MAT 310: Linear Algebra Spring 2019

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Welcome to MAT 310

Textbook: Linear Algebra Done Right by S. Axler.

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Course Overview

This course is a continuation of MAT 211. We will cover fundamentals of finite dimensional vector spaces, linear maps, dual spaces, bilinear functions, and inner products. A tentative weekly plan for the course is **here**.

Information for students willing to move up to MAT 315

We will cover approximately the same material in the first couple of weeks in MAT 310 and MAT 315. On Thursday, February 14, we will have an exam in class which will decide whether a student would stay in MAT 310 or be allowed to move up to MAT 315.

Homework

Homework assignments will be posted **here** and on

BlackBoard. Please hand them in to your recitation instructor, Yoon-Joo Kim, the following week. Please note that your recitation instructor will NOT accept late homework.

Quizzes

There will be a short quiz in your recitation session every other week. The first quiz will be given in the week of Feb 11 - Feb 15.

Exams and Grading

There will be two midterms, and a final exam (dates **here**), whose weights in the overall grade are listed below.

15% Homework

10% Quizzes

20% Midterm 1

20% Midterm 2

35% Final Exam (cumulative)

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MAT 310: Linear Algebra
Spring 2019

General Information

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

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Syllabus and Weekly Plan

Week of	Topics
Jan 28	Chapter 1. Vector spaces
Feb 4	Chapter 2. Finite-Dimensional Vector Spaces
Feb 11	Chapter 2. Finite-Dimensional Vector Spaces Exam in class on Thursday
Feb 18	Chapter 3. Linear Maps
Feb 25	Chapter 3. Linear Maps
March 4	Midterm I, Tue. March 5 Chapter 5. Eigenvalues and Eigenvectors
March 11	Chapter 5. Eigenvalues and Eigenvectors
March 18	Spring Break!
March 25	Chapter 6. Inner-Product Spaces
April 1	Chapter 6. Inner-Product Spaces
April 8	

	Midterm II Review
April 15	Midterm II, Tue. April 16 Chapter 7. Operators on Inner-Product Spaces: Unitary operators
April 22	Chapter 7. Operators on Inner-Product Spaces: Normal operators and Spectral theorem
April 29	Chapter 8. Operators on Complex Vector Spaces
May 6	Chapter 10. Trace and Determinant
	Final Exam Thursday, May 16, 5:30pm-8:00pm

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Homework

Homework 1 (due on Tuesday, Feb 12): Problems 3, 4, 8, 9, 14, and 15 of this **sheet**.

Homework 2 (due on March 5/6, depending on your recitation): Problems 2, 4, 8, 11, 12, and 14 of this **sheet**.

Homework 3 (due on March 26/27, depending on your recitation): Problems 1, 3, 7, 8, 14, and 22 of this **sheet**.

Homework 4 (due on April 2/3, depending on your recitation): Problems 2, 4, 6, 9, 10, and 12 of this **sheet**.

Homework 5 (due on April 16/17, depending on your recitation): Problems 4, 5, 6, 9, 10, 16, 22, 24, 29, and 30 of this **sheet**.

Homework 6 (due on May 7/8, depending on your recitation): Problems 1(a), 6, 7, and 11 of this **sheet**.

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Exams

There will be a **mandatory exam in class on Thursday, February 14** to determine which students would be allowed to move up to MAT 315. However, this exam will NOT contribute to the final grade.

Here is the **placement exam** with **solutions**.

There will be two midterms and a final exam. The time of these exams are as follows:

Midterm 1: Tuesday, March 5, 2:30pm-3:50pm (in class)

Here are some **practice problems** for midterm 1, and here are the **solutions**.

Here are the **solutions** to Midterm 1 problems.

Midterm 2: Tuesday, April 16, 2:30pm-3:50pm (in class)

Here are some **practice problems** for midterm 2, and here are the **solutions**.

Here are the **solutions** to Midterm 2 problems.

