MAT 360: Geometric Structures, Fall 2013

- **Instructor:** Nikita Selinger, office 4-115 Math,
- **Office hours:** MoWe 4.00-5.30pm, or by appointment.
- **Class meetings:** MoWe, 10:00-11:20pm, Frey Hall 217.

Here you can find sample solutions for homeworks of a similar course. You can also look at the final exam for that course.

The final exam is scheduled on Dec 17th, 2.15pm, in our usual classroom. Here is the list of topics recommended for a review for the final exam.

Homework 10, due Dec 2 **before class** Read the lecture notes on similarity by Oleg Viro and Olga Plamenevskaya. Read the textbook pp 143-160.
Solve the following excercises from the textbook: 400, 401, 404, 412. Prove Theorems 5 and 6 from the notes.

Homework 9, due Nov 20 **before class** Read the following lecture notes by Oleg Viro and Olga Plamenevskaya.
Solve the following excercises from the textbook: 386, 388, 389, 391, 595. Prove Theorems 10 and 11 from the notes.

Homework 8, due Nov 13 **before class** Read the textbook pp 138-150. Start reading the following lecture notes by Oleg Viro and Olga Plamenevskaya.
Solve the following excercises from the textbook: 349, 359, 371, 375, 377, 379, 383.

Homework 7, due Nov 6 **before class** Read the textbook pp 117-138.
Solve the following excercises from the textbook: 321, 324, 346, 348, 355, 356, 360.

Here are examples of previous years midterms with solutions Midterm1 and Midterm2 , and a list of topics to review.

We will have additional office hours Friday 10/18 at 4pm.

Homework 6, due Oct 16 **before class** Read the textbook pp 78-96.
Solve the following excercises from the textbook: 214, 216, 217, 223, 250, 253, 256.

Homework 5, due Oct 9 **before class** Read the textbook pp 55-81.
Solve the following excercises from the textbook: 157, 160, 171, 183, 187, 197, 205, 206.

Homework 4, due Oct 2 **before class** Read the textbook pp 49-66.
Solve the following excercises from the textbook: 130, 136, 138, 140, 146, 149, 150.

Homework 3, due Sep 25 **before class** Read the textbook pp 41-53.
Solve the following excercises from the textbook: 95, 98, 99, 101, 102, 103, 116.

Homework 2, due Sep 18 Read the textbook pp 22-41.
Solve the following excercises from the textbook: 54, 58, 67, 69, 77, 91, 92.

Homework 1, due Sep 11 . Recall the following axioms of congruence:
(i) The identity mapping (i.e. the mapping that sends all points and lines to themselves) is a congruence.
(ii) There exists a congruence that takes one given ray to any other given ray.
(iii) There exists a (non-identity) congruence keeping all the points of a given straight line fixed. This "flip"

can be done in a unique way.
Solve the following excercises from the textbook: 28, 33, 36, 39, 50.

Please note the class room has changed to Frey Hall 217.

Homework is a compulsory part of the course. Homework assignments are due each week at the beginning of the Wednesday's class. Under no circumstances will late homework be accepted.

Grading system: The final grade is the weighted average according the following weights: homework 10%, in-class tests 20%, Midterm 30%, Final 40%.

Textbook: *Kiselev's Geometry (Book I, Planimetry) Sumizdat, El Cerrito, Calif., 2006.* You can find the first 33 pages of the textbook here.

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MAT 515, Geometry for Teachers
Fall 2012

Final Exam

Name:____________________________________

This is a closed book, closed notes test. No consultations with others. Calculators are not allowed.

Please turn off and take off the desk cell phones, pagers, etc. Only the exam and pens/pencils should be on your desk.

Please explain all your answers, show all work, and give careful proofs. Answers without explanation will receive little credit.

The problems are not in the order of difficulty. You may want to look through the exam and do the easier questions first.

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1	2	3	4	5	6	Total

1. Suppose that two lines are intersected by a third one such that two corresponding angles are congruent to each other. Prove that the lines are parallel.

2. Let O be a point in the interior of a triangle $\triangle ABC$. Prove that $AO + BO + CO \leq AB + BC + AC$.

3. Find the geometric locus the feet of the perpendiculars dropped from a given point A to all lines passing through another given point B.

4. Construct a line making a given angle with a given line and tangent to a given circle. (You may use, without a detailed description, the following elementary constructions: segment and angle bisection, raising a perpendicular at a point on the line, dropping a perpendicular from a point not on the line, constructing segments and angles congruent to given ones.)

5. Prove that the hypotenuse and the shorter leg of a right triangle with acute angles 60° and 30° are commensurable.

6. Find the composition of two rotations: the rotation in the counterclockwise direction about a point A by angle 150° followed by the rotation in the counterclockwise direction about a point B by angle 100°: prove that this is a rotation, find the center and angle of this rotation.

