MAT 513: Analysis for teachers

Spring 2011

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<th>Schedule</th>
<th>TuTh 5:20pm-6:40pm</th>
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<tbody>
<tr>
<td>Office hours</td>
<td>M-W 5-6 pm or by appointment</td>
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<tr>
<td>Office</td>
<td>Math Tower 3-102</td>
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Text

William C. Bauldry, *INTRODUCTION TO REAL ANALYSIS An Educational Approach*

Course Content

Topics in differential calculus, its foundations, and its applications. This course is designed for teachers and prospective teachers of advanced placement calculus.

Homework

The homework are assigned during class, and posted on the web page. You will have about a week to complete the homework assignment. Late homework will not be accepted.

Here is the list of assignments (click on the links to open the pdf file):

- **Homework 1** Due date: *Tuesday, Feb 8*
- **Homework 2** Due date: *Tuesday, Feb 15*
- **Homework 3** Due date: *Tuesday, Feb 22*
- **Homework 4** Due date: *Thursday, March 10*
- **Homework 5** Due date: *Tuesday, March 22*
- **Homework 6** Due date: *Tuesday, March 29*
- **Homework 7** Due date: *Tuesday, April 5*
- **Homework 8** Due date: *Thursday, April 28*
- **Homework 9** Due date: *Thursday, May 5*

Grading
The grading will be weighted as follows: homework 25%, midterm I 20%, midterm II 20%, final 35%. The grades are available on blackboard.

Exams schedule

**Midterm 1**: Tuesday, March 8th (class time)

**Midterm 2**: Tuesday, April 12 (class time)

**Final**: May 19, 5:15 pm-7:45 pm

Midterm 1: suggested problems
Click [here](#). I will write some more problems soon.

Midterm I: some comments and solutions
Click [here](#).

Midterm 2: suggested problems
Click [here](#).

Final Exam: practice problems
Click [here](#).

Stony Brook University Syllabus Statement

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or [http://studentaffairs.stonybrook.edu/dss/](http://studentaffairs.stonybrook.edu/dss/). They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: [http://www.sunysb.edu/ehs/fire/disabilities.shtml](http://www.sunysb.edu/ehs/fire/disabilities.shtml).
(1) Page 117: number 2.4
(2) Page 117: number 2.6
(3) Consider the closed interval $[0, 1]$. Prove that $\varepsilon = \frac{1}{2}$ is the maximum among the numbers such that the neighborhood

$$N_\varepsilon \left( \frac{1}{2} \right) = \left( \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right)$$

is contained in $[0, 1]$.
(4) Find the accumulation point(s) of the following subset of $\mathbb{R}$:

$$S = \left\{ \frac{1}{n} \mid n = \text{positive integer} \right\}$$
(5) Let $S = \{ x \mid x \in \mathbb{Q} \text{ and } 0 < x < 1 \}$. Determine the set $\overline{S}$, the closure of $S$ in $\mathbb{R}$ (hint: use Theorem 2.5).
(6) Let $S$ be a subset of $\mathbb{R}$, and assume that $s = \text{sup} S$ exists. Prove that if $s \notin S$, then $s$ is an accumulation point for $S$. Show with examples that if $s \in S$, then $s$ can be either isolated or an accumulation point.
(7) (optional) Let $\mathbb{R}^2$ be the plane, with coordinates $(x, y)$. We say that a subset $S$ of $\mathbb{R}^2$ is open if for any point $P = (x_0, y_0) \in S$, there is a $\varepsilon > 0$ such that the set $N_\varepsilon(P) = \{(x, y) \mid \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon\}$ is contained in $S$. Show that the intersection of a finite collection of open subsets is open, and that the union of any collection of open subsets is open.
Homework 2

