MAT 560: Mathematical Physics, Fall 2006

MAT 560
References
Homeworks
Contact William
Contact Jerry

Lectures: Tuesdays and Thursdays 11:20 AM -- 12:40 PM in the Math Tower, P-131.

Instructors: Jerry Jenquin and William D. Linch III.

Office Hours: Jerry's are 10 -- 11 AM on Tuesdays and Thursdays. William's are 1:30 -- 2:30 PM on Tuesdays and Thursdays.

Suggested Prerequisites We'll be assuming familiarity with the material covered in MAT 530 and 531. Some previous exposure to physics, while helpful, is by no means necessary.

Course Content: In this first semester we will cover Classical Newtonian Mechanics, Classical Relativistic Mechanics, and Electromagnetism. Specifically we hope to cover the following:

Classical Newtonian Mechanics

- Paths in Euclidean space and Newton's 2nd Law.
- Phase space and symplectic geometry.
- Hamiltonian mechanics in the Newtonian setting.
- Variational principles, Lagrangian mechanics.
- Symmetry and Noether's theorem.
- The Euclidean group, symmetry, and conserved charges (a.k.a. Newtonian kinematics).
- Time translation, energy, and dynamics.
- Hamiltonian mechanics from Lagrangian mechanics.
- Gravitational potentials and solvable systems.

Classical Relativistic Mechanics

- Geometry on Minkowski space.
- Lagrangian for paths on Minkowski space.
- The Poincare group, symmetry, and relativistic kinematics.
- Reparameterization invariance.

Electromagnetism

- Differential forms, Stoke's theorem, currents, flows.
- Hodge star for Euclidean, Lorentzian signatures and duality.
- Electromagnetic fields and Maxwell's equations
- PDE's on Minkowski space and Poincare symmetry
- Laplace and wave equations, Green's operators, boundary conditions
- Exact solutions: propagating waves, monopoles, instantons, ...
- Lagrangian formulation of electromagnetism
- Hamiltonian theory of electromagnetism
- Gauge symmetry and connections on principal R-bundles
- Magnetic sources, Dirac charge quantization, and principal U(1)-bundles
- Generalizations in various directions.

Texts and Online Notes: Although there are no official texts for this course here's a list of references for some of the topics we'll be covering and some of the prerequisite topics. In particular we recommend the following texts to complement the lectures.

1. *Mathematical Methods of Classical Mechanics* by V.I. Arnold.
2. *A Course in Mathematics for Students of Physics* by Paul Bamberg and Shlomo Sternberg.
3. *Overview of Selected Topics in Physics* by William D. Linch III. These notes offer a treatment closer to what one would find in a physics text. It's a work in progress.

We are also fortunate to have Gabriel Drummond-Cole's TeX'd course notes, annotated with physics commentary by William.

- Lecture 1
- Lecture 3
- Lecture 4

Homework: We will provide several problem sets throughout the semester. The best way to learn the material is to attempt these problems and even come up with and solve some problems on your own.

Grades: Throughout the semester students will be expected to present homework solutions in class. The course grade will be determined solely by these presentations.

DSS advisory: If you have a physical, psychiatric, medical, or learning disability that could adversely affect your ability to carry out assigned course work, we urge you to contact the Disabled Student Services office (DSS), Educational Communications Center (ECC) Building, room 128, (631) 632-6748..

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For the most part, the lectures will be heavily influenced by the first two references. We only mention the other references for those who would like to revisit some of the prerequisite topics or look into some of the more advanced topics.

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Overview of selected topics in theoretical physics

Wm D Linch III and Jerome A. Jenquin

September 14, 2006

Part I

Classical Theory

Chapter 1

Introduction, notation, and preliminaries

1.1 Our guiding philosophy

This course is meant to be an introduction to the topics usually taught to undergraduate physics major. These are classical mechanics in both the Newtonian and Relativistic setting; classical electromagnetism; thermodynamics and statistical mechanics; and quantum mechanics. We have two audiences in mind: former physics majors who have seen the general content before and pure math majors with an interest is studying physical topics.

For the first audience, we present the physics in a way that emphasizes some of the overall mathematical structure. This approach can be of great pedagogical benefit by shining new light on old topics and preparing one for further study in field and string theory, to which many of the mathematical ideas we discuss apply.

For the second audience, the mathematical structure is there for psychological reasons, as well as pedagogical ones, softening the culture shock and yet introducing math that is interesting in its own right. We also present specific examples and solutions to get a hands-on feel for the physical ideas that they display.

In some cases, particularly when we cover quantum mechanics, some may find our mathematical approach to be vague and hand-waving at best. While this is somewhat regrettable, we will not apologize for it. One of the goals of this course is to offer the students a sense (perhaps even an intuition) for how physicists achieve progress, not in spite of eschewing mathematical rigor, but sometimes *because* of it.

1.2 Physical and mathematical preliminaries

Dimensional analysis and "naturalness"
Vector fields, differential forms, and calculus

Chapter 2

Classical mechanics

2.1 Newtonian mechanics on Euclidean space

2.1.1 Space, time and particle

Space We will generally refer to *space* meaning the 3-dimensional real vector space $\mathbb{R}^3 = \{\mathbf{x} = (x, y, z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\}$ with the *right-handed orientation* and the Euclidean inner product $\langle\cdot\,,\cdot\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, to which we will refer as *dot product*. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y}$. We describe all events by coordinate expressions as that is the language most closely related to the realization of the event. We will also switch freely between various confusing but conventional notations to describe the coordinates. For example, it is common to write $\mathbf{x} \in \mathbb{R}^3$ variously as $\mathbf{x}^i$ for $i = 1, 2, 3$ or just x^i and also $\mathbf{x} = (x, y, z)$. Note that by convention $x^1 = x$ denotes the "x-coordinate", $x^2 = y$ denotes the "y-coordinate", and $x^3 = z$ denotes the "z-coordinate". In this language, $\mathbf{x} \cdot \mathbf{y} = \sum_{i,j=1}^3 \delta_{ij} x^i y^j$ where δ_{ij} is the *Kronicker-delta*, equal to $+1$ when $i = j$ and 0 otherwise. We will use the *Einstein summation convention* meaning that when covariant and contravariant indices are repeated, a summation over the full range of the indices is implied, that is, for a vector x^i and covector p_i, $p_i x^i = \sum_{i=1}^3 p_i x^i$.