MAT360. Final topics

Here is the list of topics recommended for review.

1. Theorem about vertical angles, section 26.
2. Existence and uniqueness of perpendicular to a line from a point, sections 24, 65 and 66.
3. Theorems about isosceles triangles and their properties, sections 35, 36.
4. Congruence tests for triangles section 40.
5. Inequality between exterior and interior angles in a triangle, sections 41 - 43.
6. Relations between sides and opposite angles sections 44 - 45.
7. Triangle inequality and its corollaries, sections 48 and 49.
8. Perpendicular and slants, sections 51 - 53.
9. Segment and angle bisectors, sections 56 and 57.
10. Basic construction problems, sections 61 - 69.
11. Tests for parallel lines, section section 73.
12. The parallel postulate, sections 75 and 76.
13. Angles formed by parallel lines and a transversal, sections 77 and 78.
14. Angles with respectively parallel sides, section 78.
15. Angles with respectively perpendicular sides, section 79.
16. The sum of interior angles in a triangle, section 81.
17. The sum of interior angles in a convex polygon, section 82.
18. Properties of a parallelogram, sections 85 - 87.
19. Special types of parallelograms and their properties, sections 90 - 92.
20. The midline theorem, sections 93, 94, 95.
21. The midline of trapezoid, sections 96, 97.
22. Existence and uniqueness of a circle passing through three points, sections 103, 104.
23. Constructions that use isometries, sections 98 - 101
24. Theorems about inscribed angles, section 123.
25. Corollaries of the theorem about inscribed angles, sections 125,126.
26. Constructions using theorems about inscribed angles, sections 127 - 130, 133.
27. Inscribed and circumscribed circles, sections 136, 137.
28. Concurrency points in a triangle, sections 140 - 142.
29. Mensurability, sections 143-153.
30. Lecture notes on isometries.

31. Lecture notes on similarities.
32. Similarity of triangles 156-164.
33. Thales' theorem 170-175.
34. Pythagorean theorem 188-198.

SIMILARITY

BASED ON NOTES BY OLEG VIRO, REVISED BY OLGA PLAMENEVSKAYA

Euclidean Geometry can be described as a study of the properties of geometric figures, but not all kinds of conceivable properties. Only the properties which do not change under isometries deserve to be called geometric properties and studied in Euclidian Geometry.

Some geometric properties are invariant under transformations that belongto wider classes. One such class of transformations is similarity transformations. Roughly they can be described as transformations preserving shapes, but changing scales: magnifying or contracting.

The part of Euclidean Geometry that studies the geometric properties unchanged by *similarity transformations* is called the *similarity geometry.* Similarity geometry can be introduced in a number of different ways. The most straightforward of them is based on the notion of ratio of segments.

The similarity geometry is an integral part of Euclidean Geometry. In fact, there is no interesting phenomenon that belong to Euclidean Geometry, but does not survive a rescaling. In this sense, the whole Euclidean Geometry can be considered through the glass of the similarity geometry. Moreover, all the results of Euclidean Geometry concerning relations among distances are obtained using similarity transformations.

However, main notions of the similarity geometry emerge in traditional presentations of Euclidean Geometry (in particular, in the Kiselev textbook) in a very indirect way. Below it is shown how this can be done more naturally, according to the standards of modern mathematics. But first, in Sections 1 - 4, the traditional definitions for ratio of segments and the Euclidean distance are summarized.

1. Ratio of commensurable segments. (See textbook, sections 143-154 for a detailed treatment of this material.)

If a segment CD can be obtained by summing up of n copies of a segment AB, then we say that $\frac{CD}{AB} = n$ and $\frac{AB}{CD} = \frac{1}{n}$.

If for segments AB and CD there exists a segment EF and natural numbers p and q such that $\frac{AB}{EF} = p$ and $\frac{CD}{EF} = q$, then AB and CD are said to be *commensurable,* $\frac{AB}{CD}$ is defined as $\frac{p}{q}$ and the segment EF is called a *common measure* of AB and CD.

The ratio $\frac{AB}{CD}$ *does not depend on the common measure* EF.

This can be deduced from the following two statements.

For any two commensurable segments there exists the greatest common measure.

The greatest common measure can be found by geometric version of the *Euclidean algorithm*. (See textbook, section 146)
If EF is the greatest common measure of segments AB and CD and GH is a common measure of AB and CD, then there exists a natural number n such that $\frac{EF}{GH} = n$.

If a segment AB is longer than a segment CD and these segments are commensurable with a segment EF, then $\frac{AB}{EF} > \frac{CD}{EF}$.

