Stony Brook University
Mathematics Department
Julia Viro

Fundamental Concepts of Mathematics
MAT 511
Fall 2021

Syllabus

Course description: This course aims to develop your appreciation of the logical basis of mathematics, and to lay the foundation for subsequent courses in the program. One of our goals will be to enhance your ability to understand and construct proofs. We will discuss fundamental ideas like number, set, and function; topics to be covered are logic and proofs, mathematical induction, set theory, relations and partitions, functions, cardinality, axioms and construction for integers, rationals, and reals.

Instructor: Julia Viro (julia.viro@stonybrook.edu)

Office Hours: TuTh 5pm-6pm, 3pm-4pm in MLC (online).

Textbook: Smith, Eggen, St. Andre, *A transition To Advanced Mathematics*, Eighth Edition, Brooks/Cole.

Meetings: TuTh 6:30pm-7:50pm in Physics P124.

Homework will be assigned weekly through Blackboard (Assignments). Your solutions should be submitted to Blackboard. Each submission should contain a single pdf-file. You may use any app which consolidate your pictures in a single pdf-file (for example, CamScan). Submission in a wrong format (multiple files, jpg-format, links to the cloud, etc.) will be accepted but with reduced score. Late submission will be accepted but with reduced score.

The emphasis of the course is on writing proofs, so please try to write legibly and explain your reasoning clearly and fully. You are encouraged to discuss the homework problems with others, but your write-up must be your own work. Suspiciously similar submissions won't be graded.

Grading system: your grade for the course will be based on: homework 10%, class active participation 10%, two midterms 25% each, final exam 30%.

Make-up policy: Make-up examinations are given only for work missed due to unforeseen circumstances beyond the student's control.

The Student Accessibility Support Center (SASC) statement: If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact SASC (631) 6326748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities/asp.

Academic integrity statement: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary

Critical incident management:

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Student Conduct and Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Until/unless the latest COVID guidance is explicitly amended by SBU, during Fall 2021"disruptive behavior" will include refusal to wear a mask during classes.

For the latest COVID guidance, please refer to:

https://www.stonybrook.edu/commcms/strongertogether/latest.php

Weekly Plan (tentative)

Week 1 (Tu 8/24, Th 8/26)

Learning objectives: Introduction to logic. Propositions and predicates. Logical connectives. Truth tables. Compound propositions. Conditional and biconditional sentences. Denials. Logical identities.

Learning outcomes. A student should be able to
• identify propositions and predicates;
• understand five basic logical connectives (negation, conjunction, disjunction, implication, and equivalence);
• recognize logical connectives in colloquial expressions: and, but, though, nevertheless, either … or, etc.;
• operate with truth tables, analyze and construct propositional forms involving connectives;
• determine equivalent propositional forms;
• construct logically meaningful sentences using connectives;
• define what a tautology and contradiction mean;
• use correctly logical symbols, formulate and prove logical identities, including tautology and contradiction, de Morgan's laws, the law of excluded middle and the law of consistency;
• recognize conditional and biconditional sentences and use appropriately various colloquial expressions associated with conditionals and biconditionals (sufficient, necessary, sufficient and necessary, whenever, if and only if etc.);
• understand the difference between implication in mathematics and causation in language/everyday life;
• construct useful denials of propositional forms.

Reading: 1.1, 1.2.

Exercises: **1.1:** 2, 3, 6, 7, 9, 11, 12
1.2: 1, 2, 3, 5, 6, 7, 9, 10, 15, 16.

Week 2 (Tu 8/31, Th 9/2)

Learning objectives: Quantifiers and quantified sentences. Analyzing and constructing propositions involving several quantifiers.

Learning outcomes. A student should be able to
• understand and use correctly quantifiers(universal, existential, and unique existential) and use appropriately different colloquial expressions associated with quantifiers;
• translate propositions formulated in plain English into symbolic forms and the other way around;
• analyze and construct sentences involving several quantifiers;
• understand when quantifiers commute and when they don't;
• construct useful denials of propositional forms and quantified sentences.

Reading: 1.3.

Exercises: **1.3:** 1, 2, 3, 5, 6, 8, 9, 10, 12, 13.

Week 3 (Tu 9/7, Th 9/9)

Learning objectives: Logical structure of definitions and theorems. How to read and understand mathematical texts. Structure of a mathematical theory: basic objects, axioms, definitions and theorems. The role of proofs. Examples and counterexamples. Logic behind school mathematics.

