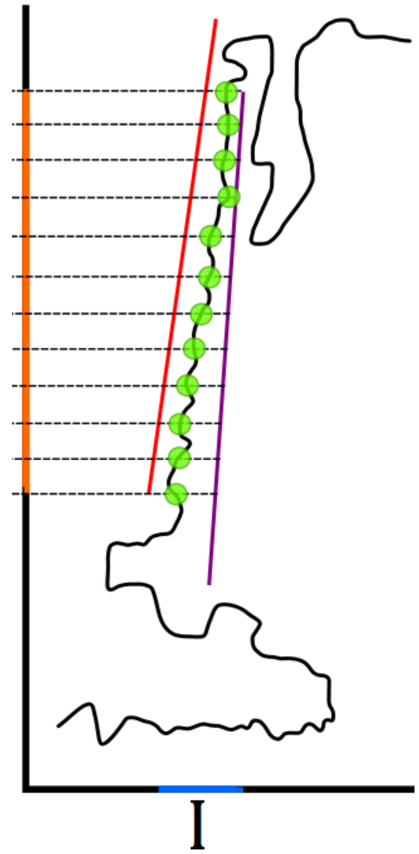
MAT 550 - Spring 2014 - Real Analysis II

Real Analysis II



[**Course Syllabus** | link to previous course: **Real Analysis I**]

Instructor's Office Hours in Math 4-117:
(most) Mondays and Wednesdays 2:30 - 4:00

Important Dates

Midterm Exams
Exam 1: Thursday, February 27, in class.
Exam 2: Thursday, April 10, in class.

Final Exam
Friday, May 16, 11:15 am - 1:45 pm

Homework

Some Remarks on Writing Mathematical Proofs
by Jack Lee

Homework #1
Due: Tuesday, February 4
Homework #2
Due: Tuesday, February 11
Homework #3
Due: Tuesday, February 18
Homework #4
Due: Tuesday, February 25
Homework #5
Due: Tuesday, March 11
Homework #6/7
Due: Tuesday, March 25
Homework #8
Due: Tuesday, April 1
Homework #9
Due: Tuesday, April 8
Homework #10
Due: Tuesday, April 22
Homework #11
Due: Tuesday, April 29
Homework #12

Due: Thursday, May 8

Last updated: April 28, 2014

MAT 550, Real Analysis II, Spring 2014 — Syllabus

Math Tower 4-130 — Tuesday, Thursday 10:00 - 11:20

This syllabus contains the policies and expectations that the instructor has established for this course. Please read the entire syllabus carefully before continuing in this course. These policies and expectations are intended to create a productive learning atmosphere for all students. Unless you are prepared to abide by these policies and expectations, you risk losing the opportunity to participate further in the course.

Instructor: Dr. Matthew Badger (badger@math.sunysb.edu)
Office: Math Tower 4-117
Office Hours: Mondays 2:30 - 4:00, Wednesdays 2:30 - 4:00, and By Appointment

Course Description

This course is the second half of a two course introduction to real analysis. Topics to be covered include: existence and uniqueness of ODEs, topology (e.g. Tychonoff's theorem, Arzelá-Ascoli theorem), functional analysis (e.g. the Hahn-Banach theorem, open mapping theorem), L^p spaces, Radon measures, Fourier analysis. Additional topics will be chosen as time permits.

Important Dates

Observed Holidays and Breaks (Spring 2014)

- March 17–23: no classes (Spring recess).

Exam Dates

- Midterm Exams, To Be Announced on the Course Website at Least One Week in Advance
- May 16: Final Exam, 11:15am – 1:45pm.

Required Resources

- **Course Webpage:** www.math.sunysb.edu/∼badger/ → Link to MAT 550
- **Lecture Notes:** Attend lectures regularly and take your own notes.
- **Textbook:** Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second Edition, Wiley, John & Sons, 1999. (Reading and homework assignments will be assigned out of this textbook. Make sure you can access a copy of the textbook.)

About Attendance

Attendance is highly encouraged. **Lectures may include material not in the textbook!** The lectures may also present material in a different order than it in the textbook.

Graded Components

- **Homework** – 40% of course average
  There will be a homework assignment due in class on most Tuesdays, starting on Tuesday, February 4. Homework assignments will be posted on the course webpage.
- **Midterm Exams** – 30% of course average,
  There will be one or more closed book, closed notes midterm exam(s) in class. The exams will be scheduled during the course, with exam dates posted at least one week in advance on the course webpage.
- **Final Exam** – 30% of course average
  There will be one closed book, closed notes final exam as scheduled by the university on Friday, May 16 from 11:15 am – 1:45 pm˙

Your *course average* will be determined by a weighted average of the graded components above. Your *final grade* for the class will be based on your course average *and* on your participation.

Late Homework Policy

A student's homework assignment shall be considered late if it is not turned in to the instructor by the end of lecture on the due date. Late homework assignments may be turned in to the instructor by the end of the first lecture after the due date, but will be penalized 40%. Late homework assignments will not be accepted more than one lecture past the due date.

Missed Exam Policy

No make-up exams will be given. If a student misses a midterm exam, then the student's final exam grade will be substituted for the missed midterm. A student must sit the final exam at the scheduled time in order to receive a passing grade in the class.

