion

We begin with the most basic example of a mechanical system, a single particle moving in space. Mathematically we model this by a function

$$x\colon \text{time} \longrightarrow \text{space}; \tag{1.1}$$

$x(t)$ is the position of the particle at time t. Time is usually taken to be the real line $\mathbb{R}$, but we will take a moment to discuss what precise structure on $\mathbb{R}$ we use. For example, it does not make physical sense to add times, so we do not want to use the vector space structure of the reals. Nor do we have a distinguished time, a zero of time, unless perhaps we consider some particular cosmological or religious model. Rather, the measurements one makes are of differences of time. Furthermore, time is homogeneous in the sense that measurements of time differences are independent of the absolute time. Finally, time differences are given by a single real number. Mathematically we model this by asking that time be a *torsor*[2] for a one-dimensional vector space V; that is, an affine space M^1 whose underlying vector space is V. There is one more piece of structure to include, as we have not yet specified the choice of *units*—seconds, hours, etc.—in which to measure time. For that we ask that M^1 carry a translation-invariant Riemannian metric, or equivalently that V be an inner product space. Then the interval between $t, t' \in M^1$ is given by the distance from t to t'. Of course, we identify $V \cong \mathbb{R}$ by choosing a unit vector in V, but notice that there are two choices, corresponding to two distinct "arrows of time." The superscript '1' reminds us that time is one-dimensional, but also indicates that soon we will encounter a generalization: Minkowski spacetime M^n of dimension n. We fix an affine coordinate t on M^1 so that $|dt| = 1$.

In general one benefit of specifying a geometric structure carefully, for example without arbitrary choices, is that the group of symmetries is then defined. The group of symmetries of $M = M^1$ is the *Euclidean group* $\mathrm{Euc}(M)$ in one dimension, i.e., the group of affine transformations which preserve the metric. An affine transformation induces a linear transformation of the underlying vector space, and this leads to an exact sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Euc}(M) \longrightarrow O(V) \longrightarrow 1. \tag{1.2}$$

[2] A *torsor* for a group G is a space T on which G acts simply transitively. In other words, given $t, t' \in T$ there is a unique $g \in G$ so that g acting on t gives t'. We could call T a principal G-bundle over a point.

The kernel V consists of time translations. Of course, $O(V) \cong \{\pm 1\}$, and correspondingly the Euclidean group is divided into two components: symmetries which project to -1 are *time-reversing*. Some systems are not invariant under such time-reversing symmetries; some are not invariant under time translations. Notice that the metric eliminates scalings of time, which would correspond to a change of units.

We take the space X in which our particle (1.1) moves to be a smooth manifold. One could wonder about the smoothness condition, and indeed there are situations in which this is relaxed, but in order that we may express mechanics and field theory using calculus we will always assume that spaces and maps are smooth (C^∞). Again we want a Riemannian metric in order that we may measure distances, and the precise scale of that metric reflects our choice of units (inches, centimeters, etc.). Usually that metric is taken to be complete; otherwise the particle might fall off the space. Physically an incomplete metric would mean that an incomplete description of the system, perhaps only a local one. Often we will take space to be the Euclidean space $\mathbb{E}^d$, that is, standard d-dimensional affine space together with the standard translation-invariant metric.

There is one more piece of data we need to specify particle motion on X: a *potential energy* function

$$V\colon X \longrightarrow \mathbb{R}. \tag{1.3}$$

Hopefully, you have some experience with potential energy in simple mechanics problems where it appears for example as the energy stored in a spring ($V(x) = kx^2/2$, where x is the displacement from equilibrium and k is the spring constant), or the potential energy due to gravity ($V(x) = mgx$, where m is the mass, g is the gravitational constant, and x is the height).

There is much to say from a physics point of view about time, length, and energy (see [**F**], for example), but in these lectures we move forward with our mathematical formulation. For the point particle we need to specify one more piece of data: the *mass* of the particle, which is a real number $m > 0$ whose value depends on the choice of units. Note that whereas the units of mass (M), length (L), and time (T) are fundamental, the units of energy (ML^2/T^2) may be expressed in terms of these.

So far we have described all potential particle motions in X as the infinite-dimensional function space $\mathcal{F} = \operatorname{Map}(M^1, X)$ of smooth maps (1.1). Actual particle trajectories are those which satisfy the differential equation

$$m\ddot{x}(t) = -V'\big(x(t)\big). \tag{1.4}$$

The dots over x denote time derivatives. As written the equation makes sense for $X = \mathbb{E}^1$; on a general Riemannian manifold the second time derivative is replaced by a covariant derivative and the spatial derivative of V by the Riemannian gradient; see (1.32). Equation (1.4) is *Newton's second law*: mass times acceleration equals force. Notice that we do not describe force as a fundamental quantity, but rather express Newton's law in terms of energy. This is an approach which generalizes to more complicated systems. Let $\mathcal{M}$ denote the space of all solutions to (1.4). It is the space of *states* of our classical system, often called the *phase space*. What structure does it have? First, it is a smooth manifold. Again, we will always treat this state space as if it is smooth, and write formulas with calculus, even though in some examples smoothness fails. Next, symmetries (1.2) of time act on $\mathcal{M}$ by composition on the right. For example, the time translation $T_s,\ s \in \mathbb{R}$ acts by

$$(T_s x)(t) = x(t-s), \qquad x \in \mathcal{M}. \tag{1.5}$$

You should check that (1.4) is invariant under time-reversing symmetries. (Velocity $\dot{x}$ changes sign under such a symmetry, but acceleration $\ddot{x}$ is preserved.) We could consider a particle moving in a time-varying potential, which would *break*[3] these symmetries. Also, isometries of X act as symmetries via composition on the left. For a particle moving in Euclidean space, there is a large group of such isometries and they play an important role, as we describe shortly. On the other hand, a general Riemannian manifold may have no isometries. These global symmetries are not required in the structure of a "classical mechanical system."

So far, then, the state space $\mathcal{M}$ is a smooth manifold with an action of time translations (and possible global symmetries). There is one more piece of data, and it is not apparent from Newton's law: $\mathcal{M}$ carries a symplectic structure for which time translations act as symplectic diffeomorphisms. We can describe it by choosing a particular time t_0, thus breaking the time-translation invariance, and consider the map

$$\begin{aligned} \mathcal{M} &\longrightarrow TX \\ x &\longmapsto \bigl(x(t_0), \dot{x}(t_0)\bigr) \end{aligned} \tag{1.6}$$

If X is a complete Riemannian manifold, then this map is a diffeomorphism. Now the Riemannian structure gives an isomorphism of vector bundles $TX \cong T^*X$, so composing with (1.6) a diffeomorphism $\mathcal{M} \to T^*X$. The symplectic structure on $\mathcal{M}$ is the pullback of the natural symplectic structure on T^*X. When we turn to the lagrangian description below, we will give a more intrinsic description of this symplectic structure.

There is an abstract framework for mathematical descriptions of physical systems in which the basic objects are *states* and *observables*. (See [**Fa**] for one account.) These spaces are in duality in that we can evaluate observables on states. Both classical and quantum systems—including statistical systems—fit into this general framework. We have already indicated that in classical mechanics the space $\mathcal{M}$ of states is a symplectic manifold, and that motion is described by a particular one-parameter group of symplectic diffeomorphisms called time translation. What are the observables? The observables of our classical system are simply functions on $\mathcal{M}$. The pairing of observables and states is then the evaluation of functions. We term this pairing "the expectation value of an observable" in a given state, though the classical theory is deterministic and the expectation value is the actual value. A typical family of real-valued observables $\mathcal{O}_{(t,f)}$ is parametrized by pairs (t,f) consisting of a time $t \in M$ and a function $f\colon X \to \mathbb{R}$; then

$$\mathcal{O}_{(t,f)}(x) = f\bigl(x(t)\bigr), \qquad x \in \mathcal{M}. \tag{1.7}$$

For example, if $X = \mathbb{E}^d$ we can take f to be the i^{th} coordinate function. Then $\mathcal{O}_{(t,f)}$ is the i^{th} coordinate of the particle at time t. Sums and products of real-valued

[3] That is, the system would not have symmetries induced from (1.2).

observables are also observables. In the classical theory we can consider observables with values in other spaces.[4] The observables (1.7) are *local* in time.

This basic structure persists in classical field theory, with the understanding that the space $\mathcal{M}$ is typically infinite-dimensional.

Some differential geometry

We quickly review some standard notions, in part to set notation and sign conventions.

Let Y be a smooth manifold. The set of differential forms on Y is a graded algebra with a differential, called the *de Rham complex*:

$$0 \to \Omega^0(Y) \xrightarrow{d} \Omega^1(Y) \xrightarrow{d} \Omega^2(Y) \xrightarrow{d} \cdots \tag{1.8}$$

A vector field ξ induces a contraction, or interior product

$$\iota(\xi)\colon \Omega^q(Y) \longrightarrow \Omega^{q-1}(Y) \tag{1.9}$$

and also a Lie derivative

$$\mathrm{Lie}(\xi)\colon \Omega^q(Y) \longrightarrow \Omega^q(Y). \tag{1.10}$$

The exterior derivative d, interior product ι, and Lie derivative are related by Cartan's formula

$$\mathrm{Lie}(\xi) = d\iota(\xi) + \iota(\xi)d. \tag{1.11}$$

Now if $E \to Y$ is a vector bundle with connection (covariant derivative) ∇, then there is an extension of the de Rham complex to differential forms with coefficients in E:

$$\Omega^0(Y;E) \xrightarrow{d_\nabla = \nabla} \Omega^1(Y;E) \xrightarrow{d_\nabla} \Omega^2(Y;E) \xrightarrow{d_\nabla} \cdots . \tag{1.12}$$

It is not a complex in general, but rather

$$d_\nabla^2 = R_\nabla \quad \in \Omega^2(Y;\mathrm{End}\,E), \tag{1.13}$$

where R_∇ is the curvature of ∇.

All of this applies to appropriate infinite-dimensional manifolds as well as finite-dimensional manifolds. In these notes we do not discuss the differential topology of infinite-dimensional manifolds. The function spaces we use are assumed to consist of smooth functions, though in a more detailed treatment we would often use completions of smooth functions.

Next, if $\phi\colon Y \to X$ is a map between manifolds, then $d\phi$ is a section of $\mathrm{Hom}(TY, \phi^*TX) \to Y$, i.e., $d\phi \in \Omega^1(Y;\phi^*TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla \quad \in \Omega^2(Y;\phi^*TX), \tag{1.14}$$

[4] In the quantum theory the values should lie in a fixed space, since the observables are integrated over the space of fields.

where T_∇ is the torsion of the connection ∇. Often ∇ is the Levi-Civita connection of a Riemannian manifold, in which case the torsion vanishes. More concretely, if $\phi\colon (-\epsilon,\epsilon)_u \times (-\epsilon,\epsilon)_t \to X$ is a smooth two-parameter map, then the partial derivatives ϕ_u and ϕ_t are sections of the pullback bundle ϕ^*TX. In the case of vanishing torsion equation (1.14) asserts

$$\nabla_{\phi_u}\phi_t = \nabla_{\phi_t}\phi_u. \tag{1.15}$$

The complex of differential forms on a product manifold $Y = Y'' \times Y'$ is naturally bigraded, and the exterior derivative d on Y is the sum of the exterior derivative d'' on Y'' and the exterior derivative d' on Y'. In computations we use the usual sign conventions for differential forms: if $\alpha = \alpha'' \wedge \alpha'$ for $\alpha'' \in \Omega^{q''}(Y'')$ and $\alpha' \in \Omega^{q'}(Y')$, then

$$d\alpha = d''\alpha'' \wedge \alpha + (-1)^{q''}\alpha'' \wedge d'\alpha'. \tag{1.16}$$

We often apply this for Y'' a function space.

We follow the standard convention in geometry that symmetry groups act on the left. For example, if $Y = \mathrm{Map}(M,X)$ and $\varphi\colon M \to M$ is a diffeomorphism, then the induced diffeomorphism of Y acts on $x \in Y$ by $x \mapsto x \circ \varphi^{-1}$. On the other hand, if $\psi\colon X \to X$ is a diffeomorphism, then the induced diffeomorphism of Y is $x \mapsto \psi \circ x$. There is an unfortunate sign to remember when working with left group actions. If $G \to \mathrm{Diff}(Y)$ is such an action, then the induced map on Lie algebras $\mathfrak{g} \to \mathfrak{X}(Y)$ is an *anti*homomorphisms. In other words, if ξ_ζ is the vector field which corresponds to $\zeta \in \mathfrak{g}$, then

$$[\xi_\zeta, \xi_{\zeta'}] = -\xi_{[\zeta,\zeta']}. \tag{1.17}$$

Hamiltonian mechanics

We begin by recalling a basic piece of symplectic geometry. Namely, if $\mathcal{M}$ is a symplectic manifold with symplectic form Ω, then there is an exact sequence

$$0 \longrightarrow H^0(\mathcal{M};\mathbb{R}) \longrightarrow \Omega^0(\mathcal{M}) \xrightarrow{\text{grad}} \mathfrak{X}_\Omega(M) \longrightarrow H^1(\mathcal{M};\mathbb{R}) \longrightarrow 0. \tag{1.18}$$

Here $\mathfrak{X}_\Omega(X)$ is the space of vector fields ξ on X which (infinitesimally) preserve Ω: $\mathrm{Lie}(\xi)\Omega = 0$. The symplectic gradient "grad" of a function $\mathcal{O}$ is the unique vector field $\xi_{\mathcal{O}}$ such that

$$d\mathcal{O} = -\iota(\xi_{\mathcal{O}})\Omega. \tag{1.19}$$

Thus any observable determines an infinitesimal group of symplectic automorphisms (so in good cases a one-parameter group of symplectic diffeomorphisms), and conversely an infinitesimal group of symplectic automorphisms which satisfies a certain cohomological constraint determines a set of observables any two elements of which differ by a locally constant function.

Now in a classical mechanical system the state space $\mathcal{M}$ carries a one-parameter group of time translations, which we assume defines a vanishing cohomology class in $\Omega^2(\mathcal{M})$, i.e., if ξ is the infinitesimal generator, we assume that $\iota(\xi)\Omega$ is exact. A choice of corresponding observable is the *negative* of a quantity we call the *energy* or *Hamiltonian* of the system. Put differently, the total energy generates a motion which is the minus time translation. For example, the energy H of a particle $x\colon M \to X$ satisfying (1.4) with the symplectic structure defined after (1.6) is[5]

$$H(x) = \frac{m}{2}|\dot{x}(t)|^2 + V\big(x(t)\big). \tag{1.20}$$

The right hand side is independent of t, so in fact defines a function of $x \in \mathcal{M}$. It is the sum of the kinetic and potential energy of the particle.

A classical system $(\mathcal{M}, H)$ consisting of a symplectic manifold $\mathcal{M}$ and a Hamiltonian H is *free* if $\mathcal{M}$ is a symplectic affine space and the motion generated by H is a one-parameter group of affine symplectic transformations, or equivalently H is at most quadratic.

Another general piece of symplectic geometry may also be expressed in terms of the exact sequence (1.18). Namely, there is a Lie algebra structure on $\Omega^0(\mathcal{M})$ so that the symplectic gradient is a Lie algebra homomorphism to the Lie algebra of vector fields (with Lie bracket). In other words, this *Poisson bracket* on functions $\mathcal{O}, \mathcal{O}'$ satisfies

$$[\xi_{\mathcal{O}}, \xi_{\mathcal{O}'}] = \xi_{\{\mathcal{O},\mathcal{O}'\}}. \tag{1.21}$$

Because the symplectic gradient has a kernel, (1.21) does not quite determine the Poisson bracket; rather we *define* it by the formula

$$\{\mathcal{O}, \mathcal{O}'\} = \xi_{\mathcal{O}}\mathcal{O}'. \tag{1.22}$$

We note that any physical system has a similar bracket on observables; e.g., in quantum mechanics it is the commutator of operators on Hilbert space.

Let $\mathcal{M}$ be a classical state space with Hamiltonian H. A *global symmetry* of the system is a symplectic diffeomorphism of $\mathcal{M}$ which preserves H. An *infinitesimal symmetry* is a vector field ξ on $\mathcal{M}$ which preserves both the symplectic form Ω and the Hamiltonian H: $\mathrm{Lie}(\xi)\Omega = \mathrm{Lie}(\xi)H = 0$. Then the observable Q which corresponds to an infinitesimal symmetry satisfies

$$\{H, Q\} = 0. \tag{1.23}$$

Quite generally, the time-translation flow on $\mathcal{M}$ induces a flow on observables which may be expressed by[6]

$$\dot{\mathcal{O}} = \{H, \mathcal{O}\}, \tag{1.24}$$

so that (1.23) is equivalent to a conservation law for Q:

$$\dot{Q} = 0. \tag{1.25}$$

[5]We compute this in Lecture 2. Here m is the mass of the particle.
[6]To understand the sign, recall that a diffeomorphism of $\mathcal{M}$ induces an action of functions on $\mathcal{M}$ using pullback by the *inverse*. On the infinitesimal level this introduces a minus sign, and it cancels the minus sign which relates H to infinitesimal time translation.

For this reason the observable Q is called the *conserved charge* associated to an infinitesimal symmetry. Again: *Symmetries lead to conservation laws.*

The symplectic point of view is fundamental in classical mechanics, but we do not emphasize it in these lectures. Rather, many of the most interesting field theories—including most of those of interest in geometry and topology—have a lagrangian description. From a lagrangian description one recovers the symplectic story, but the lagrangian description can be used in field theory when there is no distinguished time direction (which is usually the situation in geometry: an arbitrary manifold does not come equipped with a time function).

Lagrangian mechanics

Recall our description of the point particle. Its state space $\mathcal{M}$ is defined to be the submanifold of the function space $\mathcal{F} = \mathrm{Map}(M^1, X)$ cut out by equation (1.4). Ideally, we would like to describe this submanifold as the critical manifold of a function

$$S\colon \mathcal{F} \longrightarrow \mathbb{R}. \tag{1.26}$$

In that ideal world $\mathcal{M}$ would be the space of solutions to $dS = 0$. Such a function would be called the *action* of the theory, and the critical point equation the *Euler-Lagrange equation*. This is a typical situation in geometry, where we often derive interesting differential equations from variational principles, especially on compact manifolds. In our situation the function S we would like to write down is infinite on typical elements of $\mathcal{F}$, including elements of $\mathcal{M}$, due to the noncompactness of M^1. Rather, the more basic object is the *lagrangian density*, or simply *lagrangian*,

$$L\colon \mathcal{F} \longrightarrow \mathrm{Densities}(M^1) \tag{1.27}$$

which attaches to each potential particle motion $x \in \mathrm{Map}(M^1, X)$ a density[7] $L(x)$ on the line M^1. The lagrangian density is well-defined on all of $\mathcal{F}$, but its integral over the whole line may well be infinite. In this section we study its integrals over finite intervals of time, which are finite, and derive appropriate Euler-Lagrange equations. We will see that these equations can be deduced from L without consideration of S. In addition, we will construct the symplectic structure on $\mathcal{M}$. In Lecture 2 we systematize these constructions in the context of general field theories.

The lagrangian for the particle is

$$L(x) = \Big[\frac{m}{2}|\dot{x}(t)|^2 - V\big(x(t)\big)\Big]\,|dt|. \tag{1.28}$$

It is the *difference* of kinetic energy and potential energy. Also, the dependence of the lagrangian on the path x is *local* in the time variable t. (We formalize the notion of locality in the next lecture.) For each $x\colon M^1 \to X$ the right hand side is a density on M^1. For (finite) times $t_0 < t_1$ the action

$$S_{[t_0,t_1]}(x) = \int_{t_0}^{t_1} L(x) \tag{1.29}$$

[7]A density on M^1 has the form $g(t)\,|dt|$ for some function $g\colon M^1 \to \mathbb{R}$. Densities have a transformation law which corresponds to the change of variables formula for integrals.

is well-defined, whereas the integral over the whole line may be infinite. Nonetheless, we may deduce Euler-Lagrange by asking that S be stationary to first order for variations of x which are compactly supported in time.[8] Thus we choose t_0, t_1 so that a particular variation has support in $[t_0, t_1]$. Now a "variation of x" simply means a tangent vector to $\mathcal{F}$ at x. We have

$$T_x\mathcal{F} \cong C^\infty(M^1; x^*TX), \tag{1.30}$$

i.e., a tangent vector to $\mathcal{F}$ is a section over M^1 of the pullback via x of the tangent bundle TX. To see this, consider a small curve x_u, $-\epsilon < u < \epsilon$, in $\mathcal{F}$ with $x_0 = x$. It is simply a map $x\colon (-\epsilon, \epsilon) \times M^1 \to X$, and differentiating with respect to u at $u = 0$ we obtain precisely an element ζ of the right hand side of (1.30). For the moment do not impose any support condition on ζ, plug $x = x_u$ into (1.29), and differentiate at $u = 0$. We find

(1.31)
$$\begin{aligned}\frac{d}{du}\Big|_{u=0} S_{[t_0,t_1]}(x_u) &= \frac{d}{du}\Big|_{u=0} \int_{t_0}^{t_1} \Big[\frac{m}{2}|\dot x_u(t)|^2 - V\big(x_u(t)\big)\Big]\,|dt| \\ &= \int_{t_0}^{t_1} \Big[m\langle \dot x(t), \nabla_{\zeta(t)}\dot x(t)\rangle - \langle \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\Big]\,|dt| \\ &= \int_{t_0}^{t_1} \Big[m\langle \dot x(t), \nabla_{\dot x(t)}\zeta(t)\rangle - \langle \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\Big]\,|dt| \\ &= \int_{t_0}^{t_1} -\langle m\nabla_{\dot x(t)}\dot x(t) + \operatorname{grad} V\big(x(t)\big), \zeta(t)\rangle\,|dt| \\ &\qquad\qquad + m\langle \dot x(t_1), \zeta(t_1)\rangle - m\langle \dot x(t_0), \zeta(t_0)\rangle.\end{aligned}$$

In the last step we integrate by parts, and in the third we use the fact that the Levi-Civita connection ∇ is torsionfree (in the form of (1.15)). If we impose the condition that the support of ζ is compactly contained in $[t_0, t_1]$, then the boundary terms on the last line vanish. The integral that remains vanishes for all such ζ if and only if the Euler-Lagrange equation

$$m\nabla_{\dot x(t)}\dot x(t) + \operatorname{grad} V\big(x(t)\big) = 0 \tag{1.32}$$

is satisfied. Thus we have recovered Newton's law (1.4), here written for maps into a Riemannian manifold. Now let us work *on-shell*, that is, on the space $\mathcal{M} \subset \mathcal{F}$ of solutions to the Euler-Lagrange equations. Denote the differential on $\mathcal{F}$, and so the differential on the submanifold $\mathcal{M}$, as 'δ'. For each $t \in M^1$ define the 1-form $\gamma_t \in \Omega^1(\mathcal{M})$ by

$$\gamma_t(\zeta) = m\langle \dot x(t), \zeta(t)\rangle. \tag{1.33}$$

Then equation (1.31) implies that the differential of the action on-shell is

$$\delta S_{[t_0,t_1]} = \gamma_{t_1} - \gamma_{t_0} \qquad \text{on } \mathcal{M}. \tag{1.34}$$

[8]This has an analog in the quantum theory. Quantization is done around a classical solution x_0, which may or may not have finite action, but the path integral is done over field configurations x for which the "difference" of the lagrangians $L(x) - L(x_0)$ has a finite integral over M^1. The same remark applies in general field theories.

We draw two conclusions from this equation:

- The 2-form on $\mathcal{M}$

$$\Omega_t := \delta\gamma_t \tag{1.35}$$

  is independent of t.
- The 1-forms γ_t determine a principal $\mathbb{R}$-bundle $P \to \mathcal{M}$ with connection whose curvature is Ω_t.

The first statement follows simply by differentiating (1.34). Turning to the second, for each $t \in M^1$ we define $P_t \to \mathcal{M}$ to be the trivial principal $\mathbb{R}$-bundle with connection 1-form γ_t. Then given $t_0 < t_1$ equation (1.34) asserts that addition of $-S_{[t_0,t_1]}$ is an isomorphism $P_{t_0} \to P_{t_1}$ which preserves the connection. The crucial property of S which makes these trivializations consistent is its *locality* in time: for $t_0 < t_1 < t_2$ we have

$$S_{[t_0,t_2]} = S_{[t_0,t_1]} + S_{[t_1,t_2]}. \tag{1.36}$$

Finally, we remark that one can see from (1.33) that the symplectic form (1.35) has the description given after (1.6).

The Euler-Lagrange equations (1.32), the connection form (1.33), and the symplectic form (1.35) may all be defined directly in terms of the lagrangian density L; the action (1.29) is not needed. In the next lecture we develop a systematic calculus for these manipulations. In particular, we will develop formulas to compute the conserved charges associated to infinitesimal symmetries. Thus a single quantity—the lagrangian density—contains all of the information of the classical system. That is one reason why the lagrangian formulation of a classical system, if it exists, is so powerful. At a less formal level, the lagrangian formulation is a practical way to encode the physics of a complicated system. For example, it is not difficult to write the lagrangian corresponding to a mechanical system with pendulums, springs, etc: it is the total kinetic energy minus the total potential energy. (See the exercises at the end of the lecture.) Lagrangians are also practical for systems which include other physical objects: strings, membranes, fields, etc. In these lectures we focus mainly on fields.

Classical electromagnetism

Let space X be an oriented 3-dimensional Riemannian manifold. For example, we might consider $X = \mathbb{E}^3$. Then X has a *star operator* which in particular gives an isomorphism

$$*_X : \Omega^1(X) \longrightarrow \Omega^2(X) \tag{1.37}$$

whose square is the identity map. Also, we can use the metric and orientation to identify each of these spaces with the space of vector fields on X. For example, on $\mathbb{E}^3$ with standard coordinates x, y, z we have the identifications

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \longleftrightarrow a\,dx + b\,dy + c\,dz \longleftrightarrow a\,dy\wedge dz - b\,dx\wedge dz + c\,dx\wedge dy \tag{1.38}$$

for any functions a, b, c.

The electric field E is usually taken to be a time-varying vector field on X, but using (1.38) we take it instead to be a time-varying 1-form. Similarly, the magnetic field B, which is usually a time-varying vector field on X, is taken to be a time-varying 2-form instead. From one point of view these fields are maps of time M^1 into an infinite-dimensional function space, so comprise an infinite-dimensional mechanical system. A better point of view is that they are fields defined on the *spacetime* $M^1 \times X$, and map this space into a finite-dimensional one.[9] However we view this, Maxwell's law in empty space asserts that

$$\begin{aligned} dB &= 0 & dE &= -\frac{\partial B}{\partial t} \\ d{*_X}E &= 0 & c^2\, d{*_X}B &= {*_X}\frac{\partial E}{\partial t} \end{aligned} \tag{1.39}$$

where c is the speed of light. One can think of the right hand equations as evolution equations, analogous to Newton's law, and the left-hand equations as telling that we are really studying motion in a subspace of $\Omega^1(X) \oplus \Omega^2(X)$. (Note that the right hand equations preserve the subspaces defined by the left-hand equations.) To complete the description of a mechanical system we should specify a symplectic structure on the (infinite-dimensional) space of solutions $\mathcal{M}$ to (1.39).

The question arises as to whether or not there is a lagrangian description of this system. In fact, there is. Why would we like to have one? As explained earlier, the lagrangian encodes not only Maxwell's equations (1.39), but also the symplectic structure on the space of solutions. But in addition we could now try to study the combined system consisting of a particle together with an electric and magnetic field. In the combined system there is an *interaction*, which is encoded as a potential energy term in the action. Once this total lagrangian is described, we are off to the races. For example, we can fix an electric and/or magnetic field and study the motion of a *charged* particle in that background. The resulting equation of motion is called the Lorentz force law. On the other hand, one could deduce the equation for the electric and magnetic fields created by a charged particle, or system of charged particles. At a more sophisticated level one can determine the energy, momentum, etc. of the electric and magnetic fields. It is clear that lots of physics is summarized by the lagrangian! We will give the lagrangian formulation of Maxwell's equations in Lecture 3, and now conclude this first lecture by introducing a more natural setting for field theory: spacetime.

Minkowski spacetime

The electric and magnetic fields are naturally functions of four variables: $E = E(t,x,y,z)$ and $B = B(t,x,y,z)$. The domain of that function is *spacetime*[10]. See Clifford Johnson's lectures on general relativity for the physics behind the geometrization of spacetime; here we content ourselves with a mathematical description.

So far we have modeled time as the one-dimensional Euclidean space M^1 and space as a complete Riemannian manifold X. At first glance it would be natural, then, to consider spacetime as the product $M^1 \times X$ with a "partial metric" and an

[9] More precisely, they are sections of a finite-dimensional vector bundle over $M^1 \times X$.
[10] or perhaps *timespace*, since we write the time coordinate first.

isometric action of the additive group $\mathbb{R}$. (By "partial metric" we mean that we know the inner product of two vectors tangent to X or two vectors tangent to M^1, but we do not say anything about other inner products.) But partial metrics do not pull back under diffeomorphisms to partial metrics, so it not a very good geometric structure. A more physical problem is that the units are mixed: along M^1 we use time but along X we use length. We now describe two alternatives.

For a nonrelativistic spacetime we replace the metric along M^1 with a 1-form dt (or density $|dt|$) on $M^1 \times X$ which evaluates to 1 on the vector field generating the $\mathbb{R}$ action and vanishes on tangents to X. Note that dt determines a codimension one foliation of *simultaneous events* in spacetime. Putting dt as part of the structure requires that symmetries preserve this notion of simultaneity. The other piece of structure is a metric on the foliation, which is simply the Riemannian metric on X. In case $X = \mathbb{E}^{n-1}$ we obtain the *Galilean spacetime* of dimension n; its group of symmetries is the *Galilean group*.

We will focus instead on relativistic theories, in which case we extend the metrics on M^1 and X to a metric on the spacetime $M^1 \times X$. To do this we need to reconcile the disparity in units. This is accomplished by a universal constant c with units of length/time, i.e., a speed. Physically it is the speed of light. However, the spacetime metric is not positive definite; rather it is the Lorentz metric

$$c^2 ds^2_{M^1} - ds^2_X. \tag{1.40}$$

If $\dim X = n-1$, then the metric has signature $(1, n-1)$. For $X = \mathbb{E}^{n-1}$ we obtain *Minkowski spacetime* M^n.

So Minkowski spacetime M^n is a real n-dimensional affine space with underlying translation group V a vector space endowed with a nondegenerate symmetric bilinear form of signature $(1, n-1)$. To compute we often fix an affine coordinate system $x^0, x^1, \dots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.41}$$

Note that $x^0 = ct$ for t the standard time coordinate. Minkowski spacetime carries a positive density defined from the metric

$$|d^n x| = |\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \,|, \tag{1.42}$$

and each time slice $x^0 =$ constant also carries a canonical positive density

$$|d^{n-1} x| = |\, dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{1.43}$$

The vector space V has a distinguished cone, called the *lightcone*, consisting of vectors with norm zero. For $n = 1$ it degenerates to the origin, for $n = 2$ it is the union of two lines, and for $n > 2$ the set of nonzero elements on the lightcone has two components.

The group $\mathrm{Iso}(M)$ of isometries of M^n is a subgroup of the affine group, and it contains all translations by vectors in V. It maps onto the group $O(V)$ of linear transformations of V which preserve the Lorentz inner product:

$$1 \longrightarrow V \longrightarrow \mathrm{Iso}(M) \longrightarrow O(V) \longrightarrow 1. \tag{1.44}$$

The group $O(V)$ has four components if $n \geq 2$. (For $n = 1$ it is cyclic of order two.) To determine which component a transformation $T \in O(V)$ lies in we ask two questions: Is T orientation-preserving? Does T preserve or exchange the components of the nonzero elements on the lightcone? (This question needs to be refined for $n = 2$.) For elements in the identity component the answer to both questions is "yes". There is a nontrivial double cover of the identity component, called the *Lorentz group* $\mathrm{Spin}(V)$, and correspondingly a double cover of the identity component of the isometry group of M^n called the *Poincaré group* P^n:

$$1 \longrightarrow V \longrightarrow P^n \longrightarrow \mathrm{Spin}(V) \longrightarrow 1. \tag{1.45}$$

The Lorentz spin groups have some beautiful properties, some of which we will explore when we study supersymmetry. In any case this double cover is not relevant until when we introduce fermionic fields in Lecture 4.

Exercises

1. In this problem use standard affine coordinates $x^0, x^1, \dots, x^{n-1}$ on Minkowski spacetime M^n and on Euclidean space $\mathbb{E}^n$. Set the speed of light to be one, so that in the Minkowski case we can identify x^0 with a time coordinate t. Let $\partial_\mu = \partial/\partial x^\mu$ be the coordinate vector field. It is the infinitesimal generator of a translation.
   (a) Write a basis of infinitesimal generators for the orthogonal group of isometries of $\mathbb{E}^n$ which fix the origin $(0, 0, \dots, 0)$. Compute the Lie brackets.
   (b) Similarly, write a basis of infinitesimal generators for the orthogonal group of isometries of M^n which fix the origin $(0, 0, \dots, 0)$. Compute the Lie brackets. Write your formulas with 't' in place of 'x^0', and use Roman letters $i, j, \dots$ for spatial indices (which run from 1 to $n-1$).

2. In this problem we review some standard facts in symplectic geometry. Let M be a symplectic manifold.
   (a) Verify that (1.19) determines a unique vector field $\xi_{\mathcal{O}}$. Prove that (1.18) is an exact sequence.
   (b) Using (1.22) as the definition of the Poisson bracket, verify (1.21). Also, prove the Jacobi identity:
   $$\{\mathcal{O}, \{\mathcal{O}', \mathcal{O}''\}\} + \{\mathcal{O}'', \{\mathcal{O}, \mathcal{O}'\}\} + \{\mathcal{O}', \{\mathcal{O}'', \mathcal{O}\}\} = 0.$$
   (c) On a symplectic manifold of dimension $2m$ one can always find local coordinates $q^i, p_j,\ i, j = 1, \dots, m$, such that
   $$\omega = dp_i \wedge dq^i.$$
   (We use the summation convention.) Compute the Poisson brackets of the coordinate functions. Verify that on a symplectic affine space (define) functions of degree ≤ 2 form a Lie algebra under the Poisson bracket. Identify this Lie algebra. (Hint: Consider the image under the symplectic gradient.) Start with $m = 1$.

3. Let M be a smooth manifold of dimension m.
   (a) Define a canonical 1-form θ on the cotangent bundle $\pi\colon T^*M \to M$ as follows. Let $\alpha \in T^*M_m$ and $\xi \in T_\alpha(T^*M)$. Then $\theta_\alpha(\xi) = \alpha(\pi_*\xi)$. Check that $\omega := d\theta$ is a symplectic form.
   (b) Local coordinates $q^1, \dots, q^m$ on M induce local coordinates q^i, p_j on T^*M (the indices runs from 1 to m) by writing an element $\alpha \in T^*X$ as $\alpha = p_i dq^i$. Make sense of this procedure, then write θ and ω in this coordinate system.

4. (a) Check that (1.13) reproduces any other definition of curvature you may have learned. For example, if ξ, η, ζ are vector fields on Y, show that
   $$R_\nabla(\xi,\eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta.$$
   (b) Demonstrate that (1.14) and (1.15) are equivalent (in the case of vanishing torsion).

5. Let G be a Lie group. Consider the action of G on itself by left multiplication. Verify (1.17) in this case. What happens for right multiplication?

6. Let V be a finite-dimensional real vector space endowed with a nondegenerate symmetric real-valued bilinear form. We denote the pairing of vectors v, w as $\langle v, w\rangle$. Let $n = \dim V$.
   (a) Define an induced nondegenerate symmetric form on the dual V^* and on all exterior powers of both V and V^*.
   (b) The highest exterior power of a vector space is one-dimensional, and is called the *determinant line* of the vector space. An element in $\operatorname{Det} V^*$ is a *volume form* on V. There are precisely two such forms ω such that $\langle \omega, \omega\rangle = \pm 1$; they are opposite. Choose one. This amounts to fixing an *orientation* of V, which is a choice of component of $\operatorname{Det} V \setminus \{0\}$. Then the *Hodge $*$ operator*, which is a map
   $$*\colon {\textstyle\bigwedge^q} V^* \longrightarrow {\textstyle\bigwedge^{n-q}} V^*$$
   is defined implicitly by the equation
   $$\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, \omega, \qquad \alpha, \beta \in {\textstyle\bigwedge^q} V^*.$$
   Verify that the $*$ operator is well-defined.
   (c) Compute $**$. The answer depends on the signature of the quadratic form and the degree q.
   (d) Compute $*$ on a 3 dimensional vector space with a positive definite inner product (choose a basis!) and on a 4 dimensional vector space with a Lorentz inner product.

7. (a) Determine all particle motions on $X = \mathbb{E}^1$ which lead to free mechanical systems. (What are the possible potential energy functions?) For each determine the state space and one-parameter group of time translations. Generalize to $X = \mathbb{E}^n$.
   (b) A coupled system of n one-dimensional harmonic oscillators may be modeled as motion in an n-dimensional inner product space X with potential $V(x) = \frac{1}{2}|x|^2$. Investigate the equations of motion of this system. Find the state space and one-parameter group of time translations. Is this system free?

8. Write the lagrangians for the following mechanical systems with one or more particles. These systems are placed in a uniform gravitational field. The potential energy (determined only up to a constant) for a particle of mass m at height h in the gravitational field is mgh for some universal constant g. These problems are taken from *Mechanics*, by Landau and Lifshitz, a highly recommended text.

   (a) A simple pendulum of mass m and length ℓ.

   (b) A coplanar double pendulum.

   (c) A simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plain in which m_2 moves.

9. Rewrite equations (1.39) in terms of vector fields E, B, instead of forms, and verify that you get the standard version of Maxwell's equations.

LECTURE 2
Lagrangian Field Theory and Symmetries

The differential geometry of function spaces

We begin with a word about densities. Recall from Lecture 1 that the particle lagrangian gives a density on the affine time line M^1 for each path of the particle, and we investigated integrals of that density, called the action. Our motivation for using densities, rather than 1-forms, is that the many systems are invariant under time-reversing symmetries. In general, a *density* on a finite-dimensional manifold M is a tensor field which in a local coordinate system $\{x^\mu\}$ is represented by

$$\ell(x)\,|dx^1\cdots dx^n| \tag{2.1}$$

for some function ℓ. It transforms under change of coordinates by the absolute value of the Jacobian of the coordinate change. In other words, it is a twisted n-form—the twisting is by the orientation bundle. We denote the set of densities by $\Omega^{|0|}(M)$. Then we define $\Omega^{|-q|}(M)$ to be the set of twisted $(n-q)$-forms. A twisted $(n-q)$-form is the tensor product of a section of $\bigwedge^q TM$ and a density. For example, an element of $\Omega^{|-1|}(M)$ in local coordinates looks like

$$j^\mu(x)\,\frac{\partial}{\partial x^\mu}\otimes|dx^1\cdots dx^n|. \tag{2.2}$$

The twisted forms are graded as indicated; a twisted $(n-q)$-form, or $|-q|$-form, has degree $-q$. On the graded vector space of twisted forms $\Omega^{|\bullet|}(M)$ we have the usual operations of exterior differentiation d, Lie derivative $\mathrm{Lie}(\xi)$ by a vector field ξ, and interior product $\iota(\xi)$, with the Cartan formula (1.11) relating them. However, twisted forms do not form a ring; rather, twisted forms are a graded module over untwisted forms. Just as we can integrate densities over manifolds, we can integrate $|-q|$-forms over codimension q submanifolds whose normal bundle is oriented.

Continuing with a manifold M, consider the mapping space $\mathcal{F}=\mathrm{Map}(M,X)$ of smooth maps $\phi\colon M\to X$ into a manifold X. Everything we say generalizes to the case where we replace the single copy of X with a fiber bundle $E\to M$ (whose typical fiber is diffeomorphic to X, say) and $\mathcal{F}$ with its space of sections. First, note that the tangent space at ϕ to the mapping space $\mathcal{F}=\mathrm{Map}(M,X)$ is

$$T_\phi\mathcal{F}\cong\Omega^0(M;\phi^*TX). \tag{2.3}$$

To see this, consider a path ϕ_u, $-\epsilon < u < \epsilon$, in $\mathcal{F}$ such that ϕ_0 is the given map ϕ. Then the derivative in u at $u = 0$ is naturally a section of the pullback tangent bundle $\Omega^0(M; \phi^*TX)$. A crucial piece of structure is the evaluation map

$$\begin{aligned} e\colon \mathcal{F} \times M &\longrightarrow X \\ (\phi, m) &\longmapsto \phi(m). \end{aligned} \tag{2.4}$$

What distinguishes function spaces from arbitrary infinite-dimensional manifolds is this evaluation map. We obtain real-valued functions on $\mathcal{F} \times M$ by composing (2.4) with a real-valued function on X.

We express lagrangian field theory in terms of differential forms on $\mathcal{F} \times M$, except that we twist by the orientation bundle to use densities on M instead of forms. So we work in a double complex $\Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$ whose homogeneous subspace $\Omega^{p,|-q|}(\mathcal{F} \times M)$ is the space of p-forms on $\mathcal{F}$ with values in the space of twisted $(n-q)$-forms on M. Let δ be the exterior derivative on $\mathcal{F}$, d the exterior derivative[11] of forms on M, and $D = \delta + d$ the total exterior derivative. We have

$$D^2 = d^2 = \delta^2 = 0, \qquad d\delta = -\delta d. \tag{2.6}$$

We will use the following picture to depict elements in the double complex:

$$\begin{array}{c|ccc}
 & 0 & 1 & \cdots \\ \hline
|0| & & & \\
 & d\uparrow & & \\
|-1| & & \underset{\delta}{\longrightarrow} & \\
\vdots & & & \\
\end{array} \qquad \mathcal{F}$$

$$M$$

For example, a lagrangian (1.27), which to each point of $\mathcal{F}$ attaches a density on M, is an element in $\Omega^{0,|0|}(\mathcal{F} \times M)$.

There is an important subcomplex $\Omega^{\bullet,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M)$ of *local* forms. The value of a form $\alpha \in \Omega^{p,|\bullet|}(\mathcal{F} \times M)$ at a point $m \in M$ and a field $\phi \in \mathcal{F}$ on tangent vectors $\xi_1, \dots, \xi_p$ to $\mathcal{F}$ is a twisted form at m. The form α is *local* if this twisted form depends only on the k-jet at m of ϕ and the ξ_i. For example, let ζ_1, ζ_2 be fixed vector fields on M and consider $X = \mathbb{R}$, so that $\mathcal{F}$ is the space of real-valued functions on M. Then the $(0, |0|)$-form

$$L = \zeta_1\zeta_2\phi(m) \cdot \phi(m)\ |d^n x| \tag{2.7}$$

is local: $L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$. We explain the notation in detail. First, L is the product of a function $\ell = \zeta_1\zeta_2\phi(m) \cdot \phi(m)$ with the standard density $|d^n x|$. Now

[11] Our sign convention is that for $\alpha \in \Omega^p(\mathcal{F})$ and $\beta \in \Omega^{|\bullet|}(M)$,

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d\beta. \tag{2.5}$$

the function ℓ may be written in terms of the evaluation function e (2.4), which is local: $\ell = \zeta_1\zeta_2 e \cdot e$. Finite derivatives of local functions are local, as are products of local functions, so ℓ is local as claimed. In terms of the definition, at a point $m \in M$, the form L depends on the 2-jet of ϕ. On the other hand, if $m_0 \in M$ is fixed, then the function $\ell = \phi(m) \cdot \phi(m_0)$ is not local: its value at (ϕ, m) depends on the value of ϕ at m_0. Expressions similar to (2.7) are what appear in field theories; at first it is useful to rewrite them in terms of the evaluation map.

There is one nontrivial mathematical theorem which we use in our formal development. This theorem, due to F. Takens [**T**], asserts the vanishing of certain cohomology groups in the double complex of *local* forms with respect to the vertical differential d.

Theorem 2.8 (Takens). *For $p > 0$ the complex $\left(\Omega^{p,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M), d\right)$ of local differential forms is exact except in the top degree $|\bullet| = 0$.*

A proof of this result may be found in [**DF**].

Basic notions

We study *fields* on a smooth finite-dimensional manifold M, which we usually think of as spacetime. For the most part we will specialize to $M = M^n$ Minkowski spacetime, but for the general discussion it can be arbitrary. Often the spacetime M is equipped with topological or geometric structures—an orientation, spin structure, metric, etc.—which are fixed throughout. Attached to M is a space of fields $\mathcal{F}$ which are the variables for the field theory.

What is a field? We will not try to pin down a definition which works in every situation. Roughly a field is some kind of "function" on M, and many types of fields are in fact sections of some fiber bundle over M. Thus a *scalar field* is a map $\phi\colon M \to X$ for some manifold X. We also have tensor fields, spinor fields, metrics, and so on. The only real requirement is that a field be *local* in the sense that it can be cut and pasted. So, as an example of fields somewhat different than those we will encounter, fix a finite group G and consider the *category* of principal G-bundles over M. (This is more interesting when M has nontrivial fundamental group.) Since these bundles (Galois covering spaces) can be cut and pasted, they are valid fields.[12] Connections on principal bundles for a fixed structure group also form a category—morphisms are equivalences of connections—and these are the basic fields of gauge theory for arbitrary gauge groups. In setting up the general theory we will treat fields as maps $\phi\colon M \to X$ to a fixed manifold X, though the reader can easily generalize to fields which are sections of a fiber bundle $E \to M$ (and to gauge fields as well). This covers most of the examples in these lectures.

In classical mechanics one meets systems with constraints. For example a particle moving in $\mathbb{E}^3$ may be constrained to lie on the surface of a sphere. Such *holonomic* constraints are easily dealt with in our setup: simply change the manifold X to accommodate the constraint. On the other hand, our formalism does not apply as presented to nonholonomic constraints. In supersymmetric gauge theories we meet such constraints (in the superspacetime formulation).

[12] That is, the objects in the category are the fields. All morphisms in the category are invertible—the category is a *groupoid*—and ultimately one is interested in the set of equivalence classes.

Fix a spacetime M and a space of fields $\mathcal{F}$. A classical field theory is specified by two pieces of data: a *lagrangian density*

$$L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M) \tag{2.9}$$

and a *variational 1-form*

$$\gamma \in \Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M). \tag{2.10}$$

These fit into the diagram:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline |0| & L & \\ & & \\ |-1| & & \gamma \end{array} \quad \mathcal{F}$$

$$M$$

It is natural to define the *total lagrangian* as the sum of the lagrangian and the variational 1-form:

$$\mathcal{L} = L + \gamma \in \Omega^{|0|}(\mathcal{F} \times M). \tag{2.11}$$

There is a compatibility condition for L and γ, which we specify shortly, and we will see that in many cases of interest L determines γ. But first we recast the point particle of Lecture 1 in this language.

Example 2.12 (point particle). Take "spacetime" to be simply time $M = M^1$; the space of fields to be the mapping space $\mathcal{F} = \mathrm{Map}(M, X)$ of paths in a fixed Riemannian manifold X equipped with a potential energy function $V\colon X \to \mathbb{R}$; the lagrangian to be the particle lagrangian (1.28), now written as an element of $\Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$:

$$L = \left[\frac{m}{2}|\dot{x}|^2 - V(x)\right] |dt|; \tag{2.13}$$

and the variational 1-form to be the expression (1.33), now written as an element of $\Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$:

$$\begin{aligned} \gamma &= \langle \dot{x}, \delta x\rangle\ \partial_t \otimes |dt| \\ &= \langle \iota(\partial_t)dx, \delta x\rangle\ \partial_t \otimes |dt|. \end{aligned} \tag{2.14}$$

In these expressions $x\colon \mathcal{F} \times M \to X$ is the evaluation map, so that dx and δx are one-forms on $\mathcal{F} \times M$ with values in the pullback of TX via x. The reader may prefer to consider first $X = \mathbb{E}^d$, in which case there are simplifications: the covariant derivative is the ordinary derivative and $dx, \delta x$ are 1-forms with values in the constant vector space $\mathbb{R}^d$. Nothing essential to our discussion is lost in this specialization. The reader should check that (2.13) is indeed local and agrees with (1.28). In both expressions we use the standard density $|dt|$ on M^1 determined by the Riemannian metric, and we have chosen a unit length translation-invariant vector field ∂_t on M^1; it is determined up to sign. Then $\dot{x} = \partial_t x$ changes sign if we change ∂_t for its opposite. Since ∂_t appears twice in both expressions, they are invariant under this change, hence well-defined.

Though we haven't developed the theory yet, as an illustration of calculus on function spaces let us compute the differential $D\mathcal{L}$ of the total lagrangian $\mathcal{L} = L+\gamma$. First, we consider the component in degree $(1, |0|)$:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma. \tag{2.15}$$

It is convenient for computation to (finally!) fix an orientation of M^1, and using it identify the density $|dt|$ with the 1-form dt. Since we have already expressed everything without the choice of orientation, the results of the computation are necessarily independent of this choice. The orientation allows us to identify

$$dx = \dot{x}\, dt = \dot{x}\, |dt|. \tag{2.16}$$

Then the variation of the lagrangian is

$$\begin{aligned} \delta L &= m\langle \delta_\nabla \dot{x}, \dot{x}\rangle \wedge |dt| - dV \circ \delta x \wedge |dt| \\ &= m\langle \delta_\nabla dx, \dot{x}\rangle - \langle \operatorname{grad} V, \delta x\rangle \wedge dt. \end{aligned} \tag{2.17}$$

Here, as explained in Lecture 1 (see the text preceding (1.14)), the first derivative dx is a section of x^*TX, so the second derivative uses the pullback of the Levi-Civita connection. Using (2.16) we write $\gamma = m\langle \dot{x}, \delta x\rangle$, and so

$$\begin{aligned} d\gamma &= m\langle d_\nabla \dot{x} \wedge \delta x\rangle + m\langle \dot{x}, d_\nabla \delta x\rangle \\ &= -m\langle \nabla_{\dot{x}} \dot{x}, \delta x\rangle \wedge dt + m\langle \dot{x}, d_\nabla \delta x\rangle. \end{aligned} \tag{2.5}$$

The minus sign in the second line comes from commuting the 1-form dt past the 1-form δx. Since the Levi-Civita connection has no torsion, it follows from (1.14) (applied to the total differential D), and the fact that d and δ anticommute, that $\delta_\nabla dx = -d_\nabla \delta x$, so there is a cancellation in the sum

$$\begin{aligned} (D\mathcal{L})^{1,|0|} &= \delta L + d\gamma \\ &= -m\langle \nabla_{\dot{x}} \dot{x} + \operatorname{grad} V, \delta x\rangle \wedge dt. \end{aligned} \tag{2.18}$$

Newton's equation (1.32) is embedded in this formula. We also compute

$$\begin{aligned} (D\mathcal{L})^{2,|-1|} &= \delta\gamma \\ &= m\langle \delta_\nabla \dot{x} \wedge \delta x\rangle, \end{aligned} \tag{2.19}$$

where we use the vanishing of torsion again: $\delta_\nabla \delta x = 0$. This last expression is a 2-form on $\mathcal{F}$; it restricts to the symplectic form on the space $\mathcal{M}$ of solutions to Newton's law.

The formulas simplify for $X = \mathbb{E}^1$. Then we write

$$L = \frac{m}{2}\dot{x}\,dx - V(x)\,dt,$$

and so

$$\begin{aligned}\delta L &= \quad m\dot{x}\,\delta dx - V'(x)\,\delta x \wedge dt \\ &= -m\dot{x}\,d\delta x - V'(x)\,\delta x \wedge dt.\end{aligned}$$

We then set

$$\gamma = m\dot{x}\,\delta x,$$

so that

$$d\gamma = m\ddot{x}\,dt \wedge \delta x + m\dot{x}\,d\delta x.$$

Newton's law is contained in the equation

$$\delta L + d\gamma = -\left[m\ddot{x} + V'(x)\right]\,\delta x \wedge dt.$$

The reader will recognize the computation of $(D\mathcal{L})^{1,|0|}$ as a version of the computation (1.31) in Lecture 1. It encodes the integration by parts we did there. This is precisely the role of the variational 1-form γ.

For a general lagrangian $\mathcal{L} = L + \gamma$ we encode the integration by parts in the following relationship between L and γ.

Definition 2.20. A *lagrangian field theory* on a spacetime M with fields $\mathcal{F}$ is a lagrangian density $L \in \Omega^{0,|0|}_{\text{loc}}(\mathcal{F} \times M)$ and a variational 1-form $\gamma \in \Omega^{1,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ such that if $\mathcal{L} = L + \gamma$ is the total lagrangian, then $(D\mathcal{L})^{1,|0|}$ is *linear over functions* on M.

To explain "linear over functions," note first that $T_\phi\mathcal{F} \cong \Omega^0(M; \phi^*TX)$ (see (2.3)) is a module over $\Omega^0(M)$, the algebra of smooth functions on M. A form $\beta \in \Omega^{1,|\bullet|}(\mathcal{F} \times M)$ is linear over functions if for all $(\phi, m) \in \mathcal{F} \times M$ we have

$$\beta_{(\phi,m)}(f\hat{\xi}) = f(m)\beta_{(\phi,m)}(\hat{\xi}), \qquad f \in \Omega^0_M, \quad \hat{\xi} \in T_\phi\mathcal{F}. \tag{2.21}$$

The reader can verify that (2.18) satisfies this condition. More plainly: δx is linear over functions, whereas $\delta_\nabla dx$ is not. The variational 1-form γ is chosen precisely to cancel all terms where $\delta_\nabla dx$ appears. This is what the usual integration by parts accomplishes; it isolates δx in the variation of the action.

