This course will offer an introduction to the theory of Lie groups, Lie algebras, and their representations. This material is frequently used by mathematicians in a wide variety of fields. A tentative syllabus/schedule is given below, but the course may differ slightly depending on the interests of the students taking the course.

**Time/Location:** MWF 9:35am-10:30am, Physics P122

**Instructor:** Corbett Redden  
Math Tower 3-114. Phone: 632-8261. email: redden at math dot sunysb dot edu  
Office Hours: MWF 10:40 - 11:40a, or drop-in, or by appointment.


There are a number of good books on the subject, including:

- *Representation Theory: A First Course* by William Fulton and Joe Harris  
- *Lie Groups* by J.J. Duistermaat and J.A.C. Kolk  
- *Complex Semisimple Lie Algebras* by Jean-Pierre Serre  
- *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* by Brian C. Hall

**Prerequisites:** Students are expected to be familiar with most of the material of Math 530-531 (Geometry/Topology I-II) and Math 534-535 (Algebra I-II). Manifolds and group/algebra/module theory will be frequently used.

**Requirements:** There will be regular homework assignments (less than a core course, but more than a topics course) and a final exam/project. Possible topics for a final project are listed [here](#).

**Syllabus:**

<table>
<thead>
<tr>
<th>Week</th>
<th>Notes</th>
<th>Section</th>
<th>Description</th>
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<tbody>
<tr>
<td>9/1 - 9/5</td>
<td>No class 9/1</td>
<td>§2.1-</td>
<td>Definition of Lie groups, subgroups, cosets, group actions on manifolds, homogeneous spaces.</td>
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<td>§2.6</td>
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<tr>
<td>9/8 - 9/12</td>
<td>No class 9/12</td>
<td>§2.7-</td>
<td>Classical groups, exponential and logarithmic maps, Lie bracket, Lie algebras, subalgebras, ideals, stabilizers, center</td>
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<td>§3.6</td>
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<tr>
<td>9/15 - 9/19</td>
<td>Homework 1</td>
<td>§2: 6-8. §3: 1,3,9 Due Friday 9/26</td>
<td>Baker-Campbell-Hausdorff formula, Lie's Theorems (above all done in slightly different order)</td>
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<td>§3.7-</td>
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<td>§3.10</td>
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<tr>
<td>9/22 -</td>
<td>§4.1-</td>
<td></td>
<td>Representations, operations on representations, irreducible representations,</td>
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<td>Date</td>
<td>Section(s)</td>
<td>Topics</td>
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<tr>
<td>9/26</td>
<td>§4.4</td>
<td>Schur's lemma</td>
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<tr>
<td>9/29-10/3</td>
<td>§4.5-§4.7</td>
<td>Unitary representations and complete reducibility, representations of finite groups, Haar measure on compact Lie groups, characters</td>
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<tr>
<td>10/6-10/10</td>
<td>§4.7, §5.1-§5.3</td>
<td>Peter-Weyl theorem, universal enveloping algebra and Poincare-Birkhoff-Witt, commutants</td>
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<tr>
<td>10/13-10/17</td>
<td>§5.4-§5.8</td>
<td>Solvable and nilpotent Lie algebras (with Lie/Engel theorems), semisimple and reductive algebras, invariant bilinear forms, Killing form, Cartan criteria</td>
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<tr>
<td>10/20-10/24</td>
<td>§5.9-§6.3</td>
<td>Jordan decomposition, complex semisimple Lie algebras, compact groups/ algebras, complete reducibility of representations</td>
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<tr>
<td>10/27-10/31</td>
<td>§4.8, §6.4-§6.6</td>
<td>Toral subalgebras, Cartan subalgebras, root systems</td>
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<tr>
<td>11/3-11/7</td>
<td>§6.7-§7.3</td>
<td>Regular elements and Cartan subalgebras, abstract root systems, Weyl group, rank 2 root systems</td>
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<tr>
<td>11/10-11/14</td>
<td>§7.4-§7.7</td>
<td>Positive roots, simple roots, weight lattice, root lattice, Weyl chambers, simple reflections</td>
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<td>11/17-11/21</td>
<td>§7.8-§7.10</td>
<td>Dynkin diagrams, classification of root systems, classification of semisimple Lie algebras</td>
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<tr>
<td>11/24-11/28</td>
<td>§8.1-§8.2</td>
<td>Representations of semisimple Lie algebras, weight decomposition, characters, highest weight representations, Verma modules</td>
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<tr>
<td>12/1-12/5</td>
<td>§8.3-§8.6</td>
<td>Classification of irreducible finite-dimensional representations, BGG resolution, Weyl character formula</td>
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<tr>
<td>12/8-12/12</td>
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<td>Class projects/assorted topics, end of §8</td>
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<tr>
<td>12/15-12/19</td>
<td>Last class 12/15</td>
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MATH 552: LIE GROUPS AND ALGEBRAS

SUGGESTED PROJECT TOPICS

Each student enrolled in the course is expected to give a presentation during the last few classes. I will need to know ahead of time what you want to talk about so that this can be scheduled properly. You are free to talk on anything generally related to Lie groups, Lie algebras, and representation theory. Below is a list of possible suggestions, but you do not need to choose from this list and are encouraged to choose something that interests you. If you have trouble deciding, try looking through another text book and finding something that we didn’t cover but you would like to learn.