For any vector $\mathbf{x}$ we define the unit vector $\hat{\mathbf{x}} = |\mathbf{x}|^{-1}\mathbf{x}$ where $|\mathbf{x}| \equiv \sqrt{\mathbf{x} \cdot \mathbf{x}} \equiv r$. The unit vectors in the x-, y-, and z-directions are denoted $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. The orientation on space defines a *cross-product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. The right-handed orientation is the one given by the *right-hand rule* $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = +\hat{\mathbf{z}}$. This can be expressed using the totally anti-symmetric tensor ϵ_{ijk} normalized to $\epsilon_{123} = +1$, that is $\epsilon_{ijk}(\hat{\mathbf{x}})^i(\hat{\mathbf{y}})^j(\hat{\mathbf{z}})^k = 1$, in terms of which for any two vectors $\mathbf{a}$, $\mathbf{b}$, $(\mathbf{a}\times\mathbf{b})^i = \delta^{ii'}\epsilon_{i'jk}a^j b^k$.

Time and Particle In the Newtonian picture of nature, there is a universal clock defining time for all observers. We will denote this universal time by $t \in \mathbf{R}$. In general, particle motion is described, by definition, by a time-dependent vector $\mathbf{x}(t)$.[1] The *velocity* $\mathbf{v}$ of a particle is the derivative with respect to time of its position $\mathbf{v}(t) = \dot{\mathbf{x}}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)$. Its acceleration $\mathbf{a}$ is the derivative of its velocity, or the second derivative of its position $a(t) = \ddot{\mathbf{x}}(t)$. We will often drop the argument of these physical quantities, leaving their time-dependence implicit. It is also common to denote the constant values of these quantities with a 'naught', e.g. $\mathbf{x}_0$ for constant position vector. We define the linear momentum $\mathbf{p}(t)$ of a particle as the product of its mass m and velocity $\dot{\mathbf{x}}$, $\mathbf{p} = m\dot{\mathbf{x}}$.

Symmetries The space symmetry group for Newtonian mechanics is given by the Euclidean group $\mathrm{SO}(3) \ltimes \mathbb{R}^3$ where the compact factor acts on the coordinates by rotations $x^i \mapsto \Lambda^i{}_j x^j : \Lambda^i{}_j \delta_{ik} \Lambda^k{}_l = \delta_{jl}$ and the non-compact factor acts by translations $x^i \mapsto x^i + a^i$. In Newtonian mechanics the time variable does not mix with the spacial coordinates. We therefore have a separate symmetry factor $\mathbb{R}$ of translations in time $t \mapsto t + c$. The physical interpretation of these space-time symmetries is that in writing equations, the origin and orientation of the coordinate system are conventions and in particular are not physical. That is, only the relative coordinates of space-time events are physical. In general, physical quantities are invariant under the space-time symmetry group. In practice we will always fix this ambiguity by specifying the coordinate system.

Note that when physical quantities are expressed in linear-algebraic language, the transformation laws are simple, that is, linear. When a physical formalism is expressed in the way, we say that the formalism is *covariant* – in this case with respect to the space symmetry group $\mathrm{SO}(3) \ltimes \mathbb{R}^3 \times \mathbb{R}$ – and that the space-time symmetry is *manifest*. It is always the case that a covariant formalism is expressed in terms of unphysical quantities because covariance means that the symmetries are manifest which means that they are realized linearly on the variables which, in turn, means that the variables are not invariant under the symmetries and hence not physical.

2.1.2 Newton's Laws

Newton I: An object of mass m in rectilinear motion $\mathbf{x}(t) = \mathbf{v}_0 t + \mathbf{x}_0$ will stay in rectilinear motion unless acted on by a force.

[1] This is the meaning of *particle* as opposed to an extended object for which we have to specify a distribution of positions as a function of time.

This statement defines the concept of *kinematics* or geodesic motion. It is equivalent to the statement that free particle trajectories satisfy the equation

$$\ddot{\mathbf{x}} = 0. \tag{2.1}$$

Which, in turn, is equivalent to the statement that, in the absence of force, momentum is conserved

$$\dot{\mathbf{p}} = 0. \tag{2.2}$$

Note that the equation is $\mathrm{SO}(3) \ltimes \mathbb{R}^3 \times \mathbb{R}$ covariant. In a local lagrangian system, the existence of a global symmetry implies, via Noether's theorem, the existence of a conserved current (c.f. section 2.1.5). Suffice it here to say that the current associated to the translations is the momentum $\mathbf{p}$. The kinematic equation (2.2) expresses that it is *conserved*, that is, constant in time. Similarly, there is a current associated to the rotational invariance – the *angular momentum* $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Noting that $\mathbf{p} \parallel \dot{\mathbf{x}}$ and using the kinematical equation, we see that the angular momentum is conserved $\dot{\mathbf{L}} = 0$. Finally, the current associated to a shift in the time variable is $T = \frac{1}{2}m\dot{\mathbf{x}}^2 = \frac{1}{2m}\mathbf{p}^2$ and is called the (*kinetic*) *energy* (the normalization is conventional). Again, the kinematic equations imply that it is conserved.

Newton II: An object of mass m, when acted on by a force $\mathbf{F}$ will deviate from rectilinear motion with an acceleration $\mathbf{a} = \ddot{\mathbf{x}}$ according to the relation

$$\mathbf{F}(\mathbf{x}, t) = m\mathbf{a}(t) \tag{2.3}$$

or, equivalently,

$$\mathbf{F}(\mathbf{x}, t) = \dot{\mathbf{p}}(t). \tag{2.4}$$

This is the statement of *dynamics* or the deviation from geodesic motion due to an external influence. An equivalent way to express this is that the second law defines the *source* $\frac{1}{m}\mathbf{F}(\mathbf{x}, t)$ for the kinematic (read "source-less") equation $\ddot{\mathbf{x}} = 0$ or $\dot{\mathbf{p}}$ of the first law. In this sense, it defines what is meant by a force.

An important point to note is that the second law is *linear* in the force. This implies that we have the

Principle of superposition: If there are 2 forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the same particle, the effective force $\mathbf{F}_{\text{total}}$ the particle experiences is the vector sum of the individual forces $\mathbf{F}_{\text{total}} = \mathbf{F}_1 + \mathbf{F}_2$. In particular, two opposing forces of equal magnitude and opposite direction applied to the same particle produce no net dynamics.

A second important point is that the second law can be interpreted as defining the mass of an object to be the ratio of a stimulus $|\mathbf{F}|$ to the response $|\mathbf{a}|$ in its motion by $m = \frac{|\mathbf{F}|}{|\mathbf{a}|}$. In this sense, we see that m refers to an *inertial mass*, that is, a property describing its resistance to a change in motion.

Newton III: An object, when acted on by some agent with a force $\mathbf{F}_{\text{action}}$ will exert a force $\mathbf{F}_{\text{reaction}}$ on the agent of equal magnitude and opposite direction, *id est*,

$$\mathbf{F}_{\text{action}} = -\mathbf{F}_{\text{reaction}}. \tag{2.5}$$

This is a statement of linear momentum conservation during a collision. Intuitively, when pressing on an object with some force, the object presses back (otherwise, we wouldn't be able to feel it). The third law is the statement that the reaction force is of precisely the same magnitude as the applied force.[2]

Newton's law of universal gravitation Consider two objects, one of mass m_1 and the other of mass m_2. They will exert a gravitational force on one another given by

$$\mathbf{F}_{\text{gravitation}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \tag{2.6}$$

where $G \equiv \frac{1}{4\pi\kappa} \approx 6.67259 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$ is the least precisely known fundamental constant of nature.