A few comments about the variational 1-form γ:

- If the lagrangian L depends only on the 1-jet of the fields, then there is a canonical choice for γ which is characterized as being linear over functions. We always choose this γ. For such systems, then, we need only specify the lagrangian. This is the case for all examples we will meet in this course.
- If the lagrangian depends on higher derivatives, then there is more than one choice for γ, but for local lagrangians the difference between any two choices is d-exact (by Takens' Theorem 2.8). In this sense the choice of γ is not crucial.
- In many mechanics texts you will find this canonical γ written as '$p_i dq^i$'.

Given a classical field theory—a total lagrangian $\mathcal{L} = L + \gamma$ such that $(D\mathcal{L})^{1,|0|}$ is linear over functions—we define the *space of classical solutions* $\mathcal{M} \subset \mathcal{F}$ to be the space of $\phi \in \mathcal{F}$ such that the restriction of $(D\mathcal{L})^{1,|0|}$ to $\{\phi\} \times M$ vanishes:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.22}$$

This definition is motivated by the usual integration by parts manipulation in the calculus of variations. For the point particle we read off Newton's law (1.32) directly from (2.18). Notice in general that since $(D\mathcal{L})^{1,|0|}$ lies in the subcomplex of local forms, the Euler-Lagrange equations are local. Physicists term fields in $\mathcal{M} \subset \mathcal{F}$ *on-shell*, whereas the complement of $\mathcal{M}$—or sometimes all of $\mathcal{F}$—is referred to as *off-shell*.

For field theory on a general manifold M there is no Hamiltonian interpretation. The Hamiltonian story requires that we write spacetime M as time$\times$space. In that case it is appropriate to call $\mathcal{M}$ the *state space* or *phase space*.

Definition 2.23. Let $\mathcal{L} = L + \gamma$ define a lagrangian field theory. Then the associated *local symplectic form* is

$$\omega := \delta\gamma \in \Omega^{2,|-1|}_{\text{loc}}(\mathcal{F} \times M). \tag{2.24}$$

On-shell we have

$$\omega = D\mathcal{L} \qquad \text{on } \mathcal{M} \times M, \tag{2.25}$$

and so

$$D\omega = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.26}$$

We represent the on-shell data in the diagram:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \to & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \to & \omega \end{array} \qquad \mathcal{M}$$

$$M$$

In mechanics ($n = 1$) for each time t the form $\omega(t)$ is a closed 2-form on the space of classical solutions $\mathcal{M}$, since $\delta\omega = 0$. It is independent of t, since $d\omega = 0$. A theory is *nondegenerate* if ω is a symplectic form. In classical mechanics texts the nondegeneracy amounts to the nondegeneracy of the Hessian matrix with entries $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$; the symplectic form ω is usually denoted '$dp_i \wedge dq^i$' in appropriate coordinates. For field theories in higher dimensions, we obtain a symplectic structure on $\mathcal{M}$ only in the Hamiltonian situation where spacetime $=$ time $\times$ space. We discuss that later in this lecture.

The symplectic form for the point particle was computed in (2.19):

$$\omega = m\langle \delta_{\nabla}\dot{x} \wedge \delta x \rangle. \tag{2.27}$$

We conclude this general discussion with an example which involves fields. We write it quite explicitly in 2 dimensions; the reader can easily generalize to higher dimensions.

Example 2.28 (scalar field). Let $M = M^2$ denote two-dimensional Minkowski spacetime with coordinates x^0, x^1. Here x^0 is the speed of light times a standard time coordinate. For convenience fix the orientation $\{x^0, x^1\}$ and use it to identify twisted forms with forms. Note[13] that $*dx^0 = dx^1$ and $*dx^1 = dx^0$. Let $\mathcal{F} = \{\phi\colon M^{1,1} \to \mathbb{R}\}$ be the set of real scalar fields. We use the notation $\partial_\mu = \partial/\partial x^\mu$. The free (massless) lagrangian is

$$\begin{aligned} L &= \frac{1}{2} d\phi \wedge *d\phi \\ &= \frac{1}{2}|d\phi|^2\, dx^0 \wedge dx^1 \\ &= \frac{1}{2}\left\{(\partial_0\phi)^2 - (\partial_1\phi)^2\right\} dx^0 \wedge dx^1. \end{aligned} \tag{2.29}$$

From this we derive

$$\gamma = \partial_0\phi\; \delta\phi \wedge dx^1 + \partial_1\phi\; \delta\phi \wedge dx^0 \tag{2.30}$$

$$\omega = \partial_0\delta\phi \wedge \delta\phi \wedge dx^1 + \partial_1\delta\phi \wedge \delta\phi \wedge dx^0 \tag{2.31}$$

and the equation of motion

$$\partial_0^2\phi - \partial_1^2\phi = 0. \tag{2.32}$$

It is important to identify the equation of motion in this scalar field theory as a second order *wave equation*. Just as with particle motion, we can solve the wave equation—at least locally—by specifying an initial value for the field and its time derivative. The signature of the Lorentz metric manifests itself directly in this equation.

Symmetries and Noether's theorem

In lagrangian field theory there is a local version of the relationship one has in symplectic geometry between infinitesimal symmetries and conserved charges (see the text preceding (1.23)). We distinguish two types of symmetries: *manifest* and *nonmanifest.* To each symmetry we attach a local *Noether current*, and on-shell it satisfies a local version of the conservation law (1.25) in symplectic geometry. In mechanics the Noether current *is* the conserved (Noether) charge, but for general field theories we must work in the Hamiltonian framework and integrate over space to construct a conserved charge. What is important here is that we can write explicit and computable formulas for the conserved currents attached to infinitesimal symmetries.

We continue with our general setup: spacetime M, fields $\mathcal{F} = \mathrm{Map}(M, X)$, total lagrangian $\mathcal{L} = L + \gamma$, local symplectic form ω, and space of classical solutions $\mathcal{M}$. As a preliminary we extend the notion of *locality* to vector fields: a vector field $\hat{\xi}$ on $\mathcal{F}$ is *local* if for some k the value of $\hat{\xi}_\phi \in \Omega^0(M, \phi^*TX)$ at $m \in M$ depends only on the k-jet of ϕ at m. A vector field ξ on $\mathcal{F} \times M$ is said to be *decomposable and local* if it is the sum of a local vector field $\hat{\xi}$ on $\mathcal{F}$ and of a vector field η on M. Such vector fields preserve the bigrading on differential forms, and the Lie derivative by such vector fields commutes with both d and δ.

[13]See the problem sets for information about the Hodge $*$ operator.

Definition 2.33. (i) A local vector field $\hat{\xi}$ on $\mathcal{F}$ is a *generalized infinitesimal symmetry* of L if there exists

$$\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M) \tag{2.34}$$

such that

$$\mathrm{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M \tag{2.35}$$

(ii) A decomposable and local vector field ξ on $\mathcal{F} \times M$ is a *manifest infinitesimal symmetry* if

$$\mathrm{Lie}(\xi)\mathcal{L} = 0 \qquad \text{on } \mathcal{F} \times M. \tag{2.36}$$

The definition of a manifest symmetry is the usual functorial notion—a symmetry of a mathematical structure is an automorphism which preserves all the data. A generalized symmetry is *nonmanifest* if we must choose $\alpha_{\hat{\xi}}$ in (2.3) nonzero; it only preserves the lagrangian up to an exact term. Note that there is an indeterminacy for nonmanifest symmetries: we can add any closed local form to $\alpha_{\hat{\xi}}$.

Definition 2.33 encodes the notion of an infinitesimal symmetry; there is also a definition of a symmetry such that the generator of a one-parameter group of symmetries is an infinitesimal symmetry. We leave the precise formulation to the reader.

We illustrate these definitions with some examples from mechanics.

Example 2.37 (time translation: manifest). Consider the point particle as defined in Example 2.12. Time translation is the action of[14] $\mathbb{R}$ on M^1: translation by $s \in \mathbb{R}$ is $T_s(t) = t + s$. This induces an action of $\mathbb{R}$ on $\mathrm{Map}(M^1, X) \times M^1$: $T_s\bigl(x, t\bigr) = \bigl(x \circ T_s^{-1}, T_s(t)\bigr)$. Differentiating with respect to s we obtain the desired vector field ξ on $\mathcal{F} \times M$. We specify it by giving its action on the "coordinate functions", which for function space is the evaluation map:

$$\begin{aligned} \iota(\xi)dt &= 1 \\ \iota(\xi)\delta x &= -\dot{x}. \end{aligned} \tag{2.38}$$

Note the minus sign here, which easily causes confusion. The vector field ξ is local, since (2.38) only depends on the 1-jet of x at t. It is clear that T_t is a preserves the total lagrangian, since T_t preserves the evaluation map and density $|dt|$, and the lagrangian and variational 1-form are expressed in terms of these. It follows by differentiation that (2.36) is satisfied.

Example 2.39 (time translation: nonmanifest). We take $\hat{\xi}$ to be the component of the manifest infinitesimal symmetry ξ along the space of fields $\mathcal{F}$. (Since ξ is decomposable, this makes sense.) In other words,

$$\iota(\hat{\xi})\delta x = -\dot{x}. \tag{2.40}$$

[14] As in previous computations we orient time.

Then $\hat{\xi}$ is a nonmanifest symmetry of the lagrangian (2.28) with

$$\alpha_{\hat{\xi}} = -\big(\frac{m}{2}|\dot{x}|^2 - V(x)\big) + C. \tag{2.41}$$

We put in a constant C to emphasize the indeterminacy which exists for nonmanifest symmetries. Equation (2.35) follows from the fact that ξ is a manifest symmetry. For writing $\xi = \hat{\xi} + \eta$, where $\eta = \partial_t$, we have $\operatorname{Lie}(\xi)L = 0$, from which

$$\operatorname{Lie}(\hat{\xi})L = -\operatorname{Lie}(\eta)L = -d\iota(\eta)L = -d\big(\frac{m}{2}|\dot{x}|^2 - V(x)\big). \tag{2.42}$$

A few remarks about the definition:

- As we mentioned above, there is ambiguity in the definition of $\alpha_{\hat{\xi}}$ for a nonmanifest symmetry—we can add any d-exact form.
- In the definition of a generalized infinitesimal symmetry we have said nothing about $\operatorname{Lie}(\hat{\xi})\gamma$. In fact, an argument based on Takens' theorem and its close cousins implies that for some $\beta_{\hat{\xi}} \in \Omega^{1,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$, the identity

$$\operatorname{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M \tag{2.43}$$

  holds on shell. We use this identity below to see that the Noether current is a local version of the conserved charge in symplectic geometry.
- In the manifest case it is awkward to have to compute that $\operatorname{Lie}(\xi)\gamma = 0$. In good cases this follows from $\operatorname{Lie}(\xi)L = 0$, which is much easier to check. Namely, set $E = M \times X$ and suppose each element in a one-parameter group of manifest symmetries is induced from a bundle automorphism of $E \to M$ which covers a diffeomorphism of M. (This is the case in Example 2.37.) If the lagrangian L depends only on the 1-jet of the fields, and if it is preserved by the diffeomorphism of $\mathcal{F} \times M$, then it follows that γ is also preserved and we have a manifest symmetry. We use this implicitly in what follows.

For a generalized infinitesimal symmetry, we summarize (2.35) and (2.43) in an off-shell diagram and an on-shell diagram:

$$\begin{array}{c|c} & 0 \quad \mathcal{F} \\ \hline |0| & \operatorname{Lie}(\hat{\xi})L \\ & \uparrow \\ |-1| & \alpha_{\hat{\xi}} \end{array} \qquad\qquad \begin{array}{c|ccc} & 0 & & 1 \quad \mathcal{M} \\ \hline |0| & \operatorname{Lie}(\hat{\xi})L & & \\ & \uparrow & & \\ |-1| & \alpha_{\hat{\xi}} & \rightarrow & \operatorname{Lie}(\hat{\xi})\gamma \\ & & & \uparrow \\ |-2| & & & \beta_{\hat{\xi}} \end{array}$$

$$M \qquad\qquad\qquad\qquad M$$

It is easy to check that for a generalized infinitesimal symmetry $\hat{\xi}$, we have

$$\operatorname{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}}) \qquad \text{in } \Omega^{2,|-1|}(\mathcal{M} \times M). \tag{2.44}$$

This is a local version of the condition to be a Hamiltonian vector field. (Note that in mechanics—dim $M = 1$—the right hand side vanishes.j) For a manifest infinitesimal symmetry ξ, the local symplectic form is preserved on the nose: $\mathrm{Lie}(\xi)\omega = 0$.

We are ready to define the Noether current associated to an infinitesimal symmetry. Consider first the manifest case. Suppose $\xi = \hat{\xi} + \eta$ is a decomposable and local vector field on $\mathcal{F} \times M$ which is a manifest infinitesimal symmetry: $\mathrm{Lie}(\xi)\mathcal{L} = 0$. Define the *Noether current*

$$j_\xi := \big[\iota(\xi)\mathcal{L}\big]^{0,|-1|} = \iota(\hat{\xi})\gamma + \iota(\eta)L \qquad \text{(manifest symmetry).} \tag{2.45}$$

Then one can verify that on-shell we have

$$dj_\xi = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.46}$$

$$\delta j_\xi = -\iota(\hat{\xi})\omega - d\iota(\eta)\gamma \qquad \text{on } \mathcal{M} \times M, \tag{2.47}$$

Equation (2.46) is the assertion that the Noether current j_ξ is *conserved*. Equation (2.47) is the local version of the correspondence (1.19) in symplectic geometry between an infinitesimal symmetry and its associated charge. The exact term disappears upon integration over a slice at fixed time.

In the nonmanifest case there are somewhat different formulae. Thus suppose $\hat{\xi}$ is a vector field on $\mathcal{F}$ which is a generalized infinitesimal symmetry with associated $\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$. Then the Noether current is defined as

$$j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} \qquad \text{(nonmanifest symmetry).} \tag{2.48}$$

It is easy to verify from (2.35), (2.43), and the Cartan formula that in this case (2.46) and (2.47) are replaced by

$$dj_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.49}$$

$$\delta j_{\hat{\xi}} = -\iota(\hat{\xi})\omega + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M, \tag{2.50}$$

Example 2.51 (time translation: manifest). Continuing Example 2.37, we compute the Noether current from (2.45) using (2.38):

$$\begin{aligned} j_\xi &= \iota(\xi)\mathcal{L} \\ &= \iota(\xi)\Big[\big\{\frac{m}{2}|\dot{x}|^2 - V(x)\big\}\,|dt| \;+\; m\langle \dot{x}, \delta x\rangle\Big] \\ &= \frac{m}{2}|\dot{x}|^2 - V(x) - m|\dot{x}|^2 \\ &= -\big(\frac{m}{2}|\dot{x}|^2 + V(x)\big). \end{aligned} \tag{2.52}$$

Of course, this is *minus* the total energy, or Hamiltonian, of the point particle. Note that the sign agrees with our convention in Lecture 1: the Hamiltonian is the conserved charge associated to the negative of infinitesimal time translation.

Example 2.53 (time translation: nonmanifest). Continuing Example 2.39 we define the Noether current $j_{\hat{\xi}}$ associated to (2.40) and (2.41) using (2.48). A short computation gives

$$j_{\hat{\xi}} = -\Big(\frac{m}{2}|\dot{x}|^2 + V(x) + C\Big). \tag{2.54}$$

The energy is defined only up to a constant C in this picture.

Example 2.55 (linear momentum). Continuing with the point particle, we consider the special case $X = \mathbb{E}^d$ and translation in the j^{th} coordinate direction. This is an isometry of $\mathbb{E}^d$, and if it preserves the potential V, then it is manifestly a symmetry of L, as L depends only on the target metric and V. The corresponding infinitesimal symmetry is a vector field ξ_j defined by:

$$\begin{aligned} \iota(\xi_j)dt &= 0 \\ \iota(\xi_j)\delta x^i &= \delta^i_j, \end{aligned} \tag{2.56}$$

where δ^i_j has its usual meaning. One can check directly that this is a manifest symmetry in case $\partial_j V = 0$. The associated Noether current is a component of the *linear momentum*:

$$j_{\xi_j} = m\dot{x}^j. \tag{2.57}$$

Finally, we give an example from field theory.

Example 2.58 (energy for a scalar field). We continue the notation of Example 2.28. Consider infinitesimal translation in the x^0 (time) direction. (We work in units where $c = 1$.) It defines a vector field ξ on $\mathcal{F} \times M$ by

$$\begin{aligned} \iota(\xi)\delta\phi &= -\partial_0\phi \\ \iota(\xi)dx^0 &= 1 \\ \iota(\xi)dx^1 &= 0. \end{aligned} \tag{2.59}$$

Then a routine computation shows that ξ is a *manifest* symmetry: $\mathrm{Lie}(\xi)L = 0$ on $\mathcal{F} \times M$. The associated Noether current is

$$j_\xi = -\frac{1}{2}\left\{(\partial_0\phi)^2 + (\partial_1\phi)^2\right\} dx^1 - \{\partial_0\phi\,\partial_1\phi\}\, dx^0. \tag{2.60}$$

The reader should check that this current is conserved, i.e., $dj_\xi = 0$ on-shell. The equations of motion must be used. The coefficient of dx^1 is minus the energy density of the field—the time derivative is the kinetic energy and the spatial derivative the potential energy. The global energy is the integral of $-j_\xi$ over a time-slice $x^0 =$ constant, and then the coefficient of dx^0 drops out. It is there, so to speak, to guarantee that $dj_\xi = 0$.

We can also regard infinitesimal time translation as a *nonmanifest* symmetry $\hat{\xi}$ by letting it operate only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta\phi &= -\partial_0\phi \\ \iota(\hat{\xi})dx^0 &= 0 \\ \iota(\hat{\xi})dx^1 &= 0. \end{aligned} \tag{2.61}$$

Then we compute

$$\operatorname{Lie}(\hat{\xi})L = d\alpha \qquad \text{on } \mathcal{F}\times M, \tag{2.62}$$

$$\operatorname{Lie}(\hat{\xi})\gamma = \delta\alpha + d\beta \qquad \text{on } \mathcal{M}\times M, \tag{2.63}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2\right] dx^1 \in \Omega^{0,|-1|}_{\text{loc}},\\ \beta &= \partial_1\phi\,\delta\phi \qquad\qquad\qquad\quad\; \in \Omega^{1,|-2|}_{\text{loc}}. \end{aligned} \tag{2.64}$$

We leave the reader to compute the Noether current from this point of view.

Hamiltonian structures

We can study lagrangian field theory on any spacetime M, and indeed this is common in field theory, string theory, and beyond. For example, we can consider electromagnetism in a nontrivial "gravitational background", that is, on a spacetime (other than Minkowski spacetime) which satisfies Einstein's equations. In quantum field theory one often "Wick rotates" the theory on Minkowski spacetime to a theory on Euclidean space, and then generalizes M to be any Riemannian manifold. Perturbative string theory is defined in terms of correlation functions on Riemann surfaces (with Riemannian metric). In these cases there is no interpretation in terms of classical physics; the quantities of interest are the correlation functions of the quantum theory. Nonetheless, many of the concepts we discussed carry over. Of course, in differential geometry we use the calculus of variations in settings which involve no physics. All this is to emphasize that we recover a classical system—symplectic manifold of states with a distinguished one-parameter group—only in the following case.

Definition 2.65. A *Hamiltonian structure* on a lagrangian field theory is an isometry $M \cong M^1 \times N$ of spacetime to time$\times$space, where N is a Riemannian manifold and $M^1\times N$ has the Lorentz metric $c^2 ds^2_{M^1} - ds^2_N$. Also, if fields are sections of a fiber bundle $E\to M$, then we require an isomorphism of E with a fiber bundle over $M^1\times N$ which is pulled back from a fiber bundle $E_N \to N$.

This last condition means that fields take their values in a manifold which is independent of time, so it makes sense to compare fields at different times. In fact, we identify the space of fields $\mathcal{F}$ as the space of paths in a space $\mathcal{F}_N$ of fields on N. ($\mathcal{F}_N$ is the space of sections of $E_N \to N$, or in the case of a product $\mathcal{F}_N = \operatorname{Map}(N,X)$.) In that sense the Hamiltonian picture is the study of a particle moving in the infinite dimensional space $\mathcal{F}_N$.

The basic idea is to take local on-shell quantities of degree $(\bullet, |-1|)$ in the lagrangian theory and integrate over a time slice $\{t\}\times N$ to obtain global quantities on $\mathcal{M}$. Note that the integration is vacuous in mechanics, which is the case $N = \text{pt}$.

Definition 2.66. In a lagrangian field theory with Hamiltonian structure, the *symplectic form* on the phase space $\mathcal{M}$ is

$$\Omega = \int_{\{t\}\times N} \omega \;\in\; \Omega^2(\mathcal{M}). \tag{2.67}$$

Typically N is noncompact and so to ensure convergence we only evaluate Ω on tangent vectors to $\mathcal{M}$ with compact support in spatial directions, or at least with sufficient decay at spatial infinity. The hyperbolicity of the classical equations of motion implies finite propagation speed of the classical solutions, and so the decay conditions are uniform in time. From (2.26) and Stokes' theorem, it follows that the right hand side of (2.58) is independent of $t \in M^1$ and also Ω is a closed 2-form on $\mathcal{M}$. In good cases this form is nondegenerate.

Similarly, if $j \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a conserved current—that is, $dj = 0$—then the associated *charge* Q_j is

$$Q_j = \int_{\{t\}\times N} j; \tag{2.68}$$

it is a function on the space of fields $\mathcal{F}$. Noether currents are conserved currents by (2.46) and (2.49); in that case the associated charge is called a *Noether charge*. If we restrict Q_j to $\mathcal{M}$, then since $dj = 0$ the right hand side is independent of t. This is a global conservation law. Local conservation laws are obtained by considering a domain $U \subset N$. For simplicity assume the closure of U is compact with smooth boundary ∂U. Let

$$q_t = \int_{\{t\}\times U} j \tag{2.69}$$

be the total charge contained in U at time t. Write

$$j = dt \wedge j_1 + j_2, \tag{2.70}$$

where j_1, j_2 do not involve dt. Stokes' theorem applied to integration over the fibers of the projection $M^1 \times U \to M^1$ implies

$$\frac{dq_t}{dt} + \int_{\{t\}\times \partial U} j_1 = 0. \tag{2.71}$$

This says that the rate of change of the total charge in U is minus the flux through the boundary.

Exercises

1. Recall the $*$ operator from the previous problem set. Now we define it for V is a finite-dimensional real vector space with a nondegenerate bilinear form, but no choice of orientation. First, define a real line $|\operatorname{Det} V^*|$ of densities on V. (Hint: A choice of orientation gives an isomorphism $|\operatorname{Det} V^*| \cong \operatorname{Det} V$.) Then define a $|-q|$-form on V to be an element of the vector space $\bigwedge^q V \otimes |\operatorname{Det} V^*|$. Finally, construct a $*$ operator

$$*\colon \textstyle\bigwedge^q V^* \longrightarrow \bigwedge^q V \otimes |\operatorname{Det} V^*|.$$

Compute formulas in a basis in some low-dimensional examples.

2. (a) (Stokes' theorem) Recall that for a single manifold X, assumed oriented and compact with boundary, that Stokes' theorem asserts that for any differential form $\alpha \in \Omega^\bullet(X)$ we have

$$\int_X d\alpha = \int_{\partial X} \alpha.$$

Now suppose we have a family of manifolds with boundary parametrized by a smooth manifold T, i.e., a fiber bundle $\pi\colon \mathcal{X} \to T$. Here $\mathcal{X}$ is a manifold with boundary, but T is a manifold without boundary. Assume first that the relative tangent bundle is oriented. Then integration along the fibers is defined as a map

$$\int_{\mathcal{X}/T} : \Omega^q(\mathcal{X}) \longrightarrow \Omega^{q-n}(T),$$

assuming the fibers to have dimension n. To fix the signs, we remark that on a product family $\mathcal{X} = T \times X$, for a form $\alpha = \alpha_T \wedge \alpha_X$ we set $\int_{\mathcal{X}/T} \alpha = \left\{\int_X \alpha_X\right\} \cdot \alpha_T$. If you are not familiar with this map, then construct it. Let $\partial\mathcal{X} \to T$ be the family of boundaries of the fibers. Then verify Stokes' theorem, at least in the case of a product family $\mathcal{X} = T \times X$: For $\alpha \in \Omega^q(\mathcal{X})$,

$$d \int_{\mathcal{X}/T} \alpha = \int_{\mathcal{X}/T} d\alpha + (-1)^{q-n} \int_{\partial\mathcal{X}/T} \alpha.$$

(b) Extend to the case when the tangent bundle along the fibers is not oriented. Then integration maps twisted forms on $\mathcal{X}$ to untwisted forms on T. Formulate Stokes' theorem. Be careful of the signs!

(c) Verify (2.71). You may choose not to use the generalities explained in the previous parts of this exercise.

3. A system of harmonic oscillators is described as a particle moving on a finite-dimensional real inner product space X with potential $V(x) = \frac{1}{2}|x|^2$. Let A be a skew-symmetric endomorphism of X and B a symmetric endomorphism. Consider the vector field on $\mathcal{F}$ defined by

$$\iota(\hat{\xi})\delta x = Ax + B\dot{x}$$

Verify that $\hat{\xi}$ is a nonmanifest infinitesimal symmetry. What is the corresponding Noether current? Can you identify these symmetries and currents physically? Are the $\hat{\xi}$ closed under Lie bracket on $\mathcal{F}$? What about on $\mathcal{M}$?

4. (a) Consider a *complex scalar field* on Minkowski spacetime. This is a map $\Phi\colon M^n \to \mathbb{C}$ with lagrangian

$$L = \left\{|d\Phi|^2 - m^2|\Phi|^2\right\} |d^n x|$$

Here m is a real parameter, and $|\cdot|$ is the usual norm of complex numbers. Compute the variational 1-form γ and the equations of motion. Verify that multiplication by unit complex numbers acts as a manifest symmetry of the theory. Write down the corresponding manifest infinitesimal symmetry $\hat{\xi}$ and the associated Noether current.

(b) More generally, consider a scalar field $\phi\colon M^n \to X$ with values in a Riemannian manifold X with potential function $V\colon X \to \mathbb{R}$. Suppose ζ is an infinitesimal isometry (Killing vector field) on X such that $\mathrm{Lie}(\zeta)V = 0$. Construct an induced manifest infinitesimal symmetry $\hat{\xi}$ and the corresponding Noether current.

(c) As a special case of the previous, consider a free particle on $\mathbb{E}^d$ ($n = 1$ and $V = 0$) and derive the formulas for linear and angular momentum, the conserved charges associated to the Lie algebra of the Euclidean group.

5. (a) For time translation as a nonmanifest symmetry of the real scalar field in 2 dimensions (see (2.61) in Example 2.58), compute the Lie derivative of the local symplectic form.

(b) Treat infinitesimal translation in the x^1 (space) direction both as a manifest and nonmanifest symmetry. Compute the associated Noether current and Noether charge. The latter is the momentum of the field.

6. (Energy-momentum tensor) In this problem we define the energy-momentum tensor. We caution, however, that there is another definition for fields coupled to a metric (defined more or less by differentiating with respect to the metric), and that the two do not always agree. The latter is always symmetric, whereas the one considered here is not.

(a) Consider a Poincaré-invariant lagrangian field theory $\mathcal{L} = L + \gamma$ on Minkowski spacetime M^n. We use the usual coordinates and the Lorentz metric with components $g_{\mu\nu}$ and inverse metric with components $g^{\mu\nu}$. As a consequence of Poincaré invariance we have that infinitesimal translation $\partial_\mu = \partial/\partial x^\mu$ induces a manifest infinitesimal symmetry of the theory. Let *minus* the associated Noether current be

$$\Theta_{\mu\nu} * dx^\nu = \Theta_{\mu\nu} g^{\nu\nu'}\ \iota(\partial_{\nu'})|d^n x|$$

for some functions

$$\Theta_{\mu\nu}\colon \mathcal{F} \times M \longrightarrow \mathbb{R}.$$

The tensor whose components are $\Theta = (\Theta_{\mu\nu})$ is called the energy-momentum tensor. Verify the conservation law

$$\sum_\nu \partial_\nu \Theta_{\mu\nu} = 0.$$

Prove that $\Theta_{\mu\nu} = \Theta_{\nu\mu}$ if and only if the current

$$\eta \cdot \Theta = \Theta_{\mu\nu}\eta^\mu * dx^\nu$$

is conserved for every infinitesimal Lorentz transformation η.

(b) Compute the energy-momentum tensor for the real scalar field (in two or more dimensions, as you prefer). Is it symmetric?

LECTURE 3
Classical Bosonic Theories on Minkowski Spacetime

Physical lagrangians; scalar field theories

In this lecture we consider theories defined on Minkowski spacetime M^n. As usual we let $\mathcal{F}$ denote the space of fields, L the lagrangian, γ the variational 1-form, $\mathcal{M}$ the space of classical solutions, and ω the local symplectic form. (In the theories we consider γ is determined canonically from L; see the first comment following (2.21).)

We first remark that our main interest in the lagrangians we write is for their use in *quantum* field theory, not classical field theory. The classical theory makes sense, certainly, and as explained in previous lectures the lagrangian encodes a classical Hamiltonian system, once a particular time is chosen (thus breaking Poincaré invariance). In the quantum theory, the exponential e^{iS} of the *action* $S = \int_{M^n} L$, is formally integrated over[15] $\mathcal{F}$ with respect to formal measure on $\mathcal{F}$. We will not discuss the quantum theory from this path integral point of view at all, but simply mention it again to remind the reader of the context for our discussion. Also, the path integrals are "Wick rotated" to integrals over fields on Euclidean space $\mathbb{E}^n$. There is a "Wick rotation" of lagrangians to Euclidean lagrangians, and we emphasize that they do not satisfy all of the requirements of physical lagrangians on Minkowski spacetime. We will not treat Wick rotation in these lectures either.

A word about units. We already used the universal constant c in relativistic theories; it has units of velocity, so converts times to lengths. In quantum theories there is a universal constant $\hbar$, called *Planck's constant*, which has units of action: mass $\times$ length^2/time. So using both c and $\hbar$ we can convert times and lengths to masses. This is typically done in relativistic quantum field theory. Physicists usually works in units where $c = \hbar = 1$, so the conversions are not evident.

The first requirement of a physical lagrangian is that it be *real*. We have already encoded that implicitly in our notation $L \in \Omega^{0,|0|}_{\text{loc}}(\mathcal{F} \times M)$, since we always use real (twisted) forms. But we could extend the formalism to complex (twisted) forms, and indeed we often must when writing Euclidean lagrangians. Also, we have assumed that $\mathcal{F}$ is a real manifold, but sometimes it presents itself more naturally as a complex manifold. Still, our point remains that the lagrangian is real when evaluated on fields (viewed as a real manifold).

[15] As remarked in a footnote in Lecture 1, the integral is taken over fields satisfying a certain finite action condition.

The second requirement is that the lagrangian be *local*. We have already built that into our formalism, and this property remains after Wick rotating to a Euclidean lagrangian. Locality holds for *fundamental* lagrangians, that is, lagrangians which are meant to describe nature at the smallest microscopic distance scales. *Effective* lagrangians describe nature at a larger distance scale, and these are often nonlocal, though usually only local approximations are written. In any case these lectures deal only with fundamental lagrangians. In fact, typically only first derivatives of the fields occur in the lagrangian. That constraint comes from the quantum theory and is beyond the scope of this course.

A final requirement is that the lagrangian be manifestly *Poincaré-invariant*. This ensures that the Poincaré group P^n acts on $\mathcal{M}$. The Poincaré group is a subgroup of the (global) symmetry group of the theory, and the entire symmetry group in a quantum theory is usually the product of P^n and a compact Lie group. (Under certain hypotheses this is guaranteed by the Coleman-Mandula theorem.) As explained in Lecture 2, symmetries give rise to conserved quantities. The conserved quantities associated to the Poincaré group include energy and momentum; conserved quantities associated to an external compact Lie group include electric charges and other "quantum numbers."

In these lectures we only write lagrangians which satisfy the conditions outlined here. In many cases they are the most general lagrangians which satisfy them, but we will not analyze the uniqueness question.

We begin with a real-valued scalar field $\phi\colon M^n \to \mathbb{R}$. Note that in this case $\mathcal{F}$ is an infinite-dimensional vector space. We work with standard coordinates $x^0, \dots, x^{n-1}$ as in (1.41), and set $\partial_\mu = \partial/\partial x^\mu$. The "kinetic energy" term for a real scalar field is

$$\begin{aligned} L_{\mathrm{kin}} &= \frac{1}{2} d\phi \wedge * d\phi \\ &= \frac{1}{2} |d\phi|^2 \; |d^n x| \\ &= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \; |d^n x| \\ &= \frac{1}{2} \big\{ (\partial_0 \phi)^2 - \sum_{i=1}^{n-1} (\partial_i \phi)^2 \big\} \; |d^n x| \end{aligned} \tag{3.1}$$

The honest kinetic term is the first term in the last expression, the one involving the time derivative ∂_0; the terms with a minus sign are really potential energy terms. The signs are in accord with what we saw in mechanics: the lagrangian is kinetic minus potential. The various expressions make clear that L_{kin} is Poincaré-invariant, and in fact it is the Poincaré-invariant extension of the first term in (3.1), the usual kinetic energy. For this reason one often calls the entire L_{kin} the "kinetic term."

If we break Poincaré invariance and write $M^n = M^1 \times \mathbb{E}^{n-1}$ as time $\times$ space, then we can regard $\mathcal{F}$ as the space of paths in the vector space $\Omega^0(\mathbb{E}^{n-1})$. One can compute in this case that the symplectic structure on the space $\mathcal{M}$ of classical solutions is translation-invariant and the Hamiltonian is quadratic. Therefore, this is a free system: a *free real scalar field*. It is natural to ask what potential energy terms we can add to (3.1) to still have a free system. As we might expect, we can add a polynomial $V\colon \mathbb{R} \to \mathbb{R}$ of degree ≤ 2. We assume that V is bounded from below (see the discussion of energy below), so that the coefficient of the quadratic term is positive. We eliminate the linear and constant terms by assuming the minimum of V occurs at the origin and has value zero. Then the total lagrangian is

$$L = \left\{ \frac{1}{2}|d\phi|^2 - \frac{m^2}{2}\phi^2 \right\} |d^n x| \tag{3.2}$$

As written the constant m has units of inverse length. In a relativistic quantum theory it may be converted to a mass by replacing m with $mc/\hbar$, since $\hbar$ has units of *action*: mass times length squared divided by time. For this reason the potential term is called a *mass term*. The constant m is the mass of ϕ. This terminology is apparent from the quantization of the theory, as we discuss in Lecture 5. Finally, note that the lagrangian depends only on the 1-jet of ϕ, so as promised there is a canonical variational 1-form γ. We leave for the exercises a detailed computation of the variation of the lagrangian

$$\delta L = -\delta\phi \wedge \left\{ d * d\phi + m^2\phi \, |d^n x| \right\} - d\left\{ \delta\phi \wedge *d\phi \right\}, \tag{3.3}$$

the variational 1-form

$$\gamma = \delta\phi \wedge *d\phi, \tag{3.4}$$

and the local symplectic form

$$\omega = *d\delta\phi \wedge \delta\phi. \tag{3.5}$$

From this one derives the *classical field equation*, or Euler-Lagrange equation,

$$(\Box + m^2)\phi = 0. \tag{3.6}$$

Here

$$\begin{aligned} \Box &= (-1)^{n-1} * d * d \\ &= -d^* d \\ &= \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 \\ &= g^{\mu\nu}\partial_\mu \partial_\nu \end{aligned} \tag{3.7}$$

is the wave operator. We will analyze the solutions $\mathcal{M}$ in an exercise and again in Lecture 5. For now we can simply say that $\mathcal{M}$ is a real symplectic vector space which carries a representation of the Poincaré group P^n.

In nonfree theories, which of course are of greater interest than free theories, the potential is not quadratic. For example, later in the lecture we will consider a scalar field with a quartic potential. The most general model of this kind, called a *nonlinear σ-model*, starts with the data

$$\begin{array}{ll} X & \text{Riemannian manifold} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential energy function} \end{array} \tag{3.8}$$

As usual, the space of fields is the mapping space $\mathcal{F} = \operatorname{Map}(M^n, X)$. The lagrangian of this model is

$$L = \left\{ \frac{1}{2}|d\phi|^2 - \phi^* V \right\} |d^n x|. \tag{3.9}$$

Equations (3.4)–(3.6) generalize in a straightforward manner which incorporates the Riemannian structure of X. The special case $n = 1$ is the mechanical system we studied in Lecture 2—a particle moving on X—and we recover the equations we discussed there.

Hamiltonian field theory

To get a Hamiltonian interpretation of a Poincaré-invariant field theory, we break Poincaré invariance and choose an isomorphism $M^n \cong M^1 \times N$ for $N = \mathbb{E}^{n-1}$. (See Definition 2.66.) Of course, once we choose an affine coordinate system $x^0, x^1, \dots, x^{n-1}$, as we have been doing, then this splitting into time $\times$ space is determined; $x^0 = ct$ is a time coordinate and $x^1, \dots, x^{n-1}$ are coordinates on space. (We usually work in units where $c = 1$.) We consider a scalar field $\mathcal{F} = \mathrm{Map}(M^n, X)$ with values in a Riemannian manifold X, but what we say applies to other fields as well. In the Hamiltonian approach we view $\mathcal{F}$ as the space of paths in a space $\mathcal{F}_N = \mathrm{Map}(N, X)$ of fields on N:

$$\mathcal{F} \cong \mathrm{Map}(M^1, \mathcal{F}_N). \tag{3.10}$$

A *static* field $\phi \in \mathcal{F}$ is one which corresponds to a constant path under this isomorphism, and it is natural to identify $\mathcal{F}_N$ as the space of static fields.

Recall that in a mechanical system the *energy* is minus the Noether charge associated to infinitesimal time translation. More generally, in a lagrangian field theory $\mathcal{L} = L + \gamma$ we define the *energy density* to be minus the Noether current associated to infinitesimal time translation:

$$\Theta = -\iota(\xi_t)\mathcal{L}, \tag{3.11}$$

where ξ_t is infinitesimal time translation as a manifest infinitesimal symmetry. The energy at time t of a field ϕ is the integral of the energy density over the spatial slice:

$$E_\phi(t) = \int_{\{t\}\times N} \Theta(\phi). \tag{3.12}$$

For a static field the energy is constant in time. Just as typical fields in spacetime have infinite action, typical static fields have infinite energy. Define $\mathcal{FE}_N \subset \mathcal{F}_N$ to be the space of static fields of finite energy. An important point is that whereas $\mathcal{F}_N$ may have fairly trivial topology, imposing finite energy often gives a space $\mathcal{FE}_N$ of nontrivial topology.

For the scalar field the energy density is

$$\Theta(\phi) = \Big\{ \frac{1}{2}|\partial_0\phi|^2 + \sum_{i=1}^{n-1} \frac{1}{2}|\partial_i\phi|^2 + V(\phi) \Big\}\, |d^{n-1}x|, \tag{3.13}$$

where

$$|d^{n-1}x| = |dx^1 \cdots dx^{n-1}|. \tag{3.14}$$

The first term is the kinetic energy; the remaining terms are potential energy terms. Note that the kinetic term vanishes for a static field. We usually assume that the energy is bounded below, which for a scalar field means that the function V is bounded below. From (3.13) it is easy to see that fields of minimum energy are static and constant in space, i.e., constant in spacetime, and furthermore that constant must be a minimum of the potential V. Such static field configurations comprise a manifold

$$\mathcal{M}_{\mathrm{vac}} \subset \mathcal{F}\mathcal{E}_N \tag{3.15}$$

called the *moduli space of vacua*. For a real scalar field with potential V we usually assume that 0 is the minimum value of V, so

$$\mathcal{M}_{\mathrm{vac}} = V^{-1}(0). \tag{3.16}$$

For example, if $X = \mathbb{R}$ and $V = 0$ we have the theory of a massless real scalar field; in that case $\mathcal{M}_{\mathrm{vac}} \cong \mathbb{R}$. For a massive real scalar field we have $V(\phi) = \frac{m}{2}\phi^2$, so $\mathcal{M}_{\mathrm{vac}}$ is a single point $\phi = 0$. For a quartic potential

$$V(\phi) = \frac{\lambda}{8}(\phi^2 - a^2)^2 \tag{3.17}$$

(with λ, a positive real numbers), $\mathcal{M}_{\mathrm{vac}}$ consists of two points $\phi = \pm a$.

The notions of static fields, energy density, and vacuum solution extend to general field theories. Quite generally, on the space of static fields $\mathcal{F}_N$ we have the formula

$$\Theta = -\iota(\partial_t)L \qquad \text{on } \mathcal{F}_N. \tag{3.18}$$

Also, any critical point of energy on $\mathcal{F}_N$ is in fact a finite energy solution to the classical equations of motion. In particular, $\mathcal{M}_{\mathrm{vac}} \subset \mathcal{M}$. Finally, a vacuum solution is usually Poincaré-invariant. This is indeed true for the constant scalar fields. When we come to more complicated tensor fields, it is often true that Poincaré invariance already determines a unique field, which is then a unique vacuum.

A *soliton* is a static solution which is a critical point of energy, but not a minimum. Often the space $\mathcal{F}\mathcal{E}_N$ of finite energy static fields is not connected, and a soliton is a minimum energy configuration in a component where the minimum is not achieved. The two-dimensional real scalar field with quartic potential (3.17) provides an example. The space $\mathcal{F}\mathcal{E}_N$ has 4 components. The vacua are contained in two distinct components; in the other components we find solitons which minimize energy in that component.

Lagrangian formulation of Maxwell's equations

We first reformulate Maxwell's equations (1.39) in Minkowski spacetime M^4. Recall that in our previous formulation the electric field E is a time-varying 1-form on space N and the magnetic field B a time-varying 2-form on N. Now define the 2-form F on spacetime M^4 by

$$F = B - dt \wedge E. \tag{3.19}$$

Let $*$ be the $*$ operator on Minkowski spacetime and $*_N$ the $*$ operator on N; then

$$*F = \frac{1}{c} *_N E + c\,dt \wedge *_N B. \tag{3.20}$$

Maxwell's equations then simply reduce to the equations

$$\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned} \tag{3.21}$$

These equations are Poincaré-invariant, as the $*$ operator on Minkowski spacetime is.

These equations are Maxwell's equations "in a vacuum"; the true Maxwell equations allow a current $j \in \Omega^3(M)$, which is constrained to have compact spatial support and satisfy $dj = 0$. Then the second Maxwell equation is $d * F = j$. For now, though, we concentrate on the case $j = 0$.

Note that (3.21) are first-order equations, whereas the Euler-Lagrange equations we have seen have been second-order. In fact, to construct a lagrangian formulation we introduce a 1-form $A \in \Omega^1(M^4)$ and define F as a function of A:

$$F_A = dA. \tag{3.22}$$

Note that $dF_A = 0$ for all A, so that the first Maxwell equation is immediately satisfied. Now the second Maxwell equation has second order in A,

$$d * F_A = d * dA = 0, \tag{3.23}$$

and we can expect to derive it from an action principle in a similar manner to our previous examples. (Note that there are action principles which lead to first-order Euler-Lagrange equations.) Some readers will recognize these equations as analogous to the equations of Hodge theory, and the lagrangian we introduce is also analogous to the one used in Hodge theory. Namely, introduce the following lagrangian, which is a function of A:

$$L = -\frac{1}{2} F_A \wedge *F_A. \tag{3.24}$$

Note that the lagrangian—and Maxwell's equations (3.22) and (3.23)—do not change if we change A by an exact 1-form: $A \to A + df$, $f \in \Omega^0(M^4)$. So for our space of fields we take the quotient of 1-forms by exact 1-forms:

$$\mathcal{F} = \Omega^1(M^4)/d\Omega^0(M^4). \tag{3.25}$$

Notice that the differential d identifies the space of fields with the space of exact 2-forms; it maps an equivalence class of gauge fields A to the exact 2-form F_A. We leave the reader to compute δL, γ, and to derive the Euler-Lagrange equation (3.23) from this lagrangian. Also, write the lagrangian in coordinates to see that it has the form kinetic energy (time derivatives) minus potential energy (spatial derivatives). We will write more general formulas later when we discuss gauge theories.

We remark that the lagrangian formulation we have just given works as well in n-dimensional Minkowski spacetime for arbitrary n.

Principal bundles and connections

To formulate gauge theory as used in lagrangians for quantum field theory, we quickly review some more differential geometry. The account we give here is very brief.

Let M be a manifold. Fix a Lie group G. A *principal G bundle* $P \to M$ is a manifold P on which G acts freely on the right with quotient $P/G \cong M$ such that there exist local sections. If P', P are principal G bundles over M, then an isomorphism of principal bundles $\varphi\colon P' \to P$ is a smooth diffeomorphism which commutes with G and induces the identity map on M. In case $P = P'$ such automorphisms are called *gauge transformations* of P. For each M there is a category of principal G bundles and isomorphisms.

A connection on a principal G bundle $\pi\colon P \to M$ is a G-invariant distribution in TP which is transverse to the vertical distribution $\ker d\pi$. In other words, at each $p \in P$ there is a subspace $V_p \subset T_pP$ of vectors tangent to the fiber at p. A connection gives, at each p, a complementary subspace $H_p \subset T_pP$ with the restriction that the distribution H be G-invariant. The infinitesimal version of the G action identifies each V_p with the Lie algebra $\mathfrak{g}$ of G. We can express a connection as the 1-form $A \in \Omega^1(P;\mathfrak{g})$ whose value at p is the projection $T_pP \to V_p \cong \mathfrak{g}$ with kernel H_p. The G-invariance translates into an equation on A which we do not write here. Connections, being differential forms on P, pull back under isomorphisms $\varphi\colon P' \to P$. Connections form a category, and there is a set of equivalence classes under isomorphisms.

A connection A has a curvature $F_A = dA + \frac{1}{2}[A \wedge A]$, which is a 2-form on P with values in $\mathfrak{g}$. Its transformation law under G, which again we omit, indicates that F_A is a 2-form on the base M with values in the *adjoint bundle*, the vector bundle of Lie algebras associated to P via the adjoint representation of G.

We can reformulate the lagrangian picture of Maxwell's equations in terms of connections. Namely, fix the Lie group $G = \mathbb{R}$ of translations. Then the space of equivalence classes of $\mathbb{R}$ connections on M may be identified with the space of fields (3.25). In fact, applying d we identify the quotient of 1-forms by exact 1-forms with the space of exact 2-forms. On the other hand, an $\mathbb{R}$ connection has a curvature which is an exact 2-form, any exact 2-form can occur, and equivalent connections have equal curvatures. From this point of view the space of fields is a category,[16] but the field theory is formulated on the set of equivalence classes. If we interpret A as an $\mathbb{R}$-connection, then F_A is its curvature. The lagrangian (3.24) depends only on F_A, so makes sense in this new formulation.

Gauge theory

The picture of Maxwell's equations given above is adequate for the classical theory. In quantum theory, however, we encounter a new ingredient: Dirac's charge quantization law. The charge in the previous story is the spatial integral of the current j (which we set to zero for simplicity). In that theory the charge can be any value, but in the quantum theory it must be an integer multiple of some fundamental value—that's what we mean by "quantization". In these lectures we will not have time to explain the hows and whys of charge quantization. Suffice it to say that there is an interesting geometric and topological story lurking behind. We simply use the formulation of Maxwell theory in terms of $\mathbb{R}$ connections and state that in the quantum theory charge quantization is achieved by replacing the group $\mathbb{R}$ by the *compact* group $\mathbb{R}/2\pi\mathbb{Z}$. (The '$2\pi$' is put in for convenience.)

[16]in fact, a *groupoid*

Once we have phrased Maxwell theory and charge quantization in these terms—this is certainly *not* how it was done historically!—it is easy to imagine a generalization in which the group $\mathbb{R}/2\pi\mathbb{Z}$ is replaced by any compact Lie group G. This bold step was taken by Yang and Mills in 1954 much before the connection to connections was established. Therefore, we are led to formulate a lagrangian field theory based on the following data:

$$\begin{array}{ll} G & \text{compact Lie group with Lie algebra } \mathfrak{g} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant inner product on } \mathfrak{g} \end{array} \tag{3.26}$$

The theory we formulate is called *pure gauge theory*. We emphasize that in physics gauge theories with compact structure group are used in *quantum* field theory, not classical field theory. The choice of inner product incorporates *coupling constants* of the theory. For example, if G is a simple group then there is a 1-dimensional vector space of invariant inner products, any two of which are proportional. In the classical Maxwell theory, $G=\mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on the Lie algebra $\mathbb{R}$. We assume that $\langle\cdot,\cdot\rangle$ is nondegenerate; this amounts to a nondegeneracy assumption on the lagrangian (3.27) below. The space of fields is, as said above, the category of G connections over spacetime M^n, and everything we write is invariant under isomorphisms of connections. The lagrangian is

$$L = -\frac{1}{2}\langle F_A \wedge *F_A\rangle. \tag{3.27}$$

We leave the reader to compute δL and so derive the equation of motion—the *Yang-Mills equation*

$$d_A * F_A = 0, \tag{3.28}$$

the variational 1-form

$$\gamma = -\langle \delta A \wedge *F_A\rangle, \tag{3.29}$$

and the local symplectic form

$$\omega = \langle \delta A \wedge *d_A\delta A\rangle. \tag{3.30}$$

Here d_A is the differential in the extension of the de Rham complex to forms with values in the adjoint bundle (using the connection A); see (1.12).

If G is abelian, then the equations of motion are linear and the associated Hamiltonian system is free. If G is nonabelian, then the equations of motion are nonlinear. Geometers are familiar with the Yang-Mills equations (3.28) on Riemannian manifolds, where after dividing out by isomorphisms they are essentially elliptic. Here, on spacetime with a metric of Lorentz signature, the Yang-Mills equations are wave equations.

The energy density of the field A is

$$\Theta(A) = \Bigl\{\sum_{\mu<\nu}\frac{1}{2}|F_{\mu\nu}|^2\Bigr\}\,|d^{n-1}x|, \tag{3.31}$$

where $\mu, \nu = 0, \dots, n-1$ run over all spacetime indices and

$$F_A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3.32}$$

is the curvature of A. So the minimum energy is achieved when $F_{\mu\nu} = 0$ for all μ, ν; i.e., when the curvature of A vanishes. On Minkowski spacetime all such connections are isomorphic, whence the moduli space of vacua consists of a single point:

$$\mathcal{M}_{\text{vac}} = \text{pt}\,. \tag{3.33}$$

As mentioned before, we work on the space of equivalence classes of connections.

Gauged σ-models

The most general bosonic field theory we consider on M^n combines the pure gauge theory of the previous section with the σ-models considered earlier.[17] Again, these are models used in quantum theory. The data we need to specify the model is the following:

$$\begin{array}{ll} G & \text{Lie group with Lie algebra } \mathfrak{g} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant scalar product on } \mathfrak{g} \\ X & \text{Riemannian manifold on which } G \text{ acts by isometries} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential function invariant under } G \end{array} \tag{3.34}$$

An important special case has X a real vector space with positive definite inner product and G acting by orthogonal transformations. The space $\mathcal{F}$ of fields in the theory consists of pairs

$$\begin{array}{ll} A & \text{connection on a principal } G\text{-bundle } P \longrightarrow M \\ \phi & \text{section of the associated bundle } P \times_G X \longrightarrow M \end{array} \tag{3.35}$$

In the linear case the associated bundle $P \times_G X$ is a vector bundle over M. In all cases it is often convenient to view ϕ as an equivariant map $\phi\colon P \to X$. The space of fields is again best seen as a category—an isomorphism $\varphi\colon P' \to P$ of principal bundles induces an isomorphism of fields $(A', \phi') \to (A, \phi)$. As usual, we consider fields up to isomorphism. The global symmetry group of the theory is the subgroup of isometries of X which commute with the G action and preserve the potential function V.

Physicists call models with this field content a *gauged (linear or nonlinear) σ-model.*

The lagrangian we consider combines (3.27) and (3.9):

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^* V \right\} |d^n x|. \tag{3.36}$$

[17] For dimensions $n \le 4$ this is all that is usually encountered in models without gravity. However, in higher dimensions—especially in supergravity—there may be other fields which locally are differential forms of degree > 1, just as connections with structure group $\mathbb{R}$ or $\mathbb{R}/2\pi\mathbb{Z}$ are locally 1-forms.

Note that the covariant derivative d_A replaces the ordinary derivative d encountered in the pure σ-model (3.9). This term "couples" the fields A and ϕ. Quite generally, if $L = L(\phi_1, \phi_2)$ is a lagrangian which depends on fields ϕ_1, ϕ_2, then we say that the fields are *uncoupled* if we can write $L(\phi_1, \phi_2) = L_1(\phi_1) + L_2(\phi_2)$. Again, we leave the reader to derive the equations of motion, variational 1-form, etc.

The energy density of the pair (A, ϕ) works out to be

$$\Theta(A,\phi) = \Big\{ \sum_{\mu<\nu} \frac{1}{2}|F_{\mu\nu}|^2 + \sum_{\mu} \frac{1}{2}|(\partial_A)_\mu \phi|^2 + \phi^* V \Big\} \, |d^{n-1}x|. \tag{3.37}$$

We seek vacuum solutions assuming that V has a minimum at 0, so that the energy is bounded below by 0. Thus we seek solutions of zero energy. The first term implies that for a zero-energy solution A is flat, so up to equivalence is the trivial connection with zero curvature. Then we can identify covariant derivatives with ordinary derivatives, and the second term implies that ϕ must be constant for a zero-energy solution. Finally, the last term implies that the constant is in the set $V^{-1}(0)$. Now recall that we consider pairs (A, ϕ) up to equivalence. A trivial connection A has a group of automorphisms isomorphic to G. This is the group of *global gauge transformations*. Now for a vacuum solution ϕ is a constant function into $V^{-1}(0)$, and the G-action of ϕ is simply the G-action on $V^{-1}(0)$. Thus the moduli space of vacua is then a subquotient space of X:

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \,\big/\, G. \tag{3.38}$$

In certain supersymmetric field theories the manifold X is restricted to be Kähler or hyperkähler and the potential is the norm square of an appropriate moment map. In those cases $\mathcal{M}_{\text{vac}}$ is the Kähler or hyperkahler quotient. (It was in this context that the latter was in fact invented.)

