Math 552: Introduction to Lie Groups and Lie Algebras

Fall 2008
Department of Mathematics
SUNY at Stony Brook

This course will offer an introduction to the theory of Lie groups, Lie algebras, and their representations. This material is frequently used by mathematicians in a wide variety of fields. A tentative syllabus/schedule is given below, but the course may differ slightly depending on the interests of the students taking the course.

Time/Location: MWF 9:35am-10:30am, Physics P122

Instructor: Corbett Redden
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Office Hours: MWF 10:40 - 11: 40a, or drop-in, or by appointment.

Textbook: *Introduction to Lie groups and Lie algebras*, by Alexander Kirillov. Cambridge Studies in Advanced Mathematics (No. 113), 2008.

There are a number of good books on the subject, including:

- *Representation Theory: A First Course* by William Fulton and Joe Harris
- *Lie Groups* by J.J. Duistermaat and J.A.C. Kolk
- *Complex Semisimple Lie Algebras* by Jean-Pierre Serre
- *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* by Brian C. Hall

Prerequisites: Students are expected to be familiar with most of the material of Math 530-531 (Geometry/Topology I-II) and Math 534-535 (Algebra I-II). Manifolds and group/algebra/module theory will be frequently used.

Requirements: There will be regular homework assignments (less than a core course, but more than a topics course) and a final exam/project. Possible topics for a final project are listed here.

Syllabus:

Week	Notes	Section	Description
9/1 - 9/5	No class 9/1	§2.1-§2.6	Definition of Lie groups, subgroups, cosets, group actions on manifolds, homogeneous spaces.
9/8 - 9/12	No class 9/12	§2.7-§3.6	Classical groups, exponential and logarithmic maps, Lie bracket, Lie algebras, subalgebras, ideals, stabilizers, center
9/15 - 9/19	Homework 1 §2: 6-8. §3: 1,3,9 Due Friday 9/26	§3.7-§3.10	Baker-Campbell-Hausdorff formula, Lie's Theorems (above all done in slightly different order)
9/22 -		§4.1-	Representations, operations on representations, irreducible representations,

9/26		§4.4	Schur's lemma
9/29 - 10/3	No class (9/30,) 10/1	§4.5- §4.7	Unitary representations and complete reducibility, representations of finite groups, Haar measure on compact Lie groups, characters
10/6 - 10/10	(No class 10/9)	§4.7, §5.1- §5.3	Peter-Weyl theorem, universal enveloping algebra and Poincare-Birkoff-Witt, commutants
10/13 - 10/17	Homework 2 Due 10/17	§5.4- §5.8	Solvable and nilpotent Lie algebras (with Lie/Engel theorems), semisimple and reductive algebras, invariant bilinear forms, Killing form, Cartan criteria
10/20 - 10/24		§5.9- §6.3	Jordan decomposition, complex semisimple Lie algebras, compact groups/algebras, complete reducibility of representations
10/27 - 10/31		§4.8, §6.4- §6.6	Toral subalgebras, Cartan subalgebras, root systems
11/3 - 11/7	Homework 3 Due 11/7 Graph Paper	§6.7- §7.3	Regular elements and Cartan subalgebras, abstract root systems, Weyl group, rank 2 root systems
11/10 - 11/14		§7.4- §7.7	Positive roots, simple roots, weight lattice, root lattice, Weyl chambers, simple reflections
11/17 - 11/21		§7.8- §7.10	Dynkin diagrams, classification of root systems, classification of semisimple Lie algebras
11/24 - 11/28	No class (11/27,) 11/28	§8.1- §8.2	Representations of semisimple Lie algebras, weight decomposition, characters, highest weight representations, Verma modules
12/1 - 12/5		§8.3- §8.6	Classification of irreducible finite-dimensional representations, BGG resolution, Weyl character formula
12/8 - 12/12			Class projects/assorted topics, end of §8
12/15 - 12/19	Last class 12/15		

Disabilities: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.www.ehs.stonybrook.edu/fire/disabilities.shtml

MATH 552: LIE GROUPS AND ALGEBRAS

SUGGESTED PROJECT TOPICS

Each student enrolled in the course is expected to give a presentation during the last few classes. I will need to know ahead of time what you want to talk about so that this can be scheduled properly. You are free to talk on anything generally related to Lie groups, Lie algebas, and representation theory. Below is a list of possible suggestions, but you do not need to choose from this list and are encouraged to choose something that interests you. If you have trouble deciding, try looking through another text book and finding something that we didn't cover but you would like to learn.

