Math 315—Advanced Linear Algebra (Spring 2019)

Instructor: Ben McMillan
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Location: T/Th 2:30pm--3:50pm in Library N3063

Announcements

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Course Information:

The course syllabus is here. Some critical points:

- The midterms will be in class on February 14 and April 2
- My office hours this term are Tuesdays 1–2pm and Fridays 11am–12pm in the Simons Center 508. I will also be available in the MLC (S-235) on Thursdays 10:30am–11:30pm.

Schedule and Homework:

The following is a tentative schedule for the course. As homework is assigned it will be posted here. You are encouraged to work with others, but please make sure to write up solutions in your own words (this will help you on the exams and quizzes!).

Unless stated otherwise, the homework is from the corresponding section of the book.

All of the problems assigned in week n are due to your TA (at the start of recitation) in week n+1.

It will be expected that you have done the suggested reading.

Problems in square brackets are ones that I think you will benefit from, but won't be graded. You should still do them!

Week	Date	Topic(s) Covered	Reading	Homework
1	1/29	Abstract Vector Spaces	1A, 1B	1A: 1, 15 1B: 1, 6
	1/31	Subspaces, Direct Sums	1C	3, 7, 8, 10, 24
2	2/5	Spans, Polynomials	2A	1, 2, 6, 8, 11
	2/7	Linear Independence, Definition of Dimension	2B	2, 3, 5, 7, 8
3	2/12	Bases	2C	1, 3, 8, 10, 11, 16, [17]
	2/14	Midterm 1	.	Review
4	2/19	Linear Maps	3A	1, 3, 4, 8, 9 [do R--->R too!], 11
	2/21	Injectivity/Surjectivity & Kernel/Image	3B	2, 3, 7, 9, [11], 20, [21], [26], 28
5	2/26	Linear Isomorphisms	3D	3B: 17, [18] 3D: 2, 4, 5, 8, 9
	2/28	Coordinates	3C	2, 3, 8, 14
6	3/5	Equivalence Relations, quotient spaces	3E	12, 13, 16, 20
	3/7	Direct Products	3E	1, 6, [recall that a map of sets f:A--->B defines an equivalence relation ~ on A. Show that f is injective if and only if the quotient map A--->A/~ is an isomorphism.]
7	3/12	Dual Spaces	3.F	7, 9, 13
	3/14	Tensor Products	.	here
8	3/19	Spring Break	.	.
	3/21	Spring Break	.	.
9	3/26	Practice with Tensor Products	.	.
	3/28	Change of Bases	.	.
10	4/2	Midterm II	.	Review
	4/4	Trace	.	here
11	4/9	Eigenvalues/Eigenvectors	5A	11, 12, 15, 21, 25
	4/11	Eigen-decomposition	5B	1, 2, 5, 10, 11
12	4/16	Generalized Eigenvectors	5C 8A	6, 12 1, 4, 5, [12], 15, [16, 17]
	4/18	Jordan Canonical Form	8B 8D	1, 9 4, 5
13	4/23	Inner Products	6A	8, 10, 14, 18, 23
	4/25	Orthonormal Bases	6B	1, 4, 5, 11, 12
	4/30	Riesz Representation Theorem	6B	7, 8, [9], [15]

14			6C	[4], 5, 10
	5/2	Complexification	9A	[Read at least 276-278] 2, 3, 4, [17]
15	5/7	Self Adjoint operators	7A	[1, 3, 6, 8 [can you find a natural complement?], 20]
	5/9	Representations, symmetrizers and the determinant	.	.
16	5/7	Study	.	Review
	5/9	Final Exam	5:30-8:00pm	.

Math 315: Advanced Linear Algebra

Spring 2019 Stony Brook

Instructor:	Ben McMillan	**Time:**	TTh 2:30pm–3:50pm
Email:	bmcmillan@math.stonybrook.edu	**Place:**	Library N3063

Course Page: The primary webpage for this course is

math.stonybrook.edu/~bmcmillan/math315

where you will find up to date information, homework, and announcements. Please bookmark it and check back regularly.

You will also find announcements and grades on the course blackboard page.

Office Hours: Office hours are an invaluable resource, one that you really should use!