We mention that specific examples of these models have solitons, which recall are static solutions not of minimal energy. For example, in $n = 4$ dimensions the linear σ-model with $G = SO(3)$, X the standard real 3-dimensional representation, and V a quartic potential (3.17) has static *monopole* solutions.

Exercises

The lecture skipped many computations and verifications, so several of the problems ask you to fill in those gaps.

1. (a) Derive formula (3.3) for the variation of the free scalar field lagrangian.
   (b) What is the corresponding formula for the nonlinear σ-model (3.9)? What are the generalizations of (3.4) and (3.5)? Check your formulas against the formulas in Lecture 2 for the particle moving on X.
   (c) Verify formula (3.13) for the energy density of a scalar field.

2. Recall the (components of the) *energy-momentum* tensor $\Theta_{\mu\nu}$ from Problem Set 2. Note that the energy density may be expressed as $\Theta_{00} * dx^0$, and so the energy-momentum tensor is the Poincaré-invariant generalization of the energy density. Compute the energy-momentum tensor for the scalar field, and then recover (3.13).

3. In this problem you will find the solutions to the equations of motion of simple free systems, both for particles and for fields. The field in all cases is a map $\phi\colon M^n \to \mathbb{R}$ with potential $V(\phi) = \frac{1}{2}k\phi^2$.

   (a) First, consider $n = 1$. Write the Euler-Lagrange equations. What are the solutions for $k = 0$? For $k \neq 0$? (The latter is a harmonic oscillator, and is not usually called "free".)

   (b) If you didn't already do it, solve the previous problem using the 1-dimensional Fourier transform. My convention for the Fourier transform are as follows—you're welcome to use your own. Let V be an n-dimensional real vector space and $\phi\colon V \to \mathbb{C}$ a complex-valued function. Its Fourier transform $\hat{\phi}\colon V^* \to \mathbb{C}$ is a function on the dual space. The functions ϕ and $\hat{\phi}$ are related by the integrals

$$\begin{aligned} \hat{\phi}(k) &= \frac{1}{(2\pi)^{n/2}} \int_V e^{-\sqrt{-1}\langle k,x\rangle} \phi(x)\, |d^n x|, \qquad k \in V^* \\ \phi(x) &= \frac{1}{(2\pi)^{n/2}} \int_{V^*} e^{+\sqrt{-1}\langle k,x\rangle} \hat{\phi}(k)\, |d^n k|, \qquad x \in V. \end{aligned}$$

   In analysis one analyzes carefully the class of functions for which these formulas make sense; we work formally here.

   (c) Now consider both $k = 0$ and $k \neq 0$ for the case of fields (arbitrary n). What is the support of the Fourier transform of the solution to the wave equation (3.6)? The reality of the field ϕ induces a reality condition on the Fourier transform $\hat{\phi}$; what is it? The equations are Poincaré-invariant, so the Poincaré group acts on the space of solutions. What can you say about the action?

4. We study the motion of a particle on an interval $[a, b] \in \mathbb{E}^1$ with no potential energy. As a classical system this is only a local system, since a particle moving at constant velocity—a motion which solves Newton's laws—runs off the end of space in finite time. Let's describe this system instead as particle motion on the line $\mathbb{E}^1$ with a potential function V which vanishes on $[a, b]$ and is positively infinite otherwise. Study this system as the $C \to \infty$ limit of a system whose potential outside $[a, b]$ is C. You may want to smooth out the potential at the points a, b. What happens to the particle as it gets close to the endpoints of the interval? What is the moduli space of vacua $\mathcal{M}_{\text{vac}}$ in this model? I recommend that you study the quantum mechanics of this system as well. This includes the moduli space of vacua, correlation functions, their Wick rotation to Euclidean time, etc. Everything is computable and the computations illustrate features present in more complicated quantum field theories.

5. Consider a real scalar field in $n = 2$ dimensions with the quartic potential (3.17). A static field is a real-valued function on space $\mathbb{E}^1$. The space of all such is a vector space. Demonstrate that the space of *finite energy* static fields has 4 components. How are they distinguished? Find vacuum solutions. Find the soliton solutions mentioned in the lecture by writing the formula for energy and computing the critical point equation.

6. (a) Write Maxwell's equations (3.24) as a wave equation in A.

(b) Analyze the solutions using the Fourier transform. Remember that the space of fields is a quotient (3.25). This is not a particularly easy problem the first time around. Can you identify the (real) representation of the Poincaré group obtained?

7. Write the gauged σ-model for $G = \mathbb{R}$ and $X = \mathbb{E}^1$ with G acting by translations. What possible potentials V can we use? Write the equations of motion for this system. What modification to Maxwell's equations do you find?

8. (a) Write the lagrangians for gauge theory and for the gauged σ-model in coordinates so you can see what they really look like.

   (b) Fill in missing computations in the lecture: the equations of motion, local symplectic form, energy density etc. for pure gauge theory and for the gauged σ-model.

LECTURE 4
Fermions and the Supersymmetric Particle

In classical—that is, nonquantum—relativistic physics the objects which one considers are of two sorts: either particles, strings, and other extended objects moving in Minkowski spacetime M^n; or the electromagnetic field. Perhaps there are also sensible models with scalar fields. But as far as I know, there is not a good notion of a "classical fermionic field". Nonetheless, the remaining lectures incorporate fermionic fields, and in particular develop "classical supersymmetric field theory". As I have said repeatedly in these notes, although what we discuss uses the formalism of classical field theory, it is in the end used as the input into the (formal) path integral of the quantum field theory.

To write classical fermionic fields we need to bring in a new piece of differential geometry: supermanifolds. We refer the reader to John Morgan's lectures in this volume for an introduction, but recall a few key points here. The linear algebra underlying supermanifolds concerns $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $V = V^0 \oplus V^1$. The homogeneous summand V^0 is called *even*; V^1 is termed *odd*. The parity-reversed vector space $\Pi V = V^1 \oplus V^0$ has the even and odd summands interchanged. The *sign rule* extends many notions of algebra to the $\mathbb{Z}/2\mathbb{Z}$-graded world by introducing a sign when odd elements are interchanged. For example, one of the axioms of a Lie bracket (on a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra) is that $[a,b] = \mp[b,a]$, the plus sign occuring only if both a and b are odd. To understand fermionic fields, though, we have to come to grips with an odd vector space V^1 as a space, not just as an algebraic object, and for that we define it in terms of its functions. Namely, the ring of functions on V^1 is the $\mathbb{Z}/2\mathbb{Z}$-graded exterior algebra $\bigwedge^{\bullet}(V^1)^*$. (Compare: The algebraic functions on an even vector space V^0 is the symmetric algebra $\mathrm{Sym}^{\bullet}(V^0)^*$.) Since the exterior algebra contains nilpotents, we use intuition and techniques from algebraic geometry—specifically the *functor of points*—to understand the space V^1.

The supersymmetric particle

Fix a Riemannian manifold X. In the discussion of an ordinary, nonsupersymmetric particle moving on X we also have a potential energy function $V\colon X \to \mathbb{R}$. The supersymmetric version we consider forces $V = 0$. For the ordinary particle moving on X, as considered in Lecture 1, "spacetime" is simply time M^1, the field is a map $x\colon M^1 \to X$, and the lagrangian is

$$L_0 = \left\{ \frac{1}{2}|\dot{x}|^2 \right\} |dt|, \tag{4.1}$$

where t is a coordinate on M^1. We now want to add a second field to the theory, and it is a fermionic field. For simplicity we do this first in case $X = \mathbb{E}^1$, i.e., for a particle moving on a line. Then the fermionic field is a map $\psi\colon M^1 \to \Pi\mathbb{R}^1$ from time to a fixed one-dimensional odd vector space. So the space of fields is the product

$$\mathcal{F} = \mathrm{Map}(M^1, \mathbb{E}^1) \times \mathrm{Map}(M^1, \Pi\mathbb{R}^1). \tag{4.2}$$

The lagrangian for this theory is a function of the pair (x, ψ), and it is in fact the sum of the lagrangian L_0 for x and a lagrangian for ψ:

$$L = \left\{ \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\psi\dot{\psi} \right\} |dt|. \tag{4.3}$$

The fields x and ψ are uncoupled in this model. Note that whereas the bosonic kinetic term is the square of the first derivative of the field, the fermionic kinetic term—the second term in (4.3)—is the field times its first derivative. This is typical of kinetic terms in higher dimensional field theories as well as in this particle example.

It is fruitful to consider the fermionic theory on its own. Then the space of fields is

$$\mathcal{F}_1 = \mathrm{Map}(M^1, \Pi\mathbb{R}^1) \tag{4.4}$$

and the lagrangian is

$$L_1 = \left\{ \frac{1}{2}\psi\dot{\psi} \right\} |dt|. \tag{4.3}$$

To illustrate computations with fermionic fields, we compute carefully the variation of the lagrangian δL_1. As with our previous mechanics computations, we fix an orientation of time, so identify $|dt| = dt$. Then

$$\begin{aligned} \delta L_1 &= \big(\frac{1}{2}\delta\psi\,\dot{\psi} + \frac{1}{2}\psi\,\delta\dot{\psi}\big) \wedge dt \\ &= \frac{1}{2}\delta\psi \wedge d\psi + \frac{1}{2}\psi\,\delta d\psi \\ &= \frac{1}{2}\delta\psi \wedge d\psi - \frac{1}{2}\psi\,d\delta\psi \\ &= \frac{1}{2}\delta\psi \wedge d\psi - d\big(\frac{1}{2}\psi\,\delta\psi\big) + \frac{1}{2}d\psi \wedge \delta\psi \\ &= \delta\psi \wedge d\psi - d\big(\frac{1}{2}\psi\,\delta\psi\big). \end{aligned} \tag{4.5}$$

The first line is the Leibnitz rule, but already one could raise an objection. Since δ is odd and ψ is odd, why don't we pick up a sign in the second term after commuting δ past ψ? In fact, it is perfectly consistent to employ sign rules based on parity + cohomological degree. But instead, we use sign rules based on the pair (parity, cohomological degree). In this notation δ has bidegree $(0, 1)$ and ψ has bidegree $(1, 0)$. In general when commuting elements of bidegrees (p, q) and (p', q'), we pick up a sign of $(-1)^{pp'+qq'}$. The second equation of (4.5) follows simply from

$\dot\psi\, dt = d\psi$. The third uses $\delta d = -d\delta$. In the fourth we commute d past ψ and again there is no sign. Finally, in the last equation we use $d\psi \wedge \delta\psi = \delta\psi \wedge d\psi$, which holds since the bidegrees of $d\psi$ and $\delta\psi$ are both equal to $(1,1)$. So define

$$\gamma_1 = \frac{1}{2}\psi\,\delta\psi, \tag{4.6}$$

and the equation of motion is the vanishing of $\delta L_1 + d\gamma_1 = \delta\psi \wedge d\psi = -\dot\psi\,\delta\psi \wedge dt$:

$$\dot\psi = 0. \tag{4.7}$$

So the classical space of solutions $\mathcal{M}$ is identified with constant values of ψ, i.e. with the odd vector space $\Pi\mathbb{R}^1$. The symplectic form is

$$\omega_1 = \delta\gamma_1 = \delta\psi \wedge \delta\psi. \tag{4.8}$$

Since $\delta\psi$ has bidegree $(1,1)$—that is, $\delta\psi$ has odd parity and cohomological degree 1)—there is no sign commuting $\delta\psi$ past itself. Indeed this symplectic form is a nondegenerate form on the odd vector space $\Pi\mathbb{R}^1$, and can be identified with the usual inner product on $\mathbb{R}^1$.

In the full model (4.3) the fields x and ψ are completely decoupled: $L = L_0 + L_1$. It follows that

$$\gamma = \gamma_0 + \gamma_1 = \dot x\,\delta x + \frac{1}{2}\psi\,\delta\psi, \tag{4.9}$$

and the symplectic form is similarly a sum from the free particle and (4.6). The equations of motion are the simultaneous equations

$$\ddot x = \dot\psi = 0. \tag{4.10}$$

The space of solutions is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space whose even part has dimension 2 and whose odd part has dimension 1. (We remark that odd vector spaces have a nice quantization only when the dimension is even, but as we are not quantizing now we will not let that worry us unduly.)

Life is much more interesting when we consider a particle moving in a Riemannian manifold X, and then introduce the correct fermionic "partner" field ψ. In fact,

$$\psi \in \Omega^0(M^1; x^*\Pi TX), \tag{4.11}$$

a section of the pullback *odd* tangent bundle. Note again the space of fermionic fields is an (infinite-dimensional) odd vector space. The entire space of fields $\mathcal{F}$ consists of pairs (x,ψ). It is not a product, but rather the projection $(x,\psi) \mapsto x$ is well-defined, so the space of fields $\mathcal{F}$ is an odd vector bundle over the space of bosonic fields $\mathcal{F}_0 = \mathrm{Map}(M^1, X)$. The important feature is that now the fields $x\colon M^1 \to X$ and ψ are coupled in the Lagrangian:

$$L = \left\{\frac{1}{2}|\dot x|^2 + \frac{1}{2}\langle \psi, (x^*\nabla)_{\partial_t}\psi\rangle\right\} dt. \tag{4.12}$$

The second term uses the covariant derivative on $x^*\Pi TX$ induced from the Levi-Civita connection. It is an instructive exercise, not at all trivial, to analyze this model in detail: compute the variational 1-form γ, the symplectic form ω, the equations of motion, and the Hamiltonian (energy). We simply state the answers here:

$$\begin{aligned}\gamma &= \langle \dot{x}, \delta x\rangle + \frac{1}{2}\langle \psi, \delta_\nabla \psi\rangle \\ \omega &= \langle \delta_\nabla \dot{x} \wedge \delta x\rangle + \frac{1}{2}\langle \delta_\nabla\psi \wedge \delta_\nabla\psi\rangle + \frac{1}{4}\langle \psi, R(\delta x \wedge \delta x)\psi, \rangle,\end{aligned} \tag{4.13}$$

where R is the curvature of X (pulled back to M^1 using $x\colon M^1 \to X$). The equations of motion are

$$\begin{aligned}\nabla_{\dot{x}}\dot{x} &= \frac{1}{2}R(\psi,\psi)\dot{x} \\ \nabla_{\dot{x}}\psi &= 0.\end{aligned} \tag{4.14}$$

Finally, the Hamiltonian is the same as for the ordinary particle:

$$H = \frac{1}{2}|\dot{x}|^2. \tag{4.15}$$

The new feature of this model is a nonmanifest symmetry which exchanges x and ψ. In fact, there is a one-parameter family of such, and its infinitesimal generator is therefore an *odd* vector field on $\mathcal{F}$. We describe it by introducing an *auxiliary odd parameter* η, which is best understood in the "functors of points" paradigm, and then writing formulas for the *even* vector field $\hat{\zeta}$ which is the product of η and the odd vector field just mentioned:

$$\begin{aligned}\iota(\hat{\zeta})\delta x &= -\eta\psi \\ \iota(\hat{\zeta})\delta_\nabla\psi &= +\eta\dot{x}.\end{aligned} \tag{4.16}$$

Both sides of the second equation are odd sections of the pullback tangent bundle x^*TX. This infinitesimal symmetry is not manifest; rather

$$\mathrm{Lie}(\hat{\zeta})L = d\alpha_{\hat{\xi}} \tag{4.17}$$

for

$$\alpha_{\hat{\xi}} = d\big(\iota(\hat{\zeta})\gamma + \eta\langle\psi,\dot{x}\rangle\big). \tag{4.18}$$

The corresponding Noether charge is then

$$j_{\hat{\xi}} = \iota(\hat{\zeta})\gamma - \alpha_{\hat{\xi}} = -\eta\langle\psi,\dot{x}\rangle. \tag{4.19}$$

Now define

$$Q = \langle \psi, \dot{x}\rangle; \tag{4.20}$$

we use the opposite sign as in (4.19) just as the Hamiltonian is the opposite of the conserved charge associated to infinitesimal time translation. On the space of solutions to (4.14), or state space $\mathcal{M}$, the odd function Q is conserved under time evolution, just as the Hamiltonian H is. In fact, $\mathcal{M}$ is a symplectic supermanifold, and we have the Poisson bracket

$$\{Q,Q\} = -2H. \tag{4.21}$$

A brief word about supersymmetric quantum mechanics

For the ordinary point particle moving on X, say with no potential, then by fixing a time we identify the state space $\mathcal{M}_0$ with the tangent bundle TX (see (1.6)). This is the statement that a solution to Newton's law, a second order ordinary differential equation, is determined by an initial position and velocity. Now the equation of motion of the fermionic field ψ—the second equation in (4.14)—is first-order, so the solution is determined by an initial value. It follows, if X is complete, that by fixing a time we may identify the state space $\mathcal{M}$ of solutions to the simultaneous equations (4.14) as the total space of the bundle

$$\pi^*\Pi TX \longrightarrow TX, \tag{4.22}$$

where $\pi\colon TX \to X$ is the ordinary tangent bundle. This is a symplectic supermanifold. The symplectic form on each (odd) fiber of (4.22) is really the Riemannian metric on the corresponding ordinary tangent space.

Quantization—often said to be an art rather than a functor—is meant to convert symplectic (super)manifolds into (graded) Hilbert spaces and map the Poisson algebra of functions—classical observables—into the Lie algebra of self-adjoint operators—quantum observables. For the particle moving on X, whose state space is $\mathcal{M}_0 \cong TX$, there is a standard answer: The Hilbert space $\mathcal{H}^0$ is the space of L^2 functions on X and the quadratic Hamiltonian maps[18] to the second-order Laplace operator Δ. We approach the quantization of (4.22) in two steps: first quantize the fibers and then the base. Now each fiber is an odd symplectic vector space, which can be viewed as an ordinary vector space with a nondegenerate symmetric bilinear form. In fact, the bilinear form is positive definite. The Poisson algebra of functions may be identified with the Clifford algebra of that inner product. So what we seek is a representation of the Clifford algebra—a Clifford module. As we work in the super world, we take the Clifford module to be $\mathbb{Z}/2\mathbb{Z}$-graded. This works more naturally for even dimensional odd vector spaces, in which case the unique irreducible Clifford module[19] is the graded spinor representation $S^+ \oplus S^-$. To do that fiberwise over all of TX requires that X be a spin manifold. The classical system makes sense for any Riemannian manifold X, whereas for the quantization we need a spin structure on X. This is a typical situation in quantization: there is often an obstruction—or *anomaly*—which prevents the quantization. In this case the anomaly is the obstruction to putting a spin structure on X. If there is a spin structure, then combining the quantization of TX discussed earlier with the quantization of the fibers, we see that the appropriate graded Hilbert space for the supersymmetric particle is

$$\mathcal{H} = L^2(X; S^+) \oplus L^2(X; S^-), \tag{4.23}$$

the graded Hilbert space of L^2 spinor fields on X.

What are the operators corresponding to the classical observables Q and H? As before, H is the Laplace operator, but now acting on spinor fields. In the

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[18] In fact, there is an indeterminacy; we can map the Hamiltonian to $\Delta + C$ for any constant C.

[19] Recall that in ordinary quantization we also take irreducible representations: the quantization of the symplectic (p, q)-plane is, for example, the set of L^2 functions of q, not L^2 functions of both q and p.

quantization of the odd fibers, ψ becomes Clifford multiplication, and so it is not unreasonable to believe from (4.20) that Q becomes the Dirac operator. Note that the Dirac operator is odd: it exchanges even and odd elements of $\mathcal{H}$.

A study of the path integral in this model leads to a physicists' proof of the Atiyah-Singer index theorem for Dirac operators, which was discovered in the early '80s.

Superspacetime approach

Roughly speaking, the supersymmetry (4.16) is a square root of infinitesimal time translation. Namely, if we let $\hat{Q}$ be the odd vector field such that $\hat{\zeta} = \eta\hat{Q}$, and let $\hat{\xi}$ be infinitesimal time translation, then

$$[\hat{Q}, \hat{Q}] = 2\hat{\xi}. \tag{4.24}$$

we are led to wonder if there is a square root of infinitesimal time translation whose action on fields induces $\hat{Q}$, just as infinitesimal time translation on M^1 induces $\hat{\xi}$. Of course, this cannot happen on M^1, nor on any ordinary manifold. The "square" of a vector field—one-half its bracket with itself—vanishes for even vector fields, since the Lie bracket is skew-symmetric. So we replace M^1 by a supermanifold and seek an odd vector field as the square root. Then we write the fields in our theory as functions on that supermanifold so that such an odd vector field induces an action on fields.

Such a construction exists and it leads to a formulation of the superparticle in which the supersymmetry is manifest. The supermanifold we seek goes under the name *superspacetime*. (Of course, 'superspacetime' is usually rendered 'superspace'. In this example, 'supertime' would be even more appropriate.) The superspacetime formulation of supersymmetric theories goes back to the mid '70s when it was introduced by Salam and Strathdee. Be warned that not all supersymmetric theories have a superspacetime formulation. We give a general construction of superspacetime in Lecture 6. Here we simply introduce the superspacetime relevant to the superparticle.

Let $M^{1|1}$ denote the affine space whose ring of functions is

$$C^\infty(M^{1|1}) = C^\infty(M^1)[\theta]$$

for an odd variable θ. Note:

- θ is *not* an *auxiliary* odd variable—it is a *bona fide* odd function on $M^{1|1}$.
- In our *component* formulation of the superparticle we had fields x, ψ which map an ordinary manifold into a supermanifold. Now we consider a *superfield* formulation in which the field Φ maps a supermanifold into an ordinary manifold, as sketched in the nonartist's rendering below.

It is instructive to work out the space of maps $M^{1|1} \to X$ into a manifold X by considering the induced algebra homomorphism $C^\infty(X) \to C^\infty(M^{1|1})$. A single map reduces simply to a path $x = x(t)$ in X, but things become more interesting if we introduce an auxiliary odd parameter η. In other words, we consider a one-parameter family of maps with parameter η. This is equivalent to an algebra homomorphism $C^\infty(X) \to \operatorname{Spec}\mathbb{R}[\eta] \times C^\infty(M^{1|1})$. Then the pullback of a function $f \in C^\infty(X)$ may be written as $a_t(f) + \theta\eta b_t(f)$. Using the fact that the pullback is an algebra homomorphism, one can deduce that $a_t(f) = f\big(x(t)\big)$ for some path $x(t) \in X$, and $b_t(f) = V_t f$ for some path of tangent vectors $V_t \in T_{x(t)}X$. The fermionic component field $\psi(t) = \eta V_t$.

Let t be a standard affine coordinate on M^1, so that t, θ are global coordinates on $M^{1|1}$. Let

$$i\colon M^1 \hookrightarrow M^{1|1} \tag{4.25}$$

be the inclusion defined by

$$\begin{aligned} i^*t &= t \\ i^*\theta &= 0. \end{aligned} \tag{4.26}$$

We introduce a global framing of $M^{1|1}$ by the vector fields

$$\begin{aligned} \partial_t &= \frac{\partial}{\partial t} \\ D &= \partial_\theta - \theta\partial_t = \frac{\partial}{\partial\theta} - \theta\partial_t. \end{aligned} \tag{4.27}$$

Here ∂_t is even and D is odd. It is unfortunate that the vector field 'D' has the same symbol as the total differential considered in earlier parts of the lecture. *Tant pis*, there just aren't enough dees in the world! We also introduce the odd vector field

$$\tau_Q = \partial_\theta + \theta\partial_t. \tag{4.28}$$

In fact, $M^{1|1}$ is the supermanifold underlying a Lie group on which $\{\partial_t, D\}$ is a basis of left invariant vector fields and $\{\partial_t, \tau_Q\}$ a basis of right invariant vector fields. They satisfy the bracket relations

$$\begin{aligned} [D, D] &= -2\partial_t \\ [\tau_Q, \tau_Q] &= +2\partial_t \\ [D, \tau_Q] &= \quad 0. \end{aligned} \tag{4.29}$$

The vector field ∂_t commutes with both D and τ_Q.

We now formulate a field theory with spacetime $M^{1|1}$ and space of fields

$$\mathcal{F} = \{\Phi\colon M^{1|1} \longrightarrow X\}, \tag{4.30}$$

where X is our fixed Riemannian manifold. As we saw above, Φ leads to component fields x, ψ which are a path in X and an odd tangent vector field along the path We recover these *component fields* from Φ by restricting D-derivatives of the superfield to the underlying Minkowski spacetime:

$$\begin{aligned} x &:= i^*\Phi \\ \psi &:= i^*D\Phi. \end{aligned} \tag{4.31}$$

The vector fields ∂_t, τ_Q act on the field Φ directly by differentiation, and it is not hard to work out the action on component fields. For ∂_t it is simply differentiation in t, and for τ_Q we use the one parameter group $\varphi_u = \exp(u\eta\tau_Q)$ generated by $\hat{\zeta} = \eta\tau_Q$ for an odd parameter η. Recall that the action of a diffeomorphism of (super)spacetime on fields uses pullback by the *inverse* (see Example 2.37, for example):

$$\begin{aligned} \iota(\hat{\zeta})\delta x = \hat{\zeta}\cdot x &= \frac{d}{du}\Big|_{u=0}\,(\varphi_u^{-1})^* i^*\Phi \\ &= \frac{d}{du}\Big|_{u=0}\, i^*(\varphi_u^{-1})^*\Phi \\ &= -\eta i^*\tau_Q\Phi \\ &= -\eta i^*D\Phi \\ &= -\eta\psi \end{aligned} \tag{4.32}$$

Similarly, we compute

$$\begin{aligned} \iota(\hat{\zeta})\delta\psi &= i^*D\psi\tau_Q\Phi \\ &= -\eta i^*\tau_Q D\Phi \\ &= -\eta i^* D^2\Phi \\ &= \eta i^*\partial_t\Phi \\ &= \eta\dot{x} \end{aligned} \tag{4.33}$$

The lagrangian density in superspacetime is

$$\mathcal{L} = |dt|\,d\theta\,\Big\{-\frac{1}{2}\langle D\Phi, \partial_t\Phi\rangle\Big\} = |dt|\,d\theta\;\ell. \tag{4.34}$$

Here $|dt|\,d\theta$ is a bi-invariant density on $M^{1|1}$—it is invariant under ∂_t, D, τ_Q.

Before integrating we pause to point out that we have now made the supersymmetry manifest. Namely, the lagrangian $\mathcal{L}$ is invariant under the vector fields $\{\zeta, \xi\}$ on $\mathcal{F}\times M^{1|1}$ induced by $\{\eta\tau_Q, \partial_t\}$. Explicitly, we have

$$\begin{aligned} \iota(\zeta)dt &= \eta\theta & \iota(\xi)dt &= 1 \\ \iota(\zeta)d\theta &= \eta & \iota(\xi)d\theta &= 0 \\ \iota(\zeta)\delta\Phi &= -\eta\tau_Q\Phi & \iota(\xi)\delta\Phi &= -\partial_t\Phi. \end{aligned} \tag{4.35}$$

The invariance of $\mathcal{L}$ follows *a priori* from the fact that τ_Q, ∂_t commute with D, ∂_t and the fact that the density $|dt|\, d\theta$ is invariant.

Now to the integration. We define the *component lagrangian* L from the superspacetime lagrangian $\mathcal{L}$ by "integrating out" the odd variable θ. This is the *Berezin integral* which in this case amounts to

$$L = (i^* D\ell)\, dt. \tag{4.36}$$

(This is a definite finesse: We simply introduce this definition without more explanation!) So we compute

$$\begin{aligned} i^* D\ell &= -\frac{1}{2} i^* D\langle D\Phi, \partial_t \Phi\rangle \\ &= -\frac{1}{2} i^* \Big\{ \langle \nabla_D D\Phi, \partial_t\Phi\rangle - \langle D\Phi, \nabla_D \partial_t \Phi\rangle \Big\} \\ &= -\frac{1}{2} i^* \Big\{ -|\partial_t \Phi|^2 - \langle D\Phi, \nabla_{\partial_t} D\Phi\rangle \Big\} \\ &= \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}} \psi\rangle. \end{aligned} \tag{4.37}$$

Hence we recover the superparticle component lagrangian (4.12).

The reader would do well to compute that other features of the superspacetime model match the component formulation. For example, compute the action of τ_Q on the component fields using the definition (4.31) of the component fields to recover (4.16). This computation is a bit tricky, but can be viewed as a problem in ordinary differential geometry—the odd variables cause no additional difficulties. Also, you can do the classical mechanics directly on $M^{1|1}$: Compute γ, ω, the equations of motion, the supercharge, etc. This is a nontrivial exercise in calculus on supermanifolds; the component formulas in the text may be used as a check.

Exercises

1. The idea here is to analyze the lagrangian (4.12) for the supersymmetric particle. You can do it from several points of view.

   (a) First, start out working in local coordinates. That is, choose local coordinates $x^1, x^2, \dots, x^d$ on X, and write the Riemannian metric in these coordinates as $g_{ij}dx^i dx^j$. The ordinary particle lagrangian is

   $$L_0 = \Big\{ \frac{1}{2} g_{ij}\dot{x}^i \dot{x}^j \Big\}\, |dt|.$$

   Write the supersymmetric particle lagrangian L (4.12) in this notation. Note that the Christoffel symbols depend on the map x.

   (b) Now compute δL. You can explicitly take a 1-parameter family of fields x_u, ψ_u and differentiate with respect to u. Deduce the classical equations (4.14).

   (c) Do the same computation without introducing coordinates. Be careful to use δ_∇ when covariant derivatives are needed and be mindful of (1.13).

   (d) In your computations the variational 1-form γ (4.13) should be staring at you. Compute the symplectic form ω.

2. (a) Do the computations of the same quantities in superspacetime, as suggested at the end of the lecture.

   (b) Recover the formulas in components from these computations.

3. Check that $\hat{\zeta}$, as defined in (4.16), is a nonmanifest symmetry. In other words, compute $\mathrm{Lie}(\hat{\zeta})L$.

4. (a) Verify (4.24).

   (b) Verify (4.21).

5. Show that there is no way to add a potential term $-x^*V$ to the lagrangian L (4.3) in such a way that supersymmetry is maintained. You are allowed to add terms to (4.16)—change the supersymmetry transformation laws of the fields—but even allowing this it is not possible.

LECTURE 5
Free Theories, Quantization, and Approximation

Quantization of free theories: general theory

Recall that a Hamiltonian system consists of three ingredients: states, observables, and a one-parameter family of motions. In general the observables have a Lie-type bracket on them. In Lecture 1 we saw that for a classical Hamiltonian system the state space is a symplectic manifold $(\mathcal{M}, \Omega)$, the space of observables is $\Omega^0(\mathcal{M})$ with its Poisson bracket, and the motion is by a one-parameter family of symplectic diffeomorphisms generated by a Hamiltonian function $H\colon \mathcal{M} \to \mathbb{R}$. We saw in Lecture 4 that to accommodate fermions, we should allow $\mathcal{M}$ to be a supermanifold with an appropriate symplectic structure; observables may then be even or odd, but the Hamiltonian is necessarily even. Also, we saw in Lecture 3 that for field theories on Minkowski spacetime the one-parameter family of time-translations is embedded in an action of the Poincaré group on $\mathcal{M}$. In Lecture 4 we saw the first glimpses of an extension of the Poincaré group which acts in supersymmetric systems.

In a quantum mechanical system from this Hamiltonian point of view, the state space is a complex (separable) Hilbert space $\mathcal{H}$, the observables are self-adjoint operators on $\mathcal{H}$ with bracket the usual commutator of operators, and there is a Hamiltonian operator $\hat{H}$ which generates a one-parameter group of unitary transformations which represent time translation. The extension which accommodates fermions is algebraic: the state space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and there is a $\mathbb{Z}/2\mathbb{Z}$-grading on the operators—even operators preserve the grading and odd operators exchange $\mathcal{H}^0$ and $\mathcal{H}^1$. The unitary time-translations are required to be even. In a *relativistic* quantum mechanical system the unitary representation of time-translations is extended to a unitary representation of the Poincaré group. (We review the structure of the physically relevant representations later in this lecture.) In supersymmetric systems there is a larger *super Poincaré group* which acts.

The last paragraph is not quite right: The state space is the complex projective space $\mathbb{P}\mathcal{H}$ formed from a Hilbert space $\mathcal{H}$. For the free theories discussed below, we will explicitly see that appropriate groups of symmetries are only represented projectively.

In general there is no canonical way to pass back and forth between a classical system and a quantum system. Rather, there are usually parameters in a theory—whether or not it be classical or quantum—and only for certain limits of the parameters is there a reasonable correspondence. The basis for this is a *precise* correspondence for *free* theories. Recall that a classical system $(\mathcal{M},\Omega,H)$ is free if $(\mathcal{M},\Omega)$ is a symplectic affine space and H is quadratic (so generates a one-parameter family of symplectic affine transformations.)[20] We allow a "super" version of this: the affine space is a supermanifold whose underlying vector space of translations is $\mathbb{Z}/2\mathbb{Z}$-graded. For free classical systems there is a canonically associated quantum system. This has been studied in great detail mathematically, both in case $\mathcal{M}$ is finite dimensional and $\mathcal{M}$ is infinite dimensional. (Quantization in the latter case requires an additional choice.) We give a brief overview here.

Consider first the bosonic case where $(\mathcal{M}^0,\Omega^0)$ is an "ordinary" (even) symplectic affine space. The subspace of affine observables is closed under Poisson bracket; it forms a Heisenberg Lie algebra, a nontrivial central extension of the commutative algebra of translations. As well, the subspace of quadratic observables is closed under Poisson bracket; it is a central extension of the Lie algebra of affine symplectic transformations. Now what we might hope for in (free) quantization is a map

$$\text{(classical observables)}, \{\cdot,\cdot\} \longrightarrow \text{(quantum observables}, [\cdot,\cdot]) \tag{5.1}$$

which is a homomorphisms of Lie algebras. That turns out to be impossible (unless the quantization is trivial). In fact, we only demand (5.1) be a homomorphism on affine functions; it follows that it is a homomorphism on quadratic functions as well. In other words, we would like a representation of the Heisenberg algebra by self-adjoint operators, or, by exponentiation, a unitary representation of the Heisenberg group. Physicists call this a representation of the *canonical commutation relations*. It is a basic theorem that a unitary *irreducible* representation is unique, up to isomorphism.[21] In fact, the infinitesimal representation extends to a representation of quadratic observables, and on the group level we obtain a representation of a cover of the affine symplectic group. This describes the quantization in general terms.

We now give an algebraic description. It depends from the beginning on a choice: We choose both an origin for the affine space $\mathcal{M}^0$ and a *polarization* of the vector space U^0 of translations of $\mathcal{M}^0$, i.e., a decomposition

$$U^0 \cong L \oplus L' \tag{5.2}$$

of the symplectic vector space U^0 as a sum of complementary *lagrangian* subspaces. It is important that we allow a complex polarization, that is, $L, L' \subset U^0 \otimes \mathbb{C}$. Using the choice of origin we identify $\mathcal{M}^0 \cong U^0$, and then polynomial functions on $\mathcal{M}^0$ as elements of $\operatorname{Sym}^\bullet\bigl((U^0)^*\bigr)$. With these choices we take

$$\mathcal{H} = \overline{\operatorname{Sym}^\bullet(L^*) \otimes \mathbb{C}}, \tag{5.3}$$

the Hilbert space completion of the polynomial functions on one of the lagrangian subspaces. The representation on linear observables—that is, elements of $(U^0)^*$—into operators on $\operatorname{Sym}^\bullet(L^*)$ is defined by

$$\begin{aligned} \ell^* &\longmapsto \text{multiplication by } \ell^* \\ \ell'^* &\longmapsto \text{contraction with } \ell'^*, \end{aligned} \tag{5.4}$$

[20] This definition—which I allow may not be standard—has the strange consequence that a harmonic oscillator is a free system.

[21] In the infinite dimensional case we need to specify a class of polarizations to fix the representation. We will do so in our examples by requiring that energy be nonnegative.

where $\ell^* \in L^*$, $\ell'^* \in L'^*$, and the contraction uses the nondegenerate pairing of L^* and L'^* induced by the symplectic form. In the lingo of physics, ℓ^* acts as a *creation operator* and ℓ'^* acts as an *annihilation operator*. The *vacuum* is the element $1 \in \mathrm{Sym}^0(L^*)$. In field theory the 1-particle Hilbert space is $\mathrm{Sym}^1(\mathcal{H})$, the 2-particle Hilbert space $\mathrm{Sym}^2(\mathcal{H})$, etc. One can easily check that the operator bracket matches the Poisson bracket of linear functions, so we have a representation of the Heisenberg algebra. This is a purely algebraic description of the representation on a dense subspace of Hilbert space.

The odd case is exactly parallel if we work in the language of supermanifolds. So suppose (U^1, Ω^1) is an odd symplectic vector space; that is, an odd vector space equipped with a nondegenerate symmetric bilinear form, which we assume is positive definite. In short, we have a Euclidean space, viewed as being odd. Recall that the algebra of functions on U^1 is $\mathrm{Sym}\big((U^1)^*\big)$; i.e., it is the exterior algebra on the dual to the U^1 viewed as an ordinary vector space. Again it is a Poisson algebra, and the affine (linear + constant) and functions of degree ≤ 2 are each closed under Poisson brackets. The affine functions form a nontrivial central extensions of the odd vector space of linear functions with trivial bracket, by analogy with the Heisenberg algebra in the even case. The purely quadratic functions are even and under Poisson bracket form a Lie algebra isomorphic to the algebra of skew-symmetric endomorphisms of the ordinary vector space underlying U^1. As before, we ask to represent the affine functions by a homomorphism into self-adjoint operators on some $\mathbb{Z}/2\mathbb{Z}$-*graded* Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$; it follows that the representation extends to a homomorphism of quadratic functions as well.

It is useful to describe this situation as follows. If f and g are linear functions, and $O(f), O(g)$ the corresponding odd linear operators, then[22]

$$[O(f), O(g)] = i\langle f, g\rangle, \tag{5.6}$$

where $\langle \cdot, \cdot \rangle$ is the symplectic form, and $i = \sqrt{-1}$. The factor of i is there since $O(f)$ is self-adjoint, in the appropriate graded sense.[23] Better to take skew-adjoint operators $c(f) = i^{1/2}O(f)$, in which case we obtain the usual Clifford algebra relation

$$[c(f), c(g)] = -\langle f, g\rangle. \tag{5.7}$$

Then $\mathcal{H}$ is a graded module for the Clifford algebra generated by the linear functions. Note that since $\mathcal{H}$ is complex, we may as well take the complex Clifford algebra. Again: The associative algebra generated by the affine functions with the relation that the commutator be the Poisson bracket is a Clifford algebra, and the quantum Hilbert space is an irreducible graded Clifford module for this Clifford algebra.

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[22] I follow the sign rule, so that the commutator of two homogeneous elements a, b in a $\mathbb{Z}/2\mathbb{Z}$-graded algebra is

$$[a, b] = ab - (-1)^{|a|\,|b|}ba. \tag{5.5}$$

[23] The usual equation $\langle Tv, v'\rangle = \langle v, T^*v'\rangle$ which defines the adjoint picks up a sign if both T and v are odd.

If $\dim U^1$ is even (or infinite), we can describe this Clifford module, the quantum Hilbert space, in terms of polarizations as before. But here we necessarily use complex polarizations, since there are no real isotropic subspaces. Thus we write

$$U^1 \otimes \mathbb{C} \cong L_{\mathbb{C}} \oplus \overline{L}_{\mathbb{C}} \tag{5.8}$$

for a totally isotropic L. We take

$$\mathcal{H} = \overline{\mathrm{Sym}^{\bullet}(L_{\mathbb{C}}^*)} \tag{5.9}$$

as in the even case (5.3), but since $L_{\mathbb{C}}$ is odd this is the exterior algebra on the ordinary vector space underlying $L_{\mathbb{C}}^*$. If U^1 is finite dimensional, for example, then $\mathcal{H}$ is finite dimensional and there is no Hilbert space completion necessary. In any case it is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. The 1-particle subspace $\mathrm{Sym}^1(L_{\mathbb{C}}^*)$ is odd, the 2-particle subspace $\mathrm{Sym}^2(L_{\mathbb{C}}^*)$ is even, etc. The formulas for the action of linear operators are exactly the same as in (5.4). In the finite dimensional case, this is a standard construction of the Clifford module in even dimensions.

As stated above, the representation of the Poisson algebra of linear and constant functions extends to a representation of the Poisson algebra of purely quadratic functions, which recall is isomorphic to the Lie algebra of the orthogonal group. Exponentiating we obtain a representation of the double cover of the orthogonal group, called the Pin group, which should be viewed as $\mathbb{Z}/2\mathbb{Z}$-graded. The identity component is the Spin group, which operates by even unitary transformations. So the Hilbert space in this case is the Hilbert space underlying the spin representation.

Quantization of free theories: free fields

We now apply these general remarks to the case of a free scalar field, as discussed in Lecture 3. We work on Minkowski spacetime M^n, the field is a real-valued function $\phi\colon M^n \to \mathbb{R}$, and the classical field equation (3.6) is the linear wave equation (3.7):

$$(\partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 + m^2)\phi = 0, \tag{5.10}$$

where $m \geq 0$ is the mass of the field. (We work in units where $\hbar = c = 1$.) The space of solutions to this wave equation is an infinite dimensional symplectic vector space $\mathcal{M}$; the symplectic form is the integral of (3.5) over a spacelike hypersurface in M^n. We analyze (5.10) using the Fourier transform. Fix an origin in M^n, so identify ϕ as a function $\phi\colon V \to \mathbb{R}$. Its Fourier transform $\hat{\phi}\colon V^* \to \mathbb{R}$ is the coefficient function in an expansion of ϕ as a linear combination of plane waves. Namely,

$$\phi(x) = \frac{1}{(2\pi)^{n/2}} \int_{V^*} \hat{\phi}(\alpha) e^{i\langle x,\alpha\rangle}\; |d^n\alpha|, \tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between V and V^* and $|d^n\alpha|$ is the density associated to the inverse Lorentz metric on V^*. The Fourier transform $\hat{\phi}$ is a *complex*-valued function, but because ϕ is real $\hat{\phi}$ satisfies the reality condition

$$\hat{\phi}(-\alpha) = \overline{\hat{\phi}(\alpha)}. \tag{5.12}$$

Under Fourier transform derivatives become multiplication operators, so the second-order differential operator (5.10) becomes the quadratic equation

$$(|\alpha|^2 - m^2)\hat{\phi}(\alpha) = 0. \tag{5.13}$$

In other words, the support of the Fourier transform of a solution ϕ lies on the *mass shell*

$$\mathcal{O}_m = \{\alpha \in V^* : |\alpha|^2 = m^2\} \subset V^* \tag{5.14}$$

In case $m > 0$ this is a hyperbola; if $m = 0$ it is the dual lightcone. We have not specified what class of functions ϕ and $\hat{\phi}$ lie in, but at least the Fourier transform must exist. Since equation (5.10) is Poincaré-invariant, the vector space of solutions $\mathcal{M}$ is a *real* representation of the Poincaré group P^n.

According to the general discussion above, the quantization is determined by a polarization of the symplectic vector space $\mathcal{M}$. Here we use a complex polarization, a decomposition of $\mathcal{M} \otimes \mathbb{C}$ into a sum of lagrangians. Now $\mathcal{M} \otimes \mathbb{C}$ is the space of complex-valued functions on $\mathcal{O}_m$ with no reality condition. For $m > 0$ there is a decomposition

$$\mathcal{O}_m = \mathcal{O}_m^+ \cup \mathcal{O}_m^- \qquad \text{(disjoint)}, \tag{5.15}$$

where $\mathcal{O}_m^+ = \mathcal{O}_m \cap \{\alpha_0 > 0\}$ is the subset of covectors of *positive energy* and $\mathcal{O}_m^- = \mathcal{O}_m \cap \{\alpha_0 < 0\}$ the subset of covectors of negative energy. (We use linear coordinates $x^0, x^1, \dots, x^{n-1}$ on V and dual linear coordinates $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ on V^*. In appropriate units the first coordinate α_0 is energy and the remaining coordinates α_i are momenta. The norm square of α is the mass square, which is energy square minus momentum square.) The subspace L_m of $\mathcal{M}_{\mathbb{C}}$ consisting of $\hat{\phi}$ supported on $\mathcal{O}_m^+$ is lagrangian, as is the subspace $\overline{L_m}$ of $\mathcal{M}_{\mathbb{C}}$ consisting of $\hat{\phi}$ supported on $\mathcal{O}_m^-$. Thus we take the Hilbert space of the massive particle to be

$$\mathcal{H} = \overline{\mathrm{Sym}^\bullet(L_m^*)}. \tag{5.16}$$

The subspace $\mathrm{Sym}^\bullet(L_m^*)$ is called the *Fock space*; the Hilbert space is a completion. The 1-particle Hilbert space $\mathrm{Sym}^1(L_m^*)$ is an irreducible (complex) unitary representation of the Poincaré group. In the massless case the decomposition (5.15) has an additional piece:

$$\mathcal{O}_0 = \mathcal{O}_0^+ \cup \mathcal{O}_0^- \cup \{0\} \qquad \text{(disjoint)}. \tag{5.15}$$

We would like to take the lagrangian decomposition as before, so that L_0 consists of Fourier transforms supported on $\mathcal{O}_0^+$ and $\overline{L_0}$ of Fourier transforms supported on $\mathcal{O}_0^-$, but we have the bothersome 0 to worry about. In fact, it is not a problem in dimensions $n \geq 3$, but does manifest itself in two-dimensional field theory (and in quantum mechanics).

There is a similar story for other free fields. In general the Fourier transform of a solution to the linear classical equation of motion has support on a mass hyperbola or lightcone, but rather than simply being a complex-valued function it is a section of a complex vector bundle (which satisfies a reality condition). There is again a complex polarization, determined by the positive energy condition, and a similar picture of the quantum Hilbert space.

Representations of the Poincaré group

A relativistic quantum particle is defined to be an irreducible unitary positive energy representation $\mathcal{H}$ of the Poincaré group P^n. (Recall the definition (1.45) at the end of Lecture 1.) Such representations were classified by Wigner long ago, and we quickly review the construction. First, we restrict the representation to the translation subgroup V. Since V is abelian, the representation decomposes as a direct sum (really direct integral) of one-dimensional representations on which V acts by a character

$$v \longmapsto \text{multiplication by } e^{i\alpha(v)/\hbar}, \qquad \alpha \in V^*.$$

The set of infinitesimal characters α which occur are permuted by the action of the identity component of $O(V)$ on V^*. Since the representation $\mathcal{H}$ is irreducible, these infinitesimal characters form an orbit of the action. There are two types of orbits which have positive energy. These are indicated in the figure below. Notice that the axes are labeled E for energy and p_i for momentum. The mass square $m^2 = E^2/c^4 - \sum p_i^2/c^2$ is constant on an orbit and so is an invariant of an irreducible representation. The two orbit types correspond to *massless* $(m = 0)$ and *massive* $(m > 0)$ representations.

In the two-dimensional case $(n = 2)$ the massless orbit breaks up into two distinct orbits along the two rays of the positive lightcone, as indicated in the next figure. We call them *right movers* and *left movers* since the corresponding characters are functions of $ct - x$ and $ct + x$ respectively. So there is a more refined classification of massless particles in two dimensions.

Now the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit. More precisely, the total space of the bundle carries an action of $\mathrm{Spin}(V)$ covering the action (of its quotient by $\{\pm 1\}$) on the orbit. Such bundles may be constructed by fixing a point on the orbit and constructing a finite dimensional unitary representation of the stabilizer subgroup of that point, whose reductive part is called the *little group*. (Any finite dimensional representation factors through the reductive part.)

Consider first the massive case and fix the mass to be m. For convenience we set $c = 1$ and take as basepoint $(m, 0, \dots, 0)$. Then the stabilizer subgroup, or little group, is easily seen to be isomorphic to $\mathrm{Spin}(n-1)$. Thus a massive particle corresponds to a representation of $\mathrm{Spin}(n-1)$.

In the massless case we consider the basepoint $(1, 1, 0, \ldots, 0)$. The stabilizer subgroup in this case is a double cover of the Euclidean group of orientation-preserving isometries of an $(n-2)$-dimensional Euclidean space. We can see this as follows. The group $O(V)$ is the group of conformal transformations of the $(n-2)$-dimensional sphere. We can view the sphere as the set of rays in the forward lightcone. An element in the identity component of $O(V)$ acts on the forward lightcone, and the subgroup H of transformations which fix a ray R is isomorphic to the identity component of the Euclidean group of the complement plus dilations. The action on points of R is given by the dilation factor at the corresponding point of the sphere, whence the claimed stabilizer subgroup. Massless particles correspond to finite dimensional unitary representations of the corresponding subgroup $\tilde{H}$ of $\mathrm{Spin}(V)$. Such representations are necessarily trivial on the lift to $\tilde{H}$ of the subgroup of H consisting of translations. In other words, they factor through representations of a group isomorphic to $\mathrm{Spin}(n-2)$.

Representations of the little group are classified by their *spin*, (which is called *helicity* in the massless case). What do we mean by the "spin" of a representation of $\mathrm{Spin}(m)$? Suppose W is such a representation. Fix a 2-plane in $\mathbb{R}^m$ and consider the subgroup $SO(2) \subset SO(m)$ of rotations in that plane which fix the perpendicular plane. We work with the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(m)$. Restricted to $\mathrm{Spin}(2)$ the representation W decomposes as a sum of one-dimensional complex representations, on each of which $\mathrm{Spin}(2)$ acts by $\lambda \mapsto \lambda^{2j}$, where $\lambda \in \mathrm{Spin}(2)$ and j is a half-integer. (We use half-integers so that the two-dimensional tautological representation of the $SO(2)$ subgroup of $SO(m)$ is the sum of the representations $j = 1$ and $j = -1$.) The spin of the representation W is the largest $|j|$ which occurs in the decomposition. For example, the trivial representation has spin 0. It represents a *scalar particle*. The m-dimensional defining representation of $SO(m)$ has spin 1; the corresponding particle is sometimes called the *vector particle*. The reader can check that all exterior powers of this representation (except Λ^m) also have spin 1. The spin representations of $\mathrm{Spin}(m)$ have spin 1/2. One obtains higher spin by looking at the symmetric powers of the defining representation.

There are physical reasons why in interacting local quantum field theories one only sees massless particles of low spin. More precisely, in theories without gravity only massless particles of spin 0, spin 1/2, and spin 1 occur. Massless spin 1 particles only occur in *gauge theories*; a theory whose only massless particles have spin 0 and spin 1/2 is a σ-*model*. The *graviton*—the particle which mediates the gravitational force—is a massless particle of spin 2 and in theories of supergravity there are also massless particles of spin 3/2. That's it! There are no massless particles of higher spin in realistic theories.

Given a homogeneous vector bundle, we take $\mathcal{H}$ to be the space of L^2 sections of the bundle over the orbit. (There is an invariant measure on the orbit.)

- Recall that the Poincaré group P^n projects onto the identity component of $O(V)$, which consists of transformations which preserve both orientation and the splitting of the lightcone into forwards and backwards. In a local quantum field theory the *CPT theorem* states that the representation of P^n extends to a (projective) representation of the larger group which allows for orientation reversal (but still preserves the splitting of the lightcone). Elements in the new component are represented by antilinear maps. The condition that the representation extend can be stated in terms of the little group; the precise statement depends on the parity of n. For n even it says that the representation of the little group is self-conjugate, i.e., either real or quaternionic. The statement for n odd is more complicated and we omit it.
- To incorporate fermions we consider representations of the little group on $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. The even part corresponds to bosons, the odd to fermions.
- In unitary local quantum field theories there is a connection between the spin of a particle and its *statistics*—whether it is a boson or a fermion. A particle of integral spin is a boson and a particle of half-integral spin is a fermion. This spin-statistics connection is often violated in nonunitary theories, and in particular in the topological field theories which have mathematical applications.

Free fermionic fields

Given a particle representation of P^n we can ask for a free field theory whose 1-particle Hilbert space is the given representation. We saw earlier that a real scalar field gives a spin 0 particle. The spin 1 particle is the 1-particle Hilbert space associated to the lagrangian (3.24) on a quotient (3.25) of 1-forms. (This is a nice exercise.) Thus it remains for us to construct the spin 1/2 representation of Poincaré as the quantization of a free field. The spin-statistics theorem implies that it should be a fermionic field. We will not carry out the quantization in these notes—that is left for the exercises (or references)—but we will describe the theory.

Let S be any *real* spin representation of $\mathrm{Spin}(V)$. We build a theory whose space of fields is

$$\mathcal{F} = \mathrm{Map}(M^n, \Pi S), \tag{5.17}$$

a space of spinor fields on M^n. The fermionic field is odd—it is a map into the odd vector space ΠS—and so the space $\mathcal{F}$ is an infinite dimensional odd vector space. One remarkable fact about the spin group in Lorentz signature, not true in other signatures, is that the symmetric square of any real spinor representation contains a copy of the vector representation, except for $n = 2$. In fact, there always exists a symmetric Spin(V)-equivariant pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V. \tag{5.18}$$

We will not prove these facts about the spin group, but rather illustrate them in a few cases.