2. Incommensurable segments. There exist segments that are not commensurable. For example, a side and diagonal of a square are not commensurable, see textbook, section 148. Segments that are not commensurable are called *incommensurable.*

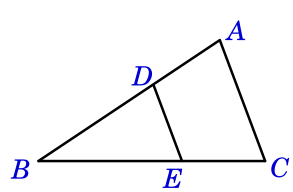
For incommensurable segments AB and CD the ratio $\frac{AB}{CD}$ is defined as the unique real number r such that

- $r < \frac{EF}{CD}$ for any segment EF, which is longer than AB and commensurable with CD;
- $\frac{EF}{CD} < r$ for any segment EF, which is shorter than AB and commensurable with CD.

3. Thales' Theorem. (See Sections 159-160 of the textbook.) *Let ABC be a triangle, D be a point on AB and E be a point on BC. If $DE \parallel AC$, then*

$$\frac{BD}{DA} = \frac{BE}{EC}.$$

$\square$



Corollary. *Under the assumptions of Thales' Theorem,*

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{DE}{AC}.$$

$\square$

The converse theorem. *Let ABC be a triangle, D be a point on AB and E be a point on BC. If*

$$\frac{BD}{DA} = \frac{BE}{EC},$$

then $DE \parallel AC$.

Proof. Through the point D, draw a line parallel to AC. Let it intersect the side BC at point E'. (We would like to show that $E = E'$.) By the direct theorem, which applies since now we are considering parallel lines,

$$\frac{BD}{DA} = \frac{BE'}{E'C}.$$

But then the hypothesis implies that

$$\frac{BE}{EC} = \frac{BE'}{E'C},$$

and then $E = E'$.

□

4. Distance. If we choose a segment AB and call it the unit, then we can assign to any other segment CD the number $\frac{CD}{AB}$, call it the *length* of CD and denote by $|CD|$.

Further, the length $|CD|$ of segment CD is called then the *distance* between points C and D and denote by $\mathrm{dist}(C, D)$. Of course, $\mathrm{dist}(C, D)$ depends on the choice of AB. Define $|CD|$ and $\mathrm{dist}(C, D)$ to be 0 if $C = D$.

The distance between points has the following properties:

- it is symmetric, $\mathrm{dist}(C, D) = \mathrm{dist}(D, C)$ for any points C, D;
- $\mathrm{dist}(C, D) = 0$ if and only if $C = D$;
- triangle inequality, $\mathrm{dist}(C, D) \leq \mathrm{dist}(C, E) + \mathrm{dist}(E, D)$.

5. Definition of similarity transformations. A map S is said to be a *similarity transformation* with *ratio* $k \in \mathbb{R}$, $k \geq 0$, if $|T(A)T(B)| = k|AB|$ for any points A, B in the plane.

Other terms may be used in the same situation: a similarity transformation may be called a *dilation*, or *dilatation*, the ratio may be also called the *coefficient* of the dilation.

General properties of similarity transformations.

1. Any isometry is a similarity transformation with ratio 1.

2. Composition $S \circ T$ of similarity transformations T and S with ratios k and l, respectively, is a similarity transformation with ratio kl.

6. Homothety. An important example of similarity transformation with ratio different from 1 is a homothety.

Definition. Let k be a positive real number, O be a point on the plane. The map which maps O to itself and any point $A \neq O$ to a point B such that the rays OA and OB coincide and $\frac{OB}{OA} = k$ is called the *homothety* centered at O with ratio k.

Composition $T \circ S$ of homotheties T and S with the same center and ratios k and l, respectively, is the homothety with the same center and the ratio kl. In particular, any homothety is invertible and the inverse transformation is the homothety with the same center and the inverse ratio.

Theorem 1. *A homothety T with ratio k is a similarity transformation with ratio k.*

Proof. We need to prove that $\frac{T(A)T(B)}{AB} = k$ for any segment AB. Consider, first, the case when O does not belong to the line AB. Then OAB is a triangle, and $OT(A)T(B)$ is also a triangle.

Assume that $k < 1$. Then $T(A)$ belongs to the segment OA, $T(B)$ belongs to OB, and since $\frac{OT(B)}{OB} = \frac{OT(A)}{OA} = k$, the converse to the Thales' theorem implies that $T(A)T(B)$ is parallel to AB. Then, by Corollary of Thales' Theorem, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

If $k > 1$, then A belongs to $OT(A)$, B to $OT(B)$, and the proof is similar.