Homework 2 is due Tuesday, February 15

(1) Page 117-118: numbers 2.14, 2.16, 2.21
(2) Using the definition of limit, prove that
\[ \lim_{x \to 1} x^2 + 1 = 2 \]
(3) Write down carefully what it means that: "the limit of \( f \) as \( x \) approaches \( a \) does not exists ". Then, consider the following function \( f : \mathbb{R} \to \mathbb{R} \):

\[ f(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
x & \text{if } x > 0 
\end{cases} \]

Prove that the limit of \( f \) as \( x \) approaches 0 does not exist.
(4) Prove Theorem 2.11 part 3 (using the definition of continuity)
(5) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Use the Intermediate Value Theorem (2.17) and the Extreme Value Theorem (2.19) to prove that the image (or range) of \( f \) is a closed bounded interval: that is, \( f([a, b]) = [c, d] \) for some real numbers \( c, d \).
Homework 3

Homework 3 is due **Thursday, February 24**

(1) Page 118: numbers 2.18, 2.29, 2.32, 2.33, 2.34
(2) For $n$ positive integer, let $f(x) = x^n$. Prove, from the definition of derivative, that $f'(x) = nx^{n-1}$
(3) Let $f : D \to \mathbb{R}$ be a continous function. Suppose that there exits $c \in D$ such that $f(c) > 0$. Prove that there exits an $\varepsilon > 0$ such that $f(x) > 0$ for any $x \in D \cap (c - \varepsilon, c + \varepsilon)$.
(4) (optional) Find the derivative of the function

$$f(x) = \frac{\tan x + e^{x^2}}{\sqrt{x+1} - \cos x}$$

(5) (optional) Find the absolute maximum and minimum value of $f(x) = x + \frac{1}{x}$ on the interval $[\frac{1}{2}, 2]$.
(6) Prove using a geometrical argument that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
**Homework 4**

Homework 4 is due **Thursday, March 11**.

**Definition.** A function $f : D \to \mathbb{R}$ is called **increasing** (respectively, **decreasing**) if $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) \geq f(x_2)$) for all $x_1, x_2$ in $D$ such that $x_1 < x_2$.

A function $f : D \to \mathbb{R}$ is called **strictly increasing** (respectively, **strictly decreasing**) if $f(x_1) < f(x_2)$ (respectively, $f(x_1) > f(x_2)$) for all $x_1, x_2$ in $D$ such that $x_1 < x_2$.

1. Let $f : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$. Prove that $f$ is decreasing on $(a, b)$ if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.
2. Let $f : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$. Prove that if $f'(x) < 0$, then $f$ is strictly decreasing on $(a, b)$.
3. Give an example of a strictly increasing differentiable function $f$ on $\mathbb{R}$ such that $f'(x) = 0$ for some $x \in \mathbb{R}$.
4. The proof of Theorem 2.33 has a small mistake: find it and fix it.
5. Page 120: numbers 2.50, 2.51, 2.52
Homework 5

Homework 5 is due Thursday, March 17.

Reference for Homework 5: Walter Rudin, *Principles of Mathematical Analysis*

(1) Repeat the proof of Theorem 1.37 to prove that there is a unique positive real number $Y$ such that $Y^2 = 2$. 

(2) Study the definition of sum of two cuts (see Theorem 1.12 and Definition 1.13).

(3) Let $\alpha$ be a cut. Theorem 1.16 states that there is one and only one cut $\beta$ such that $\alpha + \beta = 0^*$ ($0^*$ is the cut defined by 0). In the proof, $\beta$ is defined as the set of all rational numbers $p$ such that $-p$ is an upper bound of $\alpha$ but not the smallest upper bound. What goes wrong if instead we define $\beta = \{-p \mid p \in \alpha\}$?

(4) Study the definition of products of two cuts. Prove that $\alpha \beta = \beta \alpha$ for all cuts $\alpha$ and $\beta$.

(5) Prove that a convergent sequence is a Cauchy sequence.