For $n = 2$ the identity component $SO^+(V)$ of $O(V)$ is isomorphic to the multiplicative group $\mathbb{R}^{>0}$, so $\mathrm{Spin}(V) \cong \mathbb{R}^{>0} \times \mathbb{Z}/2\mathbb{Z}$. In any spinor representation the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ acts by -1. There are two inequivalent irreducible real spinor representations $S^{\pm}$ on which $\lambda \in \mathbb{R}^{>0}$ acts as $\lambda^{\pm 1}$. The vector representation $V \cong (S^+)^{\otimes 2} \oplus (S^-)^{\otimes 2}$.

For $n = 3$ we have $\mathrm{Spin}(V) \cong SL(2;\mathbb{R})$. There is a unique irreducible spinor representation S of dimension 2, the standard representation of $SL(2;R)$, and $V = \mathrm{Sym}^2(S)$.

For $n = 4$ we have $\mathrm{Spin}(V) \cong SL(2;\mathbb{C})$. There is again a unique real spinor representation S, the real 4-dimensional representation underlying the standard 2-dimensional representation S' of $SL(2;\mathbb{C})$. Let S'' be the conjugate to S'; then $V \otimes \mathbb{C} \cong S' \otimes S''$.

The exceptional isomorphisms for orthogonal groups continue up to dimension $n = 6$; in that case $\mathrm{Spin}(V) \cong SL(2;\mathbb{H})$. It is significant that dimensions 3,4,6 have Lorentz spin groups isomorphic to $SL(2;\mathbb{F})$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is even a sense in which the Lorentz spin group in $n = 10$ dimensions is $SL(2)$ over the octonions!

In any dimension either there is a unique irreducible real spinor representation S and any real spinor representation has the form $S^{\oplus N}$ for some N, or there are two distinct irreducible real spinor representations $S, \tilde{S}$ and any spinor representation has the form $S^{\oplus N} \oplus \tilde{S}^{\oplus \tilde{N}}$ for some $N, \tilde{N}$. The latter occurs in dimensions $n \equiv 2, 6 \pmod 8$.

In general, given S and the pairing (5.18) we write a kinetic lagrangian

$$L_{\text{kinetic}} = \left\{ \frac{1}{2} \tilde{\Gamma}(\psi, \partial\psi) \right\} |d^n x|. \tag{5.19}$$

Choose a basis $\{e_\mu\}$ for V and a basis $\{f^a\}$ of S. Then we can expand a spinor field ψ as

$$\psi(x) = \psi_a(x) f^a. \tag{5.20}$$

Write

$$\tilde{\Gamma}(f^a, f^b) = \tilde{\Gamma}^{\mu ab} e_\mu. \tag{5.21}$$

Then the kinetic term may be written

$$\tilde{\Gamma}(\psi, \partial\psi) = \tilde{\Gamma}^{\mu ab} \psi_a \partial_\mu \psi_b = \psi \not{D} \psi, \tag{5.22}$$

the last expression being the most common. Note that it has the same general form as the fermionic term we used in (4.3); it is a product of the fermionic field with its first derivative.

For a given spinor representation S there may or may not be a mass term possible. A mass term is specified by a quadratic function $\Pi S \to \mathbb{R}$, that is, by a skew-symmetric pairing

$$M : \textstyle\bigwedge^2 S \longrightarrow \mathbb{R}. \tag{5.23}$$

A nonzero pairing may or may not exist. Then the full lagrangian is

$$L = \Big\{ \frac{1}{2}\, \psi \not{D} \psi - \frac{1}{2}\, \psi M \psi \Big\}\, |d^n x|. \tag{5.24}$$

The 1-particle Hilbert space of this free fermionic field is a spin 1/2 representation of P^n which is the space of sections of a vector bundle of rank $\dim S/2$ over the appropriate mass shell. There is an exception to this last description in $n = 2$, where it is possible to have the Fourier transform of a massless spinor field supported on half of the lightcone; then it is a section of a vector bundle of rank $\dim S$.

The general free theory

To specify a free theory of scalar, spinor, and 1-form fields we need to give the precise field content and the masses of the fields. Since we have not developed the theory of Lorentz spinors in detail, we will be somewhat schematic about the spin 1/2 fields.

Fix a dimension n. Let $S_0, \tilde{S}_0$ be the irreducible real spinor representations, with fixed pairings (5.18); if there is only one real representation, then set $\tilde{S} = 0$. The kinetic data for the free theory is a set of even real[24] vector spaces each equipped with positive definite inner product:

$$W_0, W_{1/2}, \tilde{W}_{1/2}, W_1 \qquad \text{positive definite inner product spaces} \tag{5.25}$$

The spinor fields take values in the odd vector space

$$\mathcal{W}^{\text{odd}}_{1/2} = \Pi S_0 \otimes W_{1/2} \;\oplus\; \Pi \tilde{S}_0 \otimes \tilde{W}_{1/2}. \tag{5.26}$$

In a free theory we can have quadratic potential functions, which are mass terms and are specified by *mass matrices*. These are nonnegative quadratic forms

$$\begin{aligned} M_0 &\in \operatorname{Sym}^2(W_0^*) \\ M_{1/2} &\in \operatorname{Sym}^2\big((\mathcal{W}^{\text{odd}}_{1/2})^*\big) \\ M_1 &\in \operatorname{Sym}^2(W_1^*). \end{aligned} \tag{5.27}$$

The fields in the theory all live in linear spaces, as expected for a free theory. A field is a triple $\Phi = (\phi, \psi, \alpha)$ where

$$\begin{aligned} &\phi \colon M^n \longrightarrow W_0 \\ &\psi \colon M^n \longrightarrow \mathcal{W}^{\text{odd}}_{1/2} \\ &\alpha \in \Omega^1(M^n; W_1)/d\Omega^0(M^n; W_1) \end{aligned} \tag{5.28}$$

[24] In some dimensions we take $S_0, \tilde{S}_0$ and $W_{1/2}, \tilde{W}_{1/2}$ to be complex conjugate vector spaces.

Recall the gauge equivalence we have on 1-forms, which is the reason to divide by exact 1-forms. The free field lagrangian is quadratic. The fields are completely decoupled. For each field there is a kinetic term and a mass term:

$$L = \Big\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle \psi, \not{D}\psi\rangle - \frac{1}{2}|\alpha|^2 \\ - \frac{1}{2}M_0(\phi) - \frac{1}{2}M_{1/2}(\psi) - \frac{1}{2}M_1(\alpha) \Big\} \, |d^n x|. \tag{5.29}$$

There is an associated quantum field theory, which may be specified by its 1-particle ($\mathbb{Z}/2\mathbb{Z}$-graded) Hilbert space. The even part is a sum of spin 0 and spin 1 representations; the odd part a sum of spin 1/2 representations. To describe it we need to compute the (nonnegative real number) masses which occur and describe a vector space of particles with that mass. For the bosons ϕ, α this is straightforward: using the inner products on W_0, W_1 we may express the mass forms M_0, M_1 as nonnegative symmetric matrices and then decompose W_0, W_1 according to their eigenvalues and eigenspaces. The eigenvalues are the masses and the eigenspaces encode the multiplicity of particles with a particular mass. For the spinor field the computation of masses involves a bit more algebra of spinors.

General theory

The theories of interest are not free, of course, and there is a wide variety of physically interesting lagrangians which satisfy the basic criteria outlined at the beginning of Lecture 3. We extract a general class of lagrangians which covers many examples, just as we described a general class of bosonic theories (gauged σ-models) at the end of Lecture 3. These are theories with fields of spins 0, 1/2, and 1.

The data we need to write the fields and kinetic terms is:

(5.30)

G	Lie group with Lie algebra $\mathfrak{g}$
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
X	Riemannian manifold on which G acts by isometries
$W, \tilde{W} \longrightarrow X$	real vector bundles with metrics, connections, and orthogonal G action

Consider $\Pi S_0, \Pi\tilde{S}_0$ as constant vector bundles over X, and define

$$\mathcal{W}^{\text{odd}} = \Pi S_0 \otimes W \;\oplus\; \Pi\tilde{S}_0 \otimes \tilde{W} \longrightarrow X. \tag{5.31}$$

A field is then a triple $\Phi = (\phi, \psi, A)$, where

(5.32)

A	connection on a principal G-bundle $P \to M$
ϕ	G-equivariant map $P \to X$
ψ	G-equivariant lift of ϕ to $\mathcal{W}^{\text{odd}}$

The collection of fields is a category $\mathcal{F}$, in fact, a groupoid—all the morphisms in the category are equivalences (invertible). If we divide by these equivalences (think of them as gauge transformations), we obtain the space $\overline{\mathcal{F}}$ of equivalence classes of fields. The kinetic lagrangian is:

$$L_{\text{kinetic}} = \Big\{\frac{1}{2}|d_A\phi|^2 + \frac{1}{2}\langle\psi, D\!\!\!/_{A,\phi}\psi\rangle - \frac{1}{2}|F_A|^2\Big\}\;|d^nx|. \tag{5.33}$$

In this formula d_A is computed using the connection A and the Dirac operator $D\!\!\!/_{A,\phi}$ on M uses the connection A as well as the pullback of the connections on $W, \tilde{W}$ via the map ϕ. Note that for fixed ϕ the field ψ is a spinor field on M coupled to a vector bundle over M whose connection depends on both A and ϕ.

While the kinetic terms always look this way, there is a variety of possible potential terms. Certainly the basic one is a potential for ϕ and ψ:

$$V = V^{(0)} + V^{(2)} + \cdots \qquad G\text{-invariant section of } \operatorname{Sym}^{\text{even}}\big((\mathcal{W}^{\text{odd}})^*\big) \tag{5.34}$$

Then the lagrangian of the theory is

$$L = L_{\text{kinetic}} - V(\phi,\psi)\;|d^nx|. \tag{5.35}$$

The scalar potential $V^{(0)}\colon X \to \mathbb{R}$ is the component of V whose value lies in $\operatorname{Sym}^0\big((\mathcal{W}^{\text{odd}})^*\big)$. Mass terms for spinor fields are included in $V^{(2)}$, as are *Yukawa couplings*. The "4-fermi term" $V^{(4)}$ occurs in supersymmetric σ-models (as in (7.31)) and also in supergravity theories. One can view the total potential V as an even function on the supermanifold determined by the vector bundle $\mathcal{W}^{\text{odd}} \to X$.

There are additional possible potential terms. For example, in dimension $n = 2$ the data

$$\langle\cdot\rangle \qquad \text{Ad-invariant linear form on } \mathfrak{g} \tag{5.36}$$

determines a term

$$\langle F_A\rangle \tag{5.37}$$

in the lagrangian. This term is a 2-form, not a density, so a theory with this term in it is not invariant under isometries of M^2 which reverse the orientation; we need to fix an orientation to integrate (5.37) so to define the action.

The theory we have defined is Poincaré-invariant. There is also a commuting compact Lie group of global manifest symmetries—it is the group which preserves all of the given data. It acts by isometries on X and $\mathcal{W}^{\text{odd}}$, commutes with the G action, and preserves the potentials.

A vacuum solution of the general theory has A trivial, $\psi = 0$, and ϕ constant. (Compare with the discussion of bosonic models in Lecture 3.) Note that any vacuum solution is Poincaré-invariant. So the moduli space of vacua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{5.38}$$

Perturbation theory

In a general lagrangian field theory we can consider "small fluctuations" of the fields around any fixed $\Phi_0 \in \mathcal{F}$. The space of these small fluctuations is simply the tangent space $T_{\Phi_0}\mathcal{F}$. The idea is to construct a new lagrangian field theory whose space of fields is $T_{\Phi_0}\mathcal{F}$ and whose lagrangian is an approximation to the lagrangian L of the original theory. More precisely, the approximate lagrangian is the N^{th} order Taylor series of L at Φ_0. The quadratic approximation gives a free field theory. It is typical to consider such an approximation at a vacuum solution Φ_0, rather than at an arbitrary point of field space. The free field theory approximation may be quantized, as discussed earlier in this lecture, and that is the first step in understanding the perturbative quantum theory at the particular vacuum in question. The information in the free quantum field theory—the vector spaces (5.25) and the mass matrices (5.27)—may be read off from the geometric data. This is an exercise in differential geometry; Feynman diagrams (and more subtle quantum reasoning) enter only when we keep higher order terms in the approximate lagrangian.

We work with a general theory, as described in the previous section. Fix $\Phi_0 = (A_0, \phi_0, \psi_0)$, which we assume to be a vacuum solution. So A_0 is a trivial connection, $\psi_0 = 0$, and fixing a trivialization ϕ_0 is a constant map to X. Recall that the fields $\mathcal{F}$ form a category, and we work essentially on the space $\overline{\mathcal{F}}$ of equivalence classes. The tangent space at the equivalence class $[\Phi_0]$ of Φ_0 may be described by expressed by specifying the vector spaces (5.25). To that end, note that the infinitesimal G action on X induces a linear map

$$\rho_{\phi_0} \colon \mathfrak{g} \longrightarrow T_{\phi_0}X. \tag{5.39}$$

Then the vector spaces are

$$\begin{aligned} W_0 &= \operatorname{coker}\rho_{\phi_0} \\ W_{1/2}, \tilde{W}_{1/2} &= \text{fibers of } W, \tilde{W} \text{ at } \phi_0 \\ W_1 &= \mathfrak{g} \end{aligned} \tag{5.40}$$

We leave it as an exercise for reader to write the lagrangian to second order at $\Phi_0 = (A_0, \phi_0, 0)$ and so to derive the mass matrices

$$\begin{aligned} M_0 &= \operatorname{Hess}_{\phi_0} V^{(0)} \\ M_{1/2} &= V^{(2)}(\phi_0) \\ M_1 &= \text{pullback of metric on } T_{\phi_0}X \text{ under } \textit{action} \end{aligned} \tag{5.41}$$

This completely specifies the free field approximation at the vacuum Φ_0.

There is some terminology associated to the free field approximation:

1. Let $G_{\phi_0} \subset G$ be the stabilizer group of the G action at $\phi_0 \in X$. Physicists say, "The gauge group G is broken to G_{ϕ_0}." The 1-forms with values in its Lie algebra $\mathfrak{g}_{\phi_0} \subset \mathfrak{g}$, which is the kernel of ρ_{ϕ_0}, are the the massless 1-form fields in the free field approximation. The remaining 1-form fields are massive. The fact that these fields have a mass is called the *Higgs mechanism*.

2. Let $[\phi_0]$ denote the equivalence class of ϕ_0 in $\mathcal{M}_{\text{vac}} = V^{-1}(0)$. Assume that $[\phi_0]$ is a smooth point. Then the massless scalar fields in the free field approximation are maps into $T_{[\phi_0]}\mathcal{M}_{\text{vac}}$.
3. The global symmetry group H is broken to the subgroup H_{ϕ_0} which fixes ϕ_0. The quotient of the Lie algebras $\mathfrak{h}/\mathfrak{h}_{\phi_0}$ is a subspace of $T_{[\phi_0]}\mathcal{M}_{\text{vac}}$. Massless scalar fields in this subspace are called *Goldstone bosons*.

Supersymmetric theories put restrictions on the data which defines a theory. The restrictions depend on the dimension n and on the particular supersymmetry. In various cases it constrains X to be Kähler, hyperkähler, etc. It is useful to view theories in terms of this geometric data to keep track of the zoo of examples.

Exercises

1. In this problem consider a two-dimensional symplectic affine space, which we take to have affine coordinates p, q and symplectic form $dp \wedge dq$.

   (a) Compute the Lie algebra of functions at most quadratic in p, q. What are the symplectic gradients? Do the same for affine functions.

   (b) Quantize this space: Choose a lagrangian decomposition, build the symmetric algebra, etc.

   (c) Quantize the free particle on $\mathbb{E}^1$ and the harmonic oscillator, described as a particle moving on $\mathbb{E}^1$ subject to the potential $V(x) = (k/2)\,x^2$ for some $k > 0$.

2. (a) Quantize a $2n$-dimensional symplectic affine space explicitly. Choose coordinate functions $p_i, q^j,\ 1 \le i, j \le n$, so that the symplectic form is $dp_i \wedge dq^i$. Choose L to be the span of the $\partial/\partial q^j$, etc. You should see creation and annihilation operators explicitly.

   (b) Repeat for the odd case.

3. Verify that the subspaces in the lagrangian decomposition we used to quantize scalar fields are indeed isotropic.

4. (a) Quantize a free 1-form field. This is a bit tricky because of the gauge invariance. You need to Fourier transform solutions to the wave equation and identify them as sections of a bundle over some mass shell.

   (b) Quantize a free fermion field. The story in general requires more algebra of spinors than I have given here. So you should try some examples in 2 and 3 dimensions.

5. (a) Write the lagrangian for the general theory (5.35) as explicitly as you can. Surely you'll want to consider special cases. For example, start with no fermions. You may take X to be a vector space, and perhaps start with an abelian gauge group, or no gauge group at all. Then take the vector bundle W to be a constant vector space.

   (b) Recover all lagrangians considered in these lectures as special cases of the general theory.

6. Write the quadratic approximation to the general theory at a vacuum. Verify that you get the vector spaces and mass matrices given in (5.40) and (5.41).

7. Translate the following description of theories to the geometric data (5.30) and (5.34). What are the global symmetry groups of these models?
   (a) "A theory with gauge group $SU(2)$, a massless adjoint scalar, and a massive spinor in the vector representation."
   (b) "A nonlinear σ-model with target S^n and a circle subgroup of $SO(n+1)$ gauged."
   (c) "A theory with gauge group $SU(3) \times SU(2) \times U(1)$ with fermions in the (3,2,1) representation, fermions in the (1,2,1) representation, and scalar fields in the (1,2,1) representation."

8. Compute the moduli space of vacua and the particle content (and masses) at the vacua for each of the following theories.
   (a) Gauge group $U(1)$ and with charged complex scalar fields, i.e., $X = \mathbb{C}$ with the standard action of $U(1)$.
   (b) The same theory with potential
   $$V(\phi) = \|\phi\|^2(1 - \|\phi\|^2)^2.$$
   (c) Let $\mathbb{T}$ denote the circle group of unit norm complex numbers. Then $G = \mathbb{T} \times \mathbb{T}$ with standard inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}\sqrt{-1}\mathbb{R} \times \sqrt{-1}\mathbb{R}$. Take $X = \mathbb{C}$ with the G-action $(\lambda_1, \lambda_2) \cdot z = \lambda_1 z \lambda_2^{-1}$ and the G-invariant potential as above.
   (d) Add spinor fields to these theories. You may work in low dimensions if necessary.

LECTURE 6
Supersymmetric Field Theories

Introductory remarks and overview

We continue to work with a Poincaré-invariant quantum field theory defined on Minkowski spacetime[25] M^n, and restrict to theories of the general type outlined in the last lecture. In particular, they are theories of scalar fields, spinor fields, and gauge fields. Earlier we remarked that under certain hypotheses there is a theorem of Coleman-Mandula which asserts that the group of global symmetries of the theory has the form $P^n \times H$, where P^n is the Poincaré group and H a commuting compact Lie group. Given a theory in terms of geometric data we can read off the group H. Turning this around, we can specify H in advance and then ask for a theory with symmetry group H. For example, a free theory is determined by a set of vector spaces (5.25) and mass matrices (5.26), and imposing a symmetry group H means that the vector spaces carry a representation of H which fix the mass matrices. A similar constraint holds on the nonlinear data (5.30) for nonfree theories.

The Coleman-Mandula theorem seemed to rule out other possible symmetry groups, but in the 1970s another possibility was discovered. Namely, in each dimension n there are extensions of the Poincaré group P^n to *super Poincaré groups* $P^{n|s}$, and quantum field theories may admit these super Lie groups as symmetry groups. Furthermore, a generalization of the Coleman-Mandula theorem, due to Haag-Lopuszański–Sohnius, states that the only allowed super Lie groups are the product of a super Poincaré group $P^{n|s}$ and a compact Lie group H. In this lecture we introduce these super Lie groups.

A super Lie group is, naturally, the marriage of a supermanifold and a Lie group, or, in street lingo, "a group object in the category of supermanifolds." We do not treat supermanifolds systematically, though, and in any case will only really use the infinitesimal version. A *super Lie algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \tag{6.1}$$

equipped with a bracket operation

$$[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \tag{6.2}$$

[25] As in previous lectures, we take M^n to be an affine space with underlying vector space of translations V.

which is skew-symmetric and satisfies the Jacobi identity. The skew-symmetry and Jacobi must be written using the sign rule, and it is understood that the bracket has degree 0. Thus $[\mathfrak{g}^1, \mathfrak{g}^1] \subset \mathfrak{g}^0$ and this operation is symmetric as a map of ungraded vector spaces.

We use the notation $\mathfrak{p}^{n|s}$ for a supersymmetric extension of the Poincaré algebra $\mathfrak{p}^n = \mathrm{Lie}(P^n)$ whose odd part has dimension s. The corresponding super Poincaré group is denoted $P^{n|s}$. A theory with symmetry group $P^{n|s}$ is said to have s supersymmetries. As we see below, the odd part of $\mathfrak{p}^{n|s}$ is a real spinor representation of the Lorentz group $\mathrm{Spin}(V)$ in dimension n, so its dimension s is bounded below by a number which increases exponentially with n, roughly $2^{n/2}$.

We saw the simplest super Poincaré algebra $\mathfrak{p}^{1|1}$ in Lecture 4, e.g., equation (4.24).

Suppose now we specify a super Poincaré group $P^{n|s}$ and ask for a theory with symmetry group $P^{n|s}$. For free theories this determines a restriction on the possible vector spaces (5.25) (particle content) and masses (5.26) which can occur. Free classical theories give rise to quantum theories and in particular to the unitary representation of the Poincaré group P^n on the 1-particle Hilbert space. The restrictions on particle content and masses may be seen by requiring that this representation extend to a representation of $P^{n|s}$. It is not too difficult to catalog representations of a given $P^{n|s}$ which correspond only to free scalar fields, spinor fields, and 1-form fields. Recall that an irreducible representation of P^n may be realized on a Hilbert space of sections of a homogeneous vector bundle over a mass shell. The rank of that vector bundle is the number of *physical degrees of freedom*. An irreducible representation of $P^{n|s}$ may be realized as the space of sections of a homogeneous $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over a mass shell. One important feature is that the number of bosonic degrees of freedom equals the number of fermionic degrees of freedom, except in special two-dimensional cases (which we will not meet in these lectures).

Possible nonfree classical field theories with super Poincaré symmetry are similarly constrained. In fact, the free approximation at any vacuum gives rise to a representation of $P^{n|s}$ as above, and so the constraints on free theories give rise to constraints on nonfree theories. Furthermore, the existence of a given free theory suggests the existence of a corresponding nonfree theory. This logic was used by Nahm in the late '70s for theories with gravity to predict the existence of certain supergravity theories, which were then constructed quite rapidly.

# SUSY	maximal dimension	σ-model	gauge theory
1	2	S	S
2	3	S	S
4	4	S (Kähler)	S
8	6	✓(hyperkähler)	S
16	10		✓

Supersymmetric theories

The chart gives an overview of supersymmetric field theories. The first column shows the number of supersymmetries, which is a power of 2. The second column shows the maximum dimension of spacetime in which that number of supersymmetries may occur. Given a theory in n spacetime dimensions with s supersymmetries, there are corresponding theories in all dimensions $n' \leq n$ by a process called *dimensional reduction*. Namely, if we have a theory on M^n, and we pick a subspace $U \subset V$ of *spatial* translations—the induced metric on U is negative definite—then given a theory with space of fields $\mathcal{F}$ and lagrangian density L, we can restrict the theory to the subspace of fields invariant under translations in U. That subspace of fields can be identified with a space of fields on a lower dimensional Minkowski spacetime.

A few remarks on this table:

- In Lecture 3 we discussed two formulations of the supersymmetric particle. First, we described the theory in *components*, that is, in terms of ordinary fields. Then we gave a super(space)time formulation in terms of *superfields*. All theories with a superspacetime formulation (marked 'S' in the table) may be described in terms of either superfields or in terms of component fields. Theories with no superspacetime formulation only have a description in terms of "component fields", though these fields are not the components of anything—they are just fields on ordinary Minkowski spacetime. The component formulation of theories which have a superspacetime formulation often includes *auxiliary fields* which enter the lagrangian algebraically.
- An 'S' in the table indicates that there is a superspacetime formulation; a '✓' indicates that the theory exists, but there is no adequate superspacetime formulation (off-shell); and a blank spot in the table means that there is no theory. The blank spot can be predicted from the representation theory of the supersymmetry group.
- In a σ-model with 4 supersymmetries the target manifold X is constrained to be Kähler; in a model with 8 supersymmetries it must be hyperkähler.
- There are special theories not obtained by dimensional reduction, and even in theories which are dimensional reductions there are sometimes terms one can add to the lagrangian which do not come from the higher dimensional theory.
- Theories with a superspacetime formulation have manifest supersymmetry in the superspacetime formulation. The supersymmetry algebra closes off-shell. In the component formulation of such theories the supersymmetry is not manifest, but the algebra still closes off-shell. (For that we need to include the auxiliary fields if there are any.) If there is no superspacetime formulation, then the supersymmetry algebra does not close off-shell. (The coupled harmonic oscillators in the exercises for Lecture 2 provide an analog of these phenomena in ordinary classical mechanics.)
- The superspacetime formulations are most useful for understanding the supersymmetry, since in this way it is manifest. However, to see the physics of the theory it is often best to work in components. Some computations are easier in superspacetime, some easier in components. If there is a superspacetime formulation, then it is useful in the quantum theory to give *a priori* constraints on the possible quantum corrections. Such corrections must respect the supersymmetry, and the constraints imposed are more easily seen in superspacetime.

- The most commonly used superspacetimes are those in dimensions $n \le 4$ with $s \le 4$ supersymmetries. For theories with more supersymmetry in dimensions $n \le 4$, we can still use the $s = 4$ superspacetime to keep part of the supersymmetry manifest. Nevertheless, superspacetimes with $s > 4$ have striking applications.
- We indicated that there is a superspacetime formulation of supersymmetric gauge theories with 8 supersymmetries, but this only applies to *pure* gauge theories. The superspacetime formulation in 6 dimensions is a bit deficient in some ways; the reduction to 4 dimensions is better.
- There are superspacetime formulations of σ-models with 8 supersymmetries for special kinds of hyperkähler manifolds, but as far as I know none which works in general.

Super Minkowski spacetime and the super Poincaré group

Recall that Minkowski spacetime M^n is an affine space whose underlying vector space V of translations has a Lorentz metric, and that the Poincaré group P^n is a cover of the component of affine symmetries of M^n which preserve the metric and contains the identity. The constructions in this section provide a generalization to supermanifolds with nontrivial odd part.

Fix a dimension n. We give a general description for any n, and then describe things more explicitly for small n. To define a superspacetime we need to fix a real spin representation S, which we assume has dimension s. Recall that this is the data which we used in Lecture 5 to define the fermion field. The important extra ingredient is the symmetric pairing (5.18)

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V. \tag{6.3}$$

It was used in (5.19) to write the kinetic lagrangian for the spinor field. The supersymmetry algebra depends on a related pairing

$$\Gamma\colon S^* \otimes S^* \longrightarrow V, \tag{6.4}$$

which is used, as we will see, to write "square roots" of translations. Both Γ and $\tilde{\Gamma}$ are positive definite in the sense that once we choose a positive cone $C \subset V$ of timelike vectors, then

$$\Gamma(s^*, s^*), \tilde{\Gamma}(s, s) \in \overline{C} \tag{6.5}$$

for all $s \in S$, $s^* \in S^*$, and these quantities vanish only when the input vanishes. Furthermore, there is a Clifford relation between Γ and $\tilde{\Gamma}$ which is specified most easily in terms of bases. Let $\{P_\mu\}$ be a basis of V and $\{Q^a\}$ a basis of S, with dual basis $\{Q_a\}$ of S^*. Then we write

$$\begin{aligned} \Gamma(Q_a, Q_b) &= \Gamma^\mu_{ab} P_\mu \\ \tilde{\Gamma}(Q^a, Q^b) &= \tilde{\Gamma}^{\mu ab} P_\mu. \end{aligned} \tag{6.6}$$

Let $g_{\mu\nu}$ be the coefficients of the Lorentz metric with respect to the basis $\{P_\mu\}$, and $g^{\mu\nu}$ the coefficients of the inverse metric on V^*. Then the Clifford relation is

$$\Gamma^\mu_{ab}\, \tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\, \tilde{\Gamma}^{\mu bc} = 2 g^{\mu\nu} \delta^c_a, \tag{6.7}$$

where δ is the usual Kronecker δ-function. The theory of spin representations in Lorentz signature guarantees the existence of $\Gamma, \tilde{\Gamma}$ with these properties; in fact, Γ determines $\tilde{\Gamma}$ uniquely.

As mentioned we construct a Lie algebra in which elements of S^* act as square roots of infinitesimal translations, which are elements of V. Thus introduce the $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra

$$\mathcal{L} = V \oplus S^* \tag{6.8}$$

with V central and the nontrivial odd bracket

$$[Q_a, Q_b] = -2\Gamma^{\mu}_{ab} P_{\mu}. \tag{6.9}$$

There is a corresponding super Lie group whose underlying supermanifold is *super Minkowski spacetime*

$$M^{n|s} = M^n \times \Pi S^*. \tag{6.10}$$

Corresponding to the given bases on V and S^* are linear coordinates x^{μ} on V and θ^a on the odd supermanifold ΠS^*. So altogether x^{μ}, θ^a are global coordinates on $M^{n|s}$. The coordinate vector fields are $\partial_{\mu}, \partial/\partial\theta^a$. Now the action of the Lie algebra $\mathcal{L}$ on $M^{n|s}$ gives rise to a basis $\{\partial_{\mu}, D_a\}$ of left-invariant vector fields and a commuting basis $\{\partial_{\mu}, \tau_{Q_a}\}$ of *right* invariant vector fields. Note that ∂_{μ} is both left and right invariant since V is central. Also, as usual right invariant vector fields give rise to left actions, so that the τ_{Q_a} are part of the infinitesimal left action of $P^{n|s}$. The vector fields D_a and τ_{Q_a} are given by the formulas

$$\begin{aligned} D_a &= \frac{\partial}{\partial\theta^a} - \Gamma^{\mu}_{ab}\theta^b\partial_{\mu} \\ \tau_{Q_a} &= \frac{\partial}{\partial\theta^a} + \Gamma^{\mu}_{ab}\theta^b\partial_{\mu}. \end{aligned} \tag{6.11}$$

as the reader may check. The nontrivial brackets are

$$\begin{aligned} [D_a, D_b] &= -2\Gamma^{\mu}_{ab}\partial_{\mu} \\ [\tau_{Q_a}, \tau_{Q_b}] &= +2\Gamma^{\mu}_{ab}\partial_{\mu}. \end{aligned} \tag{6.12}$$

The brackets of the left invariant D_a are as in the Lie algebra $\mathcal{L}$; brackets of the right invariant τ_{Q_a} are opposite. (This is a general feature of right and left actions, as explained in the text preceding (1.17).) Also

$$[D_a, \tau_{Q_b}] = 0 \tag{6.13}$$

since left invariant vector fields commute with right invariant vector fields.

The super Poincaré algebra is the graded Lie algebra

$$\mathfrak{p}^{n|s} = \big(V \oplus \mathfrak{so}(V)\big) \ \oplus \ S^*. \tag{6.14}$$

Its even part is the usual Poincaré algebra. It also contains the super Lie algebra (6.8) of translations as a subalgebra. The bracket of elements of $\mathfrak{so}(V)$ and S^*

is by the infinitesimal spin representation. The super Poincaré group is the semidirect product $\mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$, expressed by the split exact sequence

$$1 \longrightarrow \exp(\mathcal{L}) \longrightarrow P^{n|s} \longrightarrow \mathrm{Spin}(V) \longrightarrow 1. \tag{6.15}$$

We mention briefly two more concepts connected with the super Poincaré algebra and illustrate them below. First, it may happen that $\mathrm{Sym}^2 S^*$ contains some copies of the trivial representation, i.e., that there is a symmetric pairing

$$S^* \otimes S^* \longrightarrow \mathbb{R}^c. \tag{6.16}$$

Then we can form new super Lie algebras by adding $\mathbb{R}^{c'}$ to the even part of (6.14) for any $c' \leq c$:

$$\tilde{\mathfrak{p}}^{n|s} = \big(V \oplus \mathbb{R}^{c'} \oplus \mathfrak{so}(V)\big) \;\oplus\; S^*. \tag{6.17}$$

The subspace $\mathbb{R}^{c'}$ is central; its elements are called *central charges*. Second, there may be outer automorphisms of $\mathfrak{p}^{n|s}$ which fix the Poincaré algebra. These are called infinitesimal *R-symmetries*; the connected group we obtain by exponentiation is the *R-symmetry group*. The R-symmetry group is compact, since the pairing Γ is positive definite.

The central charges already arise in classical field theories. This is a general feature of symplectic geometry, which is encoded in the exact sequence (1.18). Namely, if $\mathfrak{p}^{n|s}$ is a Lie algebra of symmetries of some theory, then the corresponding Lie algebra of observables is in general a central extension. In the quantum theory it is the Lie algebra of classical observables which gives rise to a Lie algebra of quantum observables—self-adjoint operators.

The infinitesimal R-symmetries, which act in a quantum theory as automorphisms of the symmetry algebra, are represented (projectively) on the Hilbert space of the theory.

Examples of super Poincaré groups

We already met $\mathfrak{p}^{1|1}$ when we discussed the supersymmetric particle. There is a single even infinitesimal translation P, which is infinitesimal time translation, and a single odd infinitesimal translation Q; the nontrivial bracket is

$$[Q, Q] = -2P \tag{6.18}$$

as above. The Lie algebra of the Lorentz group is trivial, but the Lorentz group is cyclic of order 2; its action on Q is nontrivial. (The nonidentity element of the Lorentz group maps Q to $-Q$.) There are no nontrivial R-symmetries. More conceptually, the basic real spin representation of the Lorentz group $\mathrm{Spin}(V) \cong \mathbb{Z}/2\mathbb{Z}$ is a one-dimensional space S, and we identify[26] $V \cong (S)^{\otimes(-2)}$.

There is a simple extension of this example to $\mathfrak{p}^{1|s}$ for any $s \geq 0$. Namely, let Q_a be a basis for an s-dimensional vector space of odd infinitesimal translations, and then $\{P, Q_a\}$ is a basis of $\mathfrak{p}^{1|s}$. The nontrivial brackets are

$$[Q_a, Q_a] = -2P \tag{6.19}$$

[26]The dual of a one-dimensional vector space S is often denoted $S^{\otimes(-1)}$.

for all a. More abstractly, the Q_a span a vector space S^* and the bracket is defined from a positive definite symmetric pairing

$$\Gamma\colon S^* \otimes S^* \longrightarrow \mathbb{R} \cdot P. \tag{6.20}$$

The R-symmetry algebra is the orthogonal algebra of the pairing, and the corresponding R-symmetry group is isomorphic to $Spin(s)$. For some values of s this algebra arises by dimensional reduction, as we see below.

Jump now to $n = 3$ spacetime dimensions. The Lorentz group is isomorphic to $SL(2;\mathbb{R})$ and so the basic real spin representation S has dimension 2. We identify $V = \mathrm{Sym}(S^*)$. In other words, the pairing $\Gamma\colon S^*xS^* \to V$ induces an isomorphism $\mathrm{Sym}(S^*) \cong V$. (This was also the case in our first example $\mathfrak{p}^{1|1}$, but the induced map is not usually an isomorphism.)

It is natural to label basis vectors of V as P_{ab} where the subscript is symmetric in the indices and all indices run from 1 to 2. The bracketing relations in the supersymmetry algebra are

$$[Q_a, Q_b] = -2P_{ab}. \tag{6.21}$$

Corresponding to the basis elements P_{ab} of V are the coordinate vector fields ∂_{ab}, which are related to the previous $\partial_\mu = \partial/\partial x^\mu$ by

$$\begin{aligned} \partial_{11} &= \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \\ \partial_{22} &= \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \\ \partial_{12} &= \frac{\partial}{\partial x^2}. \end{aligned} \tag{6.22}$$

Notice that ∂_{11} and ∂_{22} are lightlike vectors, whereas ∂_{12} is spacelike. In this case there are no possible central charges and the R-symmetry group is trivial. The super Poincaré algebra is denoted $\mathfrak{p}^{3|2}$ and the corresponding super Poincaré group is $P^{3|2}$.

Now we consider the dimensional reduction to $n = 2$ spacetime dimensions. If we imagine a classical field theory with an action of this algebra, then we want to restrict it to the subspace of fields on which a one-dimensional subspace T of spatial translations acts trivially. For a quantum field theory dimensional reduction means restriction to the Hilbert subspace on which the operators corresponding to elements of T act trivially. Either way, the abstract supersymmetry algebra is obtained by setting a single infinitesimal translation to zero. For $\mathfrak{p}^{3|2}$ it is convenient to pick the spatial translation ∂_{12} and set it to zero. This gives a supersymmetry algebra in 2 dimensions which is denoted $\mathfrak{p}^{2|(1,1)}$. We explain the notation now.

The Lorentz group in $n = 2$ is $\mathbb{R}^{>0} \times \mathbb{Z}/2\mathbb{Z}$. In any spin representation $\mathbb{Z}/2\mathbb{Z}$ acts nontrivially, and as pointed out in Lecture 5 there are two inequivalent one-dimensional real spin representations S^+, S^-. The group of infinitesimal translations is identified as

$$\begin{aligned} V &= (S^+)^{\otimes(-2)} \oplus (S^-)^{\otimes(-2)} \\ &= \qquad V^+ \quad \oplus \quad V^-. \end{aligned} \tag{6.23}$$

This makes clear that the superspacetime corresponding the spinor representation $S^+ \oplus S^-$ is globally a product:

$$M^{2|(1,1)} = M^{1|1} \times M^{1|1}. \tag{6.24}$$

This splitting corresponds to the splitting of the lightcone in two dimensions. For any $s^+, s^- > 0$ there is a superspacetime $M^{2|(s^+,s^-)}$ and a corresponding super Poincaré group; they are constructed starting with the spin representation $S = (S^+)^{s^+} \oplus (S^-)^{s^-}$. The existence of two distinct real spin representations explains the notation $s = (s^+, s^-)$. Similar notation is used for all $n \equiv 2, 6 \pmod 8$.

In terms of the bases written above it is customary to use '+' for the index '1' and '−' for the index '2'. Set $\partial_+ = \partial_{11}$ and $\partial_- = \partial_{22}$. Waves which are functions of x^+ are *left-moving*, and waves which are functions of x^- are right-moving. The entire picture splits as the Cartesian product of left- and right-movers.

There is a new feature of $P^{2|(1,1)}$ not encountered in $P^{3|2}$: the possibility of central charges. Namely, the symmetric square of S^* in this case is

$$\mathrm{Sym}^2\big((S^+)^* \oplus (S^-)^*\big) \cong V^+ \oplus V^- \oplus \mathbb{R}. \tag{6.25}$$

The centrally extended algebra $\tilde{\mathfrak{p}}^{2|(1,1)}$ has a single central charge Z, and the nontrivial brackets are

$$\begin{aligned} [Q_+, Q_+] &= -2\partial_+ \\ [Q_-, Q_-] &= -2\partial_- \\ [Q_+, Q_-] &= 2Z. \end{aligned} \tag{6.26}$$

Note that $\mathfrak{p}^{2|(1,1)}$ is obtained by setting $Z = 0$.

The further dimensional reduction to $n = 1$ is obtained by setting the spatial translation $\partial_+ - \partial_-$ to zero. This recovers the algebra $\mathfrak{p}^{1|2}$ constructed above. In this case there is an R-symmetry group Spin(2) which we may understand from the original 3 dimensional picture. Namely, we have now set a two-dimensional space of infinitesimal translations to zero, so broken the original Lorentz group in 3 dimensions down to a Lorentz group in 1 dimension. The infinitesimal rotations of that 2-dimensional plane, which are part of the Lorentz group in 3 dimensions, are the R-symmetries of the dimensionally reduced algebra.

Representations of the super Poincaré group

Recall that in any relativistic quantum mechanical system particles are irreducible representations of the Poincaré group P^n. In a supersymmetric theory this representation extends to a unitary representation of $P^{n|s}$ for some s depending on the amount of supersymmetry. The irreducible representations of $P^{n|s}$ are generally not irreducible when restricted to the subgroup P^n, but rather break up as a finite sum. Such collections of particles are called supersymmetric *multiplets*. The particle content of a supersymmetric theory is organized into such multiplets. There is a general story, but to understand supersymmetric quantum field theories one needs to learn the taxonomy of multiplets in various dimensions with various amounts of supersymmetry. Here we outline the general theory and then illustrate with a few low dimensional examples.

Consider the supersymmetry group $P^{n|s}$. (We sue the notation $s = (s^+, s^-)$ if $n \equiv 2, 6 \pmod 8$.) As for the Poincaré group we isolate the subgroup of ordinary translations:

$$1 \longrightarrow V \longrightarrow P^{n|s} \longrightarrow \mathrm{Spin}(V) \ltimes \Pi S^* \longrightarrow 1 \tag{6.27}$$

Suppose we are given an irreducible nonnegative energy representation of $P^{n|s}$. Restrict to V to obtain an irreducible orbit of infinitesimal characters of some mass $m \geq 0$ in V^*. The supersymmetric little groups

$$\begin{aligned} m > 0: &\qquad \mathrm{Spin}(n-1) \ltimes \Pi S^* \\ m = 0: &\qquad \mathrm{Spin}(n-2) \ltimes \Pi S^* \end{aligned} \tag{6.28}$$

are super groups; the underlying ordinary group is the little group of the Poincaré group, but there are odd parts as well. (Recall that the *little group* is defined to be the reductive part of the stabilizer; a finite dimensional unitary representation of the stabilizer factors through the little group.) The representation of $P^{n|s}$ is induced from a projective graded finite dimensional unitary representation of the little group—as before, we form a homogeneous graded hermitian vector bundle over the orbit and take the Hilbert space of sections. The fact that the representation is projective is due to the nontrivial bracket on S^*. More precisely, at $\lambda \in V^*$ we seek a representation of the central extension constructed using the quadratic form q_λ on S^* given by

$$q_\lambda(Q_1, Q_2) = \lambda\big([Q_1, Q_2]\big).$$

Recall the positivity condition on the pairing (6.4). Since λ is in the closure of the positive dual lightcone, we conclude that q_λ is negative semidefinite. In the massive case the form is negative definite, whereas in the massless case it has a kernel. Except for the exceptional case $s = 1$ this kernel has dimension $s/2$. (This follows by splitting V into the sum of a two-dimensional Lorentz space and a negative definite complement, thus reducing to the $n = 2$-dimensional case and an explicit computation.) In the massless case we work with the quotient of S^* by the kernel, and then in both cases we obtain a negative definite space. The projective representations of (3.15) we want are then graded Clifford modules

$$W = W^0 \oplus W^1 \tag{6.29}$$

for S^* (modulo the kernel in the massless case) together with an intertwining action of the appropriate Spin group.

Some remarks:

- In this construction we end up considering spinors of spinors—a Clifford module for the vector space of spinors. Also, keep in mind that S is a *real* space, whereas the representation of the little group we construct—the spinors of spinors—is *complex*. The space S is a representation of $\mathrm{Spin}(n-1)$ or $\mathrm{Spin}(n-2)$ by restriction of the $\mathrm{Spin}(V)$ action.
- Usually $\dim S$ is even and we construct the unique irreducible graded Clifford module by complexifying S and choosing a lagrangian splitting, as described in (5.8).

- The irreducible massless representations tend to be smaller than the irreducible massive representations because of the drop in dimension. This does not hold in low dimensions, however.
- The representation theory of the supersymmetry groups with central charges is similar. The important concept of a *BPS particle* is introduced there. We will not give details here, but remark that we will meet an analog for fields in the next lecture.
- Allowable representations in a local quantum field theory are constrained to be CPT invariant, as mentioned in Lecture 5.
- We choose the grading of W—which piece is even and which piece odd—according to the spin-statistics principle.
- The even and odd pieces of a Clifford module have equal dimensions. This means that *a supersymmetry multiplet has an equal number of bosonic degrees of freedom and fermionic degrees of freedom*. (The number of degrees of freedom is the dimension of the representation of the little group.) There is an exception for chiral theories in $n = 2$, however. For example, there exists a theory with symmetry group $P^{2|(1,0)}$ with one (chiral) fermion and no bosonic degrees of freedom.

Now some examples.

Example 1 ($P^{3|2}$). We first review the representations of the Poincaré group P^3. For massive representations the little group is Spin(2), and its irreducible complex representations are one-dimensional and labeled by the spin j. All of the representations are CPT invariant. The spin $j = 0$ representation is called a *massive scalar particle*, the spin $j = 1/2$, $j = -1/2$ representations *massive spinor particles*, and the spin $j = 1$, $j = -1$ representations *massive half-vector particles*. (The *massive vector particle* is the sum of the $j = 1$ and $j = -1$ representations.) In the massless case the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$. There are two irreducible complex representations, each of dimension one. The trivial one is the *massless scalar particle* and the nontrivial one is the *massless spinor particle*.

Now we turn to the supersymmetry group $P^{3|2}$. In the massive case we have in addition to the little group Spin(2) a negative definite Clifford algebra generated by S^*, which here has dimension 2. After complexification we choose a basis $\{Q_+, Q_-\}$ where Spin(2) acts on $Q_\pm$ by $j = \pm 1/2$. A physicist describes the Clifford module in the following notation. Fix a *vacuum* vector $|0\rangle$ and postulate

$$Q_-|0\rangle = 0.$$

Then the Clifford module is

$$W = \mathbb{C} \cdot |0\rangle \ \oplus \ \mathbb{C} \cdot Q_+|0\rangle.$$

To define the action of Spin(2) we need to specify a spin j for $|0\rangle$; then the spin of $Q_+|0\rangle$ is $j + 1/2$.

To summarize: A massive multiplet for $P^{3|2}$ is a pair of particles of adjacent spins j and $j + 1/2$.

We show some possibilities in the table above, which indicates the number of representations with a given spin j which occur in each multiplet. We only list

	-1	$-1/2$	0	$1/2$	1
massive scalar multiplet		1	1		
massive scalar multiplet			1	1	
massive vector multiplet	1	1		1	1

Massive multiplets for $P^{3|2}$

multiplets with spin at most 1; they are the relevant multiplets for supersymmetric theories without gravity. The multiplets are usually named after the highest spin boson. There are two *massive scalar multiplets* because of the two different possibilities for a massive spinor. In addition to the *massive vector multiplet* shown, there are also *massive half-vector multiplets*.

Recall that free classical theories can be quantized, and the 1-particle Hilbert space is a sum of particle representations. It is instructive to see how the two different spinor particles arise from free fields—the difference comes from the sign in the mass term in the lagrangian. We will not consider massive vectors in these lectures, but it is amusing to see how the massive half-vectors can be written in terms of fields—the 3-dimensional Chern-Simons term enters.

Now we consider the massless multiplets. In this case the quadratic form on S^* is degenerate, and the quotient by the kernel is one-dimensional. The irreducible complex Clifford module $W = W^0 \oplus W^1$ is still two-dimensional. The little group $\mathrm{Spin}(1) \cong \mathbb{Z}/2$ acts trivially on W^0 and nontrivially on W^1. So there is a unique massless multiplet, the *massless scalar multiplet,* which contains one scalar particle and one spinor particle.

Example 2 ($\tilde{P}^{2|(1,1)}$). This example illustrates the representation theory for a centrally extended supersymmetry group. We refer to (6.26) for the bracketing relations in the central extension $\tilde{\mathfrak{p}}^{2|(1,1)}$ of the symmetry algebra $\mathfrak{p}^{2|(1,1)}$. From these brackets we easily compute

$$\frac{1}{4}[Q_+ \pm Q_- \,,\; Q_+ \pm Q_-] = -\partial_t \pm Z, \tag{6.30}$$

where

$$2\partial_t = \partial_+ + \partial_- \tag{6.31}$$

is infinitesimal time translation. Note that the left hand side of (6.30) is the square of an odd element. Thus the quantum operators $\hat{H} \pm \hat{Z}$ corresponding to the right hand side of (6.30) are nonnegative; in other words,

$$\hat{H} \geq |\hat{Z}|. \tag{6.32}$$

Here $\hat{H}$ is the quantum hamiltonian. In an irreducible supersymmetry representation the operator $\hat{Z}$ is a constant since Z is central. For $Z = 0$ the inequality (6.32) is a special case of the general argument that the quantum hamiltonian in a supersymmetric theory is nonnegative. But with the central charge we have a stronger bound. The Poincaré invariant statement is the *BPS bound*: the *mass* is bounded below by one-half the absolute value of the central charge. Furthermore, if we have equality—the mass equal to one-half the absolute value of the central charge—then the quadratic form on the odd part of the little group has a half-dimensional kernel, just as for massless representations. (In this case that is evident in (6.30).) So we obtain special irreducible massive multiplets—*BPS multiplets*—which typically have fewer degrees of freedom than the usual massive multiplets. If this is the case, then they are stable under perturbations. Hence these BPS representations are an important source of stable particles in supersymmetric theories.

The states in a BPS representation are annihilated by 1/2 of the supersymmetry, in this example by $Q_+ + Q_-$ or $Q_+ - Q_-$. There are more complicated situations where a different fraction of the supersymmetry annihilates a representation, which is also termed BPS.

In Lecture 7 we will see a classical field configuration which satisfies a classical version of the BPS condition for the supersymmetry group $\tilde{P}^{2|(1,1)}$: it solves the equations of motion and is annihilated by half of the classical supersymmetry.

Exercises

1. (Clifford algebras) Let U be a real vector space with a positive definite inner product $\langle\cdot,\cdot\rangle$. The Clifford algebra $\mathrm{Cliff}(U)$ is the quotient of the tensor algebra by the ideal generated by
$$u \otimes u' + 2\langle u, u'\rangle \cdot 1$$
for all $u, u' \in U$.
   (a) What algebra do you get for $\dim U = 1, 2, 3$? (Hint: Choose an orthonormal basis for U and write everything explicitly in terms of that basis.)
   (b) Prove that $\mathrm{Cliff}(U)$ is finite dimensional if U is finite dimensional.
   (c) So far we have viewed $\mathrm{Cliff}(U)$ as an ordinary algebra. Show how it may be viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra. Is it (super)commutative?
   (d) Find irreducible $\mathbb{Z}/2\mathbb{Z}$-graded modules for $\mathrm{Cliff}(U)$ in low dimensions.

2. (super Lie algebras) Let
$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$$
be a super Lie algebra.
   (a) Show that the ($\mathbb{Z}/2\mathbb{Z}$-graded) skew-symmetric bracket on $\mathfrak{g}$
$$[\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}$$
is equivalent to a bracket on $\mathfrak{g}^0$, an map $\mathfrak{g}^0\otimes\mathfrak{g}^1 \to \mathfrak{g}^1$, and a symmetric pairing $\mathfrak{g}^1\otimes\mathfrak{g}^1 \to \mathfrak{g}^0$.
   (b) Show that the Jacobi identity for $\mathfrak{g}$ is equivalent to verifying that $\mathfrak{g}^0$ is a Lie algebra, the map $\mathfrak{g}^0\otimes\mathfrak{g}^1 \to \mathfrak{g}^1$ is an action, and $[Q,[Q,Q]] = 0$ for all $Q \in \mathfrak{g}^1$.
   (c) Show that $\mathfrak{p}^{n|s}$ as defined in the lecture is a super Lie algebra.

3. (a) Using (6.11), verify (6.12) and (6.13).
   (b) Use the "functor of points" point of view explained in John Morgan's lecture to construct $\exp \mathcal{L}$ as a super Lie group. In other words, for every supercommutative ring (think of $R = \mathbb{R}[\eta_1, \eta_2, \ldots, \eta_N]$ with η_i odd and mutually commuting) construct an ordinary Lie group over $\operatorname{Spec} R$. (Hint: Use the Campbell-Hausdorff formula, which terminates quickly in this case.)
   (c) Show that (6.11) are indeed left- and right-invariant vector fields on $\exp \mathcal{L}$.

4. Construct a basis for the entire super Lie algebra $\mathfrak{p}^{3|2}$ and compute all brackets.

5. In this problem you'll learn about dimensional reduction. Begin on Minkowski M^n with coordinates $x^0, x^1, \ldots, x^{n-1}$ as usual. We dimensionally reduce fields by demanding that they be translation-invariant in the x^{n-1} direction. The idea is to identify translation-invariant fields with fields on M^{n-1}.
   (a) Just to be sure you understand what we're talking about, show that a scalar field in n dimensions dimensionally reduces to a scalar field in $n-1$ dimensions.
   (b) A 1-form field in n dimensions is given by n functions of n variables. Demanding that the 1-form be translation-invariant gives n functions of $n-1$ variables. What fields do we obtain in M^{n-1}? In other words, what fields do we get by dimensionally reducing a 1-form field on M^n? (The n functions of $n-1$ variables organize into fields according to their transformation law under the Poincaré group P^{n-1}. What are these fields?)
   (c) What fields do we obtain if we dimensionally reduce a connection A on M^n for some compact gauge group A?
   (d) In both of the previous examples we have gauge symmetries. Translations act on the space $\overline{\mathcal{F}}$ of equivalence classes of all fields $\mathcal{F}$. Think through dimensional reduction in this way: identify the subspace of $\overline{\mathcal{F}}$ of equivalence classes invariant under translation in the x^{n-1} direction with the space of equivalence classes of some fields on M^{n-1}.
   (e) What happens when you dimensionally reduce a spinor field? Try some low dimensional examples, e.g. starting in $n = 3$.