My office hours this term are Tuesdays 1–2pm and Fridays 11–12pm in Simons Center 508. I will also be available in the MLC (S-235) on Thursdays 10:30–11:30pm.

You can find your TA's office hours at math.stonybrook.edu/office-hours

Textbook: The lecture will roughly follow Axler's *Linear Algebra Done Right*, and I will generally assign homework problems from the book. Later in the term we may use other resources.

Exams: The first midterm will be held in class on February 14. The second will be in class on April 2. Please let me know *soon* if you will need DSS accomodations.

The final is scheduled for Thursday, May 16, 5:30pm–8:00pm. Please ensure NOW that you won't have any scheduling conflicts with the final!

Homework: Each week I will post homework questions relevant to the week's lectures on the course webpage. You will submit it to your TA in recitation the following week.

Grading Policy: Homework: 35%, Midterm 1 : 10%, Midterm 2: 20%, Final: 35%.

Americans with Disabilities Act: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631)632–6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. `http://studentaffairs.stonybrook.edu/dss/index.html`.

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Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.

- This review is necessarily not exhaustive. I recommend looking back over the homework. I also offered exercises in lecture, which should give you an idea of what I think is interesting to ask, and what I think is reasonable to ask.
- You should be comfortable with the abstract axioms for a vector space. As always with this sort of thing, don't memorize them. Instead, seek to understand their meaning, which will then make them easy to remember and use. Good practice would be to make sure you understand how to prove from the axioms such facts such as "$0\vec{v} = \vec{0}$ for any vector $\vec{v}$ in a vector space."
- If you haven't done it, there is an exercise you should do exactly once in your life: fix a set S and let $\mathbb{R}^S$ be the set of all functions from S to R. Define addition pointwise—that is, for functions $f, g\colon S \to \mathbb{R}$, define a new function $f + g\colon S \to \mathbb{R}$ by the rule $(f + g)(s) = f(s) + g(s)$ for each $s \in S$. Likewise, define the R-multiplication on $\mathbb{R}^S$ pointwise: $(af)(s) = af(s))$. Show that $\mathbb{R}^S$ satisfies all the axioms of a vector space.
- Know the subspace criterion, and how to use it.
- For example, let $U_1, \ldots, U_m$ be a collection of subspaces of a vector space V. Show that their intersection is a subspace as well.
- Understand the definition of direct sum of subspaces, as well as how to prove direct sums. Problems 1C: 20, 23 are both good practice that didn't make it onto the assigned homework.
- The definition of span. Why is the span of a set of vectors a subspace?
- The definition of linear dependence. A list of dependent vectors has at least one redundant vector.
- The definition of linear independence. A linear relation on an independent list of vectors must be trivial.
- We spent a while on the example of polynomials. You should know it. The space $\mathcal{P}_m(\mathbb{R})$ of polynomials of degree $\leq m$ is a convenient example for us of a finite vector space that isn't $\mathbb{R}^n$, so we'll start to use it as such.

I made a slight misstatement in lecture. Recall the definition:

Definition 1. For a set S, a free vector space on S is a vector space V and a map (of sets)

$$i\colon S \longrightarrow V$$

such that the image $i(S) \subset V$ is a basis of V.

I gave the example: Let $V = \mathbb{R}^S$ be the vector space of functions from S to your ground field $\mathbb{R}$. There is a natural map $i\colon S \to V$ that sends an element $s \in S$ to the function $i_s\colon S \to \mathbb{R}$ given by the rule $i_s(s) = 1$ and $i_s(t) = 0$ for $t \neq s$.

In the case that S is an infinite set, **this is not quite a free vector space.** The issue is that $i(S)$ is not a basis when S is infinite. For example: let $S = \mathbb{N}$, the natural numbers. The set $i(\mathbb{N}) = \{i_1, i_2, i_3, \ldots\}$ does not span $\mathbb{R}^{\mathbb{N}}$. To see this, note that the constant 1-valued function in $\mathbb{R}^{\mathbb{N}}$ cannot be written as a *finite* linear combination of elements of $i(\mathbb{N})$.