(6) Prove that a Cauchy sequence is bounded.
Homework 6 is due Tuesday, March 29

(1) Given a convergent sequence $a_n$, consider the set of its values:

$$S = \{a_n \mid n \in \mathbb{N}\}$$

Prove that if $S$ is finite, then the sequence is eventually constant: $\exists N$ such that $a_n = a_N \forall n \geq N$

(2) Given a convergent sequence $a_n$, consider the set of its values:

$$S = \{a_n \mid n \in \mathbb{N}\}$$

Prove that if $S$ is infinite, then $a = \lim a_n$ is the unique accumulation point of $S$.

(3) Give an example of a sequence $a_n$, such that the set

$$S = \{a_n \mid n \in \mathbb{N}\}$$

has at two (or, if you prefer, more than two) accumulation points.

(4) Let $a_n$ be a sequence of real numbers. Consider a strictly increasing sequence

$$n_1 < n_2 < n_3 < ...$$

of positive integers. Then $a_{n_k}$ is a sequence, and it is said to be a subsequence of $a_n$.

a) Let

$$a_n = \frac{(-1)^nn + 1}{n}$$

Prove that $a_n$ is not convergent.

b) Find two subsequences of $a_n$ that are convergent.

(5) Use Bolzano-Weierstrass Theorem (2.54) to prove that if $a_n$ is a bounded sequence, there exists a convergent subsequence of $a_n$.

(6) Determine if the series is convergent or not

$$\sum_{n=0}^{\infty} \frac{n^3 - n}{n^4 + n^2 + 7}$$

(7) Determine the values of $p$ such that the series is convergent:

$$\sum_{n=0}^{\infty} n^p$$

(8) Find the sum of the series:

$$\sum_{n=0}^{\infty} \left(\frac{2^{n-1}}{3^n} + \frac{1}{n!}\right)$$
Homework 7

Homework 7 is due **Tuesday, April 5**

(1) Page 121: numbers 2.63, 2.64, 2.65, 2.66, 2.68
Homework 8

Homework 8 is due **Thursday, April 24**

(1) Compute the following limits

a) \[ \lim_{x \to \infty} \frac{3x + 5}{x - 4} \]

b) \[ \lim_{x \to -3^+} \frac{x + 2}{x + 3} \]

c) \[ \lim_{x \to \frac{\pi}{2}^+} e^{\tan x} \]

(2) Graph the function

a) \[ f(x) = \frac{x^2}{x^2 - 1} \]

b) \[ f(x) = e^{x^2} \]

c) \[ f(x) = \frac{e^x}{1 + e^x} \]

d) \[ f(x) = \sqrt{x^2 + 1} - x \]

(3) Compute the integrals

a) \[ \int_0^1 10^x \, dx \]

b) \[ \int_0^{\frac{\pi}{3}} \frac{\sin x + \sin(x\tan x)^2}{(\sec x)^2} \, dx \]

c) \[ \int_1^e \frac{dx}{x\sqrt{\ln x}} \]

d) \[ \int \frac{dx}{x^2 + 6x + 8} \]
Homework 9

Homework 8 is due Thursday, May 5

Page 121: from 2.69 to 2.73
Exam 1, suggested problems

(1) Prove, from the definition of limit, that the following limit does not exist:

\[ \lim_{x \to 0} \sin \left( \frac{1}{x^2} \right) \]

(2) Prove, using the definition of limit of a sequence, that

\[ \lim_{n \to \infty} \frac{n}{n + 1} = 1 \]

(3) Suppose \( f : (a, b) \to \mathbb{R} \) is an increasing and bounded above on the interval \((a, b)\). Prove that

\[ \sup \{ f(x) \mid x \in (a, b) \} = \lim_{x \to b^-} f(x) \]

(4) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function continuous at \( a \). Suppose that \( f(a) = 1 \). Prove that there exists a neighborhood \( N = (a - \varepsilon, a + \varepsilon) \) of \( a \) such that \( f(x) > 0 \) \( \forall x \in N \).