This formula is fundamentally different from the second law. Firstly, it introduces a constant G which is claimed to be fundamental in the sense that it is the same number no matter what material form the masses take.

Secondly and related to this, the masses $m_{1,2}$ entering it could be called *gravitational masses* since they describe a property of an object we are calling gravitation and should probably have been called gravitational charge. A priori, this is a different type of mass than the inertial mass entering the dynamical second law. Therefore, Newton's law of universal gravitation is making the bold assertion that gravitational mass and inertial mass are equivalent.

Finally, we note that setting $\mathbf{a}_2 = -\frac{Gm_2}{r^2}\hat{\mathbf{r}}$ to be the acceleration due to the gravity of the mass m_2 at a distance r from its position, we find the form $\mathbf{F}_{\text{gravitation}} = m_1\mathbf{a}_2$. Taking $m = m_{\earth}$ to be the mass of the earth and $r = r_{\earth}$ its radius, we find the famous acceleration due to gravity $g = |\mathbf{a}_{\earth}| \approx 9.8ms^{-2}$.

[2]This law causes some confusion when used in conjunction with the second to the effect that if the object pushes back with exactly the same force, the forces should cancel and there should be no resulting dynamics. Indeed, there is no relative dynamics between the hand and the object, rather, the object will accelerate relative to the ground against which we are also pushing when we try to accelerate the object.

2.1.3 Potentials

There are various drawbacks to the vector space formulation of Newtonian mechanics, not the least of which is that all defining equations are vector equations. In most cases of physical interest, drastic simplifications are made possible by switching to a description in terms of energy. Suppose the force is *holonomic* $\nabla \times \mathbf{F} = 0$. Then we can define the *potential energy function* $U(\mathbf{x}, t)$ s.t. $\mathbf{F} = -\nabla U$. The sign comes from the observation that a force acts so as to decrease the potential energy. The total energy $E = T + U$ is the sum of the kinetic and potential energy. Just as the kinetic energy was conserved in the absence of external forces, the total energy of a system is conserved when the force is the gradient of the potential energy and the latter does not depend explicitly on time: $\dot{E} = m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \nabla U \cdot \dot{\mathbf{x}} + \frac{\partial U}{\partial t} = 0$ by Newton II. This is the famous *principle of the conservation of energy*. It is very powerful because it is, in the cases in which it applies, equivalent to the second law but it is a scalar equation, making it much easier to use.

2.1.4 Hamiltonian

Very closely related to the energy formulation of Newtonian mechanics is the *Hamiltonian formalism*. In this formulation, the fundamental variables are the position x^i and the momentum p_i vs. the position and the velocity (c.f. section 2.1.5). A physical trajectory is a graph in the *phase space* $\{x^i, p_i\}_{i=1}^3$.[3] Note that the momentum is treated as a 1-form in this formulation (which, as we will soon learn, is the proper interpretation of this quantity). The dynamics is encoded in the *Hamiltonian* $H(x, p)$ which, when evaluated on a point in the phase space, is equal to the energy E introduced in section 2.1.3. In particular, it is the sum of the kinetic energy function $T(p)$ which we take to be a function only of the momentum (usually $T = \frac{1}{2m}p^2$)[4] and the potential energy function $U(x)$ with we take to depend only on the position. We can now easily show that the definition of momentum and the second law imply

Hamilton's Equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}$$

[3] In general, the space parameterized by x may be any $\mathcal{C}^2$ 3-manifold M. Then the phase space is defined to be the co-tangent bundle T^*M. From this point of view, it is easy to see the symplectic structure.

[4] On a more general space the kinetic energy function will depend on x through the metric: $T = \frac{1}{2m} g^{ij}(x) p_i p_j$.

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{2.7}$$

The form of these equations[5] displays an important aspect of the phase space, namely, its *symplectic structure*: The phase space comes equipped with its Poincaré 1-form $p_i \mathrm{d}x^i$ and therefore the symplectic 2-form $\mathrm{d}p_i \wedge \mathrm{d}x^i$. This statement is often implicit in a discussion of Hamilton's equations in which one considers transformations of the variables (x, p) which preserve the 'form' of Hamilton's equations. These *canonical transformations* are the symplectomophisms – smooth transformations on the phase space coordinates which preserve the symplectic structure.

From the Poincaré 1-form and a phase space trajectory γ (a path in phase space) we can construct the *action* (*functional*)

$$S[\gamma] = \int_\gamma p_i \mathrm{d}x^i. \tag{2.8}$$

A useful generalization of the phase space includes the time coordinate as an additional variable. This 7-dimensional space is called the *extended phase space*. Similarly to the action functional (2.8) on the un-extended phase space, from the Poincaré 1-form and the Hamiltonian function we can construct the action functional[6]

$$S[\gamma] = \int_\gamma \left[p_i \mathrm{d}x^i - H(x,p)\mathrm{d}t \right]. \tag{2.9}$$

It is important to remember that $(x^i(t), p_i(t))$ are functions of the time parameter t. As such, we are allowed to "vary" them. That is, we consider an infinitesimal deformation of the trajectory $\gamma \to \gamma' = \gamma + \delta\gamma$. The *variational* or *functional derivative* of the action functional is defined to be the linear part of $S[\gamma']$, that is

$$\frac{\delta S}{\delta \gamma} \equiv \lim_{\delta\gamma \to 0} S[\gamma + \delta\gamma]. \tag{2.10}$$

This notation δ for ∂ for the functional derivative is customary in the calculus of variations.

The path has two linearly independent variations in the x-direction and the p-direction. It is therefore possible to define the partial variations in these directions. The following notation is customary (and, hopefully, self-explanatory)

$$\delta S = \delta x^i \frac{\delta S}{\delta x^i} + \delta p_i \frac{\delta S}{\delta p_i}. \tag{2.11}$$

[5]Note that the Hamilton equations (2.7) are coupled ordinary differential equations of the *first* order which contain the same information as Newton's second law which is second order.

[6]Note that this is (the negative of) an integrated Legendre transformation of $H(x, p)$.

The action is called *stationary* when $\delta S = 0$. Since the x- and p-variations are independent, stationary action implies

$$\begin{aligned} 0 &= \frac{\delta S}{\delta x^i} = -\dot{p}_i - \frac{\partial H}{\partial x^i} \\ 0 &= \frac{\delta S}{\delta p_i} = \dot{x}^i - \frac{\partial H}{\partial p_i} \end{aligned} \tag{2.12}$$

and we recover Hamilton's equations (2.7).[7] This is the *principle of stationary action*; the physical trajectories in phase space are those which extremize (usually minimize) the action.

