COURSE DESCRIPTION

- MAT 541 (Stony Brook)
- Intermediate algebraic topology (CUNY GC)

The FALL COURSE discussed cycles and homologies by analogy with oriented closed manifolds and compact manifolds with boundary. It was important that the boundaries of homologies have collar neighborhoods so that geometric gluing was possible. The new feature beyond manifolds per se was non manifold points form a non-empty sub complex. The minimal gluing of singular chains which are cycles resulted in polyhedra where the non-manifold points have codimension three. Thus every two cycle is represented by a map of a closed surface. Cycles mapping into a target space up to homology defined by homologies mapping into the space formed a group graded by dimension. These groups were the usual homology groups of the space.

If the singularities were restricted in some way which was geometrically natural [preserved] by transversal intersection] then all but one of the axioms of usual homology theory held for the same construction using these restricted singular cycles and homologies. These are called generalized homology theories when some of the groups of a point are non-zero in some nonzero degrees. A basic paper of George Whitehead describes these theories by spatial objects [actually sequences of spaces indexed by all of the integers with maps between the suspension of one to the next]. Examples are Bott-Atiyah complex K-theory and Thom's cobordism.

The SPRING COURSE will be independent of the fall semester and will discuss new developments especially two geometric discussions of differential K -theory. However the above intuitions about cycles and homologies play an important role and will be revisited.

Generalized homology theories can be defined with coefficients and have dual versions called cohomology theories which may have cup products. One has a canonical DeRham theorem isomorphism describing the dual cohomology theory with real coefficients by differential forms labeled by the generalized cohomology groups of a point.

We will describe George Whitehead's spatial construction which allows important manipulations like coefficients and duality. We will describe the generalized DeRham theorem.

In 2004 Hopkins and Singer used the DeRham isomorphism mentioned above to construct a "differential cohomology theory" for anyone of these generalized cohomology theories which combines the latter with differential forms.

We will discuss these for two examples: ordinary cohomology and complex K theory. The ordinary differential cohomology theory was already constructed by Jeff Cheeger and Jim Simons at Stony Brook in 1972. Blaine Lawson constructed other geometric variants at Stony Brook in the early 2000's.

In 2008 axioms characterizing ordinary differential cohomology were described.

In 2009 differential K theory was described by vector bundles with connections up to Chern Simons equivalences of connections.

In 2015 a dual geometric description to that one was described by numerical functions on odd dimensional geometric cycles called differential characters for differential K theory.

The numerical invariants lie in the complexified circle and are computed by integrating certain differential forms defined by the curvature of the connection over even dimensional manifolds bounding the cycles and reducing modulo one. Chern Simons equivalences of connections imply a bit more than these differential forms are not altered.

A noteworthy corollary is that one may construct from these invariants a complete set of numerical invariants for complex vector bundles in the complex roots of unity, [in the stable range].

We will discuss these three results, and say some words about the following:

One can relate for unitary connections the general invariants to spectral invariants of dirac operators over the odd dimensional cycles and to a families index theorem for differential K theory.

The more recent work is joint with Jim Simons with an assist from Leon Takhtajan about constructing inverses of complex vector bundles with complex linear connections.