(1) Spin groups, representations of Spin(n) vs. SO(n)
(2) Canonical commutation relations in quantum mechanics and the Stone–von Neumann theorem
(3) Exceptional groups and algebras, i.e. G_2, F_4, E_6, E_7, E_8
(4) Representations of finite groups (e.g. permutation groups) [FH]
(5) The Laplace operator on S^2 and the hydrogen atom ([K] 4.9)
(6) The Casimir element/operator ([K] 6.3)
(7) Lie algebra cohomology (Chevalley–Eilenberg)
(8) Cohomology of $BU(n)$ and characteristic classes
(9) Characteristic class computations and representation theory (Borel–Hirzebruch)
(10) Borel–Weil theorem (highest weight representations as equivariant sections of vector bundles)
(11) BCG resolution ([K] 8.4)
(12) Weyl character formula ([K] 8.5)
(13) Young diagrams nd representations of $sl(n;\mathbb{C})$ ([K] 8.7, [FH])
(14) Weyl integration formula
(15) Differential geometry of Lie groups
(16) Principal G-bundles and connections/curvature

[K]=Kirillov, [FH]=Fulton–Harris.

MATH 552 HOMEWORK 2 REVISED. DUE 10/17

1. Let N be the subgroup of real upper-triangular 3×3 matrices with 1's on the diagonals; i.e. an element of N is of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{R}.$$

Let Z be the discrete normal subgroup (isomorphic to $\mathbb{Z}$) of matrices of the form

$$\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n \in \mathbb{Z}.$$

The group $H = N/Z$ is called the Heisenberg group (sometimes N is called the Heisenberg group).

(1) Show that the Hilbert space $L^2(\mathbb{R})$ (of $\mathbb{C}$-valued functions) is naturally a representation of H by

$$\left(\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} f\right)(x) := e^{-2\pi i c} e^{2\pi i b x} f(x-a)$$

(2) **Changed** Show that S^1 is naturally a compact abelian subgroup of H, and there is a short exact sequence

$$1 \to S^1 \to H \to \mathbb{R}^2 \to 1.$$

(3) Show that H has no faithful finite-dimensional representation, and therefore H is not isomorphic to a matrix group.

Hint: Use the S^1 subgroup and Lemma 4.21 in Kirillov to decompose a representation V of H. For any element in S^1, what will be the determinant of its action on V? It will help to show that any element of the central S^1 subgroup is in the commutator subgroup $[H, H]$ (where $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ is the group commutator).

Kirillov: 3.15, 4.5, 4.8

MATH 552 HOMEWORK 3. DUE 11/7

(1) (a) Show that the symmetric power representations $S^k\mathbb{C}^2$ are isomorphic, as $sl(2,\mathbb{C})$ representations, to the irreducible representations V_k of highest weight k.
(b) Write $(C^2)^{\otimes 3}$ as a sum of irreducible $sl(2,\mathbb{C})$ representations.
(c) Write $S^2\mathbb{C}^2 \otimes S^3\mathbb{C}^2$ as a sum of irreducible $sl(2,\mathbb{C})$ representations. Do the same for $S^2\mathbb{C}^2 \otimes S^5\mathbb{C}^2$. (Do you notice a pattern?)

(2) Decompose $(\mathbb{C}^3)^{\otimes 2} \otimes (\mathbb{C}^{3*})$ as a sum of irreducible $sl(3,\mathbb{C})$ representations. You may use the fact that the irreducible representation $\Gamma_{2,1}$ of highest weight $(2,1)$ has dimension 15. You can print off triangular graph paper at http://incompetech.com/graphpaper/triangle/

(3) In last homework (problem 4.5), you showed that if V is an irreducible representation of $\mathfrak{g}$, then the space of invariant symmetric bilinear forms on V is either 0 or 1-dimensional, depending on whether $V \cong V^*$ as $\mathfrak{g}$-representations.
(a) (5.6) If $\mathfrak{g}$ is a simple Lie algebra, show that the symmetric invariant bilinear form is unique up to a factor and that $\mathfrak{g} \cong \mathfrak{g}^*$ as representations of $\mathfrak{g}$.
(b) Show that any irreducible $sl(2,\mathbb{C})$ representation admits a non-zero invariant bilinear form.
(c) Show that this is not true for $sl(3,\mathbb{C})$ representations.
(d) Characterize, using the weights, when an irreducible $sl(3,\mathbb{C})$ representation is isomorphic to its dual representation (i.e. self-dual).

(4) (Optional) 5.3 Show that for $\mathfrak{g} = sl(n,\mathbb{C})$, the Killing form is given by
$$K(x,y) = 2n\operatorname{tr}(xy)$$
where tr is the usual trace of the matrix representation.