6. Let $\mathcal{H}$ be the Hilbert space underlying some nonnegative energy unitary representation of the Poincaré group P^n. Let $\mathcal{H}' \subset \mathcal{H}$ be the subspace annihilated by the infinitesimal translation operator $\hat{P}_{n-1}$. Show that the Poincaré group P^n acts on $\mathcal{H}'$. What representation do you get? The answer depends, of course, on the representation you begin with. Recall that representations are characterized by a mass and finite-dimensional unitary representation of the little group. So the dimensional reduction should be expressed as a map (mass, rep of little group in n dimensions) to (mass, rep of little group in $n-1$ dimensions).

7. Obtain $M^{1|1}$ as a dimensional reduction of $M^{2|(1,0)}$.

8. The lectures on mirror symmetry concern $P^{2|(2,2)}$. This problem is meant to familiarize you with this super Poincaré group and the underlying superspacetime $M^{2|(2,2)}$.

(a) What is the real spin representation out of which $M^{2|(2,2)}$ is constructed?

(b) It is usual to take a basis of *complex* left-invariant vector fields. The even elements are the real vector fields ∂_+, ∂_- as in the lecture. The odd vector fields are $D_+, \overline{D}_+, D_-, \overline{D}_-$ with nontrivial brackets

$$\begin{aligned} [D_+, \overline{D}_+] &= -\partial_+ \\ [D_-, \overline{D}_-] &= -\partial_- \end{aligned}$$

Verify that this *is* the super Poincaré algebra as defined in the text. What is the pairing Γ?

(c) What are the possible central extensions of this algebra by adjoining central charges?

(d) What is the Lie algebra of R-symmetries? Write the action on $P^{2|(2,2)}$.

9. Discuss the representation theory of $P^{2|(2,2)}$, without and with the possible central charges.

LECTURE 7
Supersymmetric σ-Models

In this lecture we consider one of the simplest supersymmetric field theories. Initially we consider it in dimension $n = 3$, which is the maximal dimension in which it is defined. Its dimensional reductions to $n = 1$ and $n = 1$ dimensions relate to interesting topics in geometry: Morse theory and index theory. We will see some indications of this in the classical theory, but the real power comes after quantization (which we do not discuss in these lectures).

3-dimensional theory

The supersymmetric σ-model has fields of spin 0 and spin 1/2, but no spin 1 fields. Recall from (5.30) that in general a spin 0 field is specified by a Riemannian manifold X and a spin 1/2 field by a real vector bundle $W \to X$ with metric and connection. We may also add potential terms (5.34). Theories constructed from this data are invariant under the Poincaré group P^3. Now we ask for a theory which is supersymmetric with 2 supersymmetries.[27] In other words, we ask that the larger supergroup $P^{3|2}$ act by symmetries on the theory. We expect that this leads to constraints on this basic data. In fact, we will find that $W = TX$ (where we do not include any auxiliary fields in what we mean by 'X') and that the potential (5.34) is constructed from a single real function $h\colon X \to \mathbb{R}$. Rather than begin with the general theory and attempt to derive these constraints, we simply construct the supersymmetric theory directly. There is a superspacetime formulation on $M^{3|2}$ which makes the supersymmetry manifest, and we use it to derive the theory in terms of *component fields* on M^3

Recall that in 3 dimensions $\mathrm{Spin}(V) \cong SL(2;\mathbb{R})$; we take the spin representation S to be the standard 2-dimensional representation. In Lecture 7 we defined the framing ∂_{ab}, D_a of $M^{3|2}$ by left-invariant vector fields. The indices $a, b, \dots$ run from 1 to 2, and the index ab is symmetric in a and b. We now additionally make note of the skew-symmetric form

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R}, \tag{7.1}$$

[27] There is a supersymmetric σ-model with a single supersymmetry in one dimension, which we constructed in Lecture 4 as the supersymmetric particle. There are also supersymmetric σ-models with 4 supersymmetries, where the target X is Kähler, and σ-models with 8 supersymmetries, where the target X is hyperkähler. These models exist classically in dimensions 4 and 6 respectively, so by dimensional reduction in smaller dimensions as well.

which in terms of $SL(2;\mathbb{R})$ is simply the volume form, and its dual form[28] on S^*. We choose the bases so that $\epsilon^{12} = \epsilon_{12} = 1$.

Fix a Riemannian manifold X. The field Φ in the superspacetime formulation is a scalar field on superspacetime with values in X:

$$\Phi \colon M^{3|2} \longrightarrow X.$$

Fields on superspacetime are usually called *superfields*. The left action of the super Poincaré group $P^{3|2}$ on $M^{3|2}$ induces an action on superfields by pullback. The corresponding infinitesimal action is by *right*-invariant vector fields, which we call τ_{Q_a}.

Just as we did for the supersymmetric particle (see (4.31)) we define *component fields* on Minkowski spacetime M^3 by restricting normal derivatives of the superfield to $M^3 \subset M^{3|2}$. Since the θ^a are nilpotent, a small number of normal derivatives determines the superfield completely. In the case of the supersymmetric particle on $M^{1|1}$ there is a single odd direction, so we only needed one first derivative. In $M^{3|2}$ we have two odd directions, so have two first derivatives and one (mixed) second derivative; all others vanish because of the oddness of the derivatives. The complete list of component fields is:

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a \Phi \\ F &= -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi \end{aligned} \tag{7.2}$$

Several remarks are in order:

- The leading bosonic component ϕ is a scalar field on M^3 with values in X, namely

$$\phi \colon M^3 \to X. \tag{7.3}$$

- The field ψ_a is odd, since the vector field D_a is odd. It is an odd section of the pullback tangent bundle ϕ^*TX. Together the two fields ψ_1, ψ_2 transform as a spinor field on Minkowski spacetime, since D_a correspond to basis elements of S^*, so we can combine them into

$$\psi \in \Omega^0\bigl(M^3; \Pi S \otimes \phi^*TX)\bigr). \tag{7.4}$$

- The field F is even and is also a section of the pullback tangent bundle:

$$F \in \Omega^0(M^3; \phi^*TX). \tag{7.5}$$

- The outer (leftmost) derivative in the definition of F is a covariant derivative using the Levi-Civita connection, and in detailed computations we meet

[28] Any nondegenerate symmetric or skew-symmetric bilinear form on a vector space S induces a nondegenerate bilinear form on the dual space, called the dual form. Note that in the skew case the form determines an isomorphism $S \to S^*$ only up to a sign, but that isomorphism enters twice in transferring the bilinear form to the dual, so the sign ambiguity disappears. Be warned that physicists in this circumstance often use *minus* the dual form.

multiple covariant derivatives of Φ. The fact that Levi-Civita is torsion-free means the first two derivatives acting on Φ commute, but when commuting further derivatives we pick up curvature terms. This explains why curvature terms enter the component lagrangian below.

- In case $X = \mathbb{R}$, the fields ϕ, ψ comprise precisely the field content of a massless or massive scalar multiplet in $n = 3$ dimensions. (Recall the discussion of representations of $P^{3|2}$ in Lecture 6.) That is, if we write the free massless or massive lagrangian for these fields, then upon quantization the one-particle Hilbert space is that irreducible supersymmetry multiplet. So we are led to inquire: What is F doing here? We will see that F is an *auxiliary field* in the sense that it does not contribute physical degrees of freedom in the quantization. In fact, F enters the lagrangian only algebraically—no derivatives—so the equations of motion determine F algebraically in terms of the other fields. We will eliminate F using this algebraic equation.
- Still, the presence of F allows us to write down supersymmetry transformation laws (see (7.7) below) for the component fields which close *off-shell*. In other words, we have an action of $\mathfrak{p}^{3|2}$ on the entire space of fields $\mathcal{F}$ (off-shell), not just on the solutions $\mathcal{M} \subset \mathcal{F}$ to the equation of motion. If we use the (algebraic) equation of motion to substitute for F in the transformation laws, then the algebra no longer closes off-shell.
- The component fields $\{\phi, \psi, F\}$ are often referred to as a *multiplet*, in this case a *scalar multiplet*.
- The statement that we have a complete list of component fields is the statement that the supermanifold of superfields $\mathcal{F}_\Phi$ is isomorphic to the supermanifold of component fields $\mathcal{F}_{\{\phi,\psi,F\}}$. In each case we must use the "functor of points" point of view to make sense of the statement. Also, the statement implicitly assumes that the space of maps between supermanifolds is itself an infinite dimensional supermanifold.
- We can do the classical field theory in superspacetime $M^{3|2}$ with the superfield Φ or in ordinary spacetime M^3 with the component fields ϕ, ψ, F. In the first case we do calculus on $\mathcal{F}_\Phi \times M^{3|2}$; in the second calculus on $\mathcal{F}_{\{\phi,\psi,F\}} \times M^3$. The supersymmetry is *manifest* in the superfield formulation but not in the component formulation.

Next, we write down the supersymmetry transformation law for the component fields. In superspacetime the infinitesimal supersymmetry is the induced action on Φ of $-\tau_{Q_a}$, where we use a minus sign to obtain a left action. Introduce odd parameters η^a and consider the action of the even vector field $-\eta^a \tau_{Q_a}$. This induces a vector field $\hat{\zeta}$ on the space of component fields $\mathcal{F}_{\{a,\psi,F\}}$. Quite generally, a component field f (see (7.2)) is defined by a formula

$$f = i^* D^r \Phi, \tag{7.6}$$

where 'D^r' denotes a sum of products of D_a. Then a short argument shows that under the infinitesimal supersymmetry transformation $\hat{\zeta}$, the field f transforms as

$$\hat{\zeta} f = -\eta^a i^* D_a D^r \Phi. \tag{7.7}$$

The right hand side can be rewritten in terms of component fields. This formula works in any model with a superfield formulation, not just in the $3|2$ supersymmetric

σ-model. In the case at hand we find

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a\psi_a \\ \nabla_{\hat{\zeta}}\psi_a &= \quad \eta^b(\partial_{ab}\phi - \epsilon_{ab}F) \\ \nabla_{\hat{\zeta}}F &= \quad \eta^a\big[(\partial\!\!\!/\psi)_a + \frac{1}{3}\epsilon^{bc}R(\psi_a,\psi_b)\psi_c\big], \end{aligned} \tag{7.8}$$

where R is the Riemann curvature tensor of X and the Dirac operator is

$$(\partial\!\!\!/\psi)_a = -\epsilon^{bc}\partial_{ab}\psi_c. \tag{7.9}$$

Note the appearance of the covariant derivatives in the action of the vector field $\hat{\zeta}$ on sections of the (pulled back) tangent bundle. For this reason when checking commutators of the supersymmetry transformations (7.8) for vector fields $\hat{\zeta}, \hat{\zeta}'$ we encounter curvature terms.

This computation shows off the advantages of the superspacetime formulation—the supersymmetry transformation laws of the component fields are determined *a priori*.

The lagrangian density for the supersymmetric σ-model is

$$\mathcal{L} = |d^3x|\, d^2\theta\ \frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle. \tag{7.10}$$

We have not given a detailed discussion of densities and integration on supermanifolds, and as before we finesse that point. Suffice it to say that $|d^3x|\, d^2\theta$ is a $P^{3|2}$-invariant density on $M^{3|2}$. The lagrangian function ℓ which appears after it in the lagrangian density (7.10) is obviously $P^{3|2}$-invariant, since $P^{3|2}$ acts on the left and the D_a are left-invariant. So $\mathcal{L}$ is manifestly supersymmetric—invariant under $P^{3|2}$. It is also worth mentioning that the lagrangian is constrained by asking that the kinetic term for the bosonic field be quadratic in first derivatives, as in (3.1). From this point of view each $d\theta$ counts as half of a bosonic derivative (see (7.13) below), as does each D_a. Thus (7.10) has a total of two bosonic derivatives, as it should.

We need to "integrate out the θs" to define a component lagrangian L. Quite generally, for a superspacetime model on $M^{n|s}$ the superspacetime lagrangian density is[29]

$$\mathcal{L} = |d^nx|\, d^s\theta\ \ell. \tag{7.11}$$

There is a standard notion of (Berezin) integration to integrate out the odd variables. Let

$$\pi\colon M^{n|s} \longrightarrow M^n \tag{7.12}$$

be the projection given by our construction of superspacetime and π_* the corresponding integration. Then the integral over the odd variables amounts to differentiation:

$$\pi_* = i^*\frac{\partial}{\partial\theta^s}\cdots\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}. \tag{7.13}$$

[29] To write this we need to choose an orientation on S to order the basis elements θ. That choice appears again in (7.13) below, so cancels out.

Rather than integrate, our approach is to find in each case a certain sum of products of D_a, which we denote 'D^s', such that

$$\pi_*\mathcal{L} = (i^*D^s\ell + \Delta i^*\ell)\,|d^nx| \tag{7.14}$$

for some Poincaré invariant differential operator Δ on M^n. Then instead of using $\pi_*\mathcal{L}$ as the component lagrangian, we define the component lagrangian to be

$$L = (i^*D^s\ell)\,|d^nx|. \tag{7.15}$$

A few brief remarks:

- In general this differs from the straight integration of the odd variables, as indicated by the presence of Δ. The particular D^s that we use is chosen so that the component lagrangian density L involves only first derivatives of the fields.
- In this lecture we have $\Delta = 0$. Examples with $\Delta \neq 0$ occur in superspacetimes with $s = 4$ odd directions, as we indicate in the exercises.
- Formula (7.15) is effective in that we find strings of D_a acting on superfields which we then express in terms of component fields using the commutation relations.
- The component lagrangian is supersymmetric, but the supersymmetry is not manifest. There is an explicit formula to determine the exact term by which it changes under a supersymmetry transformation. Recall from (2.48) that this term appears in the Noether current for the supersymmetry transformation, often called the *supercurrent*.

For $M^{3|2}$ we define the component lagrangian to be

$$L = i^*\big(-\frac{1}{2}\epsilon^{ab}D_aD_b\big)\ell\,|d^3x|. \tag{7.16}$$

As remarked above, this agrees with the definition using the Berezinian integral $\int d^2\theta$. For the σ-model lagrangian (7.10) we have

$$L = -\frac{1}{8}\epsilon^{ab}\epsilon^{cd}i^*D_aD_b\langle D_c\Phi, D_d\Phi\rangle. \tag{7.17}$$

To illustrate the manipulations, consider a typical term in (7.17), omitting the numerical factor:

$$i^*\langle D_1D_2D_1\Phi, D_2\Phi\rangle. \tag{7.18}$$

We emphasize that the outer two derivatives are *covariant* derivatives. When we commute them we pick up a curvature term:

$$\begin{aligned}\langle D_1D_2D_1\Phi, D_2\Phi\rangle = &-\langle D_2D_1D_1\Phi, D_2\Phi\rangle \\ &- 2\langle \partial_{12}D_1\Phi, D_2\Phi\rangle + \langle R(D_1\Phi, D_2\Phi)D_1\Phi, D_2\Phi\rangle.\end{aligned} \tag{7.19}$$

Since $D_1^2 = -\partial_{11}$ and $[\partial_{11}, D_2] = 0$, we have

$$D_2D_1D_1\Phi = -D_2\partial_{11}\Phi = -\partial_{11}D_2\Phi. \tag{7.20}$$

Restricting to M^3 we obtain

$$i^*\langle D_1 D_2 D_1 \Phi, D_2\Phi\rangle = \langle \partial_{11}\psi_2, \psi_2\rangle - 2\langle \partial_{12}\psi_1, \psi_2\rangle + \langle R(\psi_1,\psi_2)\psi_1, \psi_2\rangle. \tag{7.21}$$

Of course, there are systematic algebraic shortcuts one can develop for these manipulations. In any case, at the end of the day one obtains the component lagrangian for the supersymmetric σ-model:

$$L = \left\{\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle \psi \not{D}_\phi \psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, R(\psi_b,\psi_c)\psi_d\rangle + \frac{1}{2}|F|^2\right\}|d^3x|. \tag{7.22}$$

Some remarks:

- As promised, F appears only algebraically in the lagrangian; it is an auxiliary field. Its equation of motion is simply

$$F = 0. \tag{7.23}$$

- The spinor field ψ depends on ϕ (see (7.4)), as does the Dirac form. Also, the Dirac form is defined using a covariant derivative.
- There is a quartic potential $V^{(4)} = \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, R(\psi_b,\psi_c)\psi_d\rangle$ for the fermionic field ψ.
- The *bosonic lagrangian* is obtained by setting the fermion ψ to zero:

$$L_{\text{bos}} = \frac{1}{2}|d\phi|^2\ |d^3x| \tag{7.24}$$

  after eliminating the auxiliary field F. This is the nonlinear σ-model lagrangian (3.9) with zero potential. So we have indeed obtained a supersymmetric extension of the nonlinear σ-model with two supersymmetries in three dimensions.
- There is not much to say about this classical field theory: the moduli space of vacua is X and the global symmetry group is the group of isometries of X.

We need the formula for the *supercurrent* j_a, which is *minus* the Noether current corresponding to the supersymmetry τ_{Q_a}. Here we write the full $|-1|$-form on M^3:

$$j_a = \iota(\partial_{cd})\,|d^3x|\left\{\epsilon^{cb}\epsilon^{de}\langle \partial_{ae}\phi, \psi_b\rangle - \epsilon^{bd}\delta^c_a\langle F, \psi_b\rangle\right\}. \tag{7.25}$$

We are only interested in this on-shell, in which case we could set $F=0$, but we need the formula later in cases where $F \neq 0$. The *supercharges* are obtained by integrating j_a over a fixed time slice:

$$\begin{aligned} Q_1 &= \int_{x^0=\text{const}} \left\{\langle \partial_{11}\phi, \psi_1\rangle + \langle \partial_{12}\phi, \psi_2\rangle\right\}|dx^1dx^2| \\ Q_2 &= \int_{x^0=\text{const}} \left\{\langle \partial_{22}\phi, \psi_2\rangle + \langle \partial_{12}\phi, \psi_1\rangle\right\}|dx^1dx^2| \end{aligned} \tag{7.26}$$

In these last formulas we did set $F = 0$.

A supersymmetric potential

The theory is more fun when we add a potential term, which of course we must do in a supersymmetric way. This is easy in superspacetime. Let

$$h\colon X \longrightarrow \mathbb{R} \tag{7.27}$$

be a real-valued function on X. It is *not* the potential function V of the model; we compute V in terms of h below. Simply add the pullback of h by the superfield Φ to the superspacetime lagrangian (7.10):

$$\mathcal{L}' = |d^3x|\, d^2\theta \left\{\frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle + \Phi^*(h)\right\}. \tag{7.28}$$

It is easy to work out the contribution of the new term to the component lagrangian:

$$\left\{\langle F, \phi^* \operatorname{grad} h\rangle + \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b)\right\}|d^3x|. \tag{7.29}$$

So the new component lagrangian is the sum of (7.22) and (7.29). Now the equation of motion of F is more interesting than before:

$$F = -\phi^* \operatorname{grad} h. \tag{7.30}$$

So upon eliminating F from the component lagrangian we obtain

$$\begin{aligned} L' = \Big\{\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}_\phi\psi\rangle - \frac{1}{2}\phi^*|\operatorname{grad} h|^2 + \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) \\ + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle\psi_a, R(\psi_b, \psi_c)\psi_d\rangle\Big\}|d^3x|. \end{aligned} \tag{7.31}$$

The presence of h leads to two new terms:

- A potential energy

$$V^{(0)} = \frac{1}{2}|\operatorname{grad} h|^2 \tag{7.32}$$

  for the scalar field ϕ. Note that the potential is nonnegative, as it must be in a supersymmetric theory.
- A mass term

$$V^{(2)} = \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) \tag{7.33}$$

  for the fermions.

The bosonic lagrangian in this case is

$$L_{\text{bos}} = \left\{\frac{1}{2}|d\phi|^2 - \frac{1}{2}\phi^*|\operatorname{grad} h|^2\right\}|d^3x|, \tag{7.34}$$

a nonlinear σ-model with nonzero potential.

We also record the *on-shell* supersymmetry transformation laws for the physical fields ϕ, ψ from (7.8), using the equation of motion for F:

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a \psi_a \\ \nabla_{\hat{\zeta}}\psi_a &= \eta^b(\partial_{ab}\phi + \epsilon_{ab}\phi^* \operatorname{grad} h). \end{aligned} \tag{7.35}$$

The supercharges are given by the formulas

$$\begin{aligned} Q_1 &= \int_{x^0=\text{const}} \Big\{ \langle \partial_{11}\phi, \psi_1\rangle + \langle \partial_{12}\phi, \psi_2\rangle - \langle \phi^* \operatorname{grad} h, \psi_2\rangle \Big\} \, |dx^1 dx^2| \\ Q_2 &= \int_{x^0=\text{const}} \Big\{ \langle \partial_{22}\phi, \psi_2\rangle + \langle \partial_{12}\phi, \psi_1\rangle + \langle \phi^* \operatorname{grad} h, \psi_1\rangle \Big\} \, |dx^1 dx^2| \end{aligned} \tag{7.36}$$

Now we investigate the classical vacuum states—the minimal energy field configurations. As we discussed in Lecture 3 these occur when $\psi = 0$ and $\phi \equiv \phi_0$ is constant. If ϕ_0 is at a minimum of the potential V—in other words, $V(\phi_0) = 0$—then ϕ_0 is a critical point of h. So the moduli space of vacua is the critical point set of h. A vacuum is invariant under supersymmetry, i.e., the vector field $\hat{\zeta}$ vanishes at a vacuum, as is obvious from (7.35). Thus we say that *supersymmetry is unbroken* in these vacua. It is natural to assume that h is a Morse function, or perhaps a generalized Morse function in the sense of Bott. Then we might have critical manifolds, and at vacua which lie on such manifolds there are massless scalars and massless fermions. The fact that supersymmetry is unbroken means that these come in supersymmetric pairs, i.e., in multiplets. More generally, the masses of the scalar fields and of the fermions agree at such a vacuum. This is easily verified by computing the masses directly from the lagrangian (7.31). Or, recall our formulas (5.41). For the scalar field ϕ, the squares of the masses are the eigenvalues of the Hessian of the potential V, which at a critical point of h are the squares of the eigenvalues of $\operatorname{Hess} h$. On the other hand, the mass matrix M for the fermions is $\operatorname{Hess} h$, and so the fermion mass squares are again the squares of the eigenvalues of the Hessian.

We can also consider a vacuum which is a local minimum of V, but not a global minimum. (A critical point of V which is not a local minimum is unstable.) For such a local minimum ϕ_0 we need to shift the energy by $-V(\phi_0)$ in order to have zero energy for the vacuum. Since

$$dV = \langle \operatorname{grad} h, \nabla \operatorname{grad} h\rangle,$$

such critical points of V occur when $\operatorname{grad} h$ is nonzero but in the kernel of $\operatorname{Hess} h$. Such a point could be isolated, and at such a vacuum we might have no massless scalar fields. But there is a massless fermion since the fermion mass matrix $\operatorname{Hess} h$ has a kernel. In fact, this massless fermion can be predicted from the supersymmetry. Notice that this vacuum is *not* supersymmetric—the vector field $\hat{\zeta}$ in (7.35) does not vanish there. So we say that *supersymmetry is broken* in this vacuum. Notice that the Poincaré symmetry is always assumed to be unbroken in a vacuum, but as we see this is not true for the *super* Poincaré symmetry. Now just as broken (even) global symmetries lead to massless bosons—Goldstone bosons—so too do broken odd symmetries lead to massless fermions: *Goldstone fermions*.

Dimensional reduction to $n = 2$ dimensions

We dimensionally reduce the model in components, though we could dimensionally reduce the superspacetime formulation to obtain a superspacetime formulation in $n = 2$ dimensions.

Let's begin by collecting the formulas we need. Restrict to fields f which satisfy $\partial_{12} f = 0$. (Recall from (6.22) that ∂_{12} is infinitesimal translation along a spatial direction.) As in Lecture 6 we use a basis of *lightlike* vector fields ∂_+, ∂_-. Also, for spinors recall that we use '+' for the index '1' and '−' for the index '2'. Then from (7.35) we find the supersymmetry transformations

$$
\begin{aligned}
\hat{\zeta}\phi &= -(\eta^+\psi_+ + \eta^-\psi_-) \\
\hat{\zeta}\psi_+ &= \eta^+\partial_+\phi + \eta^-\phi^* \operatorname{grad} h \\
\hat{\zeta}\psi_- &= \eta^-\partial_-\phi - \eta^+\phi^* \operatorname{grad} h
\end{aligned}
\tag{7.37}
$$

and from (7.36) the supercharges

$$
\begin{aligned}
Q_+ &= \int_{x^0=\text{const}} \Big\{ \langle \partial_+\phi, \psi_+ \rangle - \langle \phi^* \operatorname{grad} h, \psi_- \rangle \Big\} \, |dx^1| \\
Q_- &= \int_{x^0=\text{const}} \Big\{ \langle \partial_-\phi, \psi_- \rangle + \langle \phi^* \operatorname{grad} h, \psi_+ \rangle \Big\} \, |dx^1|.
\end{aligned}
\tag{7.38}
$$

Field theory in $n = 2$ dimensions is special for many reasons, among them the fact that spatial infinity is disconnected. This has the following consequence. Let $\mathcal{FE}_{\text{bos}}$ denote the space of static bosonic field configurations of finite energy. Now the energy density of a field $\phi(x^1)$ on space is (see (3.13))

$$
\begin{aligned}
\Theta &= \Big\{ \frac{1}{2}\, |\partial_1\phi|^2 + \frac{1}{2}\, \phi^* |\operatorname{grad} h|^2 \Big\} \, |dx^1| \\
&= \frac{1}{2}\, |\partial_1\phi \pm \phi^* \operatorname{grad} h|^2 \, |dx^1| \ \mp\ d\phi^*(h).
\end{aligned}
\tag{7.39}
$$

If a field $\phi(x^1)$ on space has finite energy, then the integral of the first term is finite, in which case $\phi(x^1)$ has limits as $x^1 \to \pm\infty$ which are critical points of h. The integral over space of the topological term is plus or minus the difference

$$
Z = h\big(\phi(\infty)\big) - h\big(\phi(-\infty)\big)
\tag{7.40}
$$

of the limiting values. So the space $\mathcal{FE}_{\text{bos}}$ of finite energy field configurations splits into a disjoint union according to the limiting values at plus or minus infinity. The parameter space for the components is the Cartesian square $\pi_0 \operatorname{Crit}(h)^{\times 2}$ of the set of components of the critical point set. The *central charge* Z is constant on each component.

Example. A typical example is $X = \mathbb{R}$ with

$$
h(\phi) = \frac{\phi^3}{3} - a^2\phi, \qquad a \in \mathbb{R}.
\tag{7.41}
$$

Then we have a quartic potential

$$
V^{(0)}(\phi) = \frac{1}{2}(\phi^2 - a^2)^2
\tag{7.42}
$$

with nondegenerate zeros at $\phi = \pm a^2$. So in the $n = 2$-dimensional theory $\mathcal{FE}_{\rm bos}$ has four components.

In a "diagonal" component of $\mathcal{FE}_{\rm bos}$, where $h(\infty) = h(-\infty)$, there are field configurations of zero energy: constant fields with values in the critical point set. (For an isolated critical point there is a unique such field.) These are the classical vacua. But in components where $h(\infty) \neq h(-\infty)$ there are no vacua. In any component the energy E, which is the spatial integral of the energy density Θ (7.39), is bounded below by the absolute value of the central charge:

$$E \geq |Z|. \tag{7.43}$$

From (7.39) we see that a field configuration of minimal energy satisfies the *first order* differential equation for a flow line:

$$\partial_1 \phi \pm \phi^* \operatorname{grad} h = 0. \tag{7.44}$$

It is easy to verify that a solution to (7.44) necessarily satisfies the *second order* Euler-Lagrange equation of motion.

In general, recall that nonvacuum field configurations of locally minimal energy are called solitons. Physically a classical soliton represents a stable localized lump of energy sitting still in space. Upon quantization we might expect that it gives a particle, and because of the inequality (4.34) we can expect it to be massive. In more detail: When we quantize the theory we need to quantize the space $\mathcal{M}$ of all finite energy classical solutions to the field equations. This space divides into components as indicated above. The lowest energy state in a diagonal component is a vacuum of zero energy. The free approximation around the vacuum give massless and massive particles according to the Hessian of V, at least in perturbation theory. Nondiagonal components have no state of zero mass. Rather, the states of smallest mass saturate the inequality (7.43) (where we replace energy by mass) and form a symplectic manifold on which the Poincaré group acts. The free approximation now involves quantization of this symplectic manifold as well as quantization of the quadratic approximation in its infinite dimensional normal bundle. What we expect, then, is a collection of *quantum solitons*. They are massive, and in perturbation theory their mass is very large compared to the masses of the particles constructed from small fluctuations. Of course, this story obtains quantum corrections as we move away from the free approximation.

The discussion so far has ignored the fermion and the supersymmetry. In fact, the central charge Z appears in the supersymmetry algebra as was anticipated in (6.26), where we discussed a central extension of the supersymmetry algebra $\mathfrak{p}^{2|(1,1)}$. Now we realize the central extension classically by computing the Poisson bracket $\{Q_+, Q_-\}$ of the supersymmetry charges. The Poisson bracket of Noether charges can be computed in different ways, and is in fact the spatial integral of a Poisson bracket of Noether currents. We use the explicit formulas (7.38) and note that the nontrivial contributions come from the brackets $\{\psi_+, \psi_+\}$ and $\{\psi_-, \psi_-\}$. Note that we compute on-shell, that is, assuming the equations of motion to be satisfied. The result is

(7.45)
$$\begin{aligned}\frac{1}{2}\{Q_+, Q_-\} &= \frac{1}{2}\int_{x^0=\text{const}} \Big\{\langle \partial_+\phi, \phi^* \operatorname{grad} h\rangle - \langle \partial_-\phi, \phi^* \operatorname{grad} h\rangle\Big\}\, |dx^1| \\ &= \int_{x^0=\text{const}} \langle \partial_1\phi, \phi^* \operatorname{grad} h\rangle\, |dx^1| \\ &= \int_{x^0=\text{const}} \partial_1(\phi^* h)\, |dx^1| \\ &= Z.\end{aligned}$$

So we obtain a central extension of the supersymmetry algebra, as promised. The energy inequality (7.43) is the classical version of (6.32), and it follows from the supersymmetry algebra as in (6.30). Field configurations which saturate this classical *BPS inequality* are annihilated by the supersymmetry $\tau_{Q_+} + \tau_{Q_-}$ or by $\tau_{Q_+} - \tau_{Q_-}$. This provides an alternative derivation—using supersymmetry—of the first order equation (7.44) for minimal energy bosonic field configurations. Namely, we look at the supersymmetry transformation (7.37) and ask that the vector field $\hat{\zeta}$ vanish. Inspecting the variation of the fermions ψ_+, ψ_- we learn that such *BPS field configurations* satisfy

$$\begin{aligned}\partial_+\phi \pm \phi^* \operatorname{grad} h &= 0 \\ \pm\partial_-\phi - \phi^* \operatorname{grad} h &= 0.\end{aligned} \tag{7.46}$$

These equations are equivalent to the equations

$$\begin{aligned}\partial_0\phi &= 0 \\ \partial_1\phi \mp \phi^* \operatorname{grad} h &= 0,\end{aligned} \tag{7.47}$$

which are the equations for a static flow line. We also obtain an equation on fermions by requiring that the variation of the boson ϕ vanish in (7.37):

$$\psi_+ \pm \psi_- = 0. \tag{7.48}$$

To analyze this we must also bring in the equations of motion for the fermions, which are

$$\begin{aligned}-\partial_+\psi_- &= R(\psi_+, \psi_-)\psi_+ + \phi^*(\nabla \operatorname{grad} h)(\psi_+) \\ \partial_-\psi_+ &= -R(\psi_-, \psi_+)\psi_- + \phi^*(\nabla \operatorname{grad} h)(\psi_-).\end{aligned} \tag{7.49}$$

Combining (7.47) and (7.48) we see that the curvature terms vanish by the Bianchi identity and we are left with equations for a single fermion ψ_+:

$$\begin{aligned}\partial_0\psi_+ &= 0 \\ \partial_1\psi_+ \mp \phi^*(\nabla \operatorname{grad} h)(\psi_+) &= 0.\end{aligned} \tag{7.50}$$

Equations (7.50) are the variations of equations (7.46) in the direction ψ_+. Thus the BPS fermion field ψ_+ is an odd tangent vector to the manifold of flow lines.

Recapping: The BPS condition leads to a first order equation on bosons which implies the second order equation of motion. This demonstrates that classical fermionic fields, which are difficult to visualize in a geometric manner, can impact classical bosonic fields.

Dimensional reduction to $n = 1$

Classically, we reduce to a mechanical model by requiring the fields in the $n = 3$-dimensional model to be invariant under all spatial translations. We obtain a model with supersymmetry group $P^{1|2}$. It has a superspacetime formulation on $M^{1|2}$, which is the dimensional reduction of the superspacetime formulation in $n = 3$ dimensions. This theory has twice the minimal amount of supersymmetry. In particular, it has twice as much supersymmetry as the superparticle model considered in Lecture 4. There we indicated that the quantum Hilbert space of that model is the space of spinor fields on X. With twice as much supersymmetry we obtain instead the space of differential forms on X as the quantum Hilbert space; the $\mathbb{Z}/2$-grading is by the parity of the degree. In the minimal superparticle there is a single supercharge, and its quantization is the Dirac operator. Here there are two supercharges (7.38) and the corresponding quantum operators are, up to factors, the first order differential operators $d + d^*$ and $\sqrt{-1}(d - d^*)$. This is if we have no potential term. In case there is a potential term, then the second term in the supercharges (7.38) gives an additional term which is a combination of exterior and interior multiplication by dh. In this way we obtain the modification of the de Rham complex used by Witten in his study of Morse theory. The hamiltonian in this model is the Hodge laplacian on differential forms modified by lower order terms involving dh. The solitons we discussed in $n = 2$ dimensions are *instantons* in $n = 1$ dimension, and they enter into the Morse theory discussion.

The dimensional reduction from $n = 3$ dimensions to $n = 1$ has a global $SO(2)$ symmetry from spatial rotations. From the $n = 3$-dimensional point of view these symmetries are part of the Poincaré group; from the $n = 1$-dimensional point of view they are an example of an R-symmetry. In the quantum mechanical theory this global $SO(2)$ symmetry gives a $\mathbb{Z}$-grading on the quantum Hilbert space; it is simply the usual $\mathbb{Z}$-grading on differential forms.

Exercises

1. The Lorentz group in dimension 4 is isomorphic to $SL(2;\mathbb{C})$, so there are two complex conjugate two-dimensional spin representations S', S''. The vector representation V has complexification

$$V_{\mathbb{C}} \cong S' \otimes S''.$$

There is a unique minimal real spinor representation, which is the underlying real representation of S' (or of S''); its complexification is $S' \oplus S''$. Each of S', S'' has a skew form ϵ. Let $M^{4|4}$ be the super Minkowski spacetime built on the spin representation S.

(a) Take Q_1, Q_2 as a (complex) basis of S' and then the complex conjugate basis $\overline{Q}_{\dot{1}}, \overline{Q}_{\dot{2}}$ for S''. The corresponding left-invariant complex odd vector fields on $M^{4|4}$ are denoted D_a and $\overline{D}_{\dot{a}}$. The odd coordinates on $M^{4|4}$ corresponding to the basis are denoted θ^1, θ^2; the complex conjugates $\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}$ Use the notation $\partial_{a\dot{b}}$ for the complex even vector fields, using the isomorphism above. Write formulas for the D_a and their complex conjugates in terms of the coordinate vector fields. Write all of the brackets of the odd vector fields.

(b) Is there an infinitesimal R-symmetry?

(c) Write the vector fields $\partial_{a\dot{b}}$ in terms of the standard vector fields $\partial/\partial x^\mu$ (for some nice choice of basis; recall similar formulas (6.22) in the 3-dimensional case).

(d) Write the kinetic term for a spinor field with values in S. (Schematically the kinetic term, ignoring the factor 1/2, is $\bar{\psi}\not{D}\psi$.) Your answer should involve indices, the symbols $\epsilon^{ab}, \epsilon^{\dot{a}\dot{b}}$, etc.

(e) Integration over all four odd coordinates is defined by the first line of the formula:

$$\begin{aligned}\int d^4\theta &= i^* \frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}} \\ &= \frac{1}{2}\, i^*\, \{D_1\overline{D}_{\dot{1}}\overline{D}_{\dot{2}}D_2 + D_2\overline{D}_{\dot{2}}\overline{D}_{\dot{1}}D_1\} - \Box i^* \\ &= \frac{1}{2}\, i^* \left\{D^2\overline{D}^2 + \overline{D}^2 D^2\right\} + \Box i^*.\end{aligned}$$

Verify the equality of the first line with the remaining two. Here $\Box$ is the wave operator (3.7), which is Poincaré-invariant. Finally, $D^2 = \frac{1}{2}\epsilon^{ab}D_aD_b$ and $\overline{D}^2 = \frac{1}{2}\epsilon^{\dot{a}\dot{b}}\overline{D}_{\dot{a}}\overline{D}_{\dot{b}}$ is the complex conjugate.

(f) How does all of this reduce to 3 dimensions?

(g) How does it reduce to 2 dimensions? (You should recover what you did in Problem Set 6.)

2. (a) Write the geometric data (5.30) and (5.34) which leads to the supersymmetric σ-model lagrangian (7.31).

(b) What happens in case $X = \mathbb{R}$ and the superpotential h is quadratic? Do we get a free theory? What is the particle content (masses and spins)?

(c) Consider the previous questions in the dimensionally reduced models as well.

3. (a) Derive (7.7). To do so let $\exp(-t\eta\tau_{Q_a})$ be the one-parameter group of diffeomorphisms of $M^{n|s}$ induced from the vector field $-\eta\tau_{Q_a}$, and consider its action by pullback on (7.6). Note that it preserves the left-invariant vector fields D^r. Now differentiate at $t = 0$, use the fact that right-invariant and left-invariant vector fields commute, and the fact that τ_{Q_a} and D_a agree on $M^n \subset M^{n|s}$.

(b) Apply (7.7) to compute the supersymmetry transformation laws (7.8).

(c) Compute the commutator of infinitesimal supersymmetry transformations $\hat{\zeta}, \hat{\zeta}'$. Explain why the curvature terms enter and why they are expected.

4. (a) Complete the derivation of the component lagrangian (7.22) from the superspacetime lagrangian (7.10).

(b) Compute the equations of motion.

(c) Compute the supercurrent (7.25).

(d) Fill in the details of the computation (7.45). Compute the Poisson bracket as the action of the vector field corresponding to Q_+ on Q_-. That vector field can be read off from (7.37), though to obtain its action on $\phi^*\operatorname{grad} h$ it is easier to use the dimensional reduction of the last equation in (7.8), supplemented by (7.30). You'll need the Bianchi identity and the equations of motion as well.

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Quantum Field Theory

Sunday, September 1, 1996 (All day) to Wednesday, June 30, 1999 (All day)
(1996-1999)

- Printer-friendly version

- Calendar

A program in Quantum Field Theory for mathematicians was held at the Institute for Advanced study during the academic year 1996-97. The participants and lecturers produced lecture notes and problem sets (and some solutions to problems) throughout the year, which are stored here. This web site is in its final form as of January 21, 1999; the intention is to leave it in place indefinitely.

Contents

- Lecture notes and problem sets from the fall term
- Lecture notes and problem sets from the spring term
- You may also wish to consult the updates to this web site between 6/9/97 and 1/21/99.

Following the conclusion of the program, much of the material on this web site has been re-organized, supplemented, and polished. It is being published as:

Quantum Fields and Strings: A Course For Mathematicians (P. Deligne, P. Etingof, D.S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison and E. Witten, eds.,), 2 vols., American Mathematical Society, Providence, 1999.

Dan Freed has prepared an introductory account of supersymmetry and classical field theory, published as:

Daniel S. Freed, *Five Lectures on Supersymmetry*, American Mathematical Society, Providence, 1999.

There were two followup workshops to this program, held at the Institute for Theoretical Physics, University of California, Santa Barbara. Lecture notes and audio recordings of many of the lectures from the January, 1998 workshop are available; there is also a web site for the July-August, 1999 workshop.

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LETTER TO THE EDITOR

Invariant derivation of the Euler–Lagrange equation

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Abstract. The tangent bundle geometry is used to obtain a coordinate-free derivation of the Euler–Lagrange equation.

The Lagrangian formulation of classical mechanics is the most fundamental approach to dynamics. Nevertheless, the usual practice is to transform to the Hamiltonian form. An underlying mathematical reason is that phase space T^*Q (the cotangent bundle of configuration space Q) is *canonically* a symplectic manifold [1] whereas TQ (the tangent bundle) is not. It is this symplectic structure on phases space which gives rise to the elegant simplicity of the Hamiltonian formalism.

Although not as well known, there is also a rich geometric structure on TQ which has been studied, in particular, by Klein [2, 3] (see also Godbillion [4]). Using this structure, the Euler–Lagrange equations may be given an invariant geometric formulation directly in terms of the (pre)-symplectic geometry determined by a Lagrangian function [5] without any reference to the symplectic structure on T^*Q. This is sometimes necessary since not all Lagrangians L lead to a nice Legendre transformation (fibre derivative FL) from TQ to T^*Q. In that case there is no Hamiltonian form [5, 6].

The geometry obtained from a *degenerate* Lagrangian on TQ is *pre*-symplectic without any natural symplectic extension. The Dirac constraint algorithm [7, 8] (just as on the cotangent bundle [9–12]) can be invariantly formulated directly on TQ in terms of this presymplectic geometry [5]. A central role is played by the second-order vector field condition (i.e. $\dot{q} = v$) [13–15].

While there are geometric coordinate-free *studies* of these Lagrangian equations we know of no coordinate-free *derivation* of the Euler–Lagrange equation. Our aim is to present such an invariant derivation of the Euler–Lagrange equation from the usual starting point in physics: extremising the action determined by a Lagrangian function.

Some results, in particular the Euler–Lagrange equation, are most easily obtained by first using a coordinate system and then demonstrating suitable covariant transformation laws. Modern work, however, has shown the value of coordinate-free geometric *formulations*. Likewise there is also a value in geometrically invariant *derivations*. Not only do they directly show that the result is coordinate independent they also serve to clarify certain assumptions.

First we establish some notation and recall the basic definitions necessary for the geometric structures we will use on the tangent bundle. For more details see [2–4]. For any manifold M let τ_M denote the projection from the tangent bundle TM. A differentiable map $f: M \to Q$ induces the tangent map $Tf: TM \to TQ$. In particular, from $\tau_Q: TQ \to Q$ we obtain $T\tau_Q: T(TQ) \to TQ$ such that the diagram

$$\begin{array}{ccc} T(TQ) & \xrightarrow{T\tau_Q} & TQ \\ {\scriptstyle \tau_{TQ}}\downarrow & & \downarrow{\scriptstyle \tau_Q} \\ TQ & \xrightarrow[\tau_Q]{} & Q \end{array}$$

commutes. This diagram is the foundation for several structures. The vertical sub-bundle $V(TQ)$ of the second tangent bundle $T(TQ)$ is defined by $V(TQ) := \ker T\tau_Q$. The vertical lift $\xi_y: T_qQ \to V_y(TQ)$ is defined by

$$\xi_y(w) := \frac{\mathrm{d}}{\mathrm{d}\lambda}(y + \lambda w)$$

where $q = \tau_Q(y) = \tau_Q(w)$. From ξ we can construct the almost tangent structure

$$J_y := \xi_y \circ T\tau_Q : T_y(TQ) \to T_y(TQ)$$

which has the properties $\mathrm{Im}\, J = \ker J = V(TQ)$, hence $J^2 = 0$, and the Liouville canonical vector field V on TQ, which is defined by

$$V_y := \xi_y(y) \qquad \text{for } y \in TQ.$$

A curve $C:[a, b] \to Q$ prolongs to sections C' of TQ and C'' of $T(TQ)$. The vector field $X := C''$ on $T(TQ)$ is special in that it is second order. A second-order vector field is characterised by the property $T\tau_Q X = \tau_{TQ} X$. A direct application of the above definitions leads to the alternate characterisation

$$JX = V \tag{1}$$

which is more convenient for our purposes.

We recall that a linear endomorphism $A: TM \to TM$ induces a derivation (with grading rank 0) the interior product $i_A: \wedge^p M \to \wedge^p M$ on differential forms on M defined by

$$(i_A\beta)(X_1, \ldots, X_p) := \sum_{k=1}^{p} \beta(X_1, \ldots, AX_k, \ldots, X_p)$$

where $X_j \in TM$, with $i_A f := 0$ for any function. Further derivations may be obtained from the graded commutators with the basic derivations: the exterior derivative d and the interior product with a vector field i_X of grading rank $+1$ and -1 respectively.

On TQ this construction naturally leads to the vertical derivative

$$d_J := [i_J, d] = i_J d - d i_J.$$

It is easy to verify that

$$[d, d_J] = dd_J + d_J d = 0$$

$$[d_J, i_V] = d_J i_V + i_V d_J = i_J$$

$$[i_X, i_J] = i_X i_J - i_J i_X = i_{JX}$$

and that $d_J^2 = 0$. The remaining basic graded commutator $[i_Z, d] = i_Z d + d i_Z$ is just the Lie derivative $\mathscr{L}_Z$.

For completeness we include the purely tangent space definition of the fibre derivative $Fg: TQ \to T^*Q$ of $g: TQ \to \mathbb{R}$:

$$Fg(y) := dg(y) \circ \xi_y : T_qQ \to \mathbb{R}.$$

Remark. The fibre derivative may be used to relate the structures on TQ and T^*Q, in particular $d_Jg = Fg^*\theta$ where θ is the canonical 1-form on T^*Q. In general $dd_Jg = Fg^*\omega$ is only presymplectic whereas $\omega = d\theta$ is the natural symplectic structure on the cotangent bundle.

To each path $C:[a, b] \to Q$ a Lagrangian function $L: TQ \to \mathbb{R}$ associates an action

$$S[C] := \int_a^b L \circ C' \, \mathrm{d}t. \tag{2}$$

We wish to show the following proposition.

Proposition. For paths with fixed endpoints the action (2) is stationary for the path C iff the Euler-Lagrange equation

$$\mathscr{E} := i_X dd_J L + dE_L = 0 \tag{3}$$

is satisfied, where $X := C''$ is the Lagrangian vector field on TQ and $E_L := i_V dL - L$ is the energy.

Proof. Consider a one-parameter set of paths $C_\lambda(t)$ in Q with fixed endpoints. They determine two vector fields on TQ, the Lagrangian vector field $X := C''$ which satisfies the second-order equation condition (1) and the deviation vector field $Z := (\partial/\partial\lambda)C'_\lambda$ which is characterised by JZ vanishing at the endpoints and the properties

$$[X, Z] = 0 = [Z, V - JX]. \tag{4}$$

We wish to find the necessary and sufficient conditions for the action S to be stationary:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_0 = \int_a^b \langle dL|Z\rangle \, \mathrm{d}t = 0.$$

(i) For all Z

$$\begin{aligned} \langle dL|Z\rangle &= Z\langle dL|V\rangle - \langle dE_L|Z\rangle = Z\langle dL|V\rangle - \langle \mathscr{E}|Z\rangle + \langle dd_JL|X, Z\rangle \\ &= Z\langle dL|V\rangle - \langle \mathscr{E}|Z\rangle + X\langle d_JL|Z\rangle - Z\langle d_JL|X\rangle - \langle d_JL|[X, Z]\rangle \\ &= -\langle \mathscr{E}|Z\rangle + X\langle dL|JZ\rangle + Z\langle dL|V - JX\rangle - \langle dL|J[X, Z]\rangle \\ &= -\langle \mathscr{E}|Z\rangle + X\langle dL|JZ\rangle + \langle \mathscr{L}_Z dL|V - JX\rangle + \langle dL|[Z, V - JX]\rangle - \langle dL|J[X, Z]\rangle. \end{aligned} \tag{5}$$

Hence the vanishing of $\mathscr{E}$ is sufficient for $\mathrm{d}S/\mathrm{d}\lambda = 0$, since under the conditions (1) and (4) imposed on Z and X all of the terms vanish except

$$X\langle dL|JZ\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle dL|JZ\rangle$$

which integrates to an evaluation at the endpoints where JZ vanishes.

(ii) The above calculation does *not* show that $\mathscr{E} = 0$ is necessary since Z is not arbitrary. In particular we have the restrictions (4). To obtain an arbitrary vector field we add JW for any W to Z, then

$$\langle \mathscr{E}|Z\rangle = \langle \mathscr{E}|Z + JW\rangle - \langle \mathscr{E}|JW\rangle = \langle \mathscr{E}|Z + JW\rangle - \langle i_J\mathscr{E}|W\rangle. \tag{6}$$

Lemma.

$$i_J\mathscr{E} = i_{V-JX}\, dd_J L.$$

Proof.

$$\begin{aligned} i_J i_X dd_J L &= [i_J, i_X]dd_J L + i_X i_J dd_J L \\ &= -i_{JX} dd_J L + i_X d_J^2 L + i_X d i_J d_J L = -i_{JX} dd_J L \end{aligned}$$

and

$$\begin{aligned} i_J dE_L &= d_J E_L = d_J i_V dL - d_J L \\ &= [d_J, i_V]dL - d_J L - i_V d_J dL \\ &= -i_V d_J dL = i_V dd_J L. \end{aligned}$$

Consequently (6) becomes

$$\begin{aligned} \langle\mathscr{E}|Z\rangle &= \langle\mathscr{E}|Z+JW\rangle - \langle i_{V-JX} dd_J L|W\rangle \\ &= \langle\mathscr{E}|Z+JW\rangle + \langle i_W dd_J L|V-JX\rangle. \end{aligned}$$

Using this result (5) may be written in the form

$$\begin{aligned} \langle dL|Z\rangle = -\langle\mathscr{E}|Z+JW\rangle &+ \langle i_W dd_J L|V-JX\rangle + \langle\mathscr{L}_Z dL|V-JX\rangle - \langle dL|J[X,Z]\rangle \\ &+ \langle dL|[Z, V-JX]\rangle + X\langle dL|JZ\rangle \end{aligned}$$

for all W, Z. With the restrictions that $V-JX=0$, $[X,Z]=0$, $[Z, V-JX]=0$ and JZ vanishes at the endpoints, we have

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right|_0 = \int_a^b \langle dL|Z\rangle\,\mathrm{d}t = -\int_a^b \langle\mathscr{E}|Z+JW\rangle\,\mathrm{d}t.$$

Although Z is not completely arbitrary, $Z+JW$ is. Consequently, $\mathrm{d}S/\mathrm{d}\lambda|_0 = 0$ with $V-JX=0$ implies $\mathscr{E}=0$ and conversely $\mathscr{E}=0=V-JX$ implies $\mathrm{d}S/\mathrm{d}\lambda|_0=0$.

In general the second-order condition plays an essential and independent role. Although the Euler–Lagrange equation $\mathscr{E}=0$, via the lemma, does impose a restriction:

$$i_J\mathscr{E} = i_{V-JX}\, dd_J L = 0$$

it is not always strong enough to assure that $V-JX$ vanishes, since $dd_J L$ is only presymplectic if L is degenerate. Consequently, to guarantee that the action S be stationary we must supplement $\mathscr{E}=0$ in general with $V-JX=0$. Together these conditions are necessary and sufficient.

The form of the Euler–Lagrange equation used here corresponds to Hamilton's equations on the cotangent bundle. This form of the equation can be transcribed into

$$\mathscr{E} := i_X dd_J L + dE_L = \mathscr{L}_X d_J L - d i_X d_J L + d(i_V dL - L) = \mathscr{L}_X d_J L - dL - d i_{V-JX} dL$$

which, along with the second-order condition $V-JX$, is equivalent to

$$\mathscr{E}' := \mathscr{L}_X d_J L - dL.$$

This latter form is less convenient geometrically (it depends differentially on the Lagrangian vector field) but is more recognisable to physicists.

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Toronto Lectures on Physics

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I have been invited by the mathematicians here at Toronto to give three talks describing my joint work with the late Prof. Yuval Ne'eman in elementary particle physics. This work uses some mathematical ideas such as super Lie algebras and their representations, and the Quillen theory of superconnections. But the work is in physics, and this puts me in a quandary as to the amount of physics background that I can assume.

In order not to chase away any physicists in the audience, I will start by listing the physics problems that our approach tries to partially address. This will involve some words that may not be familiar to mathematicians, to whom I apologize. I hope to elucidate the meaning of most of these words in the course of the lectures. All of the results in these lectures are contained in the review article I wrote with Prof. Ne'eman, and which appeared in Physics Reports **406** 2005, 303 -377.

I include here some expository material (mainly for mathematicians) which I did not have time to include in the lectures; for example a review of classical electromagnetism and material on the Dirac operator.

I thank Yael Karshon for helpful comments on this text and the associated slides.

1 The "Standard Model" and some of its ills.

The "Standard Model" of the physics of particles and fields (assumed to include all known fundamental interactions except for gravity) is enormously successful, with its predictions validated by all experimental tests. In particular, the electroweak interactions seem to be correctly described by the $su(2)\times u(1)$ spontaneously broken local gauge symmetry. Although the full implementation of this (Weinberg-Salam) theory requires quantum field theory, much of its basic structure can be phrased in terms of classical field theory, see for example, Kane *Modern Elementary Particle Physics* , or, for the more mathematically inclined reader, Derdzinski *Geometry of the Standard Model of Elementary Particles.* Note that a comprehensive review intended for particle (or high energy) physicists appeared in E.S. Abers and B.W. Lee, "Gauge Theories", *Physics Reports*, **9C** no. 1 1973. So this theory has been successfully around for a long time.