The case where points A, B, O are collinear is easy, and left as exercise. □

Theorem 2. *Any similarity transformation T with ratio k of the plane is a composition of an isometry and a homothety with ratio k. The center of the homothety can be chosen arbitrarily.*

Proof. Consider a composition $T \circ H$ of T with a homothety H with ratio k^{-1} and your preferred center O. This composition is a similarity transformation with ratio $k^{-1}k = 1$, that is an isometry. Denote this isometry by I. Thus $I = T \circ H$. Notice that the homothety H is invertible: its inverse H^{-1} is the homothety with the same center and coefficient k. (Informally, scaling up can be undone by scaling down.) Now, multiply both sides of the equality $I = T \circ H$ by H^{-1} from the right hand side: $I \circ H^{-1} = T \circ H \circ H^{-1} = T$.

We have just shown that T can be represented as the composition of an isometry and a homothety, where the homothety is performed first. This argument can be modified to show that T can be also represented as a composition where the isometry is performed before the homothety. (For this, consider the composition $I' = H \circ T$, and show that this is also an isometry. Note that in general $I' \neq I$, since transformations may not commute.) □

Theorem 3. *A similarity transformation of a plane is invertible.*

Proof. By Corollary of Theorem 1, any similarity transformation T is a composition of an isometry and a homothety. A homothety is invertible, as was noticed above. An isometry of the plane is a composition of at most three reflections. Each reflection is invertible, because its composition with itself is the identity. A composition of invertible maps is invertible. □

Corollary. *The transformation inverse to a similarity transformation T with ratio k is a similarity transformation with ratio k^{-1}.*

7. Similar figures. Plane figures F_1 and F_2 are said to be *similar* if there exists a similarity transformation T such that $T(F_1) = F_2$.

Any two congruent figures are similar. In particular, any two lines are congruent and hence similar, any two rays are congruent and hence similar.

Segments are not necessarily congruent, but nonetheless any two segments are similar.

Theorem 4. *1) A figure similar to a segment is a segment, i.e. any similarity transformation maps segments to segments.*

2) Any segment can be mapped to any other by a similarity transformation, i.e. any two segments are similar.

Proof. 1) Let AB be the given segment, S the similarity transformation. By previous theorem, we can write $S = I \circ H_A$, where H_A is a homothety with center A, and I is an isometry. Because the segment AB emanates from the center of homothety, it is clear that H_A maps AB to a segment. (Note that this is far from obvious if the center of homothety lies away from the segment!) Since we know that isometries map segments to segments, we'll still get a segment after applying I.

2) Given segments AB and $A'B'$, we can find an isometry mapping A' to A, and B' to a point on the ray AB. (First find a translation mapping A' to A, and then rotate around the point $A = A'$ match rays AB and $A'B'$.) Then find a homothety with center $A = A'$ mapping one of the segments to the other one.

□

Theorem 5. *1) A figure similar to a circle is a circle, i.e. any similarity transformation maps circles to circles.*

2) Any circle can be mapped to any other by a similarity transformation, i.e. any two circles are similar.

Proof. Exercise. □

Theorem 6. *1) A figure similar to an angle is an angle.*

2)Two angles are similar if and only they are congruent.

Proof. Exercise. □

8. Similarity tests for triangles.

Theorem 7 (AA-test)***.*** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and angles $\angle B$, $\angle B'$ are congruent, then $\triangle ABC$ is similar to $\triangle A'B'C'$.*

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By ASA test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC}$. Hence, the homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 8 (SAS-test)***.*** *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and*
$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$
then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC}$, and therefore $|BE| = |B'C'|$. By SAS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 9 (SSS-test)***.*** *If in triangles ABC and $A'B'C'$*
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$$
then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Corollary of Thales' Theorem, $\frac{DB}{AB} = \frac{BE}{BC} = \frac{DE}{AC}$. Therefore $|BE| = |B'C'|$ and $|DE| = |A'C'|$. By SSS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem 10. *Conversely, suppose that triangles ABC and $A'B'C$ are similar, i.e. there exists a similarity transformation S mapping one triangle to the other. For concreteness, we assume that $S(A) = A'$, $S(B) = B'$, $S(C) = C'$. Then $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and*
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}.$$

Proof. We can represent S as the composition of a homothety H_A centered at A and an isometry, $S = I \circ H_A$. Since the center of H_A is one of the vertices of the triangle ABC, this triangle and its image, the triangle $H_A(A)H_A(B)H_A(C)$ are positioned as in Thales' theorem, so the corresponding angles of $\triangle ABC$ and $\triangle H_A(A)H_A(B)H_A(C)$ are equal, and the sides are proportional. On the other hand, the isometry I maps $\triangle H_A(A)H_A(B)H_A(C)$ to $\triangle A'B'C'$, so the latter two triangles are congruent. □

Isometries.