How do we fix this? Let

$$F(S) = \{f\colon S \to \mathbb{R} \,|\, f(s) \text{ is non-zero for only finitely many } s \in S\}.$$

This is easily seen to be a subspace of $\mathbb{R}^S$. And $i(S)$ does provide a basis, by definition. This is the correct example of a free vector space on S when S is infinite.

For another example, if we let $S = \{x^0, x^1, x^2, \ldots\}$ then it is fair to think of $F(S)$ as the formal polynomials in the variable x, while $\mathbb{R}^S$ is more or less formal power series in variable x. It's only in the latter space that you hace to worry about convergence issues (which subtleties you spent significant time on in your Calculus course.)

1. Homework Questions

It may help to read Question 1 and then do Question 2 before finishing Question 1.

1) Finish/make sure you understand/write the details of the proof that

$$\dim(V \otimes W) = \dim(V)\dim(W)$$

when V and W are finite dimensional. The steps outlined in lecture:

a) If v_i a basis for V, and w_a a basis for W, check that $v_i \otimes w_a$ spans $V \otimes W$.

b) There is a (natural, linear) map

$$\begin{aligned} V^\vee \otimes W &\longrightarrow \mathcal{L}(V, W) \\ \alpha \otimes w &\longmapsto (v \mapsto \alpha(v)w) \end{aligned}$$

Show this map is a surjection.

c) By dimension counting, the map of part b) is an isomorphism.

2) Suppose $V = W$ is finite dimensional. In the map described in part b) above, what element of $V^\vee \otimes V$ maps to the identity map in $\mathcal{L}(V, V)$?

3) Fix an element $\alpha \in V^\vee$ and $\beta \in W^\vee$. Show that the 'product' $\alpha\beta$ defines a *linear* map

$$\begin{aligned} \alpha\beta\colon V \otimes W &\longrightarrow \mathbb{R} \\ v \otimes w &\longmapsto \alpha(v)\beta(w) \end{aligned}$$

- This review is necessarily not exhaustive. I recommend looking back over the homework. I also offered exercises in lecture, which should give you an idea of what I think is interesting to ask, and what I think is reasonable to ask.
- Linear Maps—the definition should be second nature by now. The set of linear maps $\mathcal{L}(V,W)$ has a vector space structure, as well as a 'product' structure, in the sense that we can compose functions. In particular $\mathcal{L}(V,V)$ is a vector space with a multiplication (this is often called an *algebra* over the base field, which is essentially a fancy way of saying a vector space with multiplication.)
- The kernel and the image of a linear map are both important concepts you should be comfortable with by now. Including their relation to isomorphisms.
- The rank nullity theorem is an essential tool for dimension counting, which is ultimately one of the great joys of linear algebra.
- If we fix a basis of V and a basis of W, then we have a linear isomorphism between $\mathcal{L}(V,W)$ and $M_{n\times m}$, where the latter is the space of n by m matrices. If, say, $V = W$, then the isomorphism preserves products as well: it sends the composition of linear maps to the matrix product of corresponding matrices.
- In this course, we've only discussed matrices as indicial objects, the matrix A_i^j, etc. We've done a little bit of indicial calculations, including how to use Einstein summation notation.
- Invertible linear maps. The inverse map is linear. You should be comfortable using the kernel, the image and dimension counting to show that a linear map is invertible.
- The direct sum of vector spaces. The book calls them "direct products" $V \times W$, but we use the symbol $\oplus$ instead and call it direct sum to remind ourselves that we use this when we want to add vector spaces. (In particular, dimensions add.)
- Quotients of vector spaces. The map from a vector space to its quotient by a subspace. The quotient space is a vector space and the quotient map is linear. (in particular—the notion of when an operation is well defined on an equivalence class...)
- Equivalence relations. In particular, we've seen a few instances where it is important to be able to say that a function from $S/\sim$ is well defined.
- Dual vector spaces. Dual basis. Dual maps.
- We spent some time on Einstein summation notation and indices (although this is certainly an area where more needs to be said!). Bases lead to indices, and you should be comfortable expressing matrix coefficients in terms of such. A good check of your understanding is to prove that if A_i^j is the matrix associated to a linear map S and B_j^k the matrix associated to linear T, then the matrix for ST is $A_i^j B_k^j$. [Here it is implicit that each domain, codomain has a fixed basis, and they agree where needed.]
- Tensor products. We spent a lecture defining them. The important thing for working with them is the algebraic relations that we used to define the tensor product.
- The (natural) isomorphism $\mathcal{L}(V,W) \cong W \otimes V^\vee$ is useful. So is the trace map $V \otimes V^\vee \to \mathbb{R}$.