(5) Let \( f(x) \) be a polynomial function of degree 3 (that is \( f(x) = ax^3 + bx^2 + cx + d \)). Prove that the equation \( f(x) = 0 \) can have at most 3 real solutions.

(6) Now use induction to prove the general statement: if \( f(x) \) is a polynomial function of degree \( n \), then the equation \( f(x) = 0 \) has at most \( n \) real solutions.

(7) Let \( f : [a, b] \to \mathbb{R} \) be continous on \([a, b]\) and differentiable on \((a, b)\). Suppose that there exists a constant \( M > 0 \) such that \( |f'(x)| \leq M \) for all \( x \in (a, b) \). Prove that \( |f(x) - f(y)| \leq M|x - y| \), for all \( x, y \) in \([a, b]\).

(8) Let \( S \) and \( T \) be two subsets of \( \mathbb{R} \). Suppose that for all \( s \in S \) and \( t \in T \) we have \( s < t \). Prove that \( \sup S \leq \inf T \) (note: first prove that \( \sup S \) and \( \inf T \) exist!)

(9) Use the definition of derivative to compute \( f'(0) \):

\[ f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ x & \text{if } x = 0 \end{cases} \]

(10) Prove that the equation \( \cos x = x \) has exactly one solution on the interval \([0, 1]\)

(11) Calculate

\[ \lim_{n \to \infty} \sqrt{n^2 + n} - n \]
(1) For the sequence $a_n = \sqrt{4n^2 + n - 2n}$

\[ \lim_{x \to \infty} a_n \]

**Note:** If, at some point of your calculations, you use de L'Hospital rule, and you want full credits, you have to give a reason why it can be applied to a sequence.

(2) Using the definition of derivative, calculate the derivative function $f'(x)$ for

\[ f(x) = \frac{1}{x^2} \]

(3) Compute the derivative of $f(x) = xe^{(\cos x)^2} + 5x + 1$.

(4) Prove that a convergent sequence is bounded.

**Remark** Problems (1) (2) and (3) are of basic calculus type. If you missed them, you have to refresh your basic calculus knowledge. The statement of Problem (4) is proved in the book (Theorem 2.49)
(5) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Let \( a \in \mathbb{R} \). Prove that if \(-1 < f(a) < 1\), there exists a \( \delta > 0 \) such that \(-1 < f(x) < 1\) for all \( x \in (a - \delta, a + \delta) \).

**Solution**

The function \( f \) is continuous at \( a \). Let \( \varepsilon < \min\{1 - f(a), 1 + f(a)\} \), \( \varepsilon > 0 \). Then there exists a \( \delta > 0 \) such that for all \( x \in (a - \delta, a + \delta) \),

\[
|f(x) - f(a)| < \varepsilon
\]

The above estimate is equivalent to

\[
f(a) - \varepsilon < f(x) < f(a) + \varepsilon
\]

and our choice of \( \varepsilon \) implies that

\[
-1 = f(a) - f(a) - 1 < f(x) < f(a) + 1 - f(a) = 1
\]

( indeed, \( -\varepsilon > \max\{f(a) - 1, -1 - f(a)\} > -1 - f(a) \) )

**Remarks**

- We suggest to draw a graph in order to understand the choice of \( \varepsilon \) we made.
- This problem is similar to Problem 4, Exam 1 SUGGESTED PROBLEMS and Problem 3, Homework 3.

(6) Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Suppose that \( |f'(x)| < 1 \) for all \( x \in \mathbb{R} \). Prove that there exists at most one value \( a \) such that \( f(a) = a \).