Final exam: Thursday, May 16, 5:30pm-8:00pm

Here are some **practice problems** for the final exam, and here are the **solutions**.

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In all the problems, you may assume that F is the set of real numbers.

Exercises

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that
$$1/(a + bi) = c + di.$$

2. Show that
$$\frac{-1 + \sqrt{3}i}{2}$$
is a cube root of 1 (meaning that its cube equals 1).

3. Prove that $-(-v) = v$ for every $v \in V$.

4. Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

5. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

   (a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

   (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

   (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;

   (d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.

6. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

7. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

8. Prove that the intersection of any collection of subspaces of V is a subspace of V.

9. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

10. Suppose that U is a subspace of V. What is $U + U$?

11. Is the operation of addition on the subspaces of V commutative? Associative? (In other words, if U_1, U_2, U_3 are subspaces of V, is $U_1 + U_2 = U_2 + U_1$? Is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?)

12. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

13. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$U_1 + W = U_2 + W,$$
then $U_1 = U_2$.

14. Suppose U is the subspace of $\mathcal{P}(\mathbf{F})$ consisting of all polynomials p of the form
$$p(z) = az^2 + bz^5,$$
where $a, b \in \mathbf{F}$. Find a subspace W of $\mathcal{P}(\mathbf{F})$ such that $\mathcal{P}(\mathbf{F}) = U \oplus W$.

15. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$
then $U_1 = U_2$.

In all the problems, you may assume that F is the set of all real numbers.

Exercises

1. Prove that if $(v_1, \dots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

2. Prove that if $(v_1, \dots, v_n)$ is linearly independent in V, then so is the list

$$(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

3. Suppose $(v_1, \dots, v_n)$ is linearly independent in V and $w \in V$. Prove that if $(v_1 + w, \dots, v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_n)$.

4. Suppose m is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to m a subspace of $\mathcal{P}(\mathbf{F})$?

5. Prove that $\mathbf{F}^\infty$ is infinite dimensional.

6. Prove that the real vector space consisting of all continuous real-valued functions on the interval $[0, 1]$ is infinite dimensional.

7. Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $(v_1, \dots, v_n)$ is linearly independent for every positive integer n.

8. Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

9. Prove or disprove: there exists a basis (p_0, p_1, p_2, p_3) of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

10. Suppose that V is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, \dots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

11. Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

12. Suppose that $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $(p_0, p_1, \ldots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

13. Suppose U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $U \cap W = \{0\}$.

14. Suppose that U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

15. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)& \\ &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

16. Prove that if V is finite dimensional and $U_1, \ldots, U_m$ are subspaces of V, then

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

17. Suppose V is finite dimensional. Prove that if $U_1, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$

This exercise deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a finite set is written as a disjoint union of subsets, then the number of elements in the set equals the sum of the number of elements in the disjoint subsets.

Exercises

1. Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

2. Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that
$$f(av) = af(v)$$
for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is not linear.

Exercise 2 shows that homogeneity alone is not enough to imply that a function is a linear map. Additivity alone is also not enough to imply that a function is a linear map, although the proof of this involves advanced tools that are beyond the scope of this book.

3. Suppose that V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

4. Suppose that T is a linear map from V to $\mathbf{F}$. Prove that if $u \in V$ is not in $\operatorname{null} T$, then
$$V = \operatorname{null} T \oplus \{au : a \in \mathbf{F}\}.$$

5. Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V. Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

6. Prove that if $S_1, \ldots, S_n$ are injective linear maps such that $S_1 \ldots S_n$ makes sense, then $S_1 \ldots S_n$ is injective.

7. Prove that if $(v_1, \ldots, v_n)$ spans V and $T \in \mathcal{L}(V, W)$ is surjective, then $(Tv_1, \ldots, Tv_n)$ spans W.

8. Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

9. Prove that if T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that
$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\},$$
then T is surjective.

10. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

11. Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

12. Suppose that V and W are both finite dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim W \leq \dim V$.

13. Suppose that V and W are finite dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

14. Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

15. Suppose that V is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

16. Suppose that U and V are finite-dimensional vector spaces and that $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

17. Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, and C are matrices whose sizes are such that $A(B + C)$ makes sense. Prove that $AB + AC$ makes sense and that $A(B + C) = AB + AC$.

18. Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

19. Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ and that

This exercise shows that T has the form promised on page 39.

$$\mathcal{M}(T) = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix},$$

where we are using the standard bases. Prove that

$$T(x_1, \dots, x_n) = (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \dots, a_{m,1}x_1 + \cdots + a_{m,n}x_n)$$

for every $(x_1, \dots, x_n) \in \mathbf{F}^n$.

20. Suppose $(v_1, \dots, v_n)$ is a basis of V. Prove that the function $T\colon V \to \mathrm{Mat}(n, 1, \mathbf{F})$ defined by

$$Tv = \mathcal{M}(v)$$

is an invertible linear map of V onto $\mathrm{Mat}(n, 1, \mathbf{F})$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $(v_1, \dots, v_n)$.

21. Prove that every linear map from $\mathrm{Mat}(n, 1, \mathbf{F})$ to $\mathrm{Mat}(m, 1, \mathbf{F})$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathrm{Mat}(n, 1, \mathbf{F}), \mathrm{Mat}(m, 1, \mathbf{F}))$, then there exists an m-by-n matrix A such that $TB = AB$ for every $B \in \mathrm{Mat}(n, 1, \mathbf{F})$.

22. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

23. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

24. Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

25. Prove that if V is finite dimensional with $\dim V > 1$, then the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

26. Suppose n is a positive integer and $a_{i,j} \in \mathbf{F}$ for $i, j = 1, \ldots, n$. Prove that the following are equivalent:

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} a_{1,k} x_k = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} a_{n,k} x_k = 0.$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} a_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} a_{n,k} x_k = c_n.$$

Note that here we have the same number of equations as variables.

Exercises

1. Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \dots, U_m$ are subspaces of V invariant under T, then $U_1 + \cdots + U_m$ is invariant under T.

2. Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

3. Prove or give a counterexample: if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

4. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

5. Define $T \in \mathcal{L}(\mathbf{F}^2)$ by

   $$T(w, z) = (z, w).$$

   Find all eigenvalues and eigenvectors of T.

6. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

   $$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

   Find all eigenvalues and eigenvectors of T.

7. Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by

   $$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n);$$

   in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

8. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

   $$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

9. Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

10. Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbf{F} \setminus \{0\}$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

11. Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

12. Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

13. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

14. Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Prove that if $p \in \mathcal{P}(\mathbf{F})$ is a polynomial, then

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

15. Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $a \in \mathbf{C}$. Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

16. Show that the result in the previous exercise does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

17. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension j for each $j = 1, \dots, \dim V$.

18. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

19. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

These two exercises show that 5.16 fails without the hypothesis that an upper-triangular matrix is under consideration.

20. Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

21. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

22. Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Find all eigenvalues and eigenvectors of $P_{U,W}$.

Exercises

1. Prove that if x, y are nonzero vectors in $\mathbf{R}^2$, then

$$\langle x, y\rangle = \|x\|\|y\|\cos\theta,$$

where θ is the angle between x and y (thinking of x and y as arrows with initial point at the origin). *Hint:* draw the triangle formed by x, y, and $x - y$; then use the law of cosines.

2. Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \le \|u + av\|$$

for all $a \in \mathbf{F}$.

3. Prove that

$$\Big(\sum_{j=1}^{n} a_j b_j\Big)^2 \le \Big(\sum_{j=1}^{n} j{a_j}^2\Big)\Big(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\Big)$$

for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$.

4. Suppose $u, v \in V$ are such that

$$\|u\| = 3, \quad \|u + v\| = 4, \quad \|u - v\| = 6.$$

What number must $\|v\|$ equal?

5. Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x_1, x_2)\| = |x_1| + |x_2|$$

for all $(x_1, x_2) \in \mathbf{R}^2$.