Congruence mappings as isometries. The notion of *isometry* is a general notion commonly accepted in mathematics. It means a mapping which preserves distances. The word *metric* is a synonym to the word *distance*. We will study isometries of the plane. In fact, we have already encountered them, when we superimposed a plane onto itself in various ways (eg by reflections or rotations) to prove congruence of triangles and such. We now show that each isometry is a "congruence mapping" like that.

Theorem 1. *An isometry maps*
(i) straight lines to straight lines;
(ii) segments to congruent segments;
(iii) triangles to congruent triangles;
(iv) angles to congruent angles.

Proof. Let's show that an isometry S maps a segment AB to segment $S(A)S(B)$ which is congruent to AB. It is clear (from the definition of isometry) that the distance between $S(A)$ and $S(B)$ is the same as the distance between A and B. However, we need to check that the image of AB will indeed be a straight line segment. To do so, pick an arbitrary point X on AB. Then $S(A)S(B) = AB = AX + XB = S(A)S(X)+S(X)S(B)$, and by the triangle inequality the point $S(X)$ must be on the segment $S(A)S(B)$ (otherwise we would have $S(A)S(X) + S(X)S(B) > S(A)S(B)$. So image of the segment AB lies in the segment $S(A)S(B)$, and indeed, covers the whole of $S(A)S(B)$ without leaving any holes: if X' is a point on $S(A)S(B)$, find X on AB such that $XA = X'S(A)$, $XB = X'S(B)$, then $S(X) = X'$. □

Examples of isometries. We have encountered quite a few examples before: reflections, rotations, and translations are all isometries. (It is pretty easy to see that the distances are preserved in each case: for instance, a reflection R_l through the line l maps any segment AB to a symmetric, and thus congruent, segment $A'B'$.) Let's look at some examples more closely.

Translations and central symmetries. A map of the plane to itself is called a *translation* if, for some fixed points A and B, it maps a point X to a point $T(X)$ such that $ABT(X)X$ is a parallelogram. (Note the order of points!)

Here we have to be careful with the notion of parallelogram, because a parallelogram may degenerate to a figure in a line. Not any degenerate quadrilateral fitting in a line deserves to be called a parallelogram, although any two sides of such a degenerate quadrilateral are parallel. By a parallelogram we mean a sequence of four segments KL, LM, MN and MK such that KL is congruent and parallel to MN and LM is congruent and parallel to MK. This definition describes the usual parallelograms, for which congruence can be deduced from parallelness and vice versa, and the degenerate parallelograms.

Theorem 2. *For any points A and B there exists a translation mapping A to B. A translation is an isometry.*

Proof. Any three points A, B and X can be completed in a unique way to a parallelogram $ABX'X$. Define $T(X) = X'$. For any points X, Y the quadrilateral $XYT(Y)T(X)$ is a parallelogram, since $XT(X)||AB||YT(Y)$. Therefore, $XY = T(X)T(Y)$, so T is an isometry. □

Denote by T_{AB} the translation which maps A to B.

Theorem 3. *The composition of any two translations is a translation.*

Proof. Exercise. □

Theorem *3* means that $T_{BC} \circ T_{AB} = T_{AC}$.

Fix a point O. A map of the plane to itself which maps a point A to a point B such that O is a midpoint of the segment AB is called the *symmetry about a point* O.

Theorem 4. *A symmetry about a point is an isometry.*

Proof. SAS-test for congruent triangles (extended appropriately to degenerate triangles.) □

Theorem 5. *The composition of any two symmetries in a point is a translation. More precisely, $S_B \circ S_A = T_{2\overrightarrow{AB}}$, where S_X denotes the symmetry about point X.*

Proof. Exercise. □

Remark. The equality

$$S_B \circ S_A = T_{2\overrightarrow{AB}}$$

implies a couple of other useful equalities. Namely, compose both sides of this equality with S_B from the left:

$$S_B \circ S_B \circ S_A = S_B \circ T_{2\overrightarrow{AB}}$$

Since $S_B \circ S_B$ is the identity, it can be rewritten as

$$S_A = S_B \circ T_{2\overrightarrow{AB}}.$$

Similarly, but multiplying by S_A from the right, we get

$$S_B = T_{2\overrightarrow{AB}} \circ S_A.$$

Corollary. *The composition of an even number of symmetries in points is a translation; the composition of an odd number of symmetries in points is a symmetry in a point.*

Remark. In general, it is clear that a composition of isometries is an isometry: if each mapping keeps distances the same, their composition also will. It is trickier, however, to see the resulting isometry explicitly; we will prove a few theorems related to compositions of isometries. To practice with compositions, consider, for example, a reflection about a line l and a rotation by $90°$ counterclockwise about a point

$O \in l$. When composed in different order (rotation followed by reflection vs reflection followed by rotation), these yield reflections about **different** lines. The proof that the composition is a reflection can be obtained by an explicit examination of which points go where; by Theorem *6*, it suffice to examine 3 non-collinear points.