This point of view has many advantages. Firstly, it generalizes the intuitive idea that physical processes are such that they minimize the energy. Secondly, a modification of this formalism (c.f. section 2.1.5) us a very powerful tool to solve complicated concrete problems in analytical dynamics especially dynamical systems defined in terms of constrained degrees of freedom. Finally, the principle of stationary action will fit seamlessly into the description of quantum mechanical systems c.f. chapter **??**. There we will see that quantum mechanical corrections to classical mechanics have the interpretation of deviations $\delta\gamma$ of the phase space trajectories.

2.1.5 Lagrangian

A complementary formulation of Newtonian mechanics is the Lagrangian formulation. The Lagrangian formulation is a "Legendre transform of the Hamiltonian formulation". Indeed, the space replacing the phase space of Hamiltonian mechanics is the space parameterized by positions q^i and velocities $\dot{q}^i$.[8] The *Lagrangian function* $L(q^i, \dot{q}^i)$ is the Legendre transform of the Hamiltonian $H(x^i, p_i)$

$$L(q^i, \dot{q}^i) = p_i \dot{q}^i - H(x, p). \tag{2.13}$$

Plugging in the form $H = T + U$ and substituting $p_i = m\dot{q}_i$ we find that $L = T - U$. By the definition of the action (2.9), the Lagrangian function is the unintegrated action density

$$S[\gamma] = \int_\gamma L(q, \dot{q}) \tag{2.14}$$

[7] In the first equation, we have integrated the time derivative by parts. This is legal since the surface term is proportional to δx which vanishes when evaluated at the endpoints of the path (recall that we vary the path but keep the endpoints fixed).

[8] For a general space M the Lagrangian formalism is defined on the tangent bundle TM. It is conventional in this context to denote the positions by q^i instead of x^i.

where γ is re-interpreted as a section of the tangent bundle.[9] The stationary phase principle in this case implies the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{2.15}$$

The advantages of the Lagrangian formalism over the Hamiltonian one include the use of the stationary action principle to solve complicated problems in anaytical dynamics and the possibility to easily manifest Lorentz invariance in relativistic theory (c.f. **??**).

Noether's theorem Consider a time-independent infinitesimal transformation of coordinates $q^i \mapsto q'^i \approx q^i + \epsilon^i$ under which the action is invariant $S \mapsto S$, that is, a *symmetry* of the theory. Now promote the parameter $\epsilon^i \to \epsilon^i(t)$ to a function. The resulting change in the action must be proportional to $\dot{\epsilon}$ since, when ϵ is constant, the transformation is a symmetry. Given this, there must be a function $J_i(q, \dot{q}, t)$ such that $\delta S = \int \dot{\epsilon}^i J_i = \epsilon^i J_i| - \int \epsilon^i \dot{J}_i$. The first term is the "surface term" $\epsilon J|_{t_i}^{t_f}$ – the difference of the quantity ϵJ at the final time t_f and the initial time t_i. The second term vanishes by the equation of motion. (Prove it!) When ϵ is constant, we see that J is conserved $J(t_f) = J(t_i)$. Such conserved functions arising from symmetries of the theory are called *Noether currents*. In this case the symmetry is a translation and the current is $J_i = \partial L/\partial \dot{q}^i$, which is the definition of the momentum. In the absence of external forces, this is indeed conserved.

With an eye to the future we will refer to a time-independent symmetry as a *global symmetry*. Noether's theorem is the statement that for every global symmetry of the action, there is a conserved current and *vice versa*.

2.1.6 Examples

Gravitational potentials and solvable systems.
Potential theory and the need for fields.

[9] Usually this whole story is reversed: One defines the Lagrangian function as the difference between the potential and kinetic energy functions and develops the Lagrangian formalism and stationary action principle. Subsequently the Legendre transformation to the Hamiltonian function is performed. It is then proven that the resulting Hamiltonian is independent of $\dot{q}$ and the phase space picture is developed.

Index

Bibliography

2.2 Physics

[1] H. Georgi, "Waves,"

[2] Griffiths, "Quantum Mechanics,"

2.3 Math

[3] V.A. Arnold, "Classical Mechanics,"

[4] Bamberg and Sternberg, "vol. 2,"

I am no longer at Stonybrook. As of September 2013, I am a research fellow at the Institute for Basic Science Center for Geometry and Physics.

My current web page is here.

Mathematical Physics
September 6, 2006

Gabriel C. Drummond-Cole

September 6, 2006

[Course overview.]

Let's get started with Newtonian mechanics. The prerequisite is that you know what a manifold is. I won't assume Riemannian geometry. I'm trying to keep things simple, so things will be on flat affine spaces. If you want to sup it up in your head as we go along, feel free.

So classical mechanics, we're talking about the physics (and math) of a single particle moving in some Euclidean space. So if you want to play along at home it could be a Riemannian manifold. The mathematical models are paths x, maps from time $M^1 \to X$ where X is a Euclidean target space. (or possibly a Riemannian manifold). Do I need to define a Euclidean space? Sorry, I guess that's a little bit insulting.

All right, so there are two spaces that are going to color our approach to this, M^1 and X. Let's look at the structure of both of these spaces.

Okay, so time has physical significance. We attribute certain mathematical structure to it to correspond to this. In particular,

1. It's affine, meaning that it's not necessarily a vector space. How do you add two instances in time? You can't. You can talk about how much time has passed, so it's an affine space over a one dimensional vector space $T \cong \mathbb{R}$.
2. There are units, like seconds or hours. What sort of mathematical structure would units be associated with? A norm, a metric. In particular we have a translation invariant metric on affine time. In other words, T has an inner product on it.
3. We could potentially also attribute to it an orientation, a differentiation between going forward and backward. We'll hold off on that for now.

To have a cogent discussion, we want to do math, so I want to fix some affine coordinate $t : M^1 \to \mathbb{R}$. We want to choose this so that $|dt| = 1$.

The structure, excluding the third, gives us a symmetry group that we will talk about again and again. The symmetry group is the Euclidean group for M^1, which includes translations and reflections. So there is a short exact sequence $1 \to \text{Translations} \to \text{Euc}(M^1) \to O(T) \to 1$. That's the structure we associate to the domain.

Now what structure do we have in the range?

1. X is an affine space over a vector space V.
2. There is a (translation invariant) metric so that V is an inner product space. This measures distance on $\mathbb{E}^d$.