Classroom Policies

Students are expected to arrive to lecture on time and remain until the lecture is concluded. (Leaving early creates distraction and is disrespectful to the instructor and your fellow students.) Cell phones should be silenced for the duration of the lecture. Tablet and laptop computers should not be used during lecture, except for taking notes.

Disability Support Services

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services (631) 632-6748 or

`studentaffairs.stonybrook.edu/dss/`

They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

`www.sunysb.edu/facilities/ehs/fire/disabilities`

Academic Integrity

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

`www.stonybrook.edu/uaa/academicjudiciary/`

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

Syllabus Revision

The standards and requirements set forth in this syllabus may be modified at any time by the course instructor. Notice of such changes will be by announcement in class and changes to this syllabus will be posted on the course website.

This syllabus was last revised on January 21, 2014.

Math 550, Real Analysis II, Spring 2014 **Homework #1**

Due In Class: February 4, 2014

Reading: Start reading Chapter 4 in Folland's *Real Analysis (second edition)*.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercises 4.1 and 4.2.

Problem B. Exercise 4.4.

Problem C. Exercise 4.6.

Problem D. Exercise 4.10.

Problem E. Exercises 4.11 and 4.12.

Problem F. Let (X, d) be a metric space, and let $A, B \subseteq X$ be nonempty sets. The *excess of A over B*, denoted by $e_d(A, B)$, is defined by

$$e_d(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b) \in [0, \infty].$$

The *Hausdorff distance between A and B*, denoted by $\mathrm{HD}_d(A, B)$, is defined by

$$\mathrm{HD}_d(A, B) := \max\{e_d(A, B), e_d(B, A)\} \in [0, \infty].$$

Let $\mathrm{CLB}_d(X)$ denote the collection of nonempty, closed and bounded subsets of X. (Recall that a nonempty set $A \subseteq X$ is bounded if it has finite diameter, i.e. $\operatorname{diam} A := \sup_{a,a' \in A} d(a, a') < \infty$.)

Prove that $(\mathrm{CLB}_d(X), \mathrm{HD}_d)$ is a metric space. (Hint: Show that $e_d(A, C) \le e_d(A, B) + e_d(B, C)$.)

Due In Class: February 11, 2014

Reading: Continue reading Chapter 4.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 4.14

Problem B. Exercise 4.23

Problem C. Let $A \subset \mathbb{R}$. A map $f : A \to \mathbb{R}$ is *Lipschitz* if there exists a constant $L \in [0, \infty)$ such that

$$|f(a) - f(b)| \leq L|a - b| \quad \text{for all } a, b \in A. \tag{1}$$

The smallest constant L such that (1) holds is called the *Lipschitz constant* of f and is denoted by $\mathrm{Lip}(f)$. Show that for every Lipschitz map $f : A \to \mathbb{R}$ there exists a Lipschitz map $F : \mathbb{R} \to \mathbb{R}$ such that $F|_A = f$ and $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

Problem D. Let $E \subset \mathbb{R}^n$ be closed set. If $I = [x - r, x + r]$ and $\lambda > 0$, let $\lambda I = [x - \lambda r, x + \lambda r]$. Let Δ denote a maximal collection of almost disjoint closed dyadic intervals $I \subset \mathbb{R}$ such that $3I \cap E = \emptyset$. Let $d(x) = \mathrm{dist}(x, E)$. Show that

(1) $\bigcup_{I \in \Delta} I = \bigcup_{I \in \Delta} 2I = \mathbb{R} \setminus E$.
(2) $\mathrm{diam}\, I \leq d(x) \leq 4 \,\mathrm{diam}\, I$ for all $I \in \Delta$ and $x \in I$.
(3) $0.5 \,\mathrm{diam}\, I \leq d(x) \leq 4.5 \,\mathrm{diam}\, I$ for all $I \in \Delta$ and $x \in 2I$.
(4) there exists $N \in \mathbb{N}$ such that $\sum_{I \in \Delta} \chi_{2I}(x) \leq N$ for all $x \in \mathbb{R}$.

Show that there exists a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(0) = E$.
[*Hint:* To start, show that for each $I \in \Delta$ there exists a C^1 function $\phi_I : \mathbb{R} \to [0, \infty)$ such that $\phi_I(x) = 1$ for all $x \in I$ and $\phi_I(x) = 0$ for all $x \in \mathbb{R} \setminus 2I$.]

Problem E. Exercise 4.34

Math 550, Real Analysis II, Spring 2014 — Homework #3

Due In Class: February 18, 2014

Reading: Finish reading Chapter 4.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Let $X = (\mathrm{CLB}(\mathbb{R}^n), \mathrm{HD})$ denote the metric space of nonempty, closed and bounded sets in $\mathbb{R}^n$ equipped with the Hausdorff distance (defined on HW 1). Show that every uniformly bounded sequence $(F_i)_{i=1}^{\infty}$ in X has a convergent subsequence in X.

[*Remark:* There are two natural ways to interpret "uniformly bounded" in the previous sentence, but they are equivalent *a posteriori*. In your solution, pick one and state it precisely.]