Recovering an isometry from the image of three points.

Theorem 6. *An isometry of the plane can be recovered from its restriction to any triple of non-collinear points.*

Proof. Given images A', B' and C' of non-collinear points A, B, C under and isometry, let us find the image of an arbitrary point X. Using a compass, draw circles c_A and c_B centered at A' and B' of radii congruent to AX and BX, respectively. They intersect in at least one point, because segments AB and $A'B'$ are congruent and the circles centered at A and B with the same radii intersect at X. There may be two intersection point. The image of X must be one of them. In order to choose the right one, measure the distance between C and S and choose the intersection point X' of the circles c_A and c_B such that $C'X'$ is congruent to CX. □

In fact, there are exactly two isometries with the same restriction to a pair of distinct points. They can be obtained from each other by composing with the reflection about the line connecting these points.

Isometries as compositions of reflections.

Theorem 7. *Any isometry of the plane is a composition of at most three reflections.*

Proof. Choose three non-collinear points A, B, C. By theorem *6*, it would suffice to find a composition of at most three reflections which maps A, B and C to their images under a given isometry S.

First, find a reflection R_1 which maps A to $S(A)$. The axis of such a reflection is a perpendicular bisector of the segment $AS(A)$. It is uniquely defined, unless $S(A) = A$. If $S(A) = A$, one can take either a reflection about any line passing through A, or take, instead of reflection, an identity map for R_1 .

Second, find a reflection R_2 which maps segment $S(A)R_1(B)$ to $S(A)S(B)$. The axis of such a reflection is the bisector of angle $\angle R_1(B)S(A)S(B)$.

The reflection R_2 maps $R_1(B)$ to $S(B)$. Indeed, the segment $S(A)R_1(B) = R_1(AB)$ is congruent to AB (because R_1 is an isometry), AB is congruent to $S(A)S(B) = S(AB)$ (because S is an isometry), therefore $S(A)R_1(B)$ is congruent to $S(A)S(B)$. Reflection R_2 maps the ray $S(A)R_1(B)$ to the ray $S(A)S(B)$, preserving the point $S(A)$ and distances. Therefore it maps $R_1(B)$ to $S(B)$.

Triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are congruent via an isometry $S \circ (R_2 \circ R_1)^{-1} = S \circ R_1 \circ R_2$, and the isometry is identity on the side $S(AB) = R_2 \circ R_1(AB)$. Now either $R_2(R_1(C)) = C$ and then $S = R_2 \circ R_1$, or the triangles $R_2 \circ R_1(\triangle ABC)$ and $S(\triangle ABC)$ are symmetric about their common side $S(AB)$. In the former case

$S = R_2 \circ R_1$, in the latter case denote by R_3 the reflection about $S(AB)$ and observe that $S = R_3 \circ R_2 \circ R_1$. $\square$

Compositions of two reflections.

Theorem 8. *The composition of two reflections in non-parallel lines is a rotation about the intersection point of the lines by the angle equal to doubled angle between the lines. In formula:*

$$R_{AC} \circ R_{AB} = Rot_{A,2\angle BAC},$$

where R_{XY} denotes the reflection in line XY, and $Rot_{X,\alpha}$ denotes the rotation about point X by angle α.

Proof. Pick some points whose images under reflections are easy to track. From symmetries/congruent triangles in the picture, it is clear that effect of two refections is that of a rotation. Since we know that an isometry is determined by the image of 3 non-collinear points, the ir no need to consider all possible positions of the points. $\square$

Theorem 9. *The composition of two reflections in parallel lines is a translation in a direction perpendicular to the lines by a distance twice larger than the distance between the lines.*

More precisely, if lines AB and CD are parallel, and the line AC is perpendicular to the lines AB and CD, then

$$R_{CD} \circ R_{AB} = T_{2\overrightarrow{AC}}.$$

Proof. Similar to the above. $\square$

Application: finding triangles with minimal perimeters. We have considered the following problem:

Problem 1. *Given a line l and points A, B on the same side of l, find a point $C \in l$ such that the broken line ACB would be the shortest.*

Recall that a solution of this problem is based on reflection. Namely, let $B' = R_l(B)$. Then the desired C is the intersection point of l and AB'.

Notice that this problem can be reformulated as finding $C \in l$ such that the perimeter of the triangle ABC is minimal.