If you're playing along with Riemannian geometry, these are the conditions that

1. X is a smooth manifold
2. X has a Riemannian metric
3. The metric is complete so we can work globally on X.

Let's go back to the Euclidean space. Again there is an associated symmetry group $\text{Euc}(\mathbb{E}^d)$, where you have translations and then reflections and rotations. Again you have a short exact sequence

$$1 \to \underbrace{V}_{\text{translations}} \to \text{Euc}(\mathbb{E}^d) \to O(V) \to 1.$$

For a general Riemannian manifold the group of isometries will be smaller, meaning lower dimensional. Sometimes this group will even be trivial.

There is one last piece of data that we need to define classical mechanics on X. We have the model of maps from affine Euclidean time into a Riemannian manifold. The last piece of data is

- a potential energy function $\mathscr{V} : X \to \mathbb{R}$.
- A mass $n > 0$ of the particle.

A quick note on units. In this course we'll come across a few fundamental units. In classical mechanics we'll only come across mass, length, and time. Energy will have units mL^2/t^2.

Actually, these are the only basic units. This is an empirical fact. We will see that extensions of classical mechanics will introduce new fundamental constants *such as the speed of light c with units L/T (special relativity), a fundamental angular momentum $\hbar$ with units ML^2/T (quantum mechanics), etc. but must reduce to classical mechanics in the appropriate limit, e.g. $c \to \infty$ and $\hbar \to 0$. Therefore, no new* fundamental units *are introduced even when more fundamental physics is uncovered.*

Now we can finally define what classical mechanics is, or at least the classical mechanics of the system. So all togethher, for any target space X, all possible particle motions are modeled on paths $P = \mathrm{Map}(M^1, X)$. We'll assume that the paths are smooth. Don't worry about putting a Frechet topology or whatever on this.

This is again actually an empirical fact. Discontinuous paths imply the disappearance and reappearance (at a later time) of particles while a kink in the path amounts to an instantaneous change in the particle's velocity. In order to change a Newtonian particle's velocity discontinuously one must apply an infinite force. Infinite forces are considered pathological and any use of such a thing should be considered only as an approximation.

Now given the potential energy function $\mathscr{V}$ the actual particle motions are paths x such that they satisfy Newton's second law

$$m\ddot{x}(t) = -\mathscr{V}'(x(t)) = -\nabla\mathscr{V}(x(t))$$

In the physics literature this equation is variously written as $m\ddot{\mathbf{x}} = -\nabla\mathscr{V}$ as a "vector" equation or as $m\ddot{x}^i = -\partial\mathscr{V}/\partial x_i$ in "components". Here $\mathbf{x}$ is a coordinate for the point x and x^i are its components (in some orthonormal coordinate system) where the indices $i = 1, 2, 3$ label the linearly independent directions. A common notation is $(x^1, x^2, x^3) = (x, y, z)$. The index on x^i is defined to be 'up' and is lowered with the metric or its inverse. In Euclidean space the placement of the indices (upper or lower) doesn't matter as the metric $g_{ij} = \delta_{ij}$ is just the Kronicker symbol but the more general case such as in Riemannian geometry or "curvilinear coordinates" (e.g. spherical coordinates) it, of course, matters a great deal.

We're going to look at the space of solutions to this equation $\mathscr{M}$, the space of states. A solution is a state. It's also called a phase space. Let me mention some properties right off the bat.

- It will be clear soon that $\mathscr{M}$ is a smooth manifold, so we can do calculus on it.
- The affine Euclidean group for time $\mathrm{Euc}(M^1)$ acts on $\mathscr{M}$ on the right so in particular time translation acts on it. So $T_s(x)(t) = x(t - s)$.
- One other thing, the potential was a function of X. It can also be a function of time, so that the symmetry is broken. So $\mathrm{Euc}(M^1)$ no longer preserves the space of solutions.

Let me argue that this is a smooth manifold. Let's see this by breaking some symmetry. Choose an instant t_0 in time and by picking this we break the affine symmetry. Then there's a natural map $\mathscr{M} \to T\mathbb{E}^d = V \times \mathbb{E}^d$ given by $x \mapsto (\dot{x}(t), x(t))$. This is a bijection and you can just transfer the differentiable structure across.

The physical intuition behind this diffeomorphism is the intuitively obvious fact that when you want to specify a particles trajectory, it suffices to give the initial position, its initial velocity, and the potential field in which the particle motion occurs (i.e. the force which acts on it). For example, the parabolic trajectory of a baseball depends on the gravitational potential field (in this case $\mathscr{V} = mgz$ with $g \approx 9.8ms^{-2}$) the hight of the ball upon release and its velocity

(speed and direction) upon release. Newton formulated his second law to be second order in time derivatives precisely to accommodate this empirical fact.

Now we have a picture of what the space of solutions looks like. Let me give you some examples.

Example 1 *The free particle*
This is the case where it's moving in Euclidean space and $\mathscr{V} = 0$. *Then Newton's second law says* $m\ddot{x} = 0$. *Then* $\mathscr{M} = \{x(t) = p + vt | p \in \mathbb{E}^d, v \in V\}$. *So given a* t_0, *the map takes* $p + vt$ *to* (v, p). *In this case the map doesn't depend on* t.

Example 2 *This is a little less trivial but just as famous. It's the spring or harmonic oscillator.*
Implicitly you have to have a distinguished point where the spring is stable. I may as well take $X = \mathbb{R}^1$. *Then the potential energy is* $\mathscr{V} = \frac{1}{2}kx^2$, *where the units of* k *are* $\frac{M}{T^2}$.

Then Newton's second law says $m\ddot{x}(t) = -kx(t)$. *Then* $\mathscr{M} = \{p\cos(\sqrt{\frac{k}{m}}t) + \sqrt{\frac{m}{k}}v\sin(\sqrt{\frac{k}{m}}t) | p, v \in \mathbb{R}\}$. *So then* $p\cos(\sqrt{\frac{k}{m}}t) + \sqrt{\frac{m}{k}}v\sin(\sqrt{\frac{k}{m}}t) \overset{t=0}{\mapsto} (v, p)$.

There are other things I can point out about the space of solutions. We have the symmetry group of the domain that acts on the solutions. What about the symmetry group of the target? How does that naturally act on the space of solutions? it acts on the left, but only those isometries that preserve the potential. An isometry that changes the potential will not preserve the space of solutions. What's true about the two group actions? They commute. The time group will have to do with dynamics, the target group with kinematics. If $X = \mathbb{E}^d$, and $\mathscr{V} = 0$, so that we're talking about the free particle, then the entire Euclidean group acts on the space of solutions, since everything preserves the 0 potential. In particular, if A is an affine Euclidean transformation and its derivative is in $O(V)$, then $p + vt \mapsto Ap + (dA \cdot v)t$.