1. HOMEWORK 8

1) If V and W are finite dimensional vector spaces, write a (natural) isomorphism between $V^\vee \otimes W^\vee$ and $(V \otimes W)^\vee$. Prove that you map is an isomorphism.

2) Consider 3 vector spaces U, V and W of finite dimension. There is a natural map

$$\begin{aligned}\psi\colon \mathcal{L}(U, \mathcal{L}(V, W)) &\longrightarrow \mathcal{L}(U \otimes V, W)\\ (u \mapsto T_u) &\longmapsto (u \otimes v \mapsto T_u(v))\end{aligned}$$

[That is, any linear map sending each $u \in U$ to a linear map $T_u\colon V \to W$ is mapped by ψ to a linear map from $U \otimes V$ to W.] Find its inverse, and prove it is an inverse.

Check that any element $A \in \mathcal{L}(U, \mathcal{L}(V, W))$ defines a bilinear map $U \times V \to W$.

The isomorphism ψ is called the 'tensor hom adjunction', you may be interested to read more about it. For our purposes, the isomorphism is an assertion that *bilinear* maps on $U \times V$ are the same as *linear* maps on $U \otimes V$.

3) Let V be a finite dimensional vector space, and $\beta = \{e_i\}$, $\gamma = \{f_i\}$ two bases of V, with dual bases $\beta^\vee = \{e^i\}, \gamma^\vee = \{f^i\}$. There is an invertible matrix A_i^j so that $f_i = A_i^j e_j$. Show that $f^i = (A^{-1})^i_j e^j$. [In terms of indices, if A_i^j is a square matrix, its inverse $(A^{-1})^i_j$ is characterized by either the equation $(A^{-1})^k_i A^j_k = \delta^j_i$ or by $A^k_i (A^{-1})^j_k = \delta^j_i$. Don't be thrown off by the notation: if we renamed the matrix $A^{-1} = B$, then $(A^{-1})^i_j = B^i_j$ and we have, for example, $B^k_i A^j_k = \delta^j_i$.]

4) Continuing with notation from problem 3, for $V \otimes V^\vee$, we have the basis $\beta \times \beta^\vee = \{e_i \otimes e^j\}$ and the basis $\gamma \times \gamma^\vee = \{f_i \otimes f^j\}$. Fix an element $T \in V \otimes V^\vee$. In the first basis, there are coefficients $T^j_i \in \mathbb{R}$ so that $T = T^j_i e_j \otimes e^i$ and in the second basis, coefficients $\tilde{T}^j_i \in \mathbb{R}$ so that $T = \tilde{T}^j_i f_j \otimes f^i$. Show that $\tilde{T}^j_i = A^k_i T^l_k (A^{-1})^j_l$.

5) For an invertible matrix C and any matrix A, show that $\operatorname{tr}(CAC^{-1}) = \operatorname{tr}(A)$. You may use either the matrix definition of trace, or the $V \otimes V^\vee \to \mathbb{R}$ definition. Whichever you write up for the HW, you should do the other for yourself.

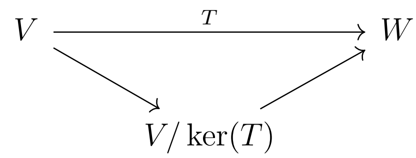
6) Fix a ground field $\mathbb{F}$. Show that $(F(\mathbb{N}))^\vee$ (where $F(\mathbb{N})$ is the free vector space on $\mathbb{N}$, which may be identified uniquely with $\{$functions from $\mathbb{N}$ to $\mathbb{F}$ that are non-zero only finitely often$\}$) is isomorphic to $\mathbb{F}^\mathbb{N}$ (the vector space of all functions $\mathbb{N} \to \mathbb{F}$). (Hint: $F(\mathbb{N})$ has a natural basis, you should probably use it.)