Problem B. Exercise 4.51

Problem C. Exercise 4.57

Problem D. Let $1 \leq m \leq n$ and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be continuous. Suppose that there exists a (weakly) increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ (onto $[0, \infty)$) such that

$$|x - a| \leq t|y - a| \Longrightarrow |f(x) - f(a)| \leq \eta(t)|f(y) - f(a)| \quad \text{for all } x, y, a \in \mathbb{R}^m.$$

Show that either f is constant or f is a homeomorphism from $\mathbb{R}^m$ onto $f(\mathbb{R}^m)$.

[*Hint:* First show that f is either constant or injective. Then show that if f is nonconstant, then f extends to a continuous map $F : \mathbb{R}^m \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$.]

Problem E. Exercise 4.64

Due In Class: February 25, 2014

Announcement: First midterm is in class on Thursday, February 27, 2014.
Reading: Start reading Chapter 5.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 4.64 (carried over from last week).

Problem B. Let $K \in C([0,1] \times [0,1], \mathbb{R})$. For $f \in C([0,1])$, define

$$Tf(x) = \int_0^1 K(x,y)f(y)dy \quad \text{for all } x \in [0,1].$$

Show that $Tf \in C([0,1], \mathbb{R})$ and $\{Tf : \|f\|_\infty \leq 1\}$ is precompact in $C([0,1])$.

Problem C. Exercise 4.67.

Problem D. Exercise 4.68.

Problem E. Exercise 4.69.

Due In Class: March 11, 2014

Reading: Continue reading Chapter 5, skipping §5.4.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 5.1.

Problem B. Exercise 5.4.

Problem C. Exercise 5.7.

Problem D. Exercise 5.22.

Problem E. Exercise 5.26.

Due In Class: March 25, 2014

Reading: Finish reading Chapter 5. (You may skip §5.4.)

This is a double assignment (half of your total score will count as Homework #6, half of your total score will count as Homework #7).

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 5.12.

Problem B. Exercise 5.18.

Problem C. Exercise 5.19.

Problem D. Exercise 5.27.

Problem E. Exercise 5.29.

Problem F. Exercise 5.32.

Problem G. Exercise 5.37.

Problem H. Exercise 5.38.

Problem I. Exercise 5.40.

Problem J. Exercise 5.41.

Due In Class: April 1, 2014

Announcement: The second midterm will be Thursday, April 10, in class.

Reading: Start reading Chapter 6

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 5.54.

Problem B. Exercise 5.55.

Problem C. Exercise 5.59.

Problem D. Exercise 5.66.

Problem E. Exercise 5.47(a) and Exercise 5.53.

Due In Class: April 8, 2014

Reminder: The second midterm will be Thursday, April 10, in class.

Reading: Continue reading Chapter 6

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 6.2.

Problem B. Exercise 6.4.

Problem C. Exercise 6.6.

Problem D. Exercise 6.7.

Problem E. Exercise 6.10.

Due In Class: April 22, 2014

Reading: Finish reading Chapter 6.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 6.20.

Problem B. Exercise 6.21.

Problem C. Exercise 6.32.

Problem D. Exercise 6.36.

Problem E. Exercise 6.39.

Due In Class: April 28, 2014

Reading: Chapter 8, Sections 1–3.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Compare and contrast the Riesz-Thorin interpolation theorem and the Marcinkiewicz Interpolation Theorem in one or two paragraphs.

Problem B. Suppose T is a sublinear operator of weak type (1,4) and strong type (2,2).

(1) What can you conclude about how T acts on L^p for $0 < p \leq \infty$ in general?

(2) What can you conclude if, in addition, you know the underlying measure spaces are finite?

Problem C. Exercise 6.41.

Problem D. Exercise 8.1.

Problem E. Exercise 8.4.

This is the last homework assignment.

Due In Class: Thursday, May 8, 2014

Reading: Chapter 8, sections 2 and 3; Chapter 7, sections 1 and 3.

Turn in the following problems. Exercise a.b refers to Exercise b in Chapter a of the textbook.

Problem A. Exercise 8.6

Problem B. Exercise 8.8.

Problem C. Exercise 8.18.

Problem D. Prove that if μ is a Radon measure on $\mathbb{R}^n$, then

$$R_\mu = \{r > 0 : \mu(\partial B(x, r)) > 0\}$$

has Lebesgue measure zero. Give an example a finite measure ν on $\mathbb{R}^n$ such that R_ν is infinite.

Problem E. Let m denote Lebesgue measure on $\mathbb{R}$.

(1) Prove that there exists a sequence $(\sigma_i)_{i=1}^\infty$ of finite measures on $\mathbb{R}$ such that
  - (i) $\sigma_i \ll m$ for all $i \geq 1$, and
  - (ii) σ_i converges weakly to m in the sense of Radon measures.

(2) Prove that there exist a sequence $(\tau_i)_{i=1}^\infty$ of finite measures on $\mathbb{R}$ such that
  - (i) $\tau_i \perp m$ for all $i \geq 1$, and
  - (ii) τ_i converges to m weakly in the sense of Radon measures.