Let me wrap up what I've said today, which isn't much. In summary, we've discerned that the space of solutions $\mathscr{M}$ to Newton's second law has the following structure:

- There's a right action by the Euclidean time group $\mathrm{Euc}(M^1)$ acting by right composition with the inverse.
- There's a left action by potential-preserving isometries of X, $\mathrm{Isom}(X, \mathscr{V})$.
- For $t_0 \in M^1$ there's a natural diffeomorphism $\mathscr{M} \overset{t_0}{\cong} TX$.

Next time I hope to make this fit in with the idea of a symplectic structure.

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Gabriel C. Drummond-Cole

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[Is the exact sequence from last time secretly Noether's theorem?]

We'll see Noether's theorem later. Let me recap what we've seen so far. So far we've seen particle motion, and the structure of the phase space $\mathscr{M}$ which are paths from the affine time to the target satisfying Newton's second law

$$\{x : M^1 \to X | \ddot{x} = -\mathscr{V}'(x(t))\}$$

This has a right action by $\mathrm{Euc}(M^1)$, a left action by $\mathrm{Isom}(X, \mathscr{V})$ and a for each $t_0 \in M^1$ a natural diffeomorphism $\mathscr{M} \overset{t_0}{\cong} TX$.

We saw something about symplectic geometry on a smooth manifold M^{2n}. This means there is a two form $\omega \in \Omega^2(M)$ such that ω^n is nowhere vanishing (nondegeneracy) and $d\omega = 0$ (closed).

The thing that will play a big role today is the symplectic gradient which takes smooth functions on a symplectic manifold into vector fields $C^\infty(M) \to \mathscr{X}(M)$ via $f \mapsto \xi_f$ characterised by $\iota(\xi_f) = df$. This gives us the Poisson bracket $\{\cdot, \cdot\}$ which makes $C^\infty(M)$ a Lie algebra. This is given by $\{f, g\} = \omega(\xi_f, \xi_g)$. Then this map ξ is a homomorphism of Lie algebras.

There is a typo in the equation above. The formula for ξ_f should be $\iota(\xi_f)\omega = \mathrm{d}f$, or in components (local coordinates) $\xi_f^i = \omega^{ij}\partial_j f$. The Poisson bracket is given in local coordinates z^i by

$$\{f, g\} \quad = \quad \frac{\partial f}{\partial z^i}\omega^{ij}\frac{\partial g}{\partial z^j}. \tag{1}$$

Note that $\xi_f = -\{f, \cdot\}$ which is handy to keep in mind.

The symplectic form on the cotangent bundle is compatible with the coordinates of the latter in the sense that it takes the form $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the basis where $(z^i) = (x^i)$ for $i = 1, \ldots, n$ and $(z^i) = (p_i)$ for $i = n+1, \ldots, 2n$. (Compare $\omega = \mathrm{d}p_i \wedge \mathrm{d}x^i$.) Then the Poisson bracket

takes the form often found in physics books

$$\{f, g\} \quad = \quad \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i}. \tag{2}$$

By the way, I should mention that the use of components such as ξ_f^i instead of the full vector ξ_f which in local coordinates x is given by $\xi_f^i(x)\partial/\partial x^i$ is actually not really coordinate dependent. As pointed out by Penrose, the expression ξ_f^i can be understood merely as a notation which keeps track of the tensorial nature of ξ_f which is that of a vector in this case. Of course this does not apply to the coordinates themselves which prompted Penrose to use a different label e.g. ξ^a vs. x^i. However, modulo such caveats the notation is a powerful way of keeping track of covariance of complicated expressions for objects composed of multiple tensor quantities with derivatives, etc. acting on them. Finally, note that the coordinate basis vector $\partial/\partial x^i \equiv \partial_i$ is written as ∂_a in Penrose's notation. Physicists generally have never hear of this Penrose notation and never distinguish a coordinate index i from an abstract index a.

The prime example of a symplectic manifold is when $M = T^*X$, the cotangent bundle. Then $\omega = d\theta$ where θ is the God-given one-form on T^*X.

Recall that in local coordinates on T^*X $\theta = p_i \mathrm{d}x^i$ where the x^i are local coordinates in the base and the momenta p_i are local coordinates on the fibre of T_x^*X.

Why is this interesting to us in the context of particle motion? These diffeomorphisms give us a relationship, but we want to get from the tangent to the cotangent bundles. So we use the Riemannian metric to get $\mathscr{M} \overset{t_0}{\cong} TX \cong T^*X$. So now the rest of this class will be spent investigating, take the natural structure on T^*X and pull it back by $\mathscr{M}$. So we have a symplectic structure for each t_0 and these could depend on the choice of t_0. So this is breaking the symmetry.

Recall that the $\mathscr{M} \cong TX$ isomorphism comes from the map $x^i(t) \mapsto (x^i(t_0), \dot{x}^i(t_0))$ which takes a solution to Newton II*, that is, a specific particle motion, and maps it to the particle coordinate at $t = t_0$ and its velocity at $t = t_0$. The second isomorphism takes $(x^i, \dot{x}^i) \mapsto (x^i, mg_{ij}(x)\dot{x}^j)$ and, in physics at least, depends explicitly on the mass parameter.*

This brings us to Hamiltonian mechanics. The goal of Hamiltonian mechanics is to encode the symmetries of our phase space into the Lie algebra of smooth functions with the Poisson bracket $(C^\infty(M), \{\cdot, \cdot\})$.

To point out, to be grandiose, where this fits in the grand scheme of physical systems, there's usually a phase space, and another space (of observables). There should be a duality $\sim$ between them, as observables are evaluated on states. In our particular situation in classical mechanics, our state space is our phase space. Our observables are the functions on our phase space. These would be things like momentum and energy that we can assign to a particular particle path.

Strictly speaking, and perhaps shockingly, these are not observables. The problem is that these quantities are coordinate dependent. The (potential) energy, for example is only defined up to

a constant since it appears with a derivative in Newton II. *Therefore, only* energy differences *are physical. Similarly, the momentum is only defined up to a constant vector and only the relative momentum between us and the particle is physical. This is the reason the observables are required to be* scalar functions *on the phase space; they are not allowed to transform non-trivially under a change of coordinates.*

One would expect that the symmetries of the phase space should translate into symmetries of the symplectic structure. Let me talk about that, and symplectomorphisms. I'm never going to write out symplectomorphism again. I probably spelled it wrong in the first place. I'll call them whatever in the future, unless you want me to call them, like Bob. That might look bad in Gabe's notes.

Weinstein coined the term symplectic, from taking the Greek equivalent for the Latin word for complex. Before it was the Abelian linear group. It sounds like a Victorian word, like perambulator. It's the Greek root for intertwined.

Let (M, ω) be a symplectic manifold. Then $\varphi \in \mathrm{Diff}(M)$ is a symplectomorphism if and only if $\varphi^*\omega = \omega$. Let me give you some examples related to the cotangent space. Since this happened automatically, we might think that any diffeomorphism from a diffeomorphism of the underlying manifold would be a symplectomorphism. That is the case.