The very success of this theory prompted a number of questions relating to its structure, hypotheses and input. The unresolved issues include

- The large number of free parameters which must be experimentally determined to serve as input into the theory such as the various gauge coupling constants (including the Weinberg angle), the parameters of Higgs potential, the coupling constants of the matter fields, the eigenvalues of the weak isospin and weak hypercharge for the chiral leptons and fermions etc.

- As a result, the theory is unable to predict the value of the mass of the Higgs particle. This meson has therefore been searched for all over the accelerator-available spectrum, from a few GeV to the 115 GeV reached at Cern in October 2000, when 9 "events" were reported at the limit of the accelerator's energy range. (These "events" constituted 2.6 standard deviations above background level, whereas 5 standard deviations are considered necessary for an accepted result that could be interpreted as evidence for the Higgs particle.) All this was before the planned closure of the machine. However when the accelerator was granted another month of operation, no further evidence was found. Several machines are expected to renew the search in the next 1- 3 years, reaching into the 100-500 GeV range.

- The lack of correlation between the quantum numbers of left and right chiral leptons and fermions.

- The *ad hoc* introduction of Higgs fields to implement spontaneous symmetry breaking.
- The fact that these Higgs fields constitute a weak isospin doublet.
- No explanation of the origin of the Higgs potential needed to achieve Goldstone-Higgs spontaneous symmetry breaking.
- No explanation of the absence of right handed neutrinos. In fact, since we now know that the neutrino is massive, we know that right handed neutrinos do exist. So we can reformulate the question as follows: Why don't the right handed neutrinos participate in the Weinberg-Salam theory?

I wish to show in these lectures how using superconnections allows an answer to some of these difficulties.

2 Problems of translation between mathematics and physics.

There are several communications difficulties between mathematicians and physicists, some more serious than others. I want to get a few of these out into the open before one group or the other disappears:

2.1 Is there an i in the structure constants of a Lie algebra?

The first barrier between the mathematics literature and most of the physics literature is the ubiquitous factor of i: The mathematical definition of a Lie algebra is that it is a vector space $\mathfrak{k}$ with a bilinear map

$$\mathfrak{k} \times \mathfrak{k} \to \mathfrak{k}$$

which is anti-symmetric and satisfies Jacobi's identity.

So the set of self-adjoint matrices under commutator bracket is *not* a Lie algebra. Indeed the commutator of two self adjoint matrices is skew adjoint. So the Lie algebra of $u(n)$ is not the space of self-adjoint matrices but rather the space of skew adjoint matrices. Indeed, if A is a skew adjoint matrix then $\exp tA$ is a one parameter group of unitary matrices. The physicists prefer to write $\exp itH$ where H is self adjoint. This is of course due to the fact that self adjoint operators are the observables of quantum mechanics, and Noether's theorem suggests that elements of the Lie algebra should correspond to observables. But the price to pay for this is to put an i in front of all brackets.

For example, the three dimensional real vector space consisting of self adjoint two by two matrices of trace zero has, as a basis, $\tau_i,\ i = 1, 2, 3$ the "Pauli matrices", where, to be absolutely sure of the factors 1/2 etc.,

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The physicists like to think of these as "generators" of $SU(2)$, i.e. as elements of the Lie algebra $su(2)$. Of course, we mathematicians would say that multiplying each of these three matrices by i gives a basis of $su(2)$. This distinction is relatively harmless, but is a nuisance for a mathematician reading a physics book or paper.

2.2 Ad invariant metrics on $u(2)$.

If we use the scalar product

$$(A, B) = 2 \operatorname{tr} AB$$

then the elements $\frac{1}{2}\tau_i$ form an orthonormal basis of our three dimensional space of self-adjoint matrices of trace zero. Since the algebra $su(2)$ is simple, the most general Ad invariant scalar product on our three dimensional space of self-adjoint matrices of trace zero must be a positive multiple of the above scalar product.

We will want to consider the four dimensional space of all two by two self adjoint matrices. (After multiplication by i this would yield the Lie algebra of $U(2)$.) So we must add the two by two identity matrix I to get a basis of this four dimensional real space. The algebra $u(2) \sim su(2) \oplus u(1)$ is not simple, but decomposes into the sum of two ideals consisting of $su(2)$ and all (real) multiples of iI. These ideals must be orthogonal under any Ad invariant metric. So there is a two parameter family of Ad invariant metrics on $u(2)$.

2.3 Ad invariant metrics, coupling constants, and the Weinberg angle.

Indeed, the most general Ad invariant metric on our four dimensional space of all self-adjoint two by two matrices can be written as

$$\frac{2}{g_2^2} \operatorname{tr}(A - \frac{1}{2}(\operatorname{tr} A)I)(B - \frac{1}{2}(\operatorname{tr} B)I) + \frac{1}{g_1^2} \operatorname{tr} A \operatorname{tr} B. \tag{1}$$

Relative to this scalar product the elements

$$\frac{g_2}{2}\tau_1,\ \frac{g_2}{2}\tau_2,\ \frac{g_2}{2}\tau_3, \frac{g_1}{2}I \tag{2}$$

form an orthonormal basis.

Notice that for traceless matrices the second term in (1) vanishes, and the first term reduces to a multiple of $2 \operatorname{tr} AB$; similarly, for multiples of I, the first term vanishes.

For mathematicians, the question is "why this strange notation, with the g_2 and g_1 occurring in the denominator?". The answer is that, in the physics literature, these constants are not regarded as parametrizing metrics on $u(2)$, but rather as "universal coupling constants". I will spend a chunk of today's lecture explaining why choosing a metric on a Lie algebra is important, and what is its physical significance.

In any event, however you want to interpret these parameters, the *Weinberg angle* θ_W is defined by

$$\frac{g_1^2}{g_2^2} = \tan^2 \theta_W.$$

It plays an important role in the theory.

2.4 What are classical fields?

A third difference between the mathematical literature and the physics literature is that in the physics literature all (classical) fields are regarded as scalar valued functions (or vector fields) or n-tuplets of scalar valued functions (or vector fields) . One must then discuss the "field transformations" under which, for example, the Lagrangian is invariant. The mathematical literature prefers a "basis free" formulation where many of the invariance properties of the Lagrangian are obvious - they are built into the formulation. The price to pay is that the fields are no longer scalar functions or n-tuplets of scalar functions but vector valued functions, or, more generally, sections of a vector bundle, or differential forms with values in a vector bundle.

This means that in the physics literature a basis of the vector space (or a basis of sections of the vector bundle) is chosen. Thus, for example, if we choose a basis $v_1, \ldots, v_n$ of a Lie algebra $\mathfrak{k}$ then the Lie bracket can be given in terms of the Cartan structure constants c_{jk}^{ℓ} where

$$[v_j, v_k] = \sum_{\ell} c_{jk}^{\ell} v_{\ell}.$$

As explained above, in the physics literature there will be an additional factor of i in front of the structure constants as understood by the mathematicians. For example, if we take the orthonormal basis of the space of traceless two by two self adjoint matrices consisting of the first three elements of (2), we find by direct computation that

$$\left[\frac{g}{2}\tau_1, \frac{g}{2}\tau_2\right] = i\frac{g^2}{2}\tau_3 = ig\frac{g}{2}\tau_3, \qquad g = g_2,$$

with a similar formula for the brackets of the remaining two elements. So relative to this basis, the structure constants are

$$C_{jk\ell} = ig\epsilon_{jk\ell}.$$

Up to an overall sign arising from slightly different conventions this is the statement about the structure constants of $SU(2)_L$ found in S. Weinberg, *The Quantum Theory of Fields*, Cambridge U. Press (1996), vol. 2. page 307 just after equation (21.3.11) giving the expression of the Lagrangian of the Yang-Mills field. So whereas for mathematicians the parameter g describes the scalar product on $su(2)$, for physicists, who write out the fields in terms of an orthonormal basis, the g appears in the structure constants and is interpreted as a "coupling constant", measuring the "strength of the interaction between the fields".

3 The permittivity of space-time is a metric on $u(1)$.

In order to bolster my contention that the metric on a Lie algebra has important physical significance, I want to review Maxwell's classical theory of electromagnetism, with special attention to units.

I will begin with two non-relativistic regimes:

3.1 Electrostatics.

The objects are:

3.1.1 The electric field.

This is a linear differential form, E, called the *electric field strength*. A point charge e experiences the force eE. The integral of E along any path gives the voltage drop along that path. The units of E are

$$\frac{\text{voltage}}{\text{length}} = \frac{\text{energy}}{\text{charge} \cdot \text{length}}.$$

Remember that force has the units energy/length and voltage has units energy/charge.

The fundamental law satisfied by E is

$$dE = 0.$$

In simply connected regions this implies the existence of a function u called the potential such that

$$E = -du.$$

3.1.2 The dielectric displacement.

This is a two form D on $\mathbb{R}^3$. Its physical significance is as follows. To determine the value of D on a (small) oriented plane element, insert two small metal plates of the shape of this plane element, touch them together and then separate them. Charges $\pm Q$ are acquired on the plates. The orientation of the plane together with the orientation of $\mathbb{R}^3$ determines which of these two separated plates is the "top" plate and the value of D is (the limit of)

$$4\pi \frac{\text{charge on the top plate}}{\text{area of the plates}}.$$

So the units of D are

$$\frac{\text{charge}}{\text{area}}.$$

Notice that this definition makes no mention of the electric field.

The fundamental law satisfied by D is **Gauss's law** which asserts that for any region U

$$\int_{\partial U} D = 4\pi \int_U \rho dx \wedge dy \wedge dz$$

where ρ is the electric charge density.

Stokes' theorem gives the infinitesimal version of Gauss's law as

$$dD = 4\pi\rho dx \wedge dy \wedge dz.$$

If E is an electric field strength and D is a dielectric displacement then $E \wedge D$ is a three form which we may integrate over $\mathbb{R}^3$ if it is of compact support or if it vanishes sufficiently rapidly at infinity. Then we can form

$$\langle D, E\rangle := \int_{\mathbf{R}^3} E \wedge D$$

which we can consider as a sort of pairing between the space of electric fields and the space of dielectric displacements. The value of this pairing has units

$$\text{volume} \cdot \frac{\text{force}}{\text{charge}} \cdot \frac{\text{charge}}{\text{area}} = \text{force} \cdot \text{length} = \text{energy}.$$

3.1.3 The dielectric operator and the dielectric coefficient.

This is a map C from the space of electric fields to the space of dielectric displacements. Later on we shall be more specific as to the form of C in terms of the three dimensional $\star$ operator. At the moment we can do with the following mild assumptions:

- C is linear.
- C is local in the sense that it E vanishes on an open set U so does $C(E)$.
- C is symmetric in the sense that

  $$\langle C(E), \hat{E}\rangle = \langle C(\hat{E}), E\rangle$$

  when both sides are defined. When both sides are defined, we set

  $$(E, \hat{E}) := \langle C(E), \hat{E}\rangle.$$

We can then define the energy of an electric field as

$$\frac{1}{2}(E, E).$$

3.1.4 The dielectric coefficient.

A more specific choice of the dielectric operator is to take

$$C(E) = \epsilon \star E$$

where $\star$ is the three dimensional star operator mapping one forms into two forms and ϵ is a function. Even more specifically, in many cases (such as for the vacuum) ϵ is a constant - called the **dielectric constant**. We will postpone the issue of units for the moment, assume that ϵ is indeed constant, and then choose our units of length so that it is absorbed into the star operator. Then the equations of electrostatics becomes

$$\begin{aligned}
E &= -du \\
D &= \star E \\
dD &= 4\pi\rho dx \wedge dy \wedge dz \\
&\text{so} \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= -4\pi\rho.
\end{aligned}$$

If $\rho = 0$ in some region, then in that region the last equation becomes Laplace's equation which we can write in coordinate free notation as

$$d \star du = 0.$$

3.1.5 Rotationally invariant solutions of Laplace's equation.

In polar coordinates we have

$$\star dr = r^2 \sin\theta d\theta \wedge d\phi.$$

So if $f = f(r)$ is defined for $r > 0$ we have

$$\begin{aligned}
df &= f'(r)dr \\
\star df &= (r^2 f'(r)) \sin\theta d\theta \wedge d\phi \\
d \star df &= \left(r^2 f'(r)\right)' dr \wedge \sin\theta d\theta \wedge d\phi \\
&\text{so} \\
d \star df = 0 &\Rightarrow \left(r^2 f'(r)\right)' = 0 \\
&\Rightarrow r^2 f'(r) = -c \text{ (a constant)} \\
&\Rightarrow \\
f(r) &= \frac{c}{r} + A
\end{aligned}$$

where c and A are constants.

The inverse square law of high-school physics drops out.

Exactly the same computation yields the Yukawa potential as the static fundamental solution of the Klein-Gordon equation: Indeed

$$d\left(\frac{e^{-mr}}{r}\right) = -e^{-mr}\frac{mr+1}{r^2}dr$$

so

$$\star d\left(\frac{e^{-mr}}{r}\right) = -e^{-mr}(mr+1)\sin\theta d\theta \wedge d\phi$$

and hence

$$\begin{aligned} d\star d\left(\frac{e^{-mr}}{r}\right) &= m^2 r e^{-mr}\sin\theta dr \wedge d\theta \wedge d\phi \\ &= m^2\left(\frac{e^{-mr}}{r}\right) r^2 \sin\theta dr \wedge d\theta \wedge d\phi. \end{aligned}$$

Thus if we think of a solution of the Klein-Gordon equation or of the Proca equation as a "transmitter" of a "force", then the value of m determines the range of this force.

3.2 Magnetoquasistatics.

3.2.1 Objects: The magnetic induction and the magnetic excitation.

The magnetic induction is a two form B on $\mathbb{R}^3$, which exerts a force on a current according to the following rule: if a charge e moves past the point P with velocity $\mathbf{v}$ then the force exerted on that charge is the covector

$$ei(\mathbf{v})B_P.$$

At each point of space the form B if $\neq 0$ will determine a direction in space: the line determined by the equation

$$i(\mathbf{w})B = 0.$$

Iron filings free to rotate but not to move will align themselves in these directions, producing the "magnetic lines of force" favored by Faraday. These are precisely the directions in which a current will feel no force.

The second item is a one form H known as the magnetic excitation. The second of Ampère's laws says that if S is any surface bounded by a curve γ, and if J is the two form representing the current flow, then

$$\int_\gamma H = 4\pi \int_S J.$$

3.2.2 The laws. 1: Faraday's law of induction.

This says that if S is a surface bounded by a curve γ then

$$-\frac{d}{dt}\int_S B = \int_\gamma E. \tag{3}$$

By Stokes the differential version of this law is

$$\frac{\partial B}{\partial t} = -dE. \tag{4}$$

If S is a closed surface bounding a region (so with no boundary curves) then Faraday's law implies that

$$\frac{d}{dt}\int_S B = 0.$$

In fact, a stronger law holds (Hertz), not only does the derivative of the integral of B over a closed surface vanish, but the integral itself does:

3.2.3 The laws. 2: There are no magnetic poles.

This says that

$$\int_S B = 0 \quad \text{for any closed surface} \ \ S.$$

By Stokes, the differential version of this law is

$$dB = 0.$$

3.2.4 The laws. 3: Ampère's law.

Recall that this says that if S is any surface bounded by a curve γ, and if J is the two form representing the current flow, then

$$\int_\gamma H = 4\pi \int_S J.$$

By Stokes' theorem, the differential version of this law is

$$dH = 4\pi J.$$

3.2.5 The force on a moving charge.

The force exerted by a magnetic field B on a moving charge $\mathbf{I} = e\mathbf{v}$ is

$$i(\mathbf{I})B.$$

3.2.6 The permeability.

There is a relation between H and B given by

$$B = \mu \star H$$

where $\star$ is the three dimensional star operator and μ is known as the permeability.

3.2.7 The units of $\int_S B$.

By Faradays' law of induction, the time derivative of this integral over a surface bounded by a curve is equal to the negative of the integral of E around that curve which has units of voltage which is energy/charge. So

$$\frac{\text{units of } \int\int_S B}{\text{time}} = \frac{\text{energy}}{\text{charge}}.$$

In "natural units", where $\hbar = 1$, energy has units of inverse time. This implies that the integral of B over a surface has units of inverse charge.

3.3 The Maxwell equations.

The laws of quasi-magnetostatics take on a very suggestive form when written in four dimensions rather than three, and when an important modification to Ampère's law is made. This modification was introduced by Maxwell.

3.3.1 The equation $dF = 0$.

We can combine the laws

$$\begin{aligned} dB &= 0 \qquad \text{(Hertz)} \\ \frac{\partial B}{\partial t} &= -dE \qquad \text{(Faraday's law of induction)} \end{aligned}$$

into the single law

$$dF = 0 \tag{5}$$

if we set

$$F = B + E \wedge dt.$$

In (5) the operator d means the four dimensional (space-time) d. The coefficient of $dx \wedge dy \wedge dz$ in (5) says that

$$d_{space} B = 0 \qquad \text{(Hertz)}$$

while the coefficient of dt in (5) gives Faraday's law of induction.

The equation $dF = 0$ implies that locally we can find a one form A (called the four potential) such that

$$dA = F. \tag{6}$$

3.3.2 The equation $dG = 4\pi j$.

In electrostatics we assumed that $J = 0$ and that the charge density ρ did not depend on t. In quasi-magnetostatics we ignored ρ. For the full equations of electromagnetism one assumes that there is a charge density and a current, and so consider the three form

$$j := \rho dx \wedge dy \wedge dz - J \wedge dt$$

on space time. "Conservation of charge" then demands that

$$dj = 0.$$

Locally this says that there is two form G such that

$$dG = 4\pi j. \tag{7}$$

(The 4π is conventional.) If we write

$$G = D - H \wedge dt, \tag{8}$$

then the dt component of (7) is

$$d_{space} H = \frac{\partial D}{\partial t} + 4\pi J.$$

So we recover Ampère's law with the modification that the "displacement current"

$$\frac{\partial D}{\partial t}$$

is added to the right hand side of Ampère's original law. The "space component" of (7) is

$$d_{space} D = 4\pi \rho dx \wedge dy \wedge dz$$

as in electrostatics.

3.4 Units.

Let us work in natural units where $\hbar = 1$ so that energy has units of inverse time.

3.4.1 The units of the integral of F over a surface.

We have already observed that the integral of B over a surface has units of inverse charge. The integral of E over a curve has units of (energy)/(charge), so the integral of $E \wedge dt$ over a surface in space time has units of

$$\frac{(\text{energy}) \times (\text{time})}{(\text{charge})} = \frac{1}{(\text{charge})}.$$

In short, the integral of F over a surface in space time has units of inverse charge.

3.4.2 The units of the integral of G over a surface.

From its definition, or from Gauss's law $dD = 4\pi\rho dx \wedge dy \wedge dz$ we see that the units of the integral of D over a surface are charge. Ampère's law

$$d_{space}H = \frac{\partial D}{\partial t} + 4\pi J$$

together with Stokes' theorem says that the integral of H over a curve has the same units as the flux of current through a surface and this has units (charge)/(time). So the integral of $H \wedge dt$ over a surface in space time also has the units of (charge). In short, the integral of G over a surface in space time has units of charge.

3.4.3 The integral of $F \wedge G$ over a four dimensional region is a scalar.

This follows from the preceding two results. In particular, this means that G is in a sense "dual" to F, the duality being given by exterior multiplication followed by integration. Of course we can not expect that the integral over all of space time will converge. We will examine this"duality" in more detail further on.

Notice that until now we have not used the metric structure of space time.

3.4.4 The units of the permittivity.

The units D are (charge)/(area). The units of E are (energy)/(charge)×(length). If there is a point-wise matrix which expresses the coefficients of D in terms of those of E its entries will have units

$$\frac{\text{charge}}{\text{area}} \times \frac{(\text{charge}) \times (\text{length})}{\text{energy}} = \frac{(\text{charge})^2}{(\text{energy}) \times (\text{length})}.$$

Indeed, the permittivity of free space is a scalar ϵ_0 given by

$$\epsilon_0 = 8.854187... \times 10^{-12}\ \frac{\text{Farad}}{\text{meter}}$$

where the Farad is a unit of capacitance:

$$1 \text{ Farad} := 1 \frac{\text{coulomb}}{\text{volt}}.$$

Since

$$1 \text{ volt} = 1\ \frac{\text{joule}}{\text{coulomb}}$$

has units of (energy)/(charge) we see that ϵ_0 has units of

$$\frac{(\text{charge})^2}{(\text{energy}) \times (\text{length})}.$$

3.4.5 The units of the permeability.

According to Ampère's law the units of H are (charge)/(length)$\times$(time).

According to Faraday's law the units of B are (energy)$\times$(time) / (charge)$\times$(length^2. If there is a point-wise matrix which expresses the coefficients of B in terms of those of H its entries will have units

$$\frac{\text{energy} \times (\text{time})^2}{(\text{charge})^2 \times (\text{length})}.$$

Indeed, the permeability of free space is a scalar μ_0 given by

$$\mu_0 = 12.566370 \ \times 10^{-7} \ \frac{\text{joule}}{(\text{amp})^2 \times (\text{meter})}.$$

Since one amp = one (coulomb)/(second) we see that μ_0 does have the above stated units.

3.4.6 $\epsilon_0 \times \mu_0 = 1/c^2$.

This was of course another of the great discoveries of Maxwell and verified by Hertz. We can see that the product of the units of the permittivity with those of the permeabilty yield units of $1/(\text{velocity})^2$, and doing the multiplication for the values of free space give the velocity of light, implying that light consists of electromagnetic propagation.

As a consequence, we can choose units in which $c = 1$ and lengths and times are measured in the same units. Special relativity with its Minkowski metric $dt^2 - dx^2 - dy^2 - dz^2$ is then an immediate consequence.

3.4.7 The permittivity and the permeability in natural units.

If we choose natural units so that $\hbar = 1$ and $c = 1$ then length has the same units as time and so energy has units of inverse length and the expression in the denominator for the units of the permeability is just a scalar. So the permittiviity has units of $(\text{charge})^2$.

Similarly, the units of the permeability become $(\text{charge})^{-2}$.

3.4.8 The fine structure constant.

The expression

$$\alpha := \frac{(\text{charge of the electron})^2}{4\pi\epsilon_0}$$

is a pure number in terms of our natural units where $\hbar = 1$ and $c = 1$ and is equal to

$$\frac{1}{137.0359...}.$$

In terms of conventional units we would write

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}.$$

4 Gauge theories.

Hermann Weyl had suggested that the true objects of general relativity should not be (semi-)Riemann metrics, but rather the associated Levi-Civita connection. And if we generalize this connection to be a conformal connection (i.e. if we enlarge the group from $O(1,3)$ to $\mathbb{R}^+ \times O(1,3)$) then we can incorporate electromagnetism.(See his classic *Raum Zeit Materie*, Springer, Berlin (1918)). The word "gauge" derives from Weyl's theory in which the length is changed by a conformal transformation.

Einstein rejected Weyl's proposal of considering a conformal connection as the underlying physical field, although Einstein himself considered the possibility that Riemannian geometry be replaced by conformal geometry as a basis for unified theories - see his article in *Preuss Akad.* 261 (1921) as well as the following notes on the "unified field theory": *loc. cit.* (1925) p. 414, (1928) p. 3, (1929) p. 3.

After the advent of quantum mechanics, Fritz London, in a short note in early 1927(F. London, "Die Theorie von Weyl und die Quantenmechanik", *Naturwiss.* **15** 187. and soon after in a longer paper, "Quantenmechanische Deutung der Theorie von Weyl," *Zeit. für Physik* **42**, 375-389 (1927), proposed a quantum mechanical interpretation of Weyl's attempt to unify electromagnetism and gravitation. The essential idea is to replace Weyl's $\mathbb{R}^+$ by $U(1)$ acting as phase transformations of the quantum mechanical state vector. The group $U(1)$ does not act on the tangent space of space time. It is "internal". The London theory for $U(1)$ was generalized to $SU(2)$ by Yang and Mills in 1954, C.N. Yang and R. Mills, "Conservation of isotopic spin and isospin gauge invariance," *Phys. Rev.* **96** 191-195 (1954).

The "field" in a Yang-Mills theory on space time is a connection on a principal bundle P.

Giving a connection on a principal bundle is the same as giving (consistently) the notion of covariant derivative on any associated bundle. The covariant derivative language is more popular in the standard physics texts. I will give a self contained review of the notions of connection and curvature in the more general setting of superconnections and supercurvature later on.

If $\mathcal{G}$ is the structure group of the bundle P and $\mathfrak{g}_0$ is the Lie algebra of $\mathcal{G}$, the curvature of such a connection is a 2-form on space-time with values in the vector bundle $\mathfrak{g}_0(P)$ associated to the adjoint representation of $\mathcal{G}$. If F is such a curvature form, and if $\star$ denotes the Hodge star operator of space time, then $\star F$ is another 2-form with values in $\mathfrak{g}_0(P)$, so

$$F \wedge \star F$$

is a 4-form with values in $\mathfrak{g}_0(P) \otimes \mathfrak{g}_0(P)$. In order to get a numerical valued 4-form which we can consider as a Lagrangian density, we need an Ad invariant scalar product on $\mathfrak{g}_0$.

For example, we have seen that the electromagnetic field F is a two form whose integral over any surface has units of inverse charge. So $F \wedge \star F$ is a 4-form with units of $1/(\text{charge})^2$. In order to get the correct Lagrangian density, we must multiply by ϵ_0, the permittivity of empty space which (in natural units) has units of $(\text{charge})^2$, so that

$$\frac{1}{2}\epsilon_0 F \wedge \star F$$

is the Lagrangian density for the electromagnetic field in empty space. If we want to consider F (strictly speaking iF) as the curvature of a connection on a $U(1)$ bundle, we see that we must consider ϵ_0 as determining a metric on $u(1)$ (different from the "natural" one regarding $u(1)$ as $i\mathbb{R}$), and this metric has deep physical significance.

In the Standard Model of the electroweak theory, the group under consideration is $U(2)$ or $SU(2) \times U(1)$ with Lie algebra $\mathfrak{g}_0 = u(2)$. As we have seen, there is a two parameter family of invariant metrics on $u(2)$ given by (1).

We repeat that we are regarding g_1 and g_2 as parameters describing possible Ad invariant scalar products on the Lie algebra $u(2)$. As such they have physical significance similar to that of the permittivity of free space in electromagnetic theory and are necessary to be able to formulate a Yang-Mills functional. In a general relativistic theory one would expect them to have a space time dependence just as the metric of space time does. The interpretation of g_1 and g_2 as "universal coupling constants" then derives from the interpretation as defining a metric.

5 The Higgs mechanism.

5.1 The Higgs mechanism in a nutshell.

The Higgs mechanism in the Standard Model of electroweak interactions is a device for breaking the $u(2) = su(2) \oplus u(1)$ symmetry of a $U(2)$ gauge theory in such a way that the three of the four components of a connection form (originally massless in a pure Yang-Mills theory) become differential forms with values in a vector bundle associated to $U(1)$ and which enter into a Lagrangian whose quadratic terms correspond to particles with positive mass. In mathematical terms this corresponds to a reduction of a principal $U(2)$ bundle to a $U(1)$ bundle.

The ingredients that go into this mechanism and into the computation of the acquired masses are the following:

- An Ad invariant positive definite metric on $u(2)$. This is needed for the original (unbroken) Yang-Mills theory. We have argued that the "universal coupling constants" that enter into the general formulation of this theory are in fact parameters which describe the possible Ad invariant metrics on $u(2)$. In general there is a two parameter family of such metrics. They are related by a certain angle θ_W known as the Weinberg angle. Our internal supersymmetry proposal will determine this angle as 30^0, or $\sin^2\theta_W = 0.25$, which is not too far from the measured value of 0.2312 ± 0.003.

- A two dimensional Hermitian vector bundle associated to the principal $U(2)$ bundle. In the general presentation of the Standard Model this vector bundle is an extraneous ingredient put in "by hand". In our theory this vector bundle is $\mathfrak{g}_1$, the odd component of a Lie super algebra bundle. The sections of this bundle are regarded as the exterior degree zero components of a superconnection. More details on this later.

- A degree-four polynomial on this vector bundle. In the general presentation this must also be provided by hand. In our theory, the quartic term of this polynomial is the super-Yang-Mills functional.

- The vector bundle $\mathfrak{g}_1$ is associated to the original $U(2)$ bundle, so $U(2)$ invariance determines the Hermitian metric up to a scalar factor. We proposed to fix this scalar by relating it to the choice of scale entering into the metric on $su(2)$. This is done by using the concept of a Hermitian Lie algebra, see S. Sternberg, J. Wolf, "Hermitian Lie algebras and metaplectic representations", *Trans. Amer. Math. Soc.* **231** 1 (1978) which relates certain Lie superalgebras to ordinary Lie algebras. Once the metric has been fixed, we can write the most general (invariant) degree four polynomial as
$$a\|\cdot\|^4 - b\|\cdot\|^2.$$

The next three steps are part of the standard Higgs mechanism, cf. for example A. Derdzinski, *Geometry of the standard model* Section 11. We summarize them here for the reader's convenience. Additional details will be given below.

- If a and b are both positive, then the quadratic polynomial
$$az^2 - bz$$
achieves its minimum at
$$z_0 = \frac{b}{2a}$$
and hence any section ψ of our vector bundle lying on the three-sphere bundle
$$\|\psi\|^2 = z_0$$
is a global minimum. Any such section is called a *vacuum state*. The reduction of the principal $U(2)$ bundle is achieved by fixing one such vacuum. For example, if the bundle is trivial and is given a trivialization which identifies it with the trivial $\mathbb{C}^2$ bundle then we may choose ψ of the form
$$\psi = \psi_0 := \begin{pmatrix} 0 \\ v \end{pmatrix}, v > 0$$
so
$$\|\psi_0\| = \sqrt{\frac{b}{2a}}.$$

- The mass of the W particle is then given as

$$m(W) = \frac{\|\psi_0\|}{\|i\tau_1\|_{u(2)}} \tag{9}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

See the discussion in Section 5.2 below. In terms of the parameter g_2 entering into the definition of the metric on $su(2)$ (see (1)) this becomes

$$m(W) = \frac{1}{2} g_2 \|\psi_0\| = \frac{1}{2} g_2 \sqrt{\frac{b}{2a}}. \tag{10}$$

- The mass of the Higgs field (see Section 5.3 below) is given by

$$m(\text{Higgs}) = 2\sqrt{b}. \tag{11}$$

This gives the value of the Higgs mass in terms of parameters entering into the Higgs model. Notice that only the coefficient of the quadratic term (b) enters into this formula, but if we know the coefficient a of the quartic term, then we can get b from $\|\psi_0\| = \sqrt{b/2a}$.

As indicated above, we will derive the value of a from the supercurvature and the metric on the superalgebra coming from a corresponding Lie algebra, see equation (15) below. Thus we are able to predict the Higgs mass from the observed experimental value of the W mass using (10) and (11) and the value of a. We will find that m(Higgs)=2m(W).

To reiterate - we make no predictions about b. We do make a prediction of a coming from the interpretation of the quartic term in the Higgs field as arising from a super-Yang-Mills Lagrangian (to be explained below). No matter what b is, the knowledge of a determines the ratio of the mass of the Higgs to the mass of the W.

5.1.1 The Weinberg angle, again.

We return to equation (1) which gives the most general ad-invariant scalar product on $u(2)$. The Weinberg angle is then defined by

$$\frac{g_1^2}{g_2^2} = \tan^2 \theta_W.$$

Thus, for example, any choice of g_1 and g_2 which leads to a value of

$$\frac{g_1^2}{g_2^2} = 1/3$$

will yield a Weinberg angle of 30 degrees.

5.1.2 Scalar products from representations.

Any faithful unitary representation r of $u(2)$ will yield a positive definite scalar product by letting the scalar product of A and B be

$$-\operatorname{tr} r(A)r(B).$$

Under our identification of $u(2)$ with self adjoint rather than skew adjoint matrices, which involves multiplication by i, we can forget about the minus sign. But we do want to allow for an overall scale factor and so consider the metric

$$A \mapsto \frac{2}{g^2} \operatorname{tr}\left(r(A)^2\right) \tag{12}$$

as being associated to the representation r. Of course the Weinberg angle will be independent of the factor g.

So any theory which singles out a preferred faithful representation of $u(2)$ will give a prediction of the Weinberg angle. Our proposal is to regard $u(2)$ as the even part of the superalgebra $su(2/1) \subset sl(2/1)$. See Section 8.1 for the definition of the Lie superalgebras $sl(m/n)$. Each of these Lie superalgebras has a fundamental (defining) representation as described in Section 8.1. In particular, this picks out a preferred faithful representation of $u(2)$ and hence gives a prediction of the Weinberg angle. We do the computation in the next section.

5.1.3 The Weinberg angle of the fundamental representation of $sl(2/1)$.

In this representation the two by two matrix A is represented by the three by three matrix

$$r(A) = \begin{pmatrix} A & 0 \\ 0 & \operatorname{tr} A \end{pmatrix}.$$

If we take $A \in su(2)$ so $\operatorname{tr} A = 0$ in (12) we get $\operatorname{tr}(r(A)^2) = \operatorname{tr}(A^2)$ from which we see that the g_2 entering into formula (1) for the metric on $u(2)$ is given by $g_2^2 = g^2$. If we take $A = I$ in (12) we get

$$r(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

so $\operatorname{tr}(r(I)^2) = 6$. So

$$\frac{2}{g^2} \cdot 6 = \frac{4}{g_1^2} \quad \text{so} \quad \frac{g_1^2}{g_2^2} = \frac{1}{3}$$

yielding a Weinberg angle of 30 degrees.

5.2 Other quadratic forms.

Given a positive definite real scalar product $(\cdot, \cdot)$ on a real vector space, any other quadratic form is given by $x \mapsto (Sx, x)$ where S is a self-adjoint operator. We can then diagonalize S. If the second quadratic form is positive semi-definite, then these eigenvalues are non-negative, and S has a unique square root $S^{\frac{1}{2}}$ with non-negative eigenvalues. For reasons of differential geometry - essentially the reduction of a $U(2)$ bundle to a $U(1)$ bundle via the choice of section of an associated bundle - these eigenvalues are identified with the masses of certain spin 1 particles in the example I will now work out. I will do the elementary linear algebra now, so we can see what is needed for mass predictions, and discuss the geometry later.

For example, consider the standard action of $u(2)$ on $\mathbb{C}^2$ and define the "second" quadratic form on $u(2)$ to be

$$q(A) := \|A\psi_0\|^2_{\mathbb{C}^2} = (A\psi_0, A\psi_0)_{\mathbb{C}^2}$$

where ψ_0 is a fixed element of $\mathbb{C}^2$, and where $(\cdot,\cdot)_{\mathbb{C}^2}$ is some $U(2)$ invariant scalar product on $\mathbb{C}^2$ (and so is some positive multiple of the standard scalar product). The corresponding bilinear form on $u(2)$ is

$$\langle A, B\rangle = \mathrm{Re}(A\psi_0, B\psi_0)_{\mathbb{C}^2}.$$

In fact, let us take

$$\psi_0 := \begin{pmatrix} 0 \\ v \end{pmatrix}, v > 0$$

as above. Then

$$\tau_1\psi_0 = \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad \tau_2\psi_0 = \begin{pmatrix} -iv \\ 0 \end{pmatrix}, \quad \tau_3\psi_0 = \begin{pmatrix} 0 \\ -v \end{pmatrix}, \text{ and } \; I\psi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Then relative to any scalar product $(\cdot,\cdot)$ on $u(2)$ we have

$$(S\tau_1, X) = \langle \tau_1, X\rangle = 0 \quad \text{for} \quad X = \tau_2, \tau_3, I.$$

If $(\cdot,\cdot)$ is any of the invariant metrics (1), then $(\tau_1, X) = 0$ for $X = \tau_2, \tau_3, I$. This shows that τ_1 is an eigenvector of S with eigenvalue $\|\psi_0\|^2/\|\tau_1\|^2_{u(2)}$. Similarly for τ_2. Sections of the line bundles corresponding to these eigenvectors are identified with the W particles. This accounts for the mass of the W as given in equation (9) above.

We have $(\tau_3 + I)\psi_0 = 0$ so $\tau_3 + I$ is an eigenvector of S with eigenvalue 0. Expressed in terms of the orthonormal basis (2) and normalized so to have length one gives

$$\frac{1}{(g_1^2 + g_2^2)^{\frac{1}{2}}}\left(g_2\frac{g_1}{2}I + g_1\frac{g_2}{2}\tau_3\right).$$

The corresponding mass zero field is then identified with the electromagnetic field.

Taking the orthogonal complement of the three eigenvectors found so far (corresponding to the W's and the electromagnetic field) gives the field of the Z particle.

All of the material in this section is part of the standard repertoire of the Higgs mechanism and is not particular to the model we propose. For instance, equation (9) is the formula in equation (11.30) A. Derdzinski, *Geometry of the standard model* for the mass of the W up to differences in notation and the fact that we are computing in natural units.

But it might be instructive to see how all this is written out in the physics literature, where "fields" are always scalar valued. In terms of the basis $\tau_1, \tau_2, \tau_3, I$ we have verified that our quadratic form is given by

$$q(X_1\tau_1 + X_2\tau_2 + X_3\tau_3 + YI) = v^2(X_1^2 + X_2^2 + (Y - X_3)^2).$$

Let us express this in terms of the coordinates in the orthonormal basis written above (and taking the standard Hermitian form on $\mathbb{C}^2$). We have

$$X_i\tau_1 = \frac{2X_i}{g_2} \cdot \frac{g_2\tau_i}{2}$$

so the coefficient W_i of $X_i\tau_i$ in terms of the normalized basis element is

$$W_i = \frac{2X_i}{g_2}$$

and hence

$$X_i = \frac{g_2}{2}W_i, i = 1, 2, 3$$

and similarly $Y = \frac{g_1}{2}B$ where B is coefficient relative to the last normalized element. So

$$Q(W_1, W_2, W_3, B) = \frac{1}{4}v^2\left(g_2^2(W_1^2 + W_2^2) + (g_2W_3 - g_1B)^2\right).$$

The rotation

$$R_{\theta_W} = \frac{1}{\sqrt{g_1^2 + g_2^2}}\begin{pmatrix} g_2 & -g_1 \\ g_1 & g_2 \end{pmatrix}$$

in the W_3, B plane brings the quadratic form to diagonal form. This is the reason for angle terminology. The W_1, W_2 and Z are considered as transmitters of the weak interaction, while the massless field is identified with the photon.

5.2.1 Experimental determination of the coupling constant g_2.

The coupling constant g_2 enters into the definition of the metric on $u(2)$ as we have seen, and is observed via the "strength" of the electro-weak interaction. We have

$$g_2 = \frac{e}{\sin\theta_W}.$$

So if $\sin\theta_W = \frac{1}{2}$ we have $g_2 = 2e$. If

$$\frac{e^2}{4\pi} \doteq \frac{1}{137}$$

then $g_2 \doteq 0.6$.

5.3 The Higgs mass.

It is assumed that the Higgs field is a section of a Hermitian vector bundle with potential $\mathcal{V}$ which has the form

$$\mathcal{V}(\psi) = f(\langle\psi,\psi\rangle)$$

where

$$f : [0,\infty) \to \mathbb{R}$$

is a smooth function with a minimum at z_0. A particular section is ψ_0 chosen with $\langle\psi_0,\psi_0\rangle = z_0$. (If, as we shall assume, the Hermitian vector bundle is a two dimensional bundle associated to a principal $U(2)$ or $SU(2) \times U(1)$ bundle this has the effect of reducing the principal bundle to a $U(1)$ bundle.)

The most general section of our vector bundle is then written as $\psi_0 + \eta$ and we consider the quadratic term in the expansion of $f(\psi_0 + \eta)$ as a function of η. It will be given by

$$\frac{1}{2}\text{Hess}(f)(\psi_0)(\eta) = 2f''(\langle\psi_0,\psi_0\rangle)(\text{Re}\langle\psi,\eta\rangle)^2.$$

For η tangent to the orbit of the action of $U(2)$ this vanishes. But for $\eta \in \mathbb{R}\psi_0$ we have $\langle\psi_0,\eta\rangle = \pm\|\psi_0\|\|\eta\|$ so for such η (known as the Higgs field) the quadratic term is

$$2z_0 f''(z_0)\|\eta\|^2.$$

We want to consider this as a mass term, which means that we want to write this quadratic expression as $\frac{1}{2}m^2\|\eta\|^2$.

If

$$f(z) = az^2 - bz$$

with a and b positive constants, then the minimum of f is achieved at

$$z_0 = \frac{b}{2a}$$

and

$$f''(z_0) = 2a.$$

So

$$2z_0 f''(z_0) = 2b.$$

So we wish to write $2b\|\eta\|^2$ as $\frac{1}{2}m^2\|\eta\|^2$ where m is the mass of the Higgs. This gives

$$m(\text{Higgs}) = 2\sqrt{b}$$

as in equation (11) above.

Once again, all of the material in this section is part of the standard repertoire of the Higgs mechanism and is not particular to the model we propose. Equation (11) is the formula in equation (11.30) of Derdzinski, *Geometry of the standard model* for the Higgs mass up to the fact that we are computing in natural units.

We will now revert to standard notation and write the Higgs field as ψ.

6 Using superconnections.

We assume that the Higgs field ψ is the degree zero piece of a superconnection for $su(2/1)$, and use this - together with an idea coming from the theory of Hermitian Lie algebras - to predict a value of a, namely

$$a = \frac{1}{8}g_2^2.$$

I will present a detailed exposition of the theory of superconnections later. But I want to get to the punch line in a hurry. So I will now show how the super-Yang-Mills Lagrangian for $su(2/1)$ makes a prediction of the factor a occurring in the f in the preceding section. In general, the Lagrangian of a super-Yang-Mills-Higgs theory will be of the form

$$(1/2)||F||^2 + ...$$

where F is the supercurvature and where .... involves the fermions, plus a quadratic term in the Higgs whose origin we leave open. The supercurvature is quadratic in the degree zero part of the superconnection, and hence the above Lagrangian, being quadratic in F, will be quartic in the degree zero part of the superconnection. So if we identify the Higgs field with this degree zero part, we get a quartic polynomial in the Higgs which derives from the underlying theory with no additional ad hoc assumptions. Here are the details of the computation:

If the Higgs field ψ is the degree zero piece of a superconnection for $su(2/1)$, then the supercurvature F will include a term $\frac{1}{2}[\psi, \psi]$ which is a section of $u(2)$ regarded as the even part of $su(2/1)$. If

$$\psi = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ \overline{x} & \overline{y} & 0 \end{pmatrix}$$

then

$$\frac{1}{2}[\psi, \psi] = \begin{pmatrix} |x|^2 & x\overline{y} & 0 \\ \overline{x}y & |y|^2 & 0 \\ 0 & 0 & |x|^2 + |y|^2 \end{pmatrix}.$$

To compute $\|F\|^2$, we need a metric on $u(2)$. In the computation of the Weinberg angle, we took the metric to be proportional to the metric induced by the fundamental representation of $sl(2/1)$. So we must use the metric

$$A \mapsto \frac{2}{g_2^2} \operatorname{tr} \left(\begin{pmatrix} A & 0 \\ 0 & \operatorname{tr} A \end{pmatrix}^2 \right)$$

so as to get the metric (1) on the $u(2)$ component. Applied to the $\frac{1}{2}[\psi, \psi]$ given above we get

$$\frac{4}{g_2^2}(|x|^2 + |y|^2)^2.$$

Taking $\frac{1}{2}$ of the above expression (as one half of the square length appears in the Lagrangian) gives the quartic term as

$$\frac{2}{g_2^2}(|x|^2 + |y|^2)^2. \tag{13}$$

6.1 The metric on the Higgs.

We need to express (13) as $a\|\psi\|^4$. To do this we must say what $\|\psi\|^2$ is. We now use the paper S. Sternberg, J. Wolf, "Hermitian Lie algebras and metaplectic representations", *Trans. Amer. Math. Soc.* **231** 1 (1978) and propose that we think of $su(2/1)$ as the real part of the object whose imaginary part is $su(3)$.

If I have time, I will explain this later.

On $su(3)$ the only invariant metrics are scalar multiples of the Killing form, and since we want the metric to reduce to the above metric on $su(2)$ we must choose $\|\psi\|^2$ as

$$\psi \mapsto \frac{2}{g_2^2}\operatorname{tr}\psi^2 = \frac{4}{g_2^2}(|x|^2 + |y|^2). \tag{14}$$

Comparing the two expressions (13) and (14) gives

$$a = \frac{1}{8}g_2^2. \tag{15}$$

Substituting this into (10) gives

$$m(W) = \sqrt{b}. \tag{16}$$

Comparing with (11) gives

$$\frac{m(\text{Higgs})}{m(W)} = 2. \tag{17}$$

This was the prediction in [N86]. For later versions of this prediction see [R98] and references cited there.

7 Superconnections.

In this section we give a self contained introduction to the theory of superconnections for the convenience of the reader. In the main, we follow the exposition given in [BGV91] with some changes in notation. For an alternative treatment see [MaSa2000].

7.1 Superspaces and superalgebras.

A **superspace** E is just a vector space with a $\mathbb{Z}_2$ grading:

$$E = E^+ \oplus E^-.$$

A **superalgebra** A is an algebra whose underlying vector space is a superspace and such that

$$A^+ \cdot A^+ \subset A^+, \quad A^- \cdot A^- \subset A^+, \;\; A^+ \cdot A^- \subset A^-, \;\; A^- \cdot A^+ \subset A^-.$$

The commutator of two homogeneous elements of A is defined as

$$[a, b] := ab - (-1)^{|a|\cdot|b|} ba.$$

We use the notation $|a| = 0$ if $a \in A^+$ and $|a| = 1$ if $a \in A^-$ and we do addition and multiplication mod 2.

A superalgebra is **commutative** if the commutator of any two elements vanishes. For example, the exterior algebra $\wedge(V)$ of a vector space is a commutative superalgebra where

$$\wedge(V)^+ := \wedge^0(V) \oplus \wedge^2(V) \oplus \wedge^4(V) \oplus \cdots,$$

and

$$\wedge(V)^- := \wedge^1(E) \oplus \wedge^3(V) \oplus \cdots.$$

7.2 The tensor product of two superalgebras.

If A and B are superspaces we make $A \otimes B$ into a superspace by

$$|a \otimes b| = |a| + |b|.$$

If A and B are superalgebras we make $A \otimes B$ into a superalgebra by

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b|\cdot|a'|} aa' \otimes bb'.$$

For example, the Clifford algebra of any vector space with a scalar product is a superalgebra, where $C(V)^+$ consists of those elements which can be written as a sum of products of an even number of elements of V and $C(V)^-$ consists of those elements which can be written as a sum of products of an odd number of elements of V. If V and W are two spaces with scalar products then the Clifford algebra of their orthogonal direct sum is the tensor product of their Clifford algebras:

$$C(V \oplus W) = C(V) \otimes C(W).$$

We will use the convention of the algebraists rather than that of the geometers in the definition of the Clifford algebra, W. Greub, *Multilinear algebra* Springer, Berlin (1978) . So if V is a vector space with a (not necessarily positive definite) scalar product then $C(V)$ is the universal algebra relative to the relations

$$uv + vu = 2(u, v)\mathbf{1}.$$

Chevalley, in is classic book does not have the factor 2 on the right hand side, because he considers fields of arbitrary characteristic, including characteristic rwo

(In N. Berline, E. Getzler, M. Vergne: *Heat Kernels and Dirac Operators*, Springer, Berlin 1991 the opposite convention (with a minus sign on the right hand side) is used.)

7.3 Lie superalgebras.

If A is an associative superalgebra the commutator of two homogeneous elements of A was defined as

$$[a,b] := ab - (-1)^{|a|\cdot|b|}ba.$$

This commutator satisfies the axioms for a **Lie superalgebra** which are

- $[a,b] + (-1)^{|a|\cdot|b|}[b,a] = 0$, and
- $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$.

It was proved in L. Corwin, Y. Ne'eman, S. Sternberg, "Graded Lie algebras in mathematics and physics", *Rev. Mod. Phy.* **47** 573 (1975) that every Lie superaglebra has a universal (associative) enveloping algebra and that the analogue of the Poincaré-Birkhoff-Witt theorem holds.

If A is a commutative superalgebra and L is a Lie superalgebra then $A \otimes L$ is again a Lie superalgebra under the usual definition:

$$[a \otimes X, b \otimes Y] := (-1)^{|X|\cdot|b|}ab \otimes [X,Y].$$

7.4 The endomorphism algebra of a superspace.

Let $E = E^+ \oplus E^-$ be a superspace. We make the algebra of all endomorphisms (= linear transformations) of E into a superalgebra by letting $\mathrm{End}(E)^+$ consist of those linear transformations which carry E^+ into E^+ and E^- into E^- while $\mathrm{End}(E)^-$ interchanges the two components. Thus a typical element of $\mathrm{End}(E)^+$ looks like

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A \in \mathrm{End}(E^+), \quad D \in \mathrm{End}(E^-)$$

while a typical element of $\mathrm{End}(E)^-$ looks like

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad B : E^- \to E^+, \quad C : E^+ \to E^-.$$

An action (or a representation) of an associative algebra A on a superspace E is a (gradation preserving) homomorphism of A into $\mathrm{End}(E)$. We then also say that E is an A module.

Similarly, a representation of a Lie superalgebra L on a superspace E is a homomorphism of L into the commutator Lie superalgebra of $\mathrm{End}(E)$. This is the same as an action of the universal enveloping algebra $U(L)$ on E. We say that E is an L module.

7.5 Superbundles.

Let $\mathcal{E} \to M$ be a bundle of superspaces over a manifold M. We call such an object a **superbundle**. So $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ where $\mathcal{E}^+ \to M$ and $\mathcal{E}^- \to M$ are vector bundles over M. We will call a section of $\mathcal{E}^+$ an even section of $\mathcal{E}$ and a section of $\mathcal{E}^-$ an odd section of $\mathcal{E}$.

If $\mathcal{E}$ and $\mathcal{F}$ are superbundles, then $\mathcal{E} \otimes \mathcal{F}$ is a superbundle. In particular, $\wedge(T^*M)$ is a superbundle where

$$\wedge(T^*M)^+ := \wedge^0(T^*M) \oplus \wedge^2(T^*M) \oplus \wedge^4(T^*M) \oplus \cdots ,$$

$$\wedge(T^*M)^- := \wedge^1(T^*M) \oplus \wedge^3(T^*M) \oplus \wedge^5(T^*M) \oplus \cdots .$$

A section of $\wedge(T^*M) \otimes \mathcal{E}$ is called an **$\mathcal{E}$-valued differential form** and the space of all $\mathcal{E}$-valued differential forms will be denoted by $\mathcal{A}(M, \mathcal{E})$. Locally any element of $\mathcal{A}(M, \mathcal{E})$ is a sum of terms of the form $\alpha \otimes s$ where α is a differential form on M and s is a section $\mathcal{E}$.

7.6 The endomorphism bundle of a superbundle.

If $\mathcal{E} \to M$ is a superbundle, then we can consider the superbundle $\mathrm{End}(\mathcal{E})$ where, at each $m \in M$ we have $\mathrm{End}(\mathcal{E})_m := \mathrm{End}(\mathcal{E}_m)$. We have an action of any section of $\mathrm{End}(\mathcal{E})$ on any section of $\mathcal{E}$. By tensor product, any element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$ acts on any element of $\mathcal{A}(M, \mathcal{E})$. In particular any element of $\mathcal{A}(M)$, i.e. any differential form acts on $\mathcal{A}(M, \mathcal{E})$ and (super)commutes with all elements of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

7.7 The centralizer of multiplication by differential forms.

Any element of $\mathcal{A}(M)$, i.e. any differential form, acts on $\mathcal{A}(M, \mathcal{E})$ and (super)commutes with all elements of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

There is an important converse to this last assertion. A differential operator on $\mathcal{A}(M, \mathcal{E})$ is by definition an operator which in local coordinates looks like

$$\sum_\gamma a_\gamma \partial^\gamma$$

where a_γ is a section of $\mathrm{End}\,\mathcal{A}(M, \mathcal{E})$ and $\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$ is a partial differentiation operator in terms of the local coordinates. Leibnitz's rule implies that if such an operator commutes with all multiplications by functions then it can't really involve any differentiations. If furthermore it commutes with the action of all elements of $\mathcal{A}(M)$ it must be given by the action of some element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$. In short: a differential operator on $\mathcal{A}(M, \mathcal{E})$ commutes with the action of $\mathcal{A}(M)$ if and only if it is given by an element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$.

7.8 Bundles of Lie superalgebras.

If $\mathfrak{g}$ is a bundle of Lie superalgebras over M then $\mathcal{A}(M, \mathfrak{g})$ is a Lie superalgebra with bracket determined fiberwise (as we have seen) by

$$[\alpha \otimes X, \beta \otimes Y] = (-1)^{|X| \cdot |\beta|} (\alpha \wedge \beta) \otimes [X, Y].$$

If $\mathcal{E}$ is a superbundle on which $\mathfrak{g}$ acts, meaning that we have a fiberwise Lie superalgebra homomorphism ρ of $\mathfrak{g}$ into the Lie superalgebra bundle $\mathrm{End}(\mathcal{E})$

(under fiberwise bracket), then we have an action of $\mathcal{A}(M,\mathfrak{g})$ on $\mathcal{A}(M,\mathcal{E})$ determined by

$$\rho(\alpha\otimes X)(\beta\otimes v)=(-1)^{|X|\cdot|\beta|}(\alpha\wedge\beta)\otimes(\rho(X)v).$$

7.9 Superconnections.

A **superconnection** on a superbundle $\mathcal{E}$ is an odd first order differential operator

$$\mathbb{A}:\mathcal{A}^{\pm}(M,\mathcal{E})\to\mathcal{A}^{\mp}(M,\mathcal{E})$$

which satisfies

$$\mathbb{A}(\alpha\wedge\theta)=d\alpha\wedge\theta+(-1)^{|\alpha|}\alpha\wedge\mathbb{A}\theta,\qquad\forall\ \alpha\in\mathcal{A}(M),\ \theta\in\mathcal{A}(M,\mathcal{E}).$$

We can write this as

$$[\mathbb{A},e(\alpha)]=e(d\alpha)\tag{18}$$

where $e(\beta)$ denotes the operation of exterior multiplication by $\beta\in\mathcal{A}(M)$.