6. Prove that if V is a real inner-product space, then

$$\langle u, v\rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}$$

for all $u, v \in V$.

7. Prove that if V is a complex inner-product space, then

$$\langle u, v\rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i}{4}$$

for all $u, v \in V$.

8. A norm on a vector space U is a function $\| \ \| \colon U \to [0, \infty)$ such that $\|u\| = 0$ if and only if $u = 0$, $\|\alpha u\| = |\alpha| \|u\|$ for all $\alpha \in \mathbf{F}$ and all $u \in U$, and $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in U$. Prove that a norm satisfying the parallelogram equality comes from an inner product (in other words, show that if $\| \ \|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \ , \ \rangle$ on U such that $\|u\| = \langle u, u\rangle^{1/2}$ for all $u \in U$).

9. Suppose n is a positive integer. Prove that

$$\Big(\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}\Big)$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

This orthonormal list is often used for modeling periodic phenomena such as tides.

10. On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x) q(x)\, dx.$$

Apply the Gram-Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

11. What happens if the Gram-Schmidt procedure is applied to a list of vectors that is not linearly independent?

12. Suppose V is a real inner-product space and $(v_1, \dots, v_m)$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $(e_1, \dots, e_m)$ of vectors in V such that

$$\operatorname{span}(v_1, \dots, v_j) = \operatorname{span}(e_1, \dots, e_j)$$

for all $j \in \{1, \dots, m\}$.

13. Suppose $(e_1, \dots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \dots, e_m)$.

14. Find an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$ (with inner product as in Exercise 10) such that the differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbf{R})$ has an upper-triangular matrix with respect to this basis.

15. Suppose U is a subspace of V. Prove that

$$\dim U^{\perp} = \dim V - \dim U.$$

16. Suppose U is a subspace of V. Prove that $U^{\perp} = \{0\}$ if and only if $U = V$.

17. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$, then P is an orthogonal projection.

18. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$, then P is an orthogonal projection.

19. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

20. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^{\perp}$ are both invariant under T if and only if $P_U T = T P_U$.

21. In $\mathbf{R}^4$, let

$$U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big).$$

Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

22. Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |2 + 3x - p(x)|^2\,dx$$

is as small as possible.

23. Find $p \in \mathcal{P}_5(\mathbf{R})$ that makes

$$\int_{-\pi}^{\pi} |\sin x - p(x)|^2\,dx$$

as small as possible. (The polynomial 6.40 is an excellent approximation to the answer to this exercise, but here you are asked to find the exact solution, which involves powers of π. A computer that can perform symbolic integration will be useful.)

24. Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

25. Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)\,dx = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

26. Fix a vector $v \in V$ and define $T \in \mathcal{L}(V, \mathbf{F})$ by $Tu = \langle u, v \rangle$. For $a \in \mathbf{F}$, find a formula for T^*a.

27. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1}).$$

Find a formula for $T^*(z_1, \dots, z_n)$.

28. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Prove that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.

29. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

30. Suppose $T \in \mathcal{L}(V, W)$. Prove that

    (a) T is injective if and only if T^* is surjective;

    (b) T is surjective if and only if T^* is injective.

31. Prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$.

32. Suppose A is an m-by-n matrix of real numbers. Prove that the dimension of the span of the columns of A (in $\mathbf{R}^m$) equals the dimension of the span of the rows of A (in $\mathbf{R}^n$).

Exercises

1. Make $\mathcal{P}_2(\mathbf{R})$ into an inner-product space by defining

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

2. Prove or give a counterexample: the product of any two self-adjoint operators on a finite-dimensional inner-product space is self-adjoint.

3. (a) Show that if V is a real inner-product space, then the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

(b) Show that if V is a complex inner-product space, then the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

4. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

5. Show that if $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

6. Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{range} T = \operatorname{range} T^*.$$

7. Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

8. Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbf{R}^3)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$.

9. Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Exercise 9 strengthens the analogy (for normal operators) between self-adjoint operators and real numbers.

10. Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

11. Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a ***square root*** of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

12. Give an example of a real inner-product space V and $T \in \mathcal{L}(V)$ and real numbers α, β with $\alpha^2 < 4\beta$ such that $T^2 + \alpha T + \beta I$ is not invertible.

This exercise shows that the hypothesis that T is self-adjoint is needed in 7.11, even for real vector spaces.

13. Prove or give a counterexample: every self-adjoint operator on V has a cube root. (An operator $S \in \mathcal{L}(V)$ is called a ***cube root*** of $T \in \mathcal{L}(V)$ if $S^3 = T$.)

14. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon,$$

then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

15. Suppose U is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that U has a basis consisting of eigenvectors of T if and only if there is an inner product on U that makes T into a self-adjoint operator.

16. Give an example of an operator T on an inner product space such that T has an invariant subspace whose orthogonal complement is not invariant under T.

This exercise shows that 7.18 can fail without the hypothesis that T is normal.

17. Prove that the sum of any two positive operators on V is positive.

18. Prove that if $T \in \mathcal{L}(V)$ is positive, then so is T^k for every positive integer k.

Name: SID:

Problem 1. (10 points) Let V be a vector space and $v \in V$ a fixed element. Demonstrate (using the properties of V as a vector space) that the element $(-1)v$ is the additive inverse of v.

Problem 2. (10 points) Suppose that U and W are subspaces of a vector space V. Prove that $U \cap W$ is also a subspace of V.

Problem 3. (10 points) Let $U_1 = \{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ and $U_1 = \{(0, y) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^2$. Show that $\mathbb{R}^2 = U_1 \oplus U_2$.

Problem 4. (10 points) Show that the elements $1, x, x^2, x^3$ span $\mathcal{P}_3$, where $\mathcal{P}_3$ is the vector space of polynomials in x of degree at most 3.

Problem 5. (10 points) Recall that $\mathcal{P}_3$ is a subspace of $\mathbb{R}^{\mathbb{R}}$, functions from $\mathbb{R}$ to $\mathbb{R}$. Use this to demonstrate that $1, x, x^2, x^3$ are linearly independent.

1) V is a vector space, and $v \in V$.

$(-1)v$ is the product of the scalar (-1) with the vector v.

Note that $1 + (-1) = 0$, in $\mathbb{R}$

$\Rightarrow \underline{(1+(-1)).v = 0.v}$, in V $\rightarrow$ (*)

Again, $0 + 0 = 0$, in $\mathbb{R}$

$\Rightarrow (0+0).v = 0.v$, in V

$\Rightarrow 0.v + 0.v = 0.v$ (distributivity)

$\Rightarrow (0.v + 0.v) + (-0.v) = 0.v + (-0.v)$

$\boxed{\text{where } (-0.v) \text{ is the additive inverse of } 0.v \text{ in } V}$

$\Rightarrow 0.v + (0.v + (-0.v)) = 0.v + (-0.v)$ [associativity]

$\Rightarrow 0.v + 0_V = 0_V$ [0_V = Additive identity in V]

$\Rightarrow \underline{0.v = 0_V} \rightarrow$ (**)

Plugging (**) in (*), ~~we get~~ and using distributivity we get: (2)

$$1.v + (-1).v = 0_V$$

$$\Rightarrow v + (-1).v = 0_V \quad \text{[As 1 is the identity of scalar multiplication]}$$

$$\Rightarrow (-v) + (v + (-1).v) = -v + 0_V$$

[where, $(-v)$ is the additive inverse of v in V]

$$\Rightarrow (-v + v) + (-1).v = -v \quad \text{[Associativity]}$$

$$\Rightarrow 0_V + (-1).v = -v$$

$$\Rightarrow \underline{(-1).v = -v}.$$

Hence, $(-1).v$ is the additive inverse of v in V. $\boxtimes$

3

2) Let U, W be subspaces of V.