**Solution**

Suppose that there are two distinct values \( x \) and \( y \), such that \( f(x) = x \) and \( f(y) = y \). We can assume that \( x < y \). Since \( f \) is differentiable on \( \mathbb{R} \), it is differentiable on \((x, y)\) and continuous on \([x, y]\). Therefore according to the Mean Value Theorem there is a number \( c \) such that

\[
\frac{f(y) - f(x)}{y - x} = f'(c)
\]

This yields the following contradiction:

\[
\frac{|f(y) - f(x)|}{|y - x|} = \frac{|y - x|}{|y - x|} = 1 = |f'(c)| < 1.
\]

**Remark** For other applications of the Mean Value Theorem, see for example Homework 5, problems 1 and 2.
Note
The following is a list of suggested problems, complimentary to Homework 5 (problems 5 and 6 only) Homework 6 and Homework 7.

(1) Construct a sequence $f_n$ of continuous functions on $[0,1]$ that satisfies both the following conditions: $f_n$ converges the function $f = 0$ pointwise but not uniformly and

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0$$

HINT: modify example 2.24...

(2) Prove that the sequence $f_n = e^{-\frac{x^2}{n}}$ converges uniformly on $[-1,1]$

(3) For a continuous function $g$ on $[0,1]$ (or any other closed interval) one can prove the following inequality

$$|\int_0^1 f(x) \, dx| \leq \int_0^1 |f(x)| \, dx$$

Let $f_n$ be a sequence of continuous functions on $[0,1]$, converging uniformly to a function $f$. Using the above inequality, prove that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx$$

(4) In class we define the following length function on the space $C([0,1])$ of continuous functions on the closed interval $[0,1]$:

$$||f|| = \sup_{x \in [0,1]} |f(x)|$$

Consider now the space $C((0,1))$ of continuous functions on the open interval $(0,1)$. What goes wrong if we try to define the length as we did for $C([0,1])$?

(5) Suppose that $\sum a_n$ is convergent, $a_n > 0$. Prove that $\sum \frac{a_n}{n}$ is convergent. If $\sum a_n$ is divergent, is it true that $\sum \frac{a_n}{n}$ is divergent?
(6) Determine whether the series is convergent or divergent:

\[ a) \quad \sum_{n=1457}^{\infty} \frac{1}{n} \]

\[ b) \quad \sum_{n=1}^{\infty} \frac{n^2 + 4}{n^4 + 5} \]

\[ c) \quad \sum_{n=0}^{\infty} e^{-n} \]

\[ d) \quad \sum_{n=0}^{\infty} e^n \]

\[ e) \quad \sum_{n=1}^{\infty} n e^{-n^2} \]

(7) Calculate the sum of the following series:

\[ \sum_{n=0}^{\infty} \frac{1 + 3^n}{4^n} \]

\[ \sum_{n=0}^{\infty} \left( \frac{3^n}{n!} + \frac{1}{2^n} \right) \]
Exam 2, suggested problems

Note
The final exam will be open book/notes. No calculators and no computers are allowed. The final will consists of some calculus type questions, and some proving questions. Here is a list of some new practice problem.

(1) Let $A$ and $B$ be non-empty and bounded from above subsets of $\mathbb{R}$. Define the set $A + B$ as

$$A + B = \{a + b \mid a \in A, b \in B\}$$

Prove that $\sup A + \sup B = \sup (A + B)$

(2) Let $f$ be a continous function on $[0, 1]$, and suppose that $f(x) > 0$ for all $x \in [0, 1]$. Prove that there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in [0, 1]$.

(HINT: suppose not, then $\min f = ...$)

(3) Let $f$ be a continous function on $\mathbb{R}$. Prove that the set

$$f^{-1}(0, 1) = \{x \mid 0 < f(x) < 1\}$$

is open.

(4) Let $f_n : D \to \mathbb{R}$ be a sequence of bounded functions converging uniformly to $f : D \to \mathbb{R}$. Prove that $f$ is bounded.

(5) Let $S \subset \mathbb{R}$, and $a_n \in S$. Prove that if $\lim_{n \to \infty} a_n = a$, then $a \in \overline{S}$.

(6) Prove the alternating series test (Theorem 1.39)