Let $\Gamma(\mathcal{E})$ denote the space of smooth sections of $\mathcal{E}$ which we can regard as a subspace of $\mathcal{A}(M,\mathcal{E})$. Then

$$\mathbb{A}:\Gamma(\mathcal{E}^{\pm})\to\mathcal{A}^{\mp}(M,\mathcal{E})$$

and $\mathbb{A}$ is completely determined by this map since

$$\mathbb{A}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{|\alpha|}\alpha\otimes\mathbb{A}s$$

for all differential forms α and sections s of $\mathcal{E}$.

Conversely, suppose that $\mathbb{A}:\Gamma(\mathcal{E}^{\pm})\to\mathcal{A}^{\mp}(M,\mathcal{E})$ is a first order differential operator which satisfies

$$\mathbb{A}(fs)=df\otimes s+f\otimes\mathbb{A}s$$

for all functions f and sections s of $\mathcal{E}$. Then we can extend $\mathbb{A}$ to $\mathcal{A}(M,\mathcal{E})$ by setting

$$\mathbb{A}(\alpha\otimes s)=d\alpha\otimes s+(-1)^{|\alpha|}\otimes s$$

without fear of running into a contradiction.

7.10 Extending superconnections to the bundle of endomorphisms.

If $\gamma\in\mathcal{A}(M,\mathrm{End}(\mathcal{E}))$ define

$$\mathbb{A}\gamma:=[\mathbb{A},\gamma].$$

We claim that $[\mathbb{A},\gamma]$ belongs to $\mathcal{A}(M,\mathrm{End}(\mathcal{E}))$. To prove this, we must check that $[\mathbb{A},\gamma]$ commutes with all $e(\alpha),\ \alpha\in\mathcal{A}(M)$. For any $\alpha\in\mathcal{A}(M)$ we have

$$\mathbb{A}\circ\gamma\circ e(\alpha)=(-1)^{|\gamma|\cdot|\alpha|}\mathbb{A}\circ e(\alpha)\circ\gamma$$

$$= (-1)^{|\gamma|\cdot|\alpha|}e(d\alpha)\circ\gamma + (-1)^{|\alpha|+|\gamma|\cdot|\alpha|}e(\alpha)\circ\mathbb{A}\circ\gamma$$

while

$$\begin{aligned}\gamma\circ\mathbb{A}\circ e(\alpha) &= \gamma\circ e(d\alpha) + (-1)^{|\alpha|}\gamma\circ e(\alpha)\circ\mathbb{A}\\ &= (-1)^{|\gamma|+|\gamma|\cdot|\alpha|}e(d\alpha)\circ\gamma + (-1)^{|\alpha|+|\alpha|\cdot|\gamma|}e(\alpha)\circ\gamma\circ\mathbb{A}\end{aligned}$$

so

$$\begin{aligned}[\mathbb{A},\gamma]\circ e(\alpha) &= A\circ\gamma\circ e(\alpha) - (-1)^{|\gamma|}\gamma\circ\mathbb{A}\circ e(\alpha)\\ &= (-1)^{|\alpha|+|\alpha|\cdot|\gamma|}e(\alpha)\circ[\mathbb{A},\gamma]\end{aligned}$$

Since $|[\mathbb{A},\gamma]| = |\gamma|+1$ this shows that $[[\mathbb{A},\gamma],e(\alpha)] = 0$ as desired.

7.11 Supercurvature.

Consider the even operator $\mathbb{A}^2$. We have, D. Quillen, "Superconnections and the Chern character", *Topology* **24** 89 (1985),

$$[\mathbb{A}^2, e(\alpha)] = \mathbb{A}\circ[\mathbb{A},e(\alpha)] + (-1)^{|\alpha|}[\mathbb{A},e(\alpha)]\circ\mathbb{A} =$$

$$\mathbb{A}\circ e(d\alpha) - (-1)^{|d\alpha|}e(d\alpha)\circ\mathbb{A} = [\mathbb{A},e(d\alpha)] = e(dd(\alpha)) = 0.$$

So $\mathbb{A}^2\in\mathcal{A}(M,\mathrm{End}(\mathcal{E}))$. We set

$$\mathbb{F} := \mathbb{A}^2$$

and call it the **curvature** of the superconnection $\mathbb{A}$.

The **Bianchi identity** says that

$$\mathbb{A}\mathbb{F} = 0.$$

Indeed $\mathbb{A}\mathbb{F}$ is defined as $[\mathbb{A},\mathbb{F}]$ and since $\mathbb{F} := \mathbb{A}^2$ is even we have

$$[\mathbb{A},\mathbb{A}^2] = \mathbb{A}\circ\mathbb{A}^2 - \mathbb{A}^2\circ\mathbb{A} = 0$$

by the associative law.

7.12 The tensor product of two superconnections.

If $\mathcal{E}$ and $\mathcal{F}$ are superbundles recall that $\mathcal{E}\otimes\mathcal{F}$ is the superbundle with grading

$$\begin{aligned}(\mathcal{E}\otimes\mathcal{F})^+ &= \mathcal{E}^+\otimes\mathcal{F}^+\oplus\mathcal{E}^-\otimes\mathcal{F}^-,\\ (\mathcal{E}\otimes\mathcal{F})^- &= \mathcal{E}^+\otimes\mathcal{F}^-\oplus\mathcal{E}^-\otimes\mathcal{F}^+.\end{aligned}$$

If $\mathbb{A}$ is a superconnection on $\mathcal{E}$ and $\mathbb{B}$ is a superconnection on $\mathcal{F}$ then $\mathbb{A}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbb{B}$ is a superconnection on $\mathcal{E}\otimes\mathcal{F}$. Thus

$$(\mathbb{A}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbb{B})(a\otimes b) := \mathbb{A}a\otimes b + (-1)^{|a|}a\otimes\mathbb{B}b.$$

A bit of computation shows that this definition is consistent and defines a superconnection on $\mathcal{E}\otimes\mathcal{F}$.

7.13 The exterior components of a superconnection.

If $\mathbb{A}$ is a superconnection on on a superbundle $\mathcal{E}$ we may break $\mathbb{A}$ into its homogeneous components $\mathbb{A}_{[i]}$ which map $\Gamma(M,\mathcal{E})$ into $\mathcal{A}^i(M,\mathcal{E})$, the space of i-forms with values in $\mathcal{E}$:

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + A_{[2]} + \cdots .$$

Let s be a section of $\mathcal{E}$ and f a function. By the above decomposition and the defining property of a superconnection we have

$$\mathbb{A}(fs) = \sum_{i=0}^{n} \mathbb{A}_{[i]}(fs)$$

and

$$\mathbb{A}(fs) = df \otimes s + f \sum_{i=0}^{n} \mathbb{A}_{[i]} s$$

where n is the dimension of M. We see that

$$\mathbb{A}_1(fs) = df \otimes s + f\mathbb{A}_{[1]} s$$

which is the defining property of an ordinary connection. Furthermore, since $\mathbb{A}_{[1]}$ has total odd degree, we see that as an ordinary connection

$$\mathbb{A}_{[1]} : \Gamma(\mathcal{E}^+) \to \Gamma(T^*M \otimes \mathcal{E}^+) \quad \text{and} \quad \mathbb{A}_{[1]} : \Gamma(\mathcal{E}^-) \to \Gamma(T^*M \otimes \mathcal{E}^-).$$

It also follows from the above comparison of the two expressions for $\mathbb{A}(fs)$ that the remaining $\mathbb{A}_{[i]}$, $i \neq 1$ are given by the action of an element of $\mathcal{A}^i(M, \mathrm{End}(\mathcal{E}))$. For example $\mathbb{A}_{[0]}$ is given by an element of $\Gamma(M, \mathrm{End}^-(\mathcal{E}))$.

7.14 A local computation.

To see what the supercurvature computation looks like in terms of a local description, let us assume that our bundle $\mathcal{E}$ is trivial, i.e. $\mathcal{E} = M \times E$ where E is a superspace. Let us also assume that $\mathbb{A}$ has only components $\mathbb{A}_{[0]}$ and $\mathbb{A}_{[1]}$. This will be the case in the physical model that we will propose.

We may thus write $\mathbb{A}_{[0]} = L \in C^\infty(M, \mathrm{End}^-(E))$ so

$$L = \begin{pmatrix} 0 & L^- \\ L^+ & 0 \end{pmatrix}, \quad L^- \in C^\infty(M, \mathrm{Hom}(E^-, E^+)),$$

$$L^+ \in C^\infty(M, \mathrm{Hom}(E^+, E^-)).$$

We may also write

$$\mathbb{A}_{[1]} = d + A, \quad A \in \mathcal{A}^1(M, \mathrm{End}(E)^+).$$

Let ∇ denote the covariant differential corresponding to the ordinary connection $\mathbb{A}_{[1]}$ Then

$$\mathbb{F} := (\mathbb{A})^2 = \mathbb{A}_{[0]}^2 + [\mathbb{A}_{[1]}, \mathbb{A}_{[0]}] + \mathbb{A}_{[1]}^2 = \mathbb{A}_{[0]}^2 + \nabla\mathbb{A}_{[0]} + F$$

where F is the curvature of $\mathbb{A}_{[1]}$. In terms of the matrix decomposition above we have

$$\mathbb{F} = \begin{pmatrix} L^-L^+ + F^+ & \nabla L^- \\ \nabla L^+ & L^+L^- + F^- \end{pmatrix}$$

where $F^\pm$ is the restriction of F to $E^\pm$. Notice that $\mathbb{F}$ is quadratic in L, and so any quadratic function of $\mathbb{F}$ will involve a quartic function of L. This will be our proposal for the quartic term entering into the Higgs mechanism.

7.15 Superconnections and principal bundles.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and G be a Lie group whose Lie algebra is $\mathfrak{g}_0$. Suppose that we have a representation of G as (even) automorphisms of $\mathfrak{g}$ whose restriction to $\mathfrak{g}_0$ is the adjoint representation of G on its Lie algebra.

We will denote the representation of G on all of $\mathfrak{g}$ by Ad.

Let $P = P_G \to M$ be a principal bundle with structure group G. Recall that this means the following:

- We are given an action of G on P. To tie in with standard notation we will denote this action by

  $$(p, a) \mapsto pa^{-1}, \quad p \in P, \quad a \in G$$

  so $a \in G$ acts on P by a diffeomorphism that we will denote by r_a:

  $$r_a : P \to P, \qquad r_a(p) = pa^{-1}.$$

  If $\xi \in \mathfrak{g}_0$, then $\exp(-t\xi)$ is a one parameter subgroup of G, and hence

  $$r_{\exp(-t\xi)}$$

  is a one parameter group of diffeomorphisms of P, and for each $p \in P$, the curve

  $$r_{\exp(-t\xi)}p = p(\exp t\xi)$$

  is a smooth curve starting at t at $t = 0$. The tangent vector to this curve at $t = 0$ is a tangent vector to P at p. In this way we get a linear map

  $$u_p : \mathfrak{g}_0 \to TP_p, \qquad u_p(\xi) = \frac{d}{dt}p(\exp t\xi)_{|t=0}. \tag{19}$$

- The action of G on P is free.

- The space P/G is a differentiable manifold M and the projection $\pi : P \to M$ is a smooth fibration.

- The fibration π is locally trivial consistent with the G action in the sense that every $m \in M$ has a neighborhood U such that there exists a diffeomorphism
$$\psi_U : \quad \pi^{-1}(U) \to U \times G$$
such that
$$\pi_1 \circ \psi = \pi$$
where
$$\pi_1 : \; U \times F \to U$$
is projection onto the first factor and if $\psi(p) = (m, b)$ then
$$\psi(r_a p) = (m, ba^{-1}).$$

Suppose that $\pi : P \to M$ is a principal fiber bundle with structure group G. Since π is a submersion, we have the sub-bundle Vert of the tangent bundle TP where $\mathrm{Vert}_p, p \in P$ consists of those tangent vectors which satisfy $d\pi_p v = 0$. From its construction, the subspace $\mathrm{Vert}_p \subset TP_p$ is spanned by the tangents to the curves $p(\exp t\xi),\ \xi \in \mathfrak{g}_o$. In other words, u_p is a surjective map from $\mathfrak{g}_0$ to Vert_p. Since the action of G on P is free, we know that u_p is injective. Putting these two facts together we conclude that

If $\pi : P \to M$ is a principal fiber bundle with structure group G then u_p is an isomorphism of $\mathfrak{g}_0$ with Vert_p for every $p \in P$.

An (ordinary) connection on a principal bundle is a choice of a “horizontal” subbundle Hor complementary to the vertical bundle which is invariant under the action of G. At any p we can define the projection
$$\mathbf{V}_p : TP_p \to \mathrm{Vert}_p$$
along Hor_p, i.e. $\mathbf{V}_p$ is the identity on Vert_p and sends all elements of Hor_p to 0. Giving Hor_p is the same as giving $\mathbf{V}_p$ and condition of invariance under G translates into
$$d(r_b)_p \circ \mathbf{V}_p = \mathbf{V}_{r_b(p)} \circ d(r_b)_p \quad \forall\ b \in G, \quad p \in P.$$
This then defines a one form ω on P with values in $\mathfrak{g}_0$:
$$\omega_p := u_p^{-1} \circ \mathbf{V}_p.$$
Invariance of the connection under G translates into
$$r_b^* \omega = \mathrm{Ad}_b\, \omega.$$
Let ξ_P be the vector field on P which is the infinitesimal generator of $r_{\exp t\xi}$. The the infinitesimal version of the preceding equation is
$$D_{\xi_P} \omega = [\xi, \omega].$$

In view of the definition of u_p as identifying ξ with the tangent vector to the curve $t \mapsto p(\exp t\xi) = r_{\exp -t\xi} p$ at $t = 0$, we see that

$$i(\xi_P)\omega = -\xi.$$

We now generalize this to superconnections: We define a **superconnection form** A to be an odd element of $\mathcal{A}(P, \mathfrak{g})$ which satisfies

$$\begin{aligned} r_b^* A &= \mathrm{Ad}_b\, A \quad \forall b \in G & (20) \\ i(\xi_P) A &= -\xi \quad \forall\, \xi \in \mathfrak{g}_0. & (21) \end{aligned}$$

The meaning of (21) is the following:

$$A = A_{[0]} + A_{[1]} + \cdots + A_{[n]}, \qquad n = \dim M$$

where $A_{[i]}$ is an i-form with values in $\mathfrak{g}_0$ if i is odd and with values in $\mathfrak{g}_1$ it i is even. Then $A_{[1]}$ is a connection form and all the other components satisfy

$$i(\xi_P) A_{[i]} = 0.$$

This condition together with (20) imply that these other components can be identified with odd i-forms on M with values in $\mathfrak{g}(P)$, the vector bundle over M associated to the representation Ad of G on $\mathfrak{g}$.

More generally, if the superspace E is a G module and also a $\mathfrak{g}$ module in a consistent way, then we can form the associated bundle

$$\mathcal{E}(M) = E(P)$$

which is a module for the associated bundle of superalgebras $\mathfrak{g}(P)$. A k-form on M with values in $\mathcal{E}$ is the same thing as a k-form σ on P with values in E which satisfies

1. $i(\xi_P)\sigma = 0 \quad \forall\, \xi \in \mathfrak{g}_0$ and
2. $r_a^*\sigma = \rho(a)\sigma$ where ρ denotes the action of G on E.

The bilinear map

$$\mathfrak{g} \times E \to E$$

given by the action of $\mathfrak{g}$ determines an exterior multiplication

$$\Omega(P, \mathfrak{g}) \times \Omega(P, E) \to \Omega(P, E)$$

which we will denote by $\diamond$. We then obtain a superconnection on $\mathcal{E}$ given by

$$\mathbb{A}\sigma = d\sigma + A \diamond \sigma. \qquad (22)$$

7.16 The Higgs field and superconnections.

In the model that we proposed in [NS90] ,[NS91], we are given a bundle of Lie superalgebras $\mathfrak{g} = \mathfrak{g}(P) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as above. If we assume that the superconnection form A has only exterior terms of degree zero and one, then $\mathbb{A}_{[0]}$ is given by the action of a section of $\mathfrak{g}_1$. We take the sections of $\mathfrak{g}_1 = \mathfrak{g}_1(P)$ to be the Higgs fields. As described above, the supercurvature is then quadratic in the Higgs field, and hence a super-Yang-Mills functional which will be be quartic in the Higgs field.

7.17 Clifford Bundles and Clifford superconnections.

Suppose that M is a semi-Riemannian manifold so that we can form the bundle of Clifford algebras $C(TM)$. Suppose that $\mathcal{F}$ is a bundle of Clifford modules. We denote the action of a section a of $C(TM)$ on a section of $\mathcal{F}$ by $c(a)$. We extend this notation to denote the action of a Clifford bundle valued differential form, i.e. an element of $\mathcal{A}(M, C(TM))$ on $\mathcal{A}(M, \mathcal{F})$ by

$$c(\alpha \otimes a)(\beta \otimes s) = (-1)^{|a| \cdot |\beta|}(\alpha \wedge \beta) \otimes c(a)s$$

on homogeneous elements.

A superconnection $\mathbb{B}$ on $\mathcal{F}$ is called a **Clifford superconnection** [BGV91] if for all sections a of $C(T(M))$ we have

$$[\mathbb{B}, c(a)] = c(\nabla a)$$

where ∇ is the covariant differential on $C(T(M))$ coming from the Levi-Civita connection on M.

Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are Clifford superconnections on $\mathcal{F}$. Then

$$[\mathbb{B} - \mathbb{B}', e(\alpha)] = 0 \quad \forall\ \alpha \in \mathcal{A}(M)$$

so $\mathbb{B} - \mathbb{B}' \in \mathcal{A}^-(M, \mathrm{End}(\mathcal{F})$. Also

$$[\mathbb{B} - \mathbb{B}', c(a)] = 0$$

implying that

$$\mathbb{B} - \mathbb{B}' \in \mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F})).$$

Conversely, if $\tau \in \mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F}))$ and $\mathbb{B}'$ is a Clifford superconnection then $\mathbb{B} = \mathbb{B}' + \tau$ is a Clifford superconnection. Thus the collection of all Clifford superconnections is an affine space modeled on the linear space $\mathcal{A}^-(M, \mathrm{End}_{C(M)}(\mathcal{F}))$.

If $\mathcal{E}$ is a superbundle and $\mathcal{F}$ is a bundle of Clifford modules then we can make $\mathcal{E} \otimes \mathcal{F}$ into a Clifford module by letting a section a of $C(TM)$ act as $1 \otimes c(a)$ where $c(a)$ denote the action of a on $\mathcal{F}$. If $\mathbb{A}$ is a superconnection on $\mathcal{E}$ then

$$[\mathbb{A} \otimes \mathbf{1}, \mathbf{1} \otimes c(a)] = 0$$

for all sections a of $C(TM)$ and so

$$[\mathbb{A} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbb{B}, \mathbf{1} \otimes c(a)] = \mathbf{1} \otimes c(\nabla a).$$

In other words, the tensor product of a superconnection with a Clifford superconnection is a Clifford superconnection.

7.18 The Dirac operator of a Clifford superconnection.

Let $\mathcal{E}$ be a Clifford module over the semi-Riemannian manifold M and let $\mathbb{A}$ be a Clifford superconnection on $\mathcal{E}$. We can associate to this data a certain first order differential operator on sections of M

$$\mathbb{D} = \mathbb{D}_{\mathbb{A}} : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$$

which generalizes the classical Dirac operator in the presence of an electromagnetic field. In order to define it we need to record a relation between the Clifford algebra and the exterior algebra.

7.18.1 The exterior algebra as a Clifford module.

Let V be a vector space with a non-degenerate scalar product $(\cdot, \cdot)$ which then defines an isomorphism of V with its dual space V^*: $v \mapsto (v, \cdot)$.

If $v \in V$ we will let $i(v) : \wedge(V) \to \wedge(V)$ denote interior product by the element $v^* \in V^*$ corresponding to V. Explicitly, $i(v)$ is the (odd) derivation on $\wedge(V)$ determined by

$$i(v)1 = 0, \qquad i(v)w = (v, w), \qquad w \in V.$$

We let $e(v) : \wedge(V) \to \wedge(V)$ denote exterior multiplication by v. If we put the standard scalar product on $\wedge(V)$ induced by the scalar product on V, it is easy to check that $i(v)$ is the transpose of $e(v)$. Since $e(v)^2 = 0$ it follows that $i(v)^2 = 0$ (as can also be checked directly from the definition) and that

$$(i(v) + e(v))^2 = i(v)e(v) + e(v)i(v) = (v, v)\mathrm{id}.$$

So $v \mapsto i(v) + e(v)$ is a Clifford map and so makes $\wedge(V)$ into a $C(V)$ module. Consider the linear map

$$\sigma : \; C(V) \to \wedge(V), \qquad x \mapsto x1$$

where $1 \in \wedge^0(V)$ under the identification of $\wedge^0(V)$ with the ground field. The element $x1$ on the extreme right means the image of 1 under the action of $x \in C(V)$. For elements $v_1, \dots, v_k \in V$ this map sends

$$\begin{array}{rcl}
v_1 & \mapsto & v_1 \\
v_1 v_2 & \mapsto & v_1 \wedge v_2 + (v_1, v_2)1 \\
v_1 v_2 v_3 & \mapsto & v_1 \wedge v_2 \wedge v_3 + (v_1, v_2)v_3 - (v_1, v_3)v_2 + (v_2, v_3)v_1 \\
v_1 v_2 v_3 v_4 & \mapsto & v_1 \wedge v_2 \wedge v_3 \wedge v_4 + (v_2, v_3)v_1 \wedge v_4 - (v_2, v_4)v_1 \wedge v_3 \\
 & & +(v_3, v_4)v_1 \wedge v_2 + (v_1, v_2)v_3 \wedge v_4 - (v_1, v_3)v_1 \wedge v_4 \\
 & & +(v_1, v_4)v_2 \wedge v_3 + (v_1, v_4)(v_2, v_3) - (v_1, v_3)(v_2, v_4) \\
+(v_1, v_2)(v_3, v_4) & & \\
\vdots & & \vdots
\end{array}$$

If the v's form an "orthonormal" basis of V then the products

$$v_{i_1}\cdots v_{i_k},\quad i_1<i_2\cdots<i_k,\ k=0,1,\ldots,n \tag{23}$$

form a basis of $C(V)$ while the

$$v_{i_1}\wedge\cdots\wedge v_{i_k},\quad i_1<i_2\cdots<i_k,\ k=0,1,\ldots,n \tag{24}$$

form a basis of $\wedge(V)$, and in fact

$$v_1\cdots v_k\mapsto v_1\wedge\cdots\wedge v_k\quad \text{if } (v_i,v_j)=0\ \forall i\neq j. \tag{25}$$

In particular, the map σ given above is an isomorphism of vector spaces.

We will let

$$\mathbf{q}:\ \wedge(V)\to C(V) \tag{26}$$

denote the inverse of σ:

$$\mathbf{q}:=\sigma^{-1}. \tag{27}$$

On a semi-Riemannian manifold we have an identification ℓ of $\Gamma(M,\wedge(T^*M))$ with $\Gamma(M,\wedge T(M))$ given by the metric. We can then apply the map $\mathbf{q}$ at each point so as to get a map (which we will also denote by $\mathbf{q}$):

$$\mathbf{q}:\Gamma(M,\wedge(TM))\to\Gamma(M,C(M)).$$

7.18.2 The Dirac operator.

Let $\mathbb{A}$ be a Clifford superconnection on the Clifford module $\mathcal{E}$. We have the following sequence of maps:

$$\begin{array}{rcl}\mathbb{A}:\Gamma(M,\mathcal{E}) & \to & \mathcal{A}(M,\mathcal{E})=\Gamma(M,\wedge(T^*M)\otimes\mathcal{E})\\ \ell\otimes\mathrm{id}:\Gamma(M,\wedge(T^*M)\otimes\mathcal{E}) & \to & \Gamma(M,\wedge(TM)\otimes\mathcal{E})\\ \mathbf{q}\otimes\mathrm{id}:\Gamma(M,\wedge(TM)\otimes\mathcal{E}) & \to & \Gamma(M,C(M)\otimes\mathcal{E})\\ \mathbf{c}:\Gamma(M,C(M)\otimes\mathcal{E}) & \to & \Gamma(M,\mathcal{E})\end{array}$$

where the last map $\mathbf{c}$ is given by the action of $C(M)$ on $\mathcal{E}$.

The composite of all these operators is the Dirac operator

$$\mathbb{D}_{\mathbb{A}}:\Gamma(M,\mathcal{E})\to\Gamma(M,\mathcal{E}) \tag{28}$$

associated to the superconnection $\mathbb{A}$.

7.18.3 A local description of the Dirac operator.

Let $x^1,\ldots,x^n$ be a local coordinate system with $dx^1,\ldots,dx^n$ the corresponding differential forms and $\partial_1,\ldots,\partial_n$ the corresponding vector fields so that the exterior differential d is given by

$$d=\sum_{i=1}^n dx^i\otimes\partial_i.$$

Let $e_1, \ldots, e_n$ be an "orthonormal" frame field over this coordinate neighborhood and $\theta^1, \ldots, \theta^n$ the dual coframe field. The most general superconnection on $\mathcal{E}$ can then be written as

$$\mathbb{A} = \sum_{i=1}^{n} dx^i \otimes \partial_i + \sum_{I \subset \{1,\ldots,n\}} \theta^I \otimes A_I$$

where

$$\theta^I := \theta_{i_1} \wedge \cdots \wedge \theta_{i_j} \quad \text{where } \; I = \{i_1, \ldots, i_j\} \quad i_1 < i_2 < \cdots < i_j \tag{29}$$

and A_I is a section of $\mathrm{End}(\mathcal{E})$. Applying $\ell \otimes \mathrm{id}$ gives

$$\sum_{i=1}^{n} \ell(dx^i) \otimes \partial_i + \sum_{I \subset \{1,\ldots,n\}} e^I \otimes A_I. \tag{30}$$

Applying $\mathbf{q} \otimes \mathrm{id}$ gives

$$\sum_{i=1}^{n} \mathbf{q}(\ell(dx^i) \otimes \partial_i + \sum_{I \subset \{1,\ldots,n\}} \mathbf{q}(e^I) \otimes A_I$$

and the applying the Clifford action gives

$$\mathbb{D}_{\mathbb{A}} = \sum_{i=1}^{n} \mathbf{c}(\mathbf{q}(\ell(dx^i))\partial_i \; + \sum_{I \subset \{1\ldots,n\}} \mathbf{c}(\mathbf{q}(e^I)) \circ A_I.$$

7.19 Clifford bundles and spinors.

So far, we have not made any assumptions about the dimension of M or about the signature of the semi-Riemann metric on M. On a complex vector space, all non-degenerate quadratic forms are equivalent. The Clifford algebra of an even dimensional complex vector space with non-degenerate quadratic form is isomorphic to $\mathrm{End}(S)$ where $S = S_+ \oplus S_-$ is known as the space of spinors. In the case of a real vector space with a negative definite scalar product, which we then complexify, there is a positive definite Hermitian form on S invariant under the group $Spin(V)$ which is the double cover in $C(V)$ of the group $SO(V)$. The spaces S_+ and S_- are orthogonal under the Hermitian form and give the (irreducible) half spin representations of $Spin(V)$. These are well known facts and can be found in standard texts such as [G78] or [BGV91].

The case of physical interest is where we are dealing with a four dimensional space with Lorentzian metric. The following is a summary of the well known facts. As it is hard to find a cogent presentation of these facts in the standard texts, we will give a more detailed presentation in the next section.

The (real) Clifford algebra $C(3,1)$ (spacelike positive, timelike negative) is isomorphic as an algebra to $\mathrm{End}(\mathbb{R}^4)$. Wedderburn's theorem then implies that this four dimensional real $C(3,1)$ module, known as the space of Majorana spinors, is unique up to canonical isomorphism, and that any $C(3,1)$ module is isomorphic to the tensor product of this module with a trivial module.

The element

$$\gamma = e_0 e_1 e_2 e_3$$

(where e_0, e_1, e_2, e_3 is an oriented orthonormal basis) satisfies

$$\gamma^2 = -1$$

and

$$\gamma a = a\gamma, \ a \in C_0(3,1), \quad \gamma b = -b\gamma, \quad b \in C_1(3,1).$$

Thus γ defines a complex structure $\mathbf{J}$ on $\mathbb{R}^4$ and the even elements of $C(3,1)$ act as linear transformations (commute with $\mathbf{J}$) while the odd elements of $C(3,1)$ act as antilinear transformations (anti-commute with $\mathbf{J}$). This complex structure allows us identify the space $\mathbb{R}^4$ of Majorana spinors with $\mathbb{C}^2$.

The group $Sl(2,\mathbb{C})$ is simply connected and is the double cover of the connected component of the Lorentz group $O(3,1)$. It preserves a complex symplectic form (a non-degenerate anti-symmetric bilinear form) which is determined up to multiplication by a non-zero complex number. Let H be the two component group in $C(3,1)$ which (double) covers the two component subgroup of $O(3,1)$ consisting of those Lorentz transformations which preserve the forward light cone. (So H includes elements which project onto "parity transformations".) Then there is a real symplectic form s on $\mathbb{R}^4$ invariant under H which is determined up to a non-zero real scalar multiple and a bilinear map $\mathfrak{j}$ from $\mathbb{R}^4$ to Minkowski which is equivariant under the action of H.

The space of Dirac spinors is the complexification of the space of Majorana spinors. It decomposes into the direct sum of the $\pm i$ eigenvalues of $\mathbf{J}$ and these are the right and left handed spinors. This is the $\mathbb{Z}_2$ structure we will be using throughout this paper. If we extend s to be a sesquilinear form on the space of *Dirac* spinors, then is is a non-degenerate Hermitian form of signature $(2,2)$ and is uniquely determined up to real scalar multiple as being invariant under H. The space of right or left handed spinors is isotropic under this Hermitian form.

7.20 Facts about Dirac spinors.

The facts collected in this section are well known to physicists. For the convenience of the mathematical reader we collect them here.

7.20.1 The element γ in general.

Let V be a real vector space with a non-degenerate quadratic form of signature (p,q) and let C be the corresponding Clifford algebra. Let

$$v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}$$

be an "orthonormal" basis so that

$$(v_i, v_i) = \begin{cases} 1 & 1 \le i \le p \\ -1 & p+1 \le i \le p+q. \end{cases}$$

Let

$$\gamma := v_1 \cdot v_2 \cdots v_{p+q}.$$

Notice that γ is determined up to sign (fixed by choosing an orientation of V) and satisfies

$$\begin{aligned} \gamma^2 &= (-1)^{\frac{1}{2}n(n-1)+q}\mathbf{1}_C \quad n = p+q = \dim V \\ \gamma v &= (-1)^{n-1} v\gamma, \quad v \in V. \end{aligned}$$

If $p = q+2$ then $n = 2(q+1)$ and

$$\frac{1}{2}n(n-1) + q = (q+1)(2q+1) + q = 2q^2 + 4q + 1$$

is odd hence

$$\gamma^2 = -\mathbf{1}_C \tag{31}$$

$$\gamma v = -v\gamma. \tag{32}$$

These equations will also hold if $p = q + r$ where $r \equiv 2 \bmod 4$.

7.20.2 Majorana spinors for $C(q+2,2)$.

By Bott periodicity (see for example [G78]) we have

$$\begin{aligned} C(p,q) \otimes C(2,0) &= C(q+2,p) \\ C(q,q) &= \mathrm{End}(\mathbb{R}^{2^q}) \\ C(2,0) &= \mathrm{End}(\mathbb{R}^2) \quad \text{hence} \\ C(q+2,q) &\cong \mathrm{End}(\mathbb{R}^{2^{q+1}}). \end{aligned}$$

Then (31) says that $\gamma \in \mathrm{End}(\mathbb{R}^{2^{q+1}})$ defines a complex structure on $\mathbb{R}^{2^{q+1}}$ and (32) implies that all the odd elements of $C = C(q+2,q)$ act as antilinear transformations and all the even elements act as linear elements on the space of **Majorana spinors**: $S = \mathbb{C}^{2^q} \sim \mathbb{R}^{2^{q+1}}$.

7.20.3 Majorana spinors in four dimensions.

We know that $Spin(3,1)$ is isomorphic to $sl(2,\mathbb{C})$. In fact, we will shortly give a an explicit realization of this fact. So there is an invariant anti-symmetric complex bilinear form on S which is invariant under $Spin(3,1)$. (Such an object is called a complex symplectic form.) In fact, there is a whole family of them determined up to multiplication by a complex number. If we enlarge the group $Spin(3,1)$ to include conjugation by time-like vectors we will find that we obtain a group G which double covers the subgroup of $O(3,1)$ which has two components consisting of the connected component $SO(3,1)$ and also the parity transformations. We will find that there is a real symplectic form s on S which is invariant under G. This will determine s up to multiplication by a non-zero real number. We will also find that s determines a quadratic map $\mathfrak{j}$ from S to vectors, and we will use this to associate a "current" to each pair of spinors.

Let e_0 be a "unit" time like vector so that $e_0^2 = -\mathbf{1}_C$. Hence e_0 is invertible in the Clifford algebra $C = C(3,1)$ and

$$e_0^{-1} = -e_0.$$

Consider the operation of conjugation by e_0 in the Clifford algebra:

$$a \mapsto e_0 a e_0^{-1} = -e_0 a e_0.$$

Acting on e_0 we get

$$e_0 \mapsto -e_0^3 = e_0.$$

Acting on a vector v perpendicular to e_0 we get

$$v \mapsto -e_0 v e_0 = +e_0^2 v = -v.$$

Thus conjugation by e_0 carries the subspace $\mathbb{R}^{3,1}$ into itself and acts there as the "parity transformation" $\mathbf{P}$:

$$\mathbf{P}e_0 = e_0, \qquad \mathbf{P}v = -v \ \text{ if } v \perp e_0.$$

For a general discussion of the "Pin group" using twisted conjugation rather than conjugation see [G78].

7.20.4 A model for the Majorana spinors.

We identify the space $V = \mathbb{R}^{1,3}$ with the space of two by two (complex) self adjoint matrices: if P and Q are self adjoint two by two matrices we define

$$||P||^2 = \det P, \quad (P,Q) = \frac{1}{2}\mathrm{tr}\ PQ^{\mathrm{a}} \tag{33}$$

where Q^{a} denotes the "adjoint" according to Cramer's rule

$$\mathrm{a}: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so

$$QQ^{\mathrm{a}} = \det\ Q\ I.$$

We have

$$\det\begin{pmatrix} t-x & y+iz \\ y-iz & t+x \end{pmatrix} = t^2 - x^2 - y^2 - z^2 \tag{34}$$

so the space of self-adjoint two by two matrices is a model of $\mathbb{R}^{1,3}$.

Let A be a two by two complex matrix. If P is self-adjoint then so is $APA^\dagger$ and the map

$$P \mapsto APA^\dagger$$

is a real linear map of the space of two by two self adjoint matrices into itself. If $\det A = 1$ then

$$\det(APA^\dagger) = \det P.$$

This shows that we have a homomorphism from $Sl(2,\mathbb{C}) \to SO(1,3)$. It is not hard to show that this homomorphism is two to one and surjective and hence gives an identification of $Spin(1,3) = Spin(3,1)$ with $Sl(2,\mathbb{C})$. We will take the space of spinors to be $\mathbf{C}^2$ regarded as a real four dimensional space. Define the anti-linear operator

$$\star : \mathbf{C}^2 \to \mathbf{C}^2, \quad \star : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\overline{y} \\ \overline{x} \end{pmatrix}.$$

Then

$$\star^2 = -I$$

and

$$\langle \star u, u \rangle = 0, \quad \forall u \in \mathbf{C}^2$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian form on $\mathbf{C}^2$. A direct verification shows that

$$\star A = A^{\mathrm{a}\dagger}\star \tag{35}$$

for any two by two complex matrix, A.

Indeed, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\star A\begin{pmatrix} x \\ y \end{pmatrix} = \star\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} -\overline{cx} - \overline{dy} \\ \overline{ax} + \overline{by} \end{pmatrix}, \quad A^{a\dagger}\star\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}\begin{pmatrix} -\overline{y} \\ \overline{x} \end{pmatrix}.$$

In particular, for self adjoint matrices, P, we have

$$\star P \star^{-1} = P^{\mathrm{a}}. \tag{36}$$

If we take

$$P = e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $P^{\mathrm{a}} = P$. On the other hand, if P is orthogonal to e_0, so that tr $P = 0$, then $P = -P^{\mathrm{a}}$. Thus conjugation by $\star$ induces the "parity transformation" on Minkowski space.

Any $A \in Sl(2,\mathbb{C})$ satisfies

$$A^{\mathrm{a}} = A^{-1}$$

and therefore for $A \in Sl(2,\mathbb{C})$ we have

$$\begin{aligned} A\mu(P)A^{-1} &= AP\star A^{-1} \\ &= APA^{\dagger}\star \\ &= \mu(APA^{\dagger}). \end{aligned}$$

The transformation

$$P \mapsto APA^{\dagger}$$

gives the action of $A \in Sl(2,\mathbb{C})$ on $P \in \mathbb{R}^{1,3}$. Thus the equation

$$A\mu(P)A^{-1} = \mu(APA^{\dagger}) \tag{37}$$

asserts that the map $\mu : \mathbb{R}^{1,3} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^2)$ is an $Sl(2,\mathbb{C})$ morphism. Observe also that in this representation the element $\gamma \mapsto \pm i$, where i denotes the usual multiplication by the complex number i on $\mathbb{C}^2$, because γ commutes with with all even elements of $C(3,1)$ and its square is -1. The choice of sign reflects the indeterminacy in the choice of γ depending on the choice of orientation in Minkowski space. In order to avoid later confusion when we complexify the space $\mathbb{C}^2$ and hence have still another notion of multiplication by i, we shall denote the element γ in our case by the neutral symbol $\mathbf{J}$.

7.20.5 Bilinear covariants for Majorana spinors.

Define the real quadratic map

$$\mathsf{j} : S = \mathbb{C}^2 \mapsto \mathbb{R}^{1,3}, \qquad \mathsf{j}(u) := u \otimes u^{\dagger}. \tag{38}$$

We have

$$\mathsf{j}(Au) = A\mathsf{j}(u)A^{\dagger} \quad \forall A \in gl(2,\mathbb{C}), \tag{39}$$

implying the equivariance of the map j for the group $Sl(2,\mathbb{C})$. Also $(u,v) = (\star v, \star u)$ $\forall u, v \in \mathbb{C}^2$ hence

$$\begin{aligned} \mathsf{j}(\star u)v &= (v, \star u)\star u \\ &= \star\{(\star u, v)u\} \\ &= \star\{(\star v, \star\star u)u\} \\ &= \star\{(-\star v, u)u\} \\ &= \star\{(\star^{-1} v, u)u\} \quad \text{so} \\ \\ \mathsf{j}(\star u) &= \star\mathsf{j}(u)\star^{-1}. \end{aligned}$$

This equation, together with (39) has the following meaning: Let G denote the subgroup of the group of all invertible real linear transformations of $\mathbb{C}^2$ generated by $Sl(2,\mathbb{C})$ and $\star$. Since

$$\star A \star^{-1} = A^{\dagger -1} \quad \forall\ A \in Sl(2,\mathbb{C}), \tag{40}$$

we see that G consists of elements of the form B or $B\star$, $\quad B \in Sl(2,\mathbb{C})$. So the group G consists of two of the four components of the group $Pin(3,1)$, the double cover of $O(3,1)$ in the Clifford algebra. Indeed G consists of those elements of $Pin(3,1)$ which (in their action on $\mathbb{R}^{3,1}$) preserve the direction of time.

$$\mathfrak{j}(\star u) = \star \mathfrak{j}(u) \star^{-1} \tag{41}$$

thus asserts that $\mathfrak{j}$ is a morphism for the "parity" action of G on Minkowski space. (This is usually expressed by saying that $\mathfrak{j}$ defines a "vector current" as opposed to an "axial current".) Notice that the time component of $\mathfrak{j}(u)$ is always non-negative. Indeed

$$\mathrm{tr}\ \mathfrak{j}(u) = ||u||^2. \tag{42}$$

This result was important to Dirac in that it allowed the interpretation of the time component of $\mathfrak{j}(u)$ as a probability density, when $\mathfrak{j}(u)$ is interpreted as a current.

The map $\mathfrak{j}$, being quadratic, defines, by polarization, a real symmetric bilinear map from $\mathbb{C}^2$ to Minkowski space:

$$\mathfrak{j}(u,v) := \frac{1}{2}(u \otimes v^\dagger + v \otimes u^\dagger).$$

We can also consider the antisymmetric form

$$b : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}^{1,3} \qquad b(u,v) := \frac{1}{2}\mathbf{J}(u \otimes v^\dagger - v \otimes u^\dagger). \tag{43}$$

(Remember that the $\mathbf{J}$ in this equation is simply multiplication by i or by $-i$ depending on the orientation. So the matrix on the right is indeed self adjoint.) "Polarizing" the argument that we gave above shows that

$$(\star u) \otimes (\star v)^\dagger = \star[u \otimes v^\dagger]\star^{-1}.$$

But

$$\mathbf{J}\star = -\star\mathbf{J}$$

so

$$b(\star u, \star v) = -\star b(u,v)\star^{-1}. \tag{44}$$

One says that "$b(u,v)$ is an axial current". Now $\mathbb{C}^2$ carries a $\mathbb{C}$ valued symplectic form invariant under $Sl(2,\mathbb{C})$ (in fact a one complex dimensional space of them). We can use the symplectic form to identify $\mathbb{C}^2$ with its dual and so define a bilinear map

$$c : \mathbb{C}^2 \times \mathbb{C}^2 \to gl(2,\mathbb{C}), \quad c(u,v)w := \omega(v,w)u$$

where ω is (a choice of) symplectic form. One choice of the symplectic form is

$$\omega(v,w) := (w, \star v).$$

Explicitly

$$(w, \star v) = v_1 w_2 - v_2 w_1.$$

For this choice we have

$$c(u, v) = u \otimes (\star v)^\dagger. \tag{45}$$

So

$$c(u, v)w = (w, \star v)u.$$

Now $(w, \star v) = (v, \star^{-1} w) = \overline{(\star^{-1} w, v)}$ so we see that this choice of c satisfies

$$c(\star u, \star v) = \star c(u, v) \star^{-1}. \tag{46}$$

Under the conjugation action of $Sl(2, \mathbb{C})$ the space $gl(2, \mathbb{C})$ decomposes as

$$gl(2, \mathbb{C}) = sl(2, \mathbb{C}) \oplus \mathbb{C}.$$

Under the action of conjugation by $\star$ we have the further decomposition

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$$

which is the ± 1 eigenvector decomposition. We can thus write

$$c = a \oplus s \oplus iq$$

where a is the $sl(2, \mathbb{C})$ component, where s is a "scalar" (transforms according to the trivial representation of G) and where q is a "pseudoscalar" (transforms according the representation which assigns $+1$ to the identity component and -1 to the other component of G). Both s and q are real valued symplectic forms on $S = \mathbb{C}^2$.

Notice that for any $P \in \mathbb{R}^{1,3}$, $\mu(P)$ is in the symplectic algebra of the symplectic form s (as are the elements of $sl(2, \mathbb{C})$). Indeed,

$$\begin{aligned}
s(\mu(P)u, v) &= \frac{1}{2} \text{ Re tr } c(P \star u, v) \\
&= \frac{1}{2} \text{Re } (P \star u, \star v) \quad \text{while} \\
s(u, \mu(P)v) &= \text{Re } (u, \star P \star v) \\
&= \frac{1}{2} \text{Re } (\star u, \star \star P \star v) \\
&= -\frac{1}{2} \text{Re } (\star u, P \star v) \\
&= -\frac{1}{2} \text{Re } (P \star u, \star v)
\end{aligned}$$

since $\star\star = -1$ and P is self adjoint. Hence

$$s(\mu(P)u, v) + s(u, \mu(P)v) = 0.$$

Therefore $\mu(P)$ determines a quadratic form

$$u \mapsto s(\mu(P)u, u)$$

on $S = \mathbb{C}^2$ since

$$s(\mu(P)u, v) = -s(u, \mu(P)v) = s(\mu(P)v, u).$$

We claim that

$$s(\mu(P)u, u) = P \cdot \mathfrak{j}(u). \tag{47}$$

Indeed, by the definition of the scalar product, by (36), by (41), and by the definition (38) of $\mathfrak{j}$ we have,

$$\begin{aligned} P \cdot \mathfrak{j}(u) &= \frac{1}{2}\text{tr } P\mathfrak{j}(u)^a \\ &= \frac{1}{2}\text{tr } P \star \mathfrak{j}(u) \star^{-1} \\ &= \frac{1}{2}\text{tr } P\mathfrak{j}(\star u) \\ &= \frac{1}{2}(P \star u, \star u) \\ &= s(\mu(P)u, u) \end{aligned}$$

since P is self adjoint implying that $(P\star u, \star u)$ is real and by definition, $s(\mu(P)u, u) = \frac{1}{2}\text{Re } (P \star u, \star u)$.

We shall see later on that the representation of G on S is absolutely irreducible, that is, remains irreducible even after complexification. But this implies that (up to non-zero real scalars) there can exist at most one G invariant real symplectic form. Since we have expressed $\mathfrak{j}$ in terms of s, we see that s, and hence $\mathfrak{j}$ are determined (up to scalar factors) by the representation of G on S.

7.20.6 The Dirac equation for Majorana spinors.

We now explain how the general notion of the Dirac operator associated to a Clifford connection specializes to yield the Dirac operator on Majorana spinors when we take the trivial connection.

Let $\mathbf{S} \to M$ be the trivial vector bundle over Minkowski space, M whose fiber is S. Let ψ be a section of $\mathbf{S}$, so we can think of ψ as a function from $M \to S$. Then $d\psi$ is a section of $T^* \otimes S$ where T^* is the cotangent bundle of M. Using the Minkowski metric, we can identify T^* with $T \sim \mathbb{R}^{1,3}$ and then apply

$$\mu : T \otimes \mathbf{S} \to \mathbf{S}.$$

So

$$\mu(d\psi)$$

is a section of $\mathbf{S}$. The physicists write $\mu(\partial)\psi$ for $\mu(d\psi)$ since, if

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is regarded as a $\mathbb{C}^2$ valued function, then

$$\mu(d\psi) = \begin{pmatrix} \partial_0 - \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \star \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The (Majorana version of the) Dirac equation is

$$\mu(d\psi) = m\psi. \tag{48}$$

If ψ is a solution of this equation, the corresponding vector field, $j(\psi)$ is called the current associated to ψ. We claim that

$$\text{div } \mathrm{j}(\psi) = 0. \tag{49}$$

Indeed

$$\begin{aligned} \text{div } \mathrm{j}(\psi) &:= \partial \cdot \mathrm{j}(\psi) \\ &= \frac{1}{2}\text{tr } (\partial)^a(\psi \otimes \psi^\dagger) \\ &= -\frac{1}{2}\text{tr } \star(\partial) \star (\psi \otimes \psi^\dagger) \\ &= -\frac{1}{2}\text{tr } \star \mu(\partial)\psi \otimes \psi^\dagger \\ &= -\frac{1}{2}m(\star\psi, \psi) \\ &= 0. \end{aligned}$$

Equation (49) expresses the "conservation of the current".

Notice that if we seek plane wave solutions to the Dirac equation

$$\psi(x) = \cos(P \cdot x + \alpha)u \quad u \in \mathbb{C}^2$$

then (48) implies that

$$||P||^2 = m^2$$

if $u \neq 0$.

We may think of d mapping sections of $\mathbf{S}$ to sections of $T^* \otimes \mathbf{S}$ as defining a flat connection on $\mathbf{S}$. We may modify this connection by considering $\mathbf{S}$ as a $U(1)$ bundle which has its own connection adding a one form and so consider the equation

$$\mu(d\psi + eA \otimes \psi) = m\psi.$$

This is the Dirac equation in the presence of an external electromagnetic field with four potential A.

7.20.7 Complexifying a vector space with a complex structure.

The space of Dirac spinors is the complexification of the space of Majorana spinors. This will involve us several times in the painful process of complexifying a real vector space with a complex structure, so we review the general construction.

Let V be a real vector space with a complex structure. That is, we are given an operator $\mathbf{J}$ on V such that $\mathbf{J}^2 = -I$. Any operator, A, on V extends as the operator $A \otimes \mathrm{id}$ on $V^{\mathbb{C}} = V \otimes \mathbb{C}$. When there is no danger of confusion we shall continue to denote this extended operator by A. Thus the (extended) operator $\mathbf{J}$ has eigenvalues $\pm i$ on $V^{\mathbb{C}}$. In other words $V^{\mathbb{C}}$ decomposes as

$$V^{\mathbb{C}} = V^{\mathbb{C}}_{+} \oplus V^{\mathbb{C}}_{-}$$

where

$$V^{\mathbb{C}}_{+} := \{u - i\mathbf{J}u, \quad u \in V\}$$

consists of all the $+i$ eigenvectors of $\mathbf{J}$ and

$$V^{\mathbb{C}}_{-} : \{u + i\mathbf{J}u, \quad u \in V\}$$

consists of all the $-i$ eigenvectors of $\mathbf{J}$.

Suppose that the operator $A$is $\mathbf{J}$ linear, meaning that $A\mathbf{J} = \mathbf{J}A$. Suppose that we choose a $\mathbf{J}$ basis of V. This means that we choose vectors $e_1, \ldots, e_n$ so that the vectors

$$e_1, \ldots, e_n, \mathbf{J}e_1, \ldots, \mathbf{J}e_n$$

form a basis of V . Relative to such a basis the assertion that A is $\mathbf{J}$ linear amounts to saying that A has the block matrix decomposition

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Now $e_1 - i\mathbf{J}e_1, \ldots, e_n - i\mathbf{J}e_n$ is a basis of $V^{\mathbb{C}}_{+}$ while $e_1 + i\mathbf{J}e_1, \ldots, e_n + i\mathbf{J}e_n$ is a basis of $V^{\mathbb{C}}_{-}$. It then follows immediately that in terms of the combined basis of $V^{\mathbb{C}}$ we have

$$A \otimes \mathrm{id} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} \text{ if } A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ is } \mathbf{J} \text{ linear.}$$

Now suppose that A is anti-$\mathbf{J}$ linear, meaning that $A\mathbf{J} = -\mathbf{J}A$. This amounts to saying that A has the block decomposition

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

and it follows that

$$A = \begin{pmatrix} 0 & a + ib \\ a - ib & 0 \end{pmatrix} \text{ if } A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ is } \mathbf{J} \text{ anti-linear.}$$

For example, let us consider the case where $V = \mathfrak{g}$ is a Lie algebra in which the Lie bracket is $\mathbf{J}$ linear. This Lie bracket extends by complexification to $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{\mathbb{C}}$, and the two subspaces $\mathfrak{g}^{\mathbb{C}}_{+}$ and $\mathfrak{g}^{\mathbb{C}}_{-}$ are subalgebras each isomorphic to $\mathfrak{g}$ under the isomorphisms

$$\xi \mapsto \frac{1}{\sqrt{2}}(\xi - i\mathbf{J}\xi), \quad \xi \mapsto \frac{1}{\sqrt{2}}(\xi + i\mathbf{J}\xi).$$

Suppose that the Lie algebra $\mathfrak{g}$ has a representation on the vector space S which carries a complex structure, $\mathbf{J}_S$, and that the complex structure on $\mathfrak{g}$ is consistent with the complex structure on S in the sense that

$$\xi(\mathbf{J}_S u) = (\mathbf{J}_{\mathfrak{g}}\xi)u.$$

where $\mathbf{J}_{\mathfrak{g}}$ denotes the complex structure on $\mathfrak{g}$. We can drop the two subscripts and write this as

$$\xi\mathbf{J}u = \mathbf{J}\xi u.$$

Then

$$(\xi - i\mathbf{J}\xi)(u + i\mathbf{J}u) = \xi u + i\mathbf{J}\xi u - i\mathbf{J}\xi u - i^2\mathbf{J}^2\xi u = 0.$$

In other words $\mathfrak{g}_+^{\mathbb{C}}$ acts trivially on $S_-^{\mathbb{C}}$ and similarly $\mathfrak{g}_-^{\mathbb{C}}$ acts trivially on $S_+^{\mathbb{C}}$. Also the action of $\mathfrak{g}_+^{\mathbb{C}}$ on $S_+^{\mathbb{C}}$ is isomorphic to the action of $\mathfrak{g}$ on S and similarly for the other component.

In the case of interest to us we see that

$$sl(2,\mathbb{C}) \otimes \mathbb{C} = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$$

and that the space of Dirac spinors, the complexification of the space of Majorana spinors, decomposes as

$$S \otimes \mathbb{C} = S_+^{\mathbb{C}} \oplus S_-^{\mathbb{C}} = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}),$$

where $\frac{1}{2}$ denotes the standard two dimensional representation of $sl(2,\mathbb{C})$ and 0 denotes the trivial representation.

Any $\mathbf{J}-$ antilinear map of S (where $\mathbf{J}$ is now γ) extends to a complex linear map of

$$\mathbf{D} := S \otimes \mathbb{C}$$

which switches the two components. In particular this applies to the operator $\star$. So we see that the group G acts irreducibly on $\mathbf{D}$ as claimed above.