Pick $\alpha_1, \alpha_2 \in U \cap W$

$$\Rightarrow \begin{cases} \alpha_1 \in U \text{ and } \alpha_1 \in W \\ \alpha_2 \in U \text{ and } \alpha_2 \in W \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 \in U, \text{ since } U \text{ is a subspace} \\ \alpha_1 + \alpha_2 \in W, \text{ since } W \text{ is a subspace} \end{cases}$$

$$\Rightarrow \underline{(\alpha_1 + \alpha_2) \in U \cap W}.$$

Now let $\lambda \in \mathbb{R}$, and $\alpha \in U \cap W$

$$\Rightarrow \lambda \in \mathbb{R}, \begin{cases} \alpha \in U. \text{ and} \\ \alpha \in W \end{cases}$$

$$\Rightarrow \begin{cases} \lambda\alpha \in U, \text{ as } U \text{ is a subspace} \\ \text{and} \\ \lambda\alpha \in W, \text{ as } W \text{ is a subspace} \end{cases}$$

$$\Rightarrow \underline{\lambda\alpha \in U \cap W}.$$

Thus, $U \cap W$ is closed under vector addition, and scalar multiplication

$\Rightarrow U \cap W$ is a subspace of V. ⊗

3) $U_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$,

$U_2 = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y \in \mathbb{R} \right\}$ are

Subspaces of $\mathbb{R}^2$.

Now, $U_1 + U_2$

$= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ 0 \end{pmatrix} \in U_1, \begin{pmatrix} 0 \\ y \end{pmatrix} \in U_2 \right\}$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x \in \mathbb{R}, y \in \mathbb{R} \right\}$

$= \mathbb{R}^2$.

Moreover, $U_1 \cap U_2$

$= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Hence, $\mathbb{R}^2 = U_1 \oplus U_2$.

$\boxtimes$

4) P_3 is the vector space of all Polynomials in x of degree ≤ 3.

Hence, any element of P_3 is of the form $a + bx + cx^2 + dx^3$

$$= a.1 + b.x + c.x^2 + d.x^3.$$

Therefore, every element of P_3 can be written as a linear combination of the vectors $1, x, x^2, x^3$ in P_3

Hence, $P_3 = \text{span}\{1, x, x^2, x^3\}$. ☒

~~5) Suppose that $1, x, x^2,$~~

5) suppose that

$(*) \to \quad a.1 + b.x + cx^2 + d.x^3 = 0$, in P_3.

[where 0 is the zero polynomial]

Then, $(*)$ must hold for every real number x.

In particular, putting $x = 0$ in $(*)$, we get : $\underline{a = 0} \to (A)$

Again, Putting $x = 1$ in ⊛, we get:

$$0.1\ b.1 + c.1^2 + d.1^3 = 0$$

$$\Rightarrow \underline{b + c + d = 0} \rightarrow (2)$$

Putting $x = -1$ in ⊛, we get:

$$0.1 + b.(-1) + c.(-1)^2 + d(-1)^3 = 0$$

$$\Rightarrow \underline{-b + c - d = 0} \rightarrow (3)$$

Adding (2) and (3), we get: $2c = 0$

$$\Rightarrow \underline{c = 0} \rightarrow (4).$$

Putting $x = 2$ in ⊛, we get

$$0.1 + b.2 + 0.2^2 + d.2^3 = 0$$

$$\Rightarrow 2b + 8d = 0 \Rightarrow 2b = -8d \Rightarrow \underline{b = -4d} \rightarrow (5)$$

Plugging (5) and (4) in (2), we get:

$$-4d + 0 + d = 0 \Rightarrow \underline{d = 0} \rightarrow (6)$$

Plugging (6) in (5), we get: $\underline{b = 0}$.

Hence, $\underline{a.1 + b.x + c.x^2 + d.x^3 = 0 \Rightarrow a = b = c = d = 0}$.