Problem 2. *Given lines l, m and a point A, find points $B \in l$ and $C \in m$ such that the perimeter of the triangle ABC is the smallest possible.*

Idea that solves Problem 2. Reflect point A through lines l and m, that is, consider points $B' = R_l(A)$ and $C' = R_m(A)$. Use these points to find B and C (how?), and prove that the resulting triangle indeed has the smallest perimeter.

Problem 3. *Given lines l, m and n, no two of which are parallel to each other. Find points $A \in l$, $B \in m$ and $C \in n$ such that triangle ABC has minimal perimeter.*

If we knew a point $A \in l$, the problem would be solved like Problem 2: we would connect points $R_m(A)$ and $R_n(A)$ and take B and C to be the intersection points of this line with m and n. So, we have to find a point $A \in l$ such that the segment $R_m(A)R_n(A)$ would be minimal.

The endpoints $R_m(A)$, $R_n(A)$ of this segment belong to the lines $R_m(l)$ and $R_n(l)$ and are obtained from the same point $A \in l$. Therefore

$$R_n(A) = R_n(R_m(R_m(A))) = R_n \circ R_m(B),$$

where $B \in R_m(l)$. So, one endpoint is obtained from another by $R_n \circ R_m$.

By Theorem *9*, $R_n \circ R_m$ is a rotation about the point $m \cap n$. We look for a point B on $R_m(l)$ such that the segment $BR_n \circ R_m(B)$ is minimal.

The closer a point to the center of rotation, the closer this point to its image under the rotation. Therefore the desired B is the base of the perpendicular dropped from $m \cap n$ to $R_m(l)$. Hence, the desired A is the base of perpendicular dropped from $m \cap n$ to l.

Since all three lines are involved in the conditions of the problem in the same way, the desired points B and C are also the endpoints of altitudes of the triangle formed by lines l, m, n.

Composition of rotations.

Theorem 10. *The composition of rotations (about points which may be different) is either a rotation or a translation.*

Prove this theorem by representing each rotation as a composition of two reflections about a line. Choose the lines in such a way that the second line in the representation of the first rotation would coincide with the first line in the representation of the second rotation. Then in the representation of the composition of two rotations as a composition of four reflections the two middle reflections would cancel and the whole composition would be represented as a composition of two reflections. The angle between the axes of these reflections would be the sum of of the angles in the decompositions of the original rotations. If this angle is zero, and the lines are parallel, then the composition of rotations is a translation by Theorem *9*. If the angle is not zero, the axes intersect, then the composition of the rotations is a rotations around the intersection point by the angle which is the sum of angles of the original rotations.

Similar tricks with reflections allows to simplify other compositions.

Glide reflections. A reflection about a line l followed by a translation along l is called a *glide reflection*. In this definition, the order of reflection and translation does not matter, because they commute: $R_l \circ T_{AB} = T_{AB} \circ R_l$ if $l \parallel AB$.

Theorem 11. *The composition of a central symmetry and a reflection is a glide reflection.*

Use the same tricks as for Theorem *10*

Classification of plane isometries.

Theorem 12. *Any isometry of the plane is either a reflection about a line, a rotation, a translation, or a gliding reflection.*

This theorem can be deduced from Theorem *7* by taking into account relations between reflections in lines. By Theorem *7*, any isometry of the plane is a composition of at most 3 reflections about lines. By Theorems *8* and *9*, a composition of two reflections is either a rotation about a point or a translation.

Lemma. A composition of three reflections is either a reflection or a gliding reflection.

Proof. We will consider two cases: 1) all three lines are parallel, 2) not all lines are parallel (although two of the three may be parallel to one another).

The first one is easier; it is pretty straightforward to see (at least in some examples) that the composition is a translation. However, since the order of reflections matters, for a precise proof we wouold have to check different cases (if the lines are all vertical, the first reflection may be done about the leftmost, the rightmost, or the middle lien, etc.) To avoid this, we proceed as follows. Notice that $R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ (R_{l_2} \circ R_{l_1})$, and the composition $R_{l_2} \circ R_{l_1}$ of two reflections in parallel lines is a translation. This translation depends only on the direction of the lines and the distance between them, ie $R_{l_2} \circ R_{l_1} = R_{l_2'} \circ R_{l_1'}$ for any two lines l_1', l_2' that are parallel to l_1, l_2 and have the same distance between them. Thus, we translate the first two lines to make the second line coincide with the third, ie choose l_1', l_2' so that $l_2' = l_3$. Then

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l_2'} \circ R_{l_1'} = R_{l_3} \circ R_{l_3} \circ R_{l_1'} = R_{l_1'}$$

since two reflections about the same line l_3 cancel. Therefore, the result is a reflection (about the line l_1').