Let us now consider the action of the real Lie algebra $sl(2,\mathbb{C})$ on Minkowski space, identified, as usual, with the space of self adjoint two by two matrices. The action is given by

$$P \mapsto \xi P + P\xi^\dagger.$$

Since every complex square matrix can be written as $P + iQ$ where P and Q are self adjoint, we see that the complexification of Minkowski space is just $gl(2,\mathbb{C})$, the space of all complex two by two matrices. Furthermore, recalling that the complex structure on $sl(2,\mathbb{C})$ is exactly multiplication by the scalar matrix, iI, we see that

$$(\mathbf{J}\xi)P = i\xi P = \xi(iP)$$

as two by two matrices and hence

$$(\xi + i\mathbf{J}\xi)P = 0.$$

Similarly

$$P(\xi^\dagger - i\mathbf{J}\xi^\dagger) = 0.$$

Thus $M^{\mathbb{C}}$ is irreducible under $sl(2,\mathbb{C})\otimes\mathbb{C}$ and is the representation $(\frac{1}{2},\frac{1}{2})$, the tensor product of the basic representation of each factor. Recall that $\mathbf{D} = S\otimes\mathbb{C}$ is the complexification of the space of Majorana spinors. We extend $\mu(P)$ by complex linearity to $\mathbf{D}$ and define

$$\gamma(P) = i\mu(P)$$

where i is now the good old fashioned complex number and so commutes with $\mu(P)$. Hence

$$\gamma(P)^2 = ||P||^2 I.$$

These are the defining relations for the Dirac "matrices". But notice that the Clifford algebra $C(1,3)$ is isomorphic to the algebra $H(2)$ of all two by two matrices over the quaternions. Hence its minimal module must have dimension eight over the real numbers. Thus the Dirac matrices have no realization as four by four real matrices. This is in contrast to the algebra $C(3,1)$ which we studied above in conjunction with the Majorana spinors. The Dirac equation is as before, namely

$$-i\gamma(\partial)\psi = \mu(\partial)\psi = m\psi.$$

But now ψ is a $\mathbf{D}$ valued function and $\mathbf{D}$ is a complex vector space so we can seek plane wave solutions of the form

$$\psi(x) = u(P)e^{iP\cdot x}.$$

Then we must have

$$\gamma(P)u(P) = mu(P)$$

which implies

$$||P||^2 = m^2$$

as before.

Thus if ψ is a general solution of the Dirac equation, its Fourier transform must be supported on the two sheeted hyperboloid $||P||^2 = m^2$. It is a fact that the space of ψ concentrated on the forward (or backward) sheet provides an irreducible unitary representation of the Poincaré group.

7.20.8 Sesquilinear covariants for Dirac spinors.

For each of the bilinear covariants defined on the space of Majorana spinors S we have a choice: we can extend it as a bilinear or as a sesquilinear form on $D\otimes D$. For example, let us extend $\mathfrak{j}$ so as to be sesquilinear. Then

$$\begin{aligned}\mathfrak{j}(u+iv) &= (u+iv)\otimes(u^\dagger - iv^\dagger)\\ &= u\otimes u^\dagger + v\otimes v^\dagger + i[v\otimes u^\dagger - u\otimes v^\dagger],\end{aligned}$$

where u and v are elements of S. The original group G acts as real linear transformations on $\mathbf{D} = S^{\mathbb{C}}$ and hence the relations

$$\mathrm{j}(Aw) = A\mathrm{j}(w)A^{\dagger}, \qquad \mathrm{j}(\star w) = \mathrm{j}(w)^{\mathrm{a}}$$

continue to hold for $w \in \mathbf{D}$ and $A \in Sl(2, C)$. Also $\mu(\partial)$ is a real operator, so if ψ is a complex (i.e. $\mathbf{D}$ valued) solution of the Dirac equation we continue to have

$$\operatorname{div}\ \mathrm{j}(\psi) = 0.$$

Notice that

$$\begin{aligned}
\operatorname{tr}\ \mathrm{j}(u + iv) &= ||u||^2 + ||v||^2 + 2i\mathrm{Im}\ (u, v) \\
&\geq ||u||^2 + ||v||^2 - 2||u||||v|| \\
&\geq 0.
\end{aligned}$$

Similarly the real symplectic form s extends to $\mathbf{D}$ as a $\mathbf{C}$ valued anti Hermitian form:

$$s(v, u) = -\overline{s(u, v)}.$$

So we can define a G invariant Hermitian form by

$$\langle u, v\rangle := is(u, v). \tag{50}$$

Since the complexification of any (real two dimensional) Lagrangian subspace of S will be a null space for $\langle\ ,\ \rangle$ we see that $\langle\ ,\ \rangle$ has signature $(2, 2)$. In fact we have the decomposition

$$\mathbf{D} = \mathbf{D}_+ \oplus \mathbf{D}_-$$

into two complex inequivalent irreducible representations of $sl(2, \mathbb{C})$ according to the $\pm i$ eigenvectors of $\mathbf{J}$. The restriction of $\langle\ ,\ \rangle$ to each component must be trivial since $\mathbb{C}^2$ admits no $sl(2, \mathbb{C})$ invariant Hermitian form. We can see this directly since

$$s(\mathbf{J}u, v) = s(u, \mathbf{J}v)$$

and

$$\mathbf{J}^2 = -I$$

imply that

$$\begin{aligned}
s(u + i\mathbf{J}u, v + i\mathbf{J}v) &= s(u, v) + s(\mathbf{J}u, \mathbf{J}v) + i[s(\mathbf{J}u, v) - s(u, \mathbf{J}v)] \\
&= 0.
\end{aligned}$$

Notice that

$$\begin{aligned}
\langle\gamma(P)u, v\rangle &= -s(?(P)u, v) \\
&= s(u, ?(P)v) \\
&= \langle u, \gamma(P)v\rangle.
\end{aligned}$$

In other words the operators $\gamma(P)$ are self adjointrelative to the Hermitian form $\langle\ ,\ \rangle$. It follows from equation (47) that

$$P \cdot \mathfrak{j}(w) = \langle \gamma(P)w, w\rangle. \tag{51}$$

The Hermitian form $\langle\ ,\rangle$ determines an antilinear map $\mathbf{D} \to \mathbf{D}^*$. The image of a spinor w is called the spinor adjoint to w and is denoted in the physics literature by putting a bar over w. Thus

$$\overline{w}(z) = \langle z, w\rangle.$$

8 Special representations of $sl(m/n)$.

8.1 The definition of the Lie superalgebras $sl(m/n)$.

We begin by recalling the definition of these superalgebras. For general facts about Lie superalgebras we refer to the book [Sch79] or the articles [CNS75] or [Kac77].

Let

$$V = V_0 \oplus V_1$$

be a supervector space with

$$\dim V_0 = m, \quad \text{and} \quad \dim V_1 = n.$$

The Lie superalgebra $sl(V_0/V_1)$ is the (commutator) Lie superalgebra of the superalgebra of all endomorphisms with supertrace zero. A typical such endomorphism has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \operatorname{tr} A = \operatorname{tr} D.$$

Here

$$A \in \operatorname{Hom}(V_0, V_0), \quad B \in \operatorname{Hom}(V_1, V_0), \quad C \in \operatorname{Hom}(V_0, V_1), \\ D \in \operatorname{Hom}(V_1, V_1).$$

Recall that those endomorphisms which preserve the grading (those with $B = C = 0$) are "even", i.e. belong to $sl(V_0/V_1)_0$ and those that reverse the grading (those with $A = D = 0$) are "odd", i.e. belong to $sl(V_0/V_1)_1$. We are assuming that the vector spaces V_0 and V_1 are finite dimensional. The structure of the Lie algebra clearly depends only on the dimensions of these spaces and hence the notation $sl(m/n)$.

Since our spaces are finite dimensional, we may identify $\operatorname{Hom}(V_1, V_0)$ with $V_0 \otimes V_1^*$. Under this identification, if $v \in V_0$ and $\xi \in V_1^*$ then $v \otimes \xi$ is identified with the rank one linear transformation given by

$$(v \otimes \xi)w = \langle \xi, w\rangle v$$

where $\langle \xi, w\rangle$ denotes the value of the linear function ξ on the vector w. These rank one linear transformations span $\mathrm{Hom}(V_1, V_0)$. Similar identifications will be made for each of the other three spaces corresponding to the entries of our block matrix. For example, we compute the (super)commutator

$$\left[\begin{pmatrix} 0 & v\otimes\xi \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x\otimes\mu & 0\end{pmatrix}\right] = \begin{pmatrix} \langle \xi, x\rangle v\otimes\mu & 0 \\ 0 & \langle \mu, v\rangle x\otimes\xi\end{pmatrix}.$$

Notice that the trace of the upper left block and the lower right block are both equal to $\langle \xi, x\rangle\cdot\langle \mu, v\rangle$. This proves that $sl(V_0, V_1)$ is indeed a Lie super subalgebra of the Lie superalgebra of $\mathrm{End}(V)$.

To save space we will write the above bracket relations (and similar ones) as follows: We write

$$sl(V_0/V_1)_0 = (V_0 \otimes V_0^*) \oplus (V_1 \otimes V_1^*)$$

and

$$sl(V_0/V_1)_1 = (V_0 \otimes V_1^*) \oplus (V_1 \otimes V_0^*).$$

Then we would write the preceding bracket relation as

$$[v\otimes\xi, x\otimes\mu] = \langle \xi, x\rangle v\otimes\mu \ \oplus \ \langle \mu, v\rangle x\otimes\xi.$$

8.2 The representation of $sl(V_0/V_1)$ on the super exterior algebra of V.

By definition, the super exterior algebra $\bigwedge(V)$ of a superspace V is

$$\bigwedge(V) := \wedge(V_0)\otimes S(V_1)$$

where $S(V_1)$ denotes the symmetric algebra of V_1 so

$$S(V_1) = \bigoplus_{k=0}^{\infty} S^k(V_1)$$

and $S^k(V_1)$ consists of homogeneous polynomials of degree k on V_1^*. The multiplication in $S(V_1)$ is the ordinary multiplication of polynomials so the elements of $S^k(V_1)$ are all declared to have even grading even if k is odd.

The Lie superalgebra $sl(V_0, V_1)$ has a natural representation on $\bigwedge(V)$. Perhaps the best way to realize this representation is by imbedding $sl(V_0, V_1)$ in the orthosymplectic algebra as the centralizer of a one dimensional subalgebra. This "Howe pair" point of view is explained by Howe in his original paper [H77]. In [NS82] we used this description in conjunction with the method of dimensional reduction. But here is a direct description:

Each $x \in V_1$ defines a multiplication operator on $S(V)$:

$$m_x : S^k(V_1) \to S^{k+1}(V_1)$$

given by

$$(m_x f)(\eta) ;= \langle \eta, x\rangle f(\eta), \quad \forall\, \eta \in V_1^*. \tag{52}$$

Each $\xi \in V_1^*$ defines a derivation D_ξ of $S(V_1)$ so

$$D_\xi(fg) = (D_\xi f)g + fD_\xi g$$

determined by

$$D_\xi 1 = 0 \quad \text{ and } \quad D_\xi x = \langle \xi, x\rangle \quad \forall\, x \in V_1 = S^1(V_1). \tag{53}$$

The standard Fock commutation relations hold, i.e.

$$D_\xi m_x - m_x D_\xi = \langle \xi, x\rangle \mathrm{id}. \tag{54}$$

Similarly, each $v \in V_0$ determines the operator of exterior multiplication by v which we currently denote by e_v and each $\mu \in V_0^*$ defines the operator on $\wedge(V_0)$ of interior multiplication by μ which we will denote by i_μ. So i_μ is the (odd) derivation of $\wedge(V_0)$:

$$i_\mu : \wedge^k(V_0) \to \wedge^{k-1}(V_0)$$

determined by

$$i_\mu(\omega_1 \wedge \omega_2) = i_\mu(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge i_\mu(\omega_2)$$

on homogeneous elements,

$$i_\mu v = \langle \mu, v\rangle \quad \forall\, v \in V_0 = \wedge^1(V_0),$$

and

$$i_\mu 1 = 0.$$

We have the supercommutation relations

$$\begin{aligned} [e_{v_1}, e_{v_2}] &= 0, \\ [i_{\mu_1}, i_{\mu_2}] &= 0, \\ [e_v, i_\mu] &= \langle \mu, v\rangle \mathrm{id}. \end{aligned}$$

In short, m and D are Bose-Einstein creation and annihilation operators while e and i are Fermi-Dirac creation and annihilation operators.

If $x \in V_1$ and $\xi \in V_1^*$ then $m_x \circ D_\xi$ is again a derivation of $S(V_1)$ since a derivation followed by a multiplication is again a derivation. In fact, it is the derivation determined by the map

$$y \mapsto \langle \xi, y\rangle x$$

on V_1 and this is just the linear transformation $x \otimes \xi$. Similarly, $e_v \circ i_\mu$ is the derivation of $\wedge(V_0)$ determined by the linear transformation $v \otimes \mu$ on V_0.

If $v \in V_0$ and $\xi \in V_1^*$ then $e_v \circ D_\xi := (e_v \otimes 1) \circ (1 \otimes D_\xi)$ is an odd derivation of $\bigwedge(V)$:

$$(e_v \circ D_\xi)(\sigma \otimes f) = v \wedge \sigma \otimes D_\xi f$$

so that

$$\begin{aligned}
e_v \circ D_\xi\left((\sigma \otimes f)(\omega \otimes g)\right) &= e_v \circ D_\xi(\sigma \wedge \omega \otimes fg) \\
&= v \wedge \sigma \wedge \omega \otimes D_\xi(fg) \\
&= v \wedge \sigma \wedge \omega \otimes ((D_\xi f)g + fD_\xi g) \\
&= v \wedge \sigma \wedge \omega \otimes (D_\xi f)g + (-1)^{|\sigma|}\sigma \wedge v \wedge \omega \otimes fD_\xi g \\
&= (e_v \circ D_\xi(\sigma \otimes f))(\omega \otimes g) \\
&\qquad +(-1)^{|\sigma \otimes f|}(\sigma \otimes f)e_v \circ D_\xi(\omega \otimes g).
\end{aligned}$$

By definition

$$e_v \circ D_\xi : \wedge^p(V_0) \otimes S^k(V_1) \to \wedge^{p+1}(V_0) \otimes S^{k-1}(V_1). \tag{55}$$

Similarly we have the odd derivation $m_x \circ i_\mu$ on $\bigwedge(V)$ and

$$m_x \circ i_\mu : \ \wedge^p(V_0) \otimes S^k(V_1) \to \wedge^{p-1}(V_0) \otimes S^{k+1}(V_1). \tag{56}$$

Also we have the even derivations $m_x \circ D_\xi$ and $e_v \circ i_\mu$ which preserve all bidegrees. We have

$$\begin{aligned}
[e_{v_1} \circ D_{\xi_1}, e_{v_2} \circ D_{\xi_2}] &= e_{v_1} \circ D_{\xi_1} \circ e_{v_2} \circ D_{\xi_2} + e_{v_2} \circ D_{\xi_2} \circ e_{v_1} \circ D_{\xi_1} \\
&= (e_{v_1}e_{v_2} + e_{v_2}e_{v_1}) \otimes D_{\xi_1}D_{\xi_2} \quad \text{since} \quad D_{\xi_2}D_{\xi_1} = D_{\xi_1}D_{\xi_2} \\
&= 0
\end{aligned}$$

and similarly

$$[m_{x_1} \circ i_{\mu_1}, m_{x_2} \circ i_{\mu_2}] = 0$$

while

$$\begin{aligned}
[e_v \circ D_\xi, i_\mu \circ m_x] &= e_v \circ i_\mu \otimes D_\xi \circ m_x + i_\mu \circ e_v \otimes m_x \circ D_\xi \\
&= \langle \xi, x\rangle e_v \circ i_\mu \otimes 1 + e_v \circ i_\mu \otimes m_x D_\xi - e_v \circ i_\mu \otimes m_x \circ D_\xi + \langle \mu, v\rangle 1 \otimes m_x D_\xi \\
&= \langle \xi, x\rangle e_v \circ i_\mu \otimes 1 + \langle \mu, v\rangle 1 \otimes m_x D_\xi.
\end{aligned}$$

This shows that $sl(V_0/V_1)$ acts as derivations of $\bigwedge(V)$ where

$$v \otimes \mu \mapsto e_v \circ i_\mu \tag{57}$$

$$x \otimes \xi \mapsto m_x \circ D_\xi \tag{58}$$

$$v \otimes \xi \mapsto e_v \circ D_\xi \tag{59}$$

$$x \otimes \mu \mapsto m_x \circ i_\mu. \tag{60}$$

Notice that for each integer k the finite dimensional subspace of $\bigwedge(V)$ given by

$$\wedge^0(V_0) \otimes S^k(V_1) \oplus \wedge^1(V_0) \otimes S^{k-1}(V_1) \oplus \cdots \oplus \wedge^n(V_0) \otimes S^{k-n}(V_1)$$

is invariant. In the above expression (and in contrast to our notation in the next section) the space $S^\ell(V_1)$ is taken to be 0 if $\ell < 0$. It is clear that each such subspace is irreducible under $sl(V_0, V_1)$. We have thus associated an irreducible representation of $sl(V_0, V_1)$ to each non-negative integer k.

If we replace the spaces of homogenous polynomials $S^k(V_1)$ by the spaces F^b of all smooth functions homogenous of degree b and defined on some fixed open cone in V_1^* with vertex at the origin (vertex not included), then we still have the multiplication operator $m_x : F^b \to F^{b+1}$ given by (52), the derivation operator $D_\xi : F^b \to F^{b-1}$ given by (53) and the commutation relations (54) continue to hold. If dim $V_1 > 1$ and the cone is non-empty these spaces are infinite dimensional. But if V_1 is one dimensional something special happens.

8.3 Special representations of $sl(m/1)$.

We suppose that $V_1 = \mathbb{C}$. We now let $S^b = S^b(V_1)$ denote the one dimensional space with basis element p_b. Now b can be any complex number. For $x \in V_1$ define

$$m_x : S^b \to S^{b+1}$$

by

$$m_x p_b = x p_{b+1}. \tag{61}$$

For $\xi \in V_1^*$ define

$$D_\xi : S^b \to S^{b-1}$$

by

$$D_\xi p_b = b\xi p_{b-1}. \tag{62}$$

The commutation relation (54) continues to hold (where $\langle \xi, x\rangle$ is simply the product ξx). So the ingredients that we needed to construct the representations of $sl(m/n)$ in the preceding section are all present. In this way, [NS80], we have associated a finite dimensional representation of $sl(m/1)$ on

$$\wedge^0(V_0) \otimes S^b \oplus \wedge^1(V_0) \otimes S^{b-1} \oplus \cdots \oplus \wedge^m(V_0)_\otimes S^{b-m} \tag{63}$$

for each complex number b and these representations are irreducible unless b is a non-negative integer with $0 < b < m$. Since all the spaces S^a are one dimensional, all of these representation are on a space of dimension 2^m, the same dimension as that of the exterior algebra.

Each of the summands in (63) is invariant and irreducible under $sl(m/1)_0$. It will be useful for future computations to record the action of a diagonal matrix on each of these components: The action of the diagonal matrix

$$\left(\begin{array}{cccc|c} u_1 & 0 & \cdots & 0 & \\ 0 & u_2 & \cdots & 0 & \\ \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & \cdots & u_m & 0 \\ \hline 0 & 0 & \cdots & 0 & U \end{array}\right), \quad U = u_1 + u_2 + \cdots + u_m$$

is as follows:

On the one dimensional space $\wedge^0(V_0) \otimes S^b$ it is multiplication by

$$bU.$$

If $v_1, \dots v_m$ is the basis in terms of which the above matrix is diagonal, the action on $\wedge^1(V_0) \otimes S^{b-1}$ is diagonal with basis $v_1 \otimes p_{b-1}, \dots, v_m \otimes p_{b-1}$ with eigenvalues

$$u_1 + (b-1)U, \dots, u_m + (b-1)U,$$

and in general, the action on $\wedge^q(V_0) \otimes S^{b-q}$ is diagonal with basis

$$(v_{i_1} \wedge \cdots \wedge v_{i_q}) \otimes p_{b-q}, \quad i_1 < \cdots < i_q \tag{64}$$

and corresponding eigenvalues

$$u_{i_1} + \cdots + u_{i_q} + (b-q)U. \tag{65}$$

In tabulating computations we will usually use some shorthand for the eigenvectors (64). For example we do not need to include the $\otimes p_{b-q}$ since this is determined by the representation. We will also shorten the notation for the wedge product and simply write

$$i_1 i_2 \dots i_q$$

for the eigenvector (64).

9 $sl(2/1)$ and the electroweak isospins and hypercharges.

In [NS80] we showed how to derive the various values of the weak isospin and hypercharge by choosing the appropriate elements of $sl(2/1)$ and then choosing various parameters for b in (63). In particular, we predicted the existence of the right handed neutrino which occurs with weak isospin and hypercharge zero, and does not participate to first order in the weak interaction. With the recent discovery that the neutrino has positive mass [Fu98] this expectation has been justified.

The choice of the weak isospin and hyperchange elements of $sl(2/1)$ are (up to the pervasive factor of i):

$$I_3 = \left(\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad Y = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array} \right). \tag{66}$$

We will tabulate below the weak isospin and hypercharge values corresponding to the leptons ($b = 0$) and the quarks ($b = \frac{2}{3}$) and their anti-particles ($b = 1$ corresponding to the anti-leptons and $b = \frac{1}{3}$ corresponding to the anti-quarks).

In the full geometrical theory, we would take the tensor product of the superbundle associated to these representations $su(2/1)$ with the bundle of Dirac spinors which has the $\mathbb{Z}_2$ gradation according to chirality. From the tables below it will follow that all the particles have the same total degree (in the tensor product) which is opposite to the total degrees of the anti-particles.

9.1 $b = 0$ - the leptons.

We get the lepton assignments by choosing the parameter $b = 0$ in (63). For the reader's convenience we have also tabulated the electric charge

$$Q = I_3 + \frac{1}{2}Y.$$

$$\begin{array}{c|c|cc|c} \text{leptons } (b=0) & \wedge^0(V_0) & \wedge^1(V_0) & & \wedge^2(V_0) \\ \text{basis elements} & \emptyset & 1 & 2 & 12 \\ \hline & & & & \\ I_3 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ Y & 0 & -1 & -1 & -2 \\ Q & 0 & 0 & -1 & -1 \\ \text{particle} & \nu_R & \nu_L & e_L & e_R \end{array} \tag{67}$$

Notice that the gradation of the superspace on which the representation takes place corresponds to chirality - the first and third columns which correspond to $\wedge^+(V_0) = \wedge^0(V_0) \otimes S^0 \oplus \wedge^2(V_0) \otimes S^{-2}$ corresponds to right handed particles while $\wedge^-(V_0) = \wedge^1(V_0) \otimes S^{-1}$ corresponds to left handed particles. Notice also that the entire even subalgebra $sl(2/1)_0$ acts trivially on $\wedge^0(V_0) \otimes S^0$ corresponding to the right handed neutrino.

9.2 $b = \frac{2}{3}$ - the quarks.

The choice $b = \frac{2}{3}$ gives the electroweak isospin and hypercharge assignments for quarks:

$$\begin{array}{c|c|cc|c} \text{quarks } (b=\frac{2}{3}) & \wedge^0(V_0) & \wedge^1(V_0) & & \wedge^2(V_0) \\ \text{basis elements} & \emptyset & 1 & 2 & 12 \\ \hline & & & & \\ I_3 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ & & & & \\ Y & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ & & & & \\ Q & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \text{particle} & u_R & u_L & d_L & d_R \end{array} \tag{68}$$

Once again observe the relation between the gradation and chirality

9.3 $b = 1$ - the anti-leptons.

The choice $b = 1$ gives the anti-lepton assignment:

anti-leptons ($b = 0$) basis elements	$\wedge^0(V_))$ $\emptyset$	$\wedge^1(V_0)$ 1	2	$\wedge^2(V_0)$ 12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	2	1	1	0
Q	1	1	0	0
particle	$(\overline{e_R})_L$	$(\overline{e_L})_R$	$(\overline{\nu}_L)_R$	$(\overline{\nu_R})_L$

(69)

Again there is a correspondence between gradation and chirality (the opposite from that of the leptons). Notice again that the entire even subalgebra acts trivially on $\wedge^2$.

9.4 $b = \frac{1}{3}$ - the anti-quarks.

Finally the choice $b = \frac{1}{3}$ gives the anti-quark assignment:

anti-quarks ($b = \frac{1}{3}$) basis elements	$\wedge^0(V_0)$ $\emptyset$	$\wedge^1(V_0)$ 1	2	$\wedge^2(V_0)$ 12
I_3	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
Y	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$
Q	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
particle	$(\overline{d_R})_L$	$(\overline{d_L})_R$	$(\overline{u_L})_R$	$(\overline{u_R})_L$

(70)

10 Using $sl(m/1)$ for $m = 3, 5,$ and $5 + n$.

10.1 $m = 3$ - unifying quarks and leptons.

We showed in [NS80] that if we take

$$I_3 = \left(\begin{array}{ccc|c} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{ccc|c} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \end{array}\right) \tag{71}$$

then we get the correct isospins and hypercharges if we combine the anti-leptons and quarks into the single eight dimensional representation of $sl(3/1)$ with $b = \frac{2}{3}$ and if we combine the leptons and anti-quarks in the single eight dimensional representation with $b = \frac{1}{2}$. We refer to [NS80] for details.

10.2 $m = 5$ - including color.

We showed in [NS80] that if we choose

$$I_3 = \left(\begin{array}{ccccc|c} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{72}$$

then the single 32 dimensional representation given by

$$b = 2$$

gives the correct isospin and hypercharge assignments to the right and left handed up and down quarks in three colors and the right and left handed electrons and neutrino (so 16 in all) and their antiparticles (yielding 32). Again the chirality and the gradation match up: All the elements of $\wedge^+$ have eigenvalues corresponding to left handed particles and all the elements of $\wedge^-$ have eigenvalues corresponding to right handed particles We refer to the Appendix in [NS80] for the list of all 32 eigenvalues.

There is something special about the value $b = \frac{m-1}{2}$ (for example the value $b = 2$ in our current case of $m = 5$. Indeed, as pointed out in the note added in proof in [NS80], the space $\wedge^m(V_0) \otimes S^{-1}$ is acted on trivially by the even part of $sl(m/1)$, i.e. has a canonical trivialization. This means that the natural multiplication

$$\left(\wedge^k \otimes S^{b-k}\right) \otimes \left(\wedge^{m-k} \otimes S^{b-m+k}\right) \to \wedge^m \otimes S^{2b-m}$$

can be thought of as invariant bilinear form on the space of the representation corresponding to $b = \frac{m-1}{2}$. Notice that the particles and the anti-particles of any given species occur in the components $\wedge^k$ and $\wedge^{5-k}$in the representation. If m is odd then either k or $m-k$ is even, so the above bilinear form is symmetric.

In this set up all the particles and anti-particles have the same total tensor degree. What the meaning of the opposite total degree is in this formulation (whether "ghosts" or some other meaning) was left open to speculation.

10.3 $m = 5 + n$ - accomodating 2^n generations.

It was shown in [NS80] that generational symmetry can be achieved if we enlarge the superalgebra $sl(5/1)$ to $sl(5 + n/1)$. This would be a theory with 2^n or 2^{n+1} generations . At the time, this seemed inappropriate since the number of generations was observed to be at least three, and was thought to be less than four based on arguments from the Z width. In [NS91] it was argued that if the neutrinos had positive mass, especially if the neutrinos in the higher generations were heavy, then a fourth generation is not excluded.

The idea is that the weak isospin $su(2)$ and the color $su(3)$ are regarded as commuting subalgebras of the even part of $sl(m/1)$ where $m = 5 + n$ while the generational behavior is produced by an $sl(n/1)$ sub Lie superalgebra.

The I_3 assignment for $sl(5 + n/1)$ is the diagonal matrix

$$\mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0|0) \qquad (n+4) \text{ zeros in all}$$

while the hypercharge assignment is

$$Y = \mathrm{diag}\left(\frac{-n}{4+n}, \frac{-n}{4+n}, \left(\frac{4}{4+n}\right)_{n \text{ times}}, \left(\frac{4-2n}{3(4+n)}\right)_{3 \text{ times}} \middle| \frac{4}{4+n}\right). \quad (73)$$

and the preferred representation is given by $b = \frac{5+n-1}{2}$.

We will discuss the model with four generations in the next two sections.

11 $sl(7/1)$ - unifying color and four generations.

In this section we show how the value $b = 3$ can accommodate four generations of particles with the correct isospin and hypercharge values provided that we reverse the chirality assignments in two out of the four generations. Our fundamental superbundle will be the tensor product of the spin bundle with the bundle associated to this 128 dimensional representation. So this means that all particles will correspond to the same total degree as indicated above. The tables here follow the tables (42)-(45) in [NS91]. We need a name (or at least a letter) for the particles in the fourth generation, and we have tentatively chosen σ for the analogue of the electron and x and y for the analogue of the u and d quark. Also, we have made the choice that $\wedge^0 \otimes S^3$ has left handed chirality. This then determines that all the spaces with $\wedge^k \otimes S^{3-k}$ are

left handed when k is even and are right handed when k is odd. In [NS80] the choice of $m = 7$ was made in order to accommodate the possibility of ghost fields. An assignment of particles without ghosts and which fits better with the theory of Clifford superconnections will be presented in the next section.

As usual, the element I_3 is given by

$$\left(\begin{array}{ccccccc|c}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

In accordance with (73) the hypercharge is given by

$$\left(\begin{array}{ccccccc|c} -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{array}\right). \tag{74}$$

Then the eigenvalues on $\wedge^k(V_0) \otimes S^{b-k}$ (and particle assignments) are given as follows:

$\wedge^0 \otimes S^3$	$\emptyset$
Y	2
I_3	0
particle	$(\overline{e_R})_L$

$\wedge^1 \otimes S^2$	1	2	3	4	5	6	7
Y	1	1	2	2	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0
particle	$(\overline{e_L})_R$	$(\overline{\nu_{eL}})_R$	$(\overline{\mu_L})_R$	$(\overline{\tau_L})_R$	u_R	u_R	u_R

Notice the opposite chirality assignments (as compared to the electron) to the μ and τ. This is somewhat arbitrary at the moment. We could make this opposite assignment to the third and fourth generation as opposed to the second and third.

In the next tables we will conjoin the color entries, so write ${}_{2;5,6,7}$ instead of having three columns f_{25}, f_{26}, f_{27}.

$\wedge^2 \otimes S^1$	12	13	14	1:5,6,7	23	24	2;5,6,7	34	3;5,6,7	4;5,6,7	56,57,67
Y	0	1	1	$\frac{1}{3}$	1	1	$\frac{1}{3}$	2	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{2}{3}$
I_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0
particle	$(\overline{\nu_{eR}})_L$	$(\overline{\mu_R})_L$	$(\overline{\tau_R})_L$	u_L	$(\overline{\nu_{\mu R}})_L$	$(\overline{\nu_{\tau R}})_L$	d_L	$(\overline{\sigma_R})_L$	c_L	t_L	$(\overline{d_R})_L$

All 35 particle assignments in the next table of eigenvalues for $\wedge^3 \otimes S^0$ are right handed. To save space we no longer indicate this in the table.

$\wedge^3 \otimes S^0$	123	124	12;5,6,7	134	13;5,6,7	14;5,6,7	1;56,67,67	234	23:5,6,7	24;5,6,7	2;56,57,67
Y	0	0	$-\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
I_3	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
part.	$\overline{\nu_{\tau L}}$	$\nu_{\sigma R}$	y_R	$\overline{\sigma_L}$	c_R	t_R	$\overline{y_L}$	$\overline{\nu_{\sigma_L}}$	s_R	b_R	$\overline{x_L}$

$\wedge^3 \otimes S^0$	34:5,6,7	3:56,57,67	4,56,57,67	567
Y	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0
I_3	0	0	0	0
particle	x_R	$\overline{s_L}$	$\overline{b_L}$	$\overline{\nu_{\mu L}}$

The particles in the remaining four components of our 128 dimensional representation will be the anti-particles of the ones we have already seen, and paired with them under the bilinear form. So the 35 dimensional component $\wedge^4 \otimes S^{-1}$ gives following table of left handed particles:

$\wedge^4 \otimes S^{-1}$	1234	123;5,6,7	124:5,6,7	12;56,57,67	134;5,6,7	13:56,57,67	14;56,57,67	1567
Y	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1
I_3	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
particle	$\nu_{\mu L}$	b_L	s_L	$\overline{x_R}$	x_L	$\overline{b_R}$	$\overline{s_R}$	$\nu_{\sigma L}$

$\wedge^4 \otimes S^{-1}$	234;5,6,7	23;56.57,57	24;56,57,67	2567	34;56,57,67	3567	4567
Y	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	$\frac{2}{3}$	0	0
I_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
particle	y_L	$\overline{t_R}$	$\overline{c_R}$	σ_L	$\overline{y_R}$	$\overline{\nu_{\sigma R}}$	$\nu_{\tau L}$

The 21 dimensional component $\wedge^5 \otimes S^{-2}$ gives the following table of right handed particles:

$\wedge^5 \otimes S^{-2}$	1234;5,6,7	123:56,57,67	124;56,57,67	12567	134:56,57,67	13567	14567
Y	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	-2	$-\frac{1}{3}$	-1	-1
I_3	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
particle	d_R	$\overline{t_L}$	$\overline{c_L}$	σ_R	$\overline{d_L}$	$\nu_{\tau R}$	$\nu_{\mu R}$

$\wedge^5 \otimes S^{-2}$	234;56,57,67	23567	24567	34567
Y	$-\frac{1}{3}$	-1	-1	0
I_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
particle	$\overline{u_L}$	τ_R	μ_R	ν_{eR}

The 7 dimensional component $\wedge^6 \otimes S^{-3}$ gives the following table of left handed particles:

$\wedge^6 \otimes S^{-3}$	1234;56,57,67	123567	124567	134567	234567
Y	$-\frac{4}{3}$	-2	-2	-1	-1
I_3	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
particle	$\overline{u_R}$	τ_L	μ_L	ν_{eL}	e_L

Finally there is the one dimensional $\wedge^7 \otimes S^{-4}$ giving the right handed particle

$\wedge^7 \otimes S^{-4}$	1234567
Y	-2
I_3	0
particle	e_R

12 $sl(6/1)$.

If ghosts are not required, we use $sl(6/1)$ to accommodate four generations:

For $sl(6/1)$ we have $b = \frac{5}{2}$

$$I_3 = \operatorname{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0 \,\big|\, 0)$$

and

$$Y = \operatorname{diag}(-\frac{1}{5}, -\frac{1}{5}, \frac{4}{5}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15} \,\Big|\, \frac{4}{5}).$$

We will assign both left and right handed spinors to each subrepresentation so that we get four families of particles with both even and odd total gradings:

$\wedge^0 \otimes S^{\frac{5}{2}}$	$\emptyset$
Y	2
I_3	0
particle (L)	$\overline{e_R}$
particle (R)	$\overline{\tau_L}$

$\wedge^1 \otimes S^{\frac{3}{2}}$	1	2	3	4, 5, 6
Y	1	1	2	$\frac{4}{3}$
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (R)	$\overline{e_L}$	$\overline{\nu_{eL}}$	$\overline{\mu_L}$	u_R
particle (L)	$\overline{\tau_R}$	$\overline{\nu_{\tau R}}$	$\overline{\sigma_R}$	c_L

$\wedge^2 \otimes S^{\frac{1}{2}}$	12	13	1; 4, 5, 6	23	2; 4, 5, 6	3; 4, 5, 6	45, 46, 56
Y	0	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{2}{3}$
I_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (L)	$\overline{\nu_{eR}}$	$\overline{\mu_R}$	u_L	$\overline{\nu_{\mu R}}$	d_L	t_L	$\overline{d_R}$
particle (R)	$\overline{\nu_{\tau L}}$	$\overline{\sigma_L}$	c_R	$\overline{\nu_{\sigma L}}$	s_R	x_R	$\overline{s_L}$

$\wedge^3 \otimes S^{-\frac{1}{2}}$	123	12;4,5,6	13;4,5,6	1;45,46,56	23:4,5,6	2;45,46,56	3,45,46,56	456
Y	0	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0
I_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
particle (R)	$\nu_{\sigma R}$	y_R	t_R	$\overline{y_L}$	b_R	$\overline{x_L}$	$\overline{b_L}$	$\overline{\nu_{\mu L}}$
particle (L)	$\nu_{\mu L}$	b_L	x_L	$\overline{b_R}$	y_L	$\overline{t_R}$	$\overline{y_R}$	$\overline{\nu_{\sigma R}}$

$\wedge^4 \otimes S^{-\frac{3}{2}}$	123;4,5,6	12;45,46,56	13;45,46,56	1456	23:45,46,56	2456	3456
Y	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	-1	$-\frac{1}{3}$	-1	0
I_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
particle (L)	s_L	$\overline{x_R}$	$\overline{s_R}$	$\nu_{\sigma L}$	$\overline{c_R}$	σ_L	$\nu_{\tau L}$
particle (R)	d_R	$\overline{t_L}$	$\overline{d_L}$	$\nu_{\mu R}$	$\overline{u_L}$	μ_R	ν_{eR}

$\wedge^5 \otimes S^{-\frac{5}{2}}$	123;45,46,56	12456	13456	23456
Y	$-\frac{4}{3}$	-2	-1	-1
I_3	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
particle (R)	$\overline{c_L}$	σ_R	$\nu_{\tau R}$	τ_R
particle (L)	$\overline{u_R}$	μ_L	ν_{eL}	e_L

$\wedge^6 \otimes S^{-\frac{7}{2}}$	123456
Y	-2
I_3	0
particle (L)	τ_L
particle (R)	e_R

Notice that the relation between these assignments and those of the preceding section are

$$\begin{aligned}\wedge^0_7 &= {}_L\wedge^0_6,\\ \wedge^1_7 &= {}_R\wedge^0_6 \oplus {}_R\wedge^1_6,\\ \wedge^2_7 &= {}_L\wedge^1_6 \oplus {}_L\wedge^2_6,\end{aligned}$$

etc.

13 Hermitian Lie algebras.

In this section we explain the notion of a Hermitian Lie algebra which was introduced in [SW78] and which we used above to determine the metric on the Higgs field.

13.1 The Lie superalgebra $su(2/1)$ and the Lie algebra $su(3)$.

We illustrate the notion by the relevant example. It is the special case of section 2A of [SW78] corresponding to the case $k = 0, \ell = 2, a = 0, b = 1$ of that section.

For

$$z = \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\overline{w}_1 & -\overline{w}_2 & 0 \end{pmatrix}$$

we let

$$H(z,w) = izw,$$

and this equals

$$i\begin{pmatrix} -z_1\overline{w}_1 & -z_1\overline{w}_2 & 0 \\ -z_2\overline{w}_1 & -z_2\overline{w}_2 & 0 \\ 0 & 0 & -w_1\overline{z}_1 - w_2\overline{z}_2 \end{pmatrix} = i\begin{pmatrix} -z\otimes w^\dagger & 0 \\ 0 & -\overline{\langle z,w\rangle} \end{pmatrix}.$$

The right hand side is an element of $gl(2,\mathbb{C}) \oplus gl(1,\mathbb{C})$.

If we are given a hermitian form on $\mathbb{C}^n$ we define the complex conjugation on $gl(n,\mathbb{C})$ to be

$$\xi \mapsto \xi^* := -\xi^\dagger$$

where $\xi^\dagger$ denotes the adjoint of ξ relative to the hermitian form. Then the "real subspace", i.e. the set of matricies fixed by this complex conjugation is $u(n)$.

On $gl(2,\mathbb{C}) \oplus gl(1,\mathbb{C})$ we put the standard complex structure on $gl(2,\mathbb{C})$ but the conjugate complex structure on $gl(1,\mathbb{C})$. This means that we can write

$$H(z,w) = -iz\otimes w^\dagger \oplus i\langle z,w\rangle 1.$$

Then

$$H(z,w)^* = -iw\otimes z^\dagger \oplus i\langle w,z\rangle 1 = H(w,z).$$

So $H(z,w)$ is a hermitian form with values in the complexification of $u(2)\oplus u(1)$ and satisfies

$$H(w,z) = H(z,w)^*. \tag{75}$$

Since commutator is a derivation of multiplication (of matrices) we have $[M, zw] = [M, z]w + z[M, w]$ so if we define the action of $\xi \in u(2) \oplus u(1)$ on the space of z's to be commutator we have

$$[\xi, H(z,w)] = H(\xi z, w) + H(z.\xi w), \qquad \xi \in \mathfrak{g}_0, \quad z, w \in V \tag{76}$$

where

$$\mathfrak{g}_0 = u(2) \oplus u(1)$$

and where

$$V \sim \mathbb{C}^2$$

denotes the set of all matrices of the form

$$\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}.$$

Explicitly,

$$\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & z \\ -z^\dagger & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & Az - Bz \\ -(Az - Bz)^\dagger & 0 \end{pmatrix}.$$

We can write this more simply as an action on $\mathbb{C}^2$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} z = Az - Bz, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

So

$$H(u,v)w = -i\langle w, v\rangle u + i\langle v, u\rangle w.$$

Therefore if we take the cyclic sum we get zero:

$$H(u,v)w + H(v,w)u + H(w,u)v = 0. \tag{77}$$

Now

$$2\,\mathrm{Im}\, H(z,w) = \frac{1}{i}[H(z,w) - H(z,w)^*] = \frac{1}{i}[H(z,w) + H(w,z)^\dagger]$$

$$= \begin{pmatrix} -z \otimes w^\dagger + w \otimes z^\dagger & 0 \\ 0 & -\langle z, w\rangle + \langle w, z\rangle \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z}_1 & -\overline{z}_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\overline{w}_1 & -\overline{w}_2 & 0 \end{pmatrix}\right].$$

Thus if we define $\mathfrak{g}_o := u(2) \oplus u(1)$ and $\mathfrak{g}_1 = V = \mathbb{C}^2$ then $2\,\mathrm{Im}\,H$ makes $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into the Lie algebra $u(3)$.

On the other hand,

$$2\mathrm{Re}H(z,w) = H(z,w) + H(w,z)^* = H(z,w) + H(w,z) = i(zw + wz)$$

is i times the anti-commutator of z and w. Since z and w are skew-adjoint their anti-commutator is self-adjoint, so multiplying by i gives a skew-adjoint matrix. So ReH makes $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into the super Lie algebra $u(2/1)$.

13.2 The general definition.

So the general definition of a **Hermitian Lie algebra** is as follows: We start with a real Lie algebra $\mathfrak{g}_0$ which is represented on a complex vector space $\mathfrak{g}_1$. We let $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \otimes \mathbb{C}$ which is a complex Lie algebra with a preferred complex conjugation $w \mapsto w^*$ so that $\mathfrak{g}_0$ consists of the real subspace, i.e. those w which are fixed under this complex conjugation. We assume that there is sesquilinear map

$$H : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0^{\mathbb{C}}$$

which satisfies (75), (76), and (77). For the convenience of the reader we collect these conditions here:

- (75): H is Hermitian - $H(w,z) = H(z,w)^*$.
- (76): H is equivariant - $[\xi, H(z,w)] = H(\xi z, w) + H(z.\xi w), \qquad \xi \in \mathfrak{g}_0, \quad z, w \in \mathfrak{g}_1$, and
- (77): Complex Jacobi - $H(u,v)w + H(v,w)u + H(w,u)v = 0$.

When this happens we make $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into an ordinary Lie algebra using the imaginary part of H as the Lie bracket of two elements of $\mathfrak{g}_1$, and we make $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ into a Lie superalgebra using the real part of H as the superbracket of two elements of $\mathfrak{g}_1$.

It is this relation between Lie algebras and Lie superalgebras that we use to fix the metric on the Higgs field regarded as sections of a bundle associated to $\mathfrak{g}_1$.

13.3 The unitary algebras.

Let $m = k + \ell$ be integers and let V_0 be an m-dimensional complex vector space endowed with a (pseudo) Hermitian form of signature (ℓ, k). For example we might take

$$V_0 = \mathbb{C}^{k,\ell}$$

be complex m space with the Hermitian form

$$\langle z, w \rangle = -\sum j = 1^k z_j \overline{w}_j + \sum_{j=k+1}^{m} z_j \overline{w}_j.$$

Let $c = a + b$ be integers and V_1 a c-dimensional vector space with a (pseudo) Hermitian form of signature (b, a). Put the direct sum Hermitian on $V = V_0 \oplus V_1$. Then

$$\mathfrak{g} = u(V),$$

the unitary algebra of V is an ordinary Lie algebra. Then we have the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_0$ is the subalgebra

$$\mathfrak{g}_0 = u(V_0) \oplus u(V_1)$$

and $\mathfrak{g}_1$ can be identified with the complex vector space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_0)$. (see [SW78] section 2). Then there is a structure of a Hermitian Lie algebra on $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ whose imaginary part gives $u(V_0 \oplus V_1)$.

The real part gives a class of Lie superalgebras which are called Hermitian superalgebras in [SS85]. They can be viewed as a real form of the complex Lie superalgebra $gl(V_0/V_1)$. If write the most general element of $gl(V_0/V_1) = \mathrm{End}(V)_0$ where $V = V_0 \oplus V_1$ in the block form as

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

then the condition to belong to our Hermitian superalgebra is that

$$A \in u(V_0) \quad \text{and} \qquad D \in u(V_1).$$

If we write the most general element of $\mathrm{End}(V)_1$ as

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

then the condition to belong to our superalgebra is

$$\langle Cv_0, v_1 \rangle_1 = i \langle v_0, Bv_1 \rangle_0 \qquad \forall\ v_0 \in V_0,\ v_1 \in V_1.$$

See [SS85] page 4.

13.4 $su(2, 2/1)$ and the superconformal superalgebra of Wess and Zumino.

The supersymmetry studied in this paper is purely internal and related to the chirality gradation as we have seen. So it is not of the "superspace" variety. Nevertheless we should point out that the superalgebra $su(2, 2/1)$ is nothing other than the superalgebra of Wess and Zumino [CNS75] and [GGRS83] where the odd part of the superalgebra is regarded as the "square root" of the conformal algebra of flat space time. We follow the presentation in [SW75] and [SS85].

Let $V_0 = \mathbb{C}^{2,2}$ be four dimenisonal complex space equipped with a Hermitian form of signature $(2,2)$. To fix the Ideas let us assume that the form is given by

$$\langle z, w\rangle = w^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} z = z_1\overline{w_3} + z_2\overline{w_4} + z_3\overline{w_1} + z_4\overline{w_2}$$

where I is the two by two identity matrix.

The condition that a four by four matrix A belongs to $u(V_0)$ is that

$$A\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A^\dagger.$$

If we break A up into blocks of two by two matrices we see that the condition is that A be of the form

$$\begin{pmatrix} X & P \\ Q & -X^\dagger \end{pmatrix}$$

where X is an arbitrary complex two by two matrix and where $P = -P^\dagger$ and $Q = -Q^\dagger$.

The fifteen dimensional algebra $su(2,2)$ is known to be isomorphic to the conformal algebra $o(2,4)$. Under the above description of the matrix A, the condition to belong to $su(2,2)$ is that $\mathrm{Imtr}\, X = 0$. We can regard matrices of the form

$$\begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}, \quad P = -P^\dagger$$

as consisting of translations, and we may denote the set of all such matrices as $\mathfrak{g}^2$. We can regard the matrices of the form

$$\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}, \quad Q = -Q^\dagger$$

as consisting of those conformal vector fields whose expression is purely quadratic at a specified choice of origin and denote the set of such elements as $\mathfrak{g}^{-2}$. The set of elements of $su(2,2/1)$ of the form

$$\left(\begin{array}{cc|c} X & 0 & 0 \\ 0 & -X^\dagger & 0 \\ \hline 0 & 0 & 2i\mathrm{Im\ tr}\ X \end{array}\right)$$

will be denoted by $\mathfrak{g}^0$. If we impose the additional condition that $\mathrm{tr}\, A = 0$ which is the same as $\mathrm{Im\ tr}\, X = 0$ we get an element of $su(2,2)$ which acts as a linear conformal vector field on space time, i.e. as an infinitesimal Lorentz tranformation plus a scale transformation. The purely imaginary scalar matrices act trivially on space time but non-trivially on the odd part of the superalgebra which can be identified with the space of Dirac spinors.

The full algebra $su(2,2/1)$ consists of matrices of the form

$$\left(\begin{array}{cc|c} X & P & u \\ Q & -X^\dagger & v \\ \hline iv^\dagger & iu^\dagger & 2i\mathrm{Im\,tr}\, J \end{array}\right), \quad P = -P^\dagger,\ Q = -Q^\dagger,\ \ u \in \mathbb{C}^2,\ \ v \in \mathbb{C}^2. \quad (78)$$

If $X \in sl(2,\mathbb{C})$ then

$$\left[\left(\begin{array}{cc|c} X & 0 & 0 \\ 0 & -X^\dagger & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 0 & v \\ \hline iv^\dagger & iu^\dagger & 0 \end{array}\right)\right] = \left(\begin{array}{cc|c} 0 & 0 & Xu \\ 0 & 0 & -X^\dagger v \\ -iv^\dagger X & iu^\dagger X^\dagger & 0 \end{array}\right).$$

We see that u transforms as $u \mapsto Xu$ and v transforms as $v \mapsto -X^\dagger v$.

So we have a $\mathbb{Z}$ gradation more refined than the $\mathbb{Z}_2$ gradation:

$$\mathfrak{g}_0 = \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2$$

and

$$\mathfrak{g}_1 = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$$

is identified with the right and left handed spinors. We refer to [SS85] for details.

14 Renormalization of the supergroup couplings and the Higgs mass.

For couplings given solely by the internal supergroup, i.e. by the quotient $su(2/1)/[su(2) \otimes u(1)]$, there is no known non-renormalization theorem. These couplings are θ_W and a, the coefficient of the quartic. In the sequel, we show that unitarity is preserved by appropriate BRST equations, so that we can apply the renormalization group (RG)equations to estimate the corrections. We follow a linearized treatment as an approximation [HLN96] .

In one case – the angle θ_W – we have the group value $(sin\theta_W)^2 = 0.25$ and may compare it to the experimentally observed value 0.229 ± 0.005. The supergroup prediction fits, but only very roughly. One therefore evaluates the energy level $q^2 = E_s$ at which the fit becomes precise, finding $E_s \sim 5$TeV. This may possibly be the level at which a larger symmetry structure breaks down, with SU(2/1) as the residual internal supersymmetry.

One can now invert the procedure, to estimate the renormalization effects for the Higgs potential quartic coefficient a. The supergroup value is assumed to hold at the energy $E_s = 5$TeV and one then evaluates the correction for a at $E \sim 100$GeV. This corrected value can then be used to reevaluate the predicted Higgs mass, i.e. obtain the value of that mass after the inclusion of renormalization effects.

The coefficients of the renormalization group equation depend only on the field contents of the theory, which is the same as in SU(2)×U(1). One can therefore apply the Standard Model calculation. For the gauge couplings, the renormalization group equations are given by [HLN96];

$$\frac{1}{[g_i(M)]^2} - \frac{1}{[g_i(M_0)]^2} + 2t_i \ln \frac{M}{M_0}\ , \qquad i = 1, 2, 3, \tag{79}$$

where

$$\begin{aligned}
t_1 &= \frac{1}{12\pi^2}(-\frac{5}{3}N_g - \frac{1}{8}),\\
t_2 &= \frac{1}{12\pi^2}(-N_g - \frac{1}{8} + \frac{11}{2}),\\
t_3 &= \frac{1}{16\pi^2}(-\frac{4}{3}N_g + 11),
\end{aligned}$$

N_g is the number of generations, and g_1, g_2, g_3 denote the gauge couplings of U(1), SU(2), SU(3), respectively.
For the (top-quark) Yukawa-Higgs coupling g_t and the quartic Higgs coupling a, RGE are given by [SZ86];

$$\frac{dg_t}{dM} = \frac{1}{16\pi^2 M}\left\{\frac{9}{2}g_t^3 - (\frac{17}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)g_t\right\}, \tag{80}$$

$$\begin{aligned}
\frac{da}{dM} = \frac{1}{16\pi^2 M}&\left\{24a^2 + 12ag_t^2 - 6g_t^4 - 3(g_1^2 + 3g_2^2)a\right.\\
&\left.+\frac{3}{8}\left[(g_1^2 + g_2^2)^2 + 2g_2^4\right]\right\},
\end{aligned} \tag{81}$$

These equations were solved numerically, setting the su(2/1) value of a as initial condition holding at $E_s = 5$TeV and taking $M_t = 174$GeV in The low energy range($E \sim 100$ to 200GeV). Assuming three generations ($N_g = 3$), with $\alpha_Q^{-1} = 128.80 \pm .05$, $\alpha_2^{-1} = 29.5 \pm .6$, $\alpha_3^{-1} = 8.332$, where $\alpha_i^{-1} = \frac{4\pi}{g_i^2}$ and $\frac{1}{g_Q^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}$

In section 1.4 we discussed the mass of the Higgs field, as related to that of the W bosons gauging SU(2),

$$(M(\Phi))^2 = \frac{2a}{g^2}(M_W)^2 = 4(M_W)^2, \quad M(\Phi) = 2M_W \tag{82}$$

In solving the equations, the relation $g_t(M) = \frac{\sqrt{2}}{v}M_t = \frac{\sqrt{2}}{246}M_t$ was used, where $v =< 0|\Phi^0|0 >= 246GeV$. The outcome was a reduction of the predicted Higgs meson mass down to 130 ± 6GeV. Note that while there is at least one other theory predicting the Higgs mass - ordinary supersymmetry - su(2/1) is the only one that does not require the existence of a large number of new particles.

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