If $M = T^*X,\ \omega = d\theta$ and $\varphi \in \mathrm{Diff}(X)$, then

$$\varphi : T^*X \xrightarrow{\sim} T^*X$$

by $(x, p) \mapsto (\varphi(x), (d_x\varphi^{-1}) * p)$. So to check that this is a symplectomorphism, you just check that this preserves θ.

Let us denote the symplectomorphism on the coordinates z^i of M by $\varphi : z^i \mapsto \tilde{z}^i(z)$. In the case of the tangent bundle this gives $(x^i, p_j) \mapsto (\tilde{x}^i, \frac{\partial x^k}{\partial \tilde{x}^j} p_k)$ which is just the statement that p_i is a 1-form. It is then obvious that $\theta = p_i \mathrm{d}x^i$ is invariant.

Let's look at a subclass where $X = \mathbb{E}^d$, so $M = V^* \times \mathbb{E}^d$ and let $\varphi = A \in \mathrm{Euc}(\mathbb{E}^d)$. So for $x \in \mathbb{E}^d$ and $p \in V^*$ then $\Phi(x, p) = (Ax, (dA^{-1})^* p)$ is a symplectomorphism.

In the linear category last time this is analogous to the subgroup, we said $GL(L) \subset Sp(L \oplus L^*)$, and this is the general analogue of this linear statement.

The reason I harped on these examples is because when we talked about particles, there are transformations on the target space. The diffeomorphisms will give us special symplectomorphisms on the phase space.

Now I want to talk about infinitessimal symplectomorphisms. So $\xi \in \mathscr{X}(M)$ is an infinitessimal symplectomorphism if and only if $\mathrm{Lie}(\xi)\omega = 0$. This leads us to a special subset of vector fields $\mathscr{X}_\omega = \{\xi \in \mathscr{X}(M) | \mathrm{Lie}(\xi)\omega = 0\}$. This sits inside $\mathscr{X}(M)$ as a subalgebra, preserving the Lie bracket.

So as long as you stick with diffeomorphisms isotopic to the identity, these are the same requirements.

Maybe it is a good little exercise to show that the linear part of the finite symplectomorphism is the infinitesimal symplectomorphism. That is, check explicitly that $(\varphi^* - 1)\omega \approx \mathrm{Lie}(\xi)\omega$.

Okay, now the symplectic gradient. For any $f \in C^\infty(M)$ I get a vector field ξ_f. I claim that this lives in $\mathscr{X}_\omega(M)$. To see this note that

$$(Lie(\xi_f)\omega) = d \circ \iota(\xi_f)\omega + \iota(\xi_f)d\omega = 0,$$

because $\iota(\xi_f)\omega = df$ and $d\omega = 0$.

So what if I want to look at a particular symmetry. Can I find corresponding obervables? Does every infinitessimal symmetry have a corresponding observable? The answer will depend on H^1. The short answer is no. The long answer brings up the exact sequence

$$f \longrightarrow \xi_f$$

$$0 \longrightarrow H^0_{dR} \longrightarrow \Omega^0(M) \longrightarrow \mathscr{X}_\omega \longrightarrow H^1_{dR} \longrightarrow 0$$

$$\xi \longrightarrow [\iota(\xi)\omega]$$

So if $[\iota(\xi)\omega] \neq 0$ then ξ has no corresponding observables. If $\xi \in \mathscr{X}_\omega$ has an observable, it has many, only unique up to the constants.

This is not what a physicist would call many since it is as small as possible without being trivial. As we will probably see soon, the ambiguity inherent in some potentials can be hugh sometimes involving an infinite number of functions. These ambiguities called gauge invariances *have become one of the central themes in theoretical physics.*

Okay, now time translation. In classical mechanics, there is always a distinguished one-parametery group of time translations. Let's just assume for now that ξ_t is the corresponding infinitessimal generator of time translation, that is, $\iota(\xi)\omega$ is exact.

So we have a choice of corresponding observables. Pick one, up to a constant. Finally we meet the energy. This is the Hamiltonian, which in the sense of these infinitessimal symmetries, is negative the corresponding observable for time translation. In other words, energy, once you take the symplectic gradient, it generates motion which is the negative of time translation. So that is $\{x \mapsto x \circ T_s, s \in \mathbb{R}\}$.

This is an example of our previous observation that $\xi_f = -\{f, \cdot\}$. *The statement that there is a distinguished 1-parameter group of translations is equivalent to the statement that there is a distinguished observable* $f = H$, *the Hamiltonian. Time translation is usually written as* $\xi_t = \frac{\mathrm{d}}{\mathrm{d}t}$ *so that our formula becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\{H, \cdot\}. \tag{3}$$

Exercise: Hamilton's equations *follow by plugging the coordinates x^i and p_i into the equation above. Write down Hamilton's equations. Given the relation $p^i = m\dot{x}^i$, show that Hamilton's equations are equivalent to Newton* II.

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Recall that for certain $\xi \in \mathscr{X}_\omega(\mathscr{M})$ (those for which the topological obstruction disappears) there exists a corresponding observable $\mathscr{O}_\xi \in C^\infty(\mathscr{M})$ such that $-d\mathscr{O}_\xi = \iota(\xi)_\omega$.

This describes the infinitessimal symmetry ξ via $\xi f = \{\mathscr{O}_\xi, f\}$ for any $f \in C^\infty(\mathscr{M})$.

We got as far as saying that there are a particular set of symmetries we're concerned with. There are the infinitessimal symmetries of time translation, $\zeta \in \mathscr{X}_\omega(\mathscr{M})$, and this has the observable $\mathscr{O}_\zeta$ where $-\mathscr{O}_\zeta$ is the energy or Hamiltonian. For a path x, $H(x) = \mathfrak{m}2|\dot{x}(t)|^2 + V(x(t))$. For $x \in \mathscr{M}$ this is independent of t.

[Is that obvious?]

Yes, I'll get to it in a second. That's where we left off last time. Any questions?

Before I finish off Hamiltonian dynamics, let me make some tangential but useful remarks about observables. Most of the observables we see in this class will be something like $\mathscr{O}_{(t,f)}$, defined for any time $t \in M^1$ and $f : X \to \mathbb{R}$. Then

$$\mathscr{O}_{(t,f)}(x) = f(x(t)).$$

Let me give another example, two examples.

1. If $X = \mathbb{E}^d$, then we can take $f = x^i$, and in this case $\mathscr{O}_{(t,f)}$ the x^i coordinate of the particle at time t.
2. The Hamiltonian, this is the energy of the particle at time t.