1. Spin groups, representations of Spin(n) vs. SO(n)
2. Canonical commutation relations in quantum mechanics and the Stone–von Neumann theorem
3. Exceptional groups and algebras, i.e. $G_2, F_4, E_6, E_7, E_8$
4. Representations of finite groups (e.g. permutation groups) [FH]
5. The Laplace operator on $S^2$ and the hydrogen atom ([K] 4.9)
6. The Casimir element/operator ([K] 6.3)
7. Lie algebra cohomology (Chevalley–Eilenberg)
8. Cohomology of $BU(n)$ and characteristic classes
9. Characteristic class computations and representation theory (Borel–Hirzebruch)
10. Borel–Weil theorem (highest weight representations as equivariant sections of vector bundles)
11. BCG resolution ([K] 8.4)
12. Weyl character formula ([K] 8.5)
13. Young diagrams nd representations of $sl(n; \mathbb{C})$ ([K] 8.7, [FH])
14. Weyl integration formula
15. Differential geometry of Lie groups
16. Principal $G$-bundles and connections/curvature

1. Let $N$ be the subgroup of real upper-triangular $3 \times 3$ matrices with 1’s on the diagonals; i.e. an element of $N$ is of the form
\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
\]
$a, b, c \in \mathbb{R}$.

Let $Z$ be the discrete normal subgroup (isomorphic to $\mathbb{Z}$) of matrices of the form
\[
\begin{pmatrix}
1 & 0 & n \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
n \in \mathbb{Z}.
\]

The group $H = N/Z$ is called the Heisenberg group (sometimes $N$ is called the Heisenberg group).

(1) Show that the Hilbert space $L^2(\mathbb{R})$ (of $\mathbb{C}$-valued functions) is naturally a representation of $H$ by
\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix} f(x) := e^{-2\pi ic} e^{2\pi ibx} f(x - a)
\]

(2) Changed Show that $S^1$ is naturally a compact abelian subgroup of $H$, and there is a short exact sequence
\[
1 \to S^1 \to H \to \mathbb{R}^2 \to 1.
\]

(3) Show that $H$ has no faithful finite-dimensional representation, and therefore $H$ is not isomorphic to a matrix group.

Hint: Use the $S^1$ subgroup and Lemma 4.21 in Kirillov to decompose a representation $V$ of $H$. For any element in $S^1$, what will be the determinant of its action on $V$? It will help to show that any element of the central $S^1$ subgroup is in the commutator subgroup $[H, H]$ (where $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ is the group commutator).

Kirillov: 3.15, 4.5, 4.8
MATH 552 HOMEWORK 3. DUE 11/7

(1) (a) Show that the symmetric power representations $S^k \mathbb{C}^2$ are isomorphic, as $sl(2, \mathbb{C})$ representations, to the irreducible representations $V_k$ of highest weight $k$.
(b) Write $(\mathbb{C}^2)^{\otimes 3}$ as a sum of irreducible $sl(2, \mathbb{C})$ representations.
(c) Write $S^2 \mathbb{C}^2 \otimes S^3 \mathbb{C}^2$ as a sum of irreducible $sl(2, \mathbb{C})$ representations. Do the same for $S^2 \mathbb{C}^2 \otimes S^5 \mathbb{C}^2$. (Do you notice a pattern?)

(2) Decompose $(\mathbb{C}^3)^{\otimes 2} \otimes (\mathbb{C}^3^*)$ as a sum of irreducible $sl(3, \mathbb{C})$ representations. You may use the fact that the irreducible representation $\Gamma_{2,1}$ of highest weight $(2, 1)$ has dimension 15. You can print off triangular graph paper at http://incompetech.com/graphpaper/triangle/

(3) In last homework (problem 4.5), you showed that if $V$ is an irreducible representation of $g$, then the space of invariant symmetric bilinear forms on $V$ is either 0 or 1-dimensional, depending on whether $V \cong V^*$ as $g$-representations.
(a) (5.6) If $g$ is a simple Lie algebra, show that the symmetric invariant bilinear form is unique up to a factor and that $g \cong g^*$ as representations of $g$.
(b) Show that any irreducible $sl(2, \mathbb{C})$ representation admits a non-zero invariant bilinear form.
(c) Show that this is not true for $sl(3, \mathbb{C})$ representations.
(d) Characterize, using the weights, when an irreducible $sl(3, \mathbb{C})$ representation is isomorphic to its dual representation (i.e. self-dual).

(4) (Optional) 5.3 Show that for $g = sl(n, \mathbb{C})$, the Killing form is given by
$K(x, y) = 2n \text{tr}(xy)$
where tr is the usual trace of the matrix representation.