- This review is necessarily not exhaustive. Think of it as something like the minimum of what I hope you walk away from the course knowing of. I recommend looking back over the homework. I also offered exercises in lecture, which should give you an idea of what I think is interesting to ask, and what I think is reasonable to ask. The questions are posted on blackboard.
- Linear Maps are of course fundamental to the course. Their definition. Given choices of bases, you get an isomorphism from the space of linear maps to the space of matrices. This isomorphism even preserves multiplication (it is a map of algebras). If you change basis, the isomorphism changes, but in a controlled way: by conjugation by the change of basis matrix.
- Another way to consider linear maps 'in a basis' is by the isomorphism $\mathcal{L}(V, W) \cong W \otimes V^\vee$. A choice of bases of V and W gives you a basis of $W \otimes V^\vee$ (use the dual basis of V). The coefficients of an element in this basis are exactly the matrix entries in that basis.
- Knowing when a linear map is invertible is a big deal. You can either explicitly construct an inverse, or you can show injectivity + surjectivity. If you know in advance that dimensions are equal, then either of injectivity or surjectivity proves the other.
- Eigenvalues, eigenvectors and generalized eigenvectors. Eigenspaces are useful, as they provide invariant subspaces of an operator. In good cases (such as no repeated eigenvalues over $\mathbb{C}$, or a self-adjoint operator) you can completely describe an operator by using an eigenbasis. In an eigenbasis (if it exists), the matrix of an operator is diagonal. (We say an operator is diagonalizable if it has an eigenbasis.)
- Over $\mathbb{C}$, even operators that don't have an eigenbasis have something close: a basis of generalized eigenvectors. You should understand the definition, but (for example) if you understand the generalized eigenvectors of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

  then you understand everything that can happen. In shorter notation, this matrix is $J(0,1) \oplus J(2,1) \oplus J(2,1) \oplus J(3,3) \oplus J(3,1)$, where $J(\lambda, r)$ stands for the $r \times r$ Jordan block with eigenvalue λ.
- The generalized eigenspaces of an operator T are T invariant. Even better, they have a canonical T-invariant complement. This means that we can (still working over $\mathbb{C}$) inductively split the domain of T into a direct sum of generalized eigenspaces. By understanding T on each component, you arrive at the Jordan Normal form theorem.
- The Jordan Normal form theorem is a very powerful statement, giving in some sense a complete understanding of operators on finite dimensional vector spaces. Once you know the statement, many facts have proofs that begin "Without loss of generality, choose a basis so that T is in Jordan normal form...". This is a valid strategy on the test.
- On the other hand, the full version of Rank-Nullity gives a fairly complete picture of linear maps between distinct vector spaces. By full version, I mean the statement that a linear map $T \in \mathcal{L}(V, W)$ can be written as a canonical composition of a surjection

followed by an inclusion:



What the book calls rank-nullity follows from the diagram: $\dim(V) = \dim(\ker T) + \dim(V/\ker(T))$ but also $V/\ker(T) \cong \mathrm{img}(T)$.

- As mentioned above, invariant subspaces of an operator $T \in \mathcal{L}(V, V)$ are very useful. Especially useful are invariant splittings $V = U \oplus W$, where both U and W are invariant.
- We spent a while discussing tensor products. I recommend revisiting the homeworks involving them. It is likely that you will find the problems much easier by this point. The definition of tensor product is maybe different from definitions you've seen before. More important than the definition is the properties that tensor products have. The algebra of how they work ($v \otimes (w + w') = v \otimes w + v \otimes w'$ etc.) is the key point, the reason they are useful. Bases of V and W multiply to give a basis of $V \otimes W$. Make sure you are comfortable with the homework problems that were given about tensor products.
- The trace map.
- Inner products. Know how to use positive definitivity. Orthonormal bases are a key tool. As is the Riesz-Representation Theorem.
- Self adjoint operators and the spectral theorem. They have an eigenbasis, even over $\mathbb{R}$!
- To prove the spectral theorem, we had to learn about complexification. The complexification of a real vector space is its tensor product with $\mathbb{C}$ (which is a real 2-dimensional vector space.)