A jet of a function is essentially its Taylor series. The first type of observable depended on the 0-jet of the path; the Hamiltonian depends on the 1-jet. $\mathscr{O}_{(t,f)}$ is local in time, meaning it only depends on finitely many of these, only depends on a small neighborhood of a given time.

So what's the upshot? The structure on $\mathscr{M}$ is as follows. We have what is called a Hamiltonian system. That's the phase space with its symplectic structure and our function H, so

the couple $(\mathscr{M}, H)$. So this is a symplectic manifold and a distinguished observable (energy) such that $-\xi_H$ is the infinitessimal time translation.

Let's look at the symmetries of this extended structure. Global symmetries are symplectomorphisms that preserve H. The infinitessimal symmetries are infintessimal symplectomorphisms $\xi \in \mathscr{X}(\mathscr{M})$ such that $\mathrm{Lie}(\xi)H = 0$. Now we're going to look at these symmetries in terms of observables.

Here "extended structure" should not be confused with the (closely related) "extended phase space" which is used in the construction of the Hamiltonian form of the variational principle. Extended phase space is just the Cartesian product of the original phase space (the cotangent bundle T^*X of the configuration space X) with the time $M^1 \cong \mathbb{R}$.

So if Q is an observable that corresponds to an infinitessimal symmetry, then we have the following relation: $\{H, Q\} = 0$. Now, for any observable, never mind that it's a symmetry of any system, time translation flow on phase space is induced by H. So we get that $\dot{\mathscr{O}} = \{H, \mathscr{O}\}$.

This equation deserves its own line:

$$\frac{\mathrm{d}\mathscr{O}}{\mathrm{d}t} = \{H, \mathscr{O}\} \ . \tag{1}$$

In the operator formalism of quantum mechanics this will be the equivalent of the Schrödinger equation.

Exercise: *Show that the classical evolution equation (1) is equivalent to Hamilton's equations.*

Recall that you already showed that Hamilton's equations (with the condition that the momentum is given in terms of the velocity by $p^i = m\dot{x}^i$ or, equivalently, that the Hamiltonian factorizes as $H = \frac{1}{2m}p^2 + V(x)$) are equivalent to Newton's second law. Therefore, once we fix the identification $TX \cong T^*X$, the evolution equation (1) is equivalent to Newton II.

Now we use the observable energy to tell us how things change with time. So now, thus, what we can conclude, assuming that the observable is a symmetry of the Hamiltonian system, for Q, if Q induces a symmetry of the Hamiltonian system, then we havea conservation law. We have that $\dot{Q} = \{H, Q\} = \mathrm{Lie}(-\xi_H)Q = \mathrm{Lie}(\xi_Q)H = 0$.

So look at $\dot{H}$. This is $\{H, H\}$ which is zero. So H is conserved. Such observables, here's more jargon, are called, and this is why I used Q, are called conserved charges.

So here's the big idea, big enough to put in a box. Symmetries imply conservation laws.

Exercise 1 *Compute these conserved charges. The physical situation is the free particle in Euclidean space. We have the huge symmetry group, which is the isometries of $\mathbb{E}^d$.*

Compute the conserved charges for translations and for rotations. These will be momentum and angular momentum.

The term linear momentum *is sometimes used to distinguish these two types of momenta.*

Okay, let's talk about Lagrangian mechanics. For particles we have solutions to Newton's second law, $\mathscr{M} \subset \mathscr{P} = \mathrm{Map}(M^1, X)$. The idea of Lagrangian mechanics is to describe $\mathscr{M}$ as the critical submanifold of a function $S : \mathscr{P} \to \mathbb{R}$.

This function S is called the action, and $\mathscr{M}$ would be paths x such that $\delta S(x) = d_{\mathscr{P}} S(x) = 0$. So δ is the exterior derivative on $\mathscr{P}$.

This is the variational principle: we want the action to be stationary with respect to variations $\delta x^i(t)$ (which form a basis of $H^*(\mathscr{P})$) in the path $x^i(t)$. This philosophy can be motivated in various ways with various degrees of rigor. One such (rigorless) way is the following. In Newtonian mechanics, particle motion tends to minimize the potential energy; if a ball is sitting on an inclined plane it will roll to the bottom. The action principle is the precise embodiment of this intuition.

[Is this why physicists want a path integral?]

That's for quantum mechanics.

The Path Integral and the Principle of Least Action: A preview *Quantum theory introduces a fundamental unit of action $\hbar$ called Planck's constant. The path integral Z is the probability amplitude for a particle at position $\mathbf{x}_i$ at time t_i to be found at a $\mathbf{x}_f$ at a later time t_f. It is given (schematically) by*

$$Z = \int_{\mathbf{x}(t_i)=\mathbf{x}_i}^{\mathbf{x}(t_f)=\mathbf{x}_f} [\mathrm{d}\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} S[\mathbf{x}(t)]\right) \tag{2}$$

The boundary conditions on the path are indicated by the "limits of integration" and $[\mathrm{d}\mathbf{x}(t)]$ is a "measure" on the space of paths $\mathscr{P}$. This formula expresses the fact that the probability of finding a particle at $\mathbf{x}_f$ at time t_f given that it was at $\mathbf{x}_i$ at time t_i is given by a sum over all paths (a.k.a. "histories") with these boundary conditions weighted by a unimodular complex number whose phase is the action ($\hbar = 1$ in natural units). Now consider the classical limit *$\hbar \to 0$. When the action is away from its stationary point, any small deviation in the path causes wild fluctuations in the exponential with "frequency" $\frac{1}{\hbar} \to \infty$. The claim is that these fluctuations average to 0 so that the path integral has, in the classical limit, support only on those paths for which the action is extremal, that is $\delta S = 0$. The principle of least action therefore follows naturally from the quantum principle of "sum over histories".*

So these equations, call these x paths, they satisfy what are called Euler-Lagrange equations. We'll eventually see that these are just Newton's second law. Let me just continue with the philosophical baloney. This sort of variational principle is also found in geometry, where it used to obtain nice PDEs, like the harmonic PDE.

The terms "Newton's second law", "Euler-Lagrange equation" and "Hamilton's equations" are all examples of "equations of motion". The phrase "equation of motion" or EOM *is used interchangeably (and non-commitally) with any of these.*

The Lagrangian approach gives us back our phase space, but it gives us a lot more than that. The symplectic form was borrowed and depended on a time t. In the Lagrangian

approach, we'll get, the information embedded in this Lagrangian mechanics, which are the Euler Lagrange equations and the submanifold $\mathscr{M}$, but also a family of one-forms on $\mathscr{M}$ parameterized by time. Finally, these one-forms will give us the symplectic structure naturally, and that won't depend on t.

I've kept you guys ten minutes long, I apologize. But in this sense, physicists equate "theory" with a particular Lagrangian, which has all of this information in it.

[I thought it was the action?]

That's the integral of the Lagrangian, which I think is more basic.