Learning outcomes. A student should be able to
• understand the logical structure of a definition; in particular, understand mathematical definitions as biconditional sentences;
• understand the logical structure of a mathematical theorem and distinguish the formulation (statement) of a theorem from its proof;
• distinguish a definition from a theorem and example using the logical criteria;
• identify definitions, theorems, and examples in an unknown mathematical text;
• express known definitions and theorems in appropriate logical forms, both in words and symbols;
• comprehend the structure of a mathematical theory: identify the basic objects, axioms and theorems;
• explain the role of proofs in mathematics;
• distinguishing the formulation (statement) of a theorem and its proof and recognize the difference between motivation and proof;
• understand the roles of examples and counterexamples;
• recognize incorrect proofs;
• work with actual mathematical text (excerpts;)
• identify different items of mathematical narrative (definitions, theorems, examples, motivations, etc.) in school curriculum;
• understand logical schemes behind solving equations and inequalities.

Logical structure of definitions and theorems. How to read and understand mathematical texts. Logic behind school mathematics.

Reading: 1.1-1.3, and p. 64-66.

Exercises: Handouts

Week 4 (Th 9/14, Th 9/16)

Recitations.

Week 5 (Tu 9/21, Th 9/23)

Learning objectives: Proof techniques: direct proof, proof by contraposition, proof by contradiction, proof by exhaustion. Strategies for constructing proofs.

Learning outcomes. A student should be able to
• understand, construct and write proofs of different types: direct proof, proof by contraposition, proof by contradiction, proof by exhaustion;
• evaluate pros and contra of different proof techniques;
• make comparative analysis of various proofs of the same fact;
• recognize and avoid typical logical mistakes of affirming the consequent and denying the antecedent.

Reading: 1.4-1.7.

Exercises: **1.4:** 1, 4, 5, 6, 7, 8, 9, 10, 11
1.5: 2, 3, 4, 5, 6, 7, 8, 10, 12
1.6: 1, 2, 3, 4, 6, 7
1.7: 1, 3, 4, 5, 6, 7, 8, 23.

Week 6 (Tu 9/28, Th 9/30)

Learning objectives: Principle of mathematical induction in various forms: induction, strong induction, well-ordering principle.

Learning outcomes. A student should be able to
• understand, construct and write proofs using principle of mathematical induction in different forms (induction, strong induction, well-ordering principle);
• identify situations when a proof by induction is suitable and situations when it is not;
• conduct proofs by induction of various statements from combinatorics, algebra, geometry and analysis.

Reading: 2.4, 2.5.

Exercises: **2.4:** 6, 7, 8, 10, 13
2.5: 2, 4, 5, 6, 13.

Week 7 (Tu 10/5, Th 10/7)

Recitation and Midterm 1.

Week 8 (Th 10/12 no classes (Fall break), Th 10/14)

Learning objectives: Basic notions of set theory: set and its elements, empty set, subset, intersection, union, difference and complement. Families of sets. Relations between logical and set-theoretical operations. Set-theoretic identities. Maps: definitions and notations. Basic terminology associated with maps: domain, codomain, image and preimage. Examples of maps: functions in one variable, numerical sequences, identity map, constant map.

Learning outcomes. A student should be able to
• fluently operate with basic notions of set theory: set and its elements, empty set, subset, intersection, union, difference and complement;
• understand the role of Venn diagrams as illustrations and counterexamples;
• relate logical and set-theoretical operations, like negation and complement, conjunction and intersection etc.;
• formulate and prove set-theoretical identities;
• **optional:** understand the concept of families of sets and give several examples of families of sets;
• give definition of the power set and list several properties of the power set;
• define a map from one set to another, know synonyms for the word *map*;
• understand and fluently use basic terminology associated with maps: domain, codomain, range, image and preimage of a set, graph of a map;
• provide examples of maps from different parts of mathematics;
• be familiar with special maps: identity map and constant map.

Reading: 2.1-2.5.

Exercises: **2.1:** 1, 2, 4, 5, 6, 8, 14, 15, 16, 18, 19
2.2: 2, 3, 5, 9, 10, 12, 13, 16, 18
2.3: 1, 2, 6, 7, 10, 16, 17
2.4: 1, 2, 3, 5.

Week 9 (Tu 10/19, Th 10/21)

Learning objectives: Composition of maps: definition and properties. Special maps. Power set and the set of all maps from the set to a two-element set. The set of all maps $X \to Y$. Injections, surjections and bijections. Definition and properties of inverse map. Equivalence between invertibility and bijectivity. Cartesian product of sets. Coordinate projections and fibers. Graph of a map. Relations. Functions of several variables as functions on a Cartesian product. Equivalence relations and partitions. Quotient sets. Constructions of integers and rational numbers. Construction of complex numbers.