If the three lines are not all parallel, then the second line l_2 is not parallel to l_1 or l_3. Let's suppose l_1 and l_2 are not parallel (the other case is very similar). Then the composition $R_{l_2} \circ R_{l_1}$ of reflections about intersecting lines is a rotation (that depends only on the point where the lines intersect, and the angle at which they intersect). So the lines l_1, l_2 can be rotated simultaneously about their intersection point by the same angle without changing the composition.

By an appropriate rotation, make the second line l_2 perpendicular to the third line l_3 (which is not rotated), ie replace l_1, l_2 by l_1', l_2' so that $R_{l_2} \circ R_{l_1} = R_{l_2'} \circ R_{l_1'}$, and $l_2' \perp l_3$.

Then by rotating these two perpendicular lines l_2', l_3 about their intersection point, make the middle line l_2 parallel to the line l_1. That is, we replace the lines l_2', l_3 by lines l_2'', l_3'' so that

$$R_{l_3} \circ R_{l_2} \circ R_{l_1} = R_{l_3} \circ R_{l_2'} \circ R_{l_1'} = R_{l_3''} \circ R_{l_2''} \circ R_{l_1'}.$$

Now, the configuration of lines consists of two parallel lines and a line perpendicular to them: l_1', l_2'' are parallel, l_3'' is perpendicular to them both. The composition of reflections $R_{l_2''} \circ R_{l_1'}$ is a translation by a vector perpendicular to these two lines (and thus parallel to the third); so $R_{l_3''} \circ (R_{l_2''} \circ R_{l_1'})$ is a glide symmetry. But the composition of these three reflections is the same as the composition of reflections about the original three lines.

Properties of the four types of isometries. We have just seen that any isometry of the plane belongs to one of the four types. How do we detect to which type it belongs? In particular, it may seem a bit mysterious that while composition of 3 reflections is a reflection or glide reflection, a composition of two isometries can never be a reflection, but only a rotation or translation. This can be explained as follows. Suppose our plane lies in the 3-space (as a horizontal xy-plane), and its top is painted black, its bottom white. Suppose that the reflections are done by rotating the plane around the line (axis of reflection) in the 3-space. Then after a reflection, the white side will be on top, the black side on the bottom. Notice that the colors will flip this way if we perform any odd number of reflections, but after an even number of reflections the colors do not flip. (Eg after two reflections, the top will be black again, the bottom white.) By contrast, rotations and translations do not flip the colors. This explains why the composition of two reflections can be a rotation or translation, but never a reflection.

Another fundamental characteristic of an isometry is the points that it leaves fixed. For instance, a rotation doesn't move the center (but moves any other point); a reflection fixes every point of its axis. We summarize these properties in the chart below.

type of isometry	points that stay fixed	flips colors?
rotation	the center	no
reflection	every point on axis	yes
translation	none	no
glide reflection	none	yes

These properties help detect the type of isometry. In particular, the chart shows that a glied reflection cannot belong to any of the other three types.

□

MAT360. Midterm topics

Here is the list of topics recommended for review.

1. Theorem about vertical angles, section 26.
2. Existence and uniqueness of perpendicular to a line from a point, sections 24, 65 and 66.
3. Theorems about isosceles triangles and their properties, sections 35, 36.
4. Congruence tests for triangles section 40.
5. Inequality between exterior and interior angles in a triangle, sections 41 - 43.
6. Relations between sides and opposite angles sections 44 - 45.
7. Triangle inequality and its corollaries, sections 48 and 49.
8. Perpendicular and slants, sections 51 - 53.
9. Segment and angle bisectors, sections 56 and 57.
10. Basic construction problems, sections 61 - 69.
11. Tests for parallel lines, section section 73.
12. The parallel postulate, sections 75 and 76.
13. Angles formed by parallel lines and a transversal, sections 77 and 78.
14. Angles with respectively parallel sides, section 78.
15. Angles with respectively perpendicular sides, section 79.
16. The sum of interior angles in a triangle, section 81.
17. The sum of interior angles in a convex polygon, section 82.
18. Properties of a parallelogram, sections 85 - 87.
19. Special types of parallelograms and their properties, sections 90 - 92.
20. The midline theorem, sections 93, 94, 95.
21. The midline of trapezoid, sections 96, 97.
22. Existence and uniqueness of a circle passing through three points, sections 103, 104.
23. Constructions that use isometries, sections 98 - 101
24. Theorems about inscribed angles, section 123.
25. Corollaries of the theorem about inscribed angles, sections 125,126.
26. Constructions using theorems about inscribed angles, sections 127 - 130, 133.
27. Inscribed and circumscribed circles, sections 136, 137.
28. Concurrency points in a triangle, sections 140 - 142.