Learning outcomes. A student should be able to
• define a composition of maps and list its properties;
• define inclusion map, restrictions of a map;
• define and list properties of the characteristic function of a set
• provide definitions of injections, surjections and bijections. List synonyms for these words;
• provide definitions of inverse map;
• list basic examples of functions and their inverse: exponential and logarithmic, tangent and arctangent, etc.;
• state and prove equivalence of invertibility and bijectivity.
• know the definition and properties of the Cartesian product of sets;
• describe coordinate projections;
• define graph os a map;
• define and give examples of a relation from one set to another and a relation on a set

• discuss properties associated with a binary relation on a set: reflexivity, irreflexivity, symmetry, antisymmetry, transitivity;
• define equivalence relation on a set;
• provide five examples of equivalence relations;
• define partition of a set and establish connection between equivalence relations and partitions of a set;
• describe equivalence classes, the quotient set, and the quotient map.

Reading: 3.1-3.3, 4.1-4.5.

Exercises: **3.1:** 2, 3, 4
3.2: 1, 5, 6, 8, 19
3.3: 2, 3, 4, 7
4.1: 3, 4, 6, 7, 11, 12, 15, 16, 19
4.2: 1, 2, 3, 4, 5, 8, 12, 16, 18
4.3 1, 2, 3, 5, 9, 10, 11
4.4: 1, 3, 5, 6, 8
4.5: 2, 3, 4, 8, 10, 11.

Week 10 (Tu 10/26, Th 10/28)

Learning objectives: Congruence classes. Modular arithmetic.

Learning outcomes. A student should be able to
• define congruence modulo m and prove that this is an equivalence relation;
• define congruence classes;
• define operations of addition and multiplication on congruence classes;
• give definition of a ring;
• prove that $\mathbb{Z}_m$ is a ring;
• use modular arithmetic for solving various divisibility problems and control of calculations;
• define ring homomorphism and prove that the canonical projection $\mathbb{Z} \to \mathbb{Z}_m$ is a ring homomorphism.

Reading: 3.4.

Exercises: **3.4:** 1, 2, 3, 5, 6, 7, 8, 12

Week 11 (Tu 11/2, Th 11/4)

Recitation and Midterm 2.

Week 12 (Tu 11/9, Th 11/11)

Learning objectives: Number systems. Peano's axioms. Integers, rational, real and complex numbers as quotient sets. Definitions of equipotent sets and cardinality of a set. Finite and infinite sets. Finite arithmetic. Pigeonhole principle.

Learning outcomes. A student should be able to
• define which sets are called equipotent;
• define the cardinality of a set;
• explain what the cardinal numbers of the empty set and a singleton are;

• explain why natural numbers and integers have the same cardinality;
• define which sets are called finite and infinite;
• formulate (**optional:** and prove) the pigeonhole principle and its corollaries;
• solve problems using the pigeonhole principle;

Reading: 5.1

Exercises: **5.1:** 2, 3, 4, 5, 12, 20, 21.

Week 13 (Tu 11/16, Th 11/18)

Learning objectives: Denumerable, countable and uncountable sets. Examples of infinite sets of the same and different cardinalities. Hilbert's Grand Hotel. Cantor theorem about non-equipotency of a set and its power set. Denumerable arithmetic. Countable and uncountable sets.

Learning outcomes. A student should be able to
• explain which sets are called denumerable, countable and uncountable;
• formulate and prove basic facts about arithmetics of finite sets (addition, multiplication, inclusion-exclusion);
• count the number of permutations of a finite set;
• explain how to define the product of infinite cardinal numbers;
• formulate and prove basic facts about arithmetics of denumerable sets;
• understand why $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and their subsets, disjoint unions and direct products are denumerable;
• state and prove Cantor's theorem about uncountability of $\mathbb{R}$;
• prove that an open interval in uncountable;

Reading: 5.2-5.4.

Exercises: **5.2:** 1, 2, 3, 4, 5, 7
5.3: 2, 9, 10.

Week 14 (Tu 11/23, Th 11/25 no classes: Thanksgiving break)

Learning objectives: Cantor-Shröder-Bernstein theorem. Ordering of cardinal numbers. Cantor's theorem about uncountability of $\mathbb{R}$. Continuum hypotheses.

Learning outcomes. A student should be able to
• define inequalities for cardinal numbers and understand the ordering of cardinal numbers;
• state the Continuum hypothesis;
• state and prove Cantor's theorem about cardinalities of a set and its power set;
• state Cantor-Shröder-Bernstein theorem and use it for problem solving;
• prove that the power set of natural numbers is uncountable.

Reading: 5.4.

Exercises: **5.4:** 1, 2, 3, 4, 5, 7, 8.

Week 15 (Tu 11/30, Th 12/2)

Final Review.

Th 12/9 at 5:30pm-8:00pm. Final exam.