$\Rightarrow \{1, x, x^2, x^3\}$ is a L.I. set in P_3. ⊠

(1) *A Non-standard Vector Space Structure on* $\mathbb{R}^2$.

Show that $(\mathbb{R}^2, \mathbb{R}, \oplus, \odot)$ with the operations defined as follows is a vector space.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 + 2 \end{bmatrix}$$

$$c \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx - c + 1 \\ cy + 2c - 2 \end{bmatrix}$$

Here, $+$, $-$ denote the usual addition and subtraction of real numbers.

(2) *Linear Independence.*

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a linearly independent set of vectors in V.

(a) Show that $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ is also a linearly independent set in V.

(b) Prove or disprove: $\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1\}$ is a linearly independent set in V.

(3) *Hidden Linear Dependence.*

Recall that the vector space $\mathcal{P}_m$ of real polynomials of degree at most m has dimension $(m+1)$. Let $\{p_0, p_1, \cdots, p_m\}$ be a set of polynomials in $\mathcal{P}_m$ such that $p_i'(1) = 0$, for all $i = 0, 1, \cdots, m$.

Prove that $\{p_0, p_1, \cdots, p_m\}$ is a linearly dependent set in $\mathcal{P}_m$.

(4) *Linear Dependence and Span.*

Suppose that $\{v_1, \cdots, v_n\}$ is a linearly independent set in V and $w \in V$. Prove that if $\{v_1 + w, \cdots, v_n + w\}$ is a linearly dependent set, then $w \in \operatorname{Span}(v_1, \cdots, v_n)$.

(5) *A subspace of* $\mathrm{Mat}_3(\mathbb{R})$.

Show that the set V of all real 3×3 upper triangular matrices[1] is a subspace of $\mathrm{Mat}_3(\mathbb{R})$. Find a basis for V, and give its dimension.

(6) *Finding a Basis.*

Let $\mathcal{P}_3$ be the vector space of real polynomials of degree at most 3 (with respect to usual addition of polynomials and multiplication of scalars with polynomials). Let V be the subspace of $\mathcal{P}_3$ defined as:

$$V = \{f(x) \in \mathcal{P}_3 : f(0) = f(1),\ f''(0) = f''(1)\}.$$

Find a basis for V.

(7) *Describing Linear Maps.*

(a) Let $T : V \to W$ be a linear map, and $\{\alpha_1, \cdots, \alpha_n\}$ a basis for V. Show that the range of T is the subspace of W spanned by the vectors $T(\alpha_1), \cdots, T(\alpha_n)$.

(b) Using the previous part, describe explicitly a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

(8) *Linear or Not?*

If

$$\alpha_1 = (1, -1),\ \alpha_2 = (2, -1),\ \alpha_3 = (-3, 2),$$

and

$$\beta_1 = (1, 0),\ \beta_2 = (0, 1),\ \beta_3 = (1, 1),$$

[1]A square matrix is *upper diagonal* if all its entries below the principal diagonal are 0.

is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\alpha_i) = \beta_i$, for $i = 1, 2, 3$?

(9) *Image of a Linearly Independent Set under a Linear Map.*

Suppose that $T : V \to W$ is an injective linear map, and $\{v_1, \cdots, v_n\}$ is a linearly independent set in V. Prove that $\{T(v_1), \cdots, T(v_n)\}$ is a linearly independent set in W.

(10) *An Application of Rank-Nullity Theorem.*

Prove that if $T : \mathbb{R}^4 \to \mathbb{R}^2$ is a linear map such that $\mathrm{Null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 5x_2, \text{ and } x_3 = 7x_4\}$, then T is surjective.

1) Set $V = \mathbb{R}^2$. Here, $c, c_1, c_2 \in \mathbb{R}$.

- Associativity and Commutativity of $\oplus$ is straight forward.

- Additive identity is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Indeed, for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we've that

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x+1-1 \\ y-2+2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Additive inverse of $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\begin{pmatrix} 2-x \\ -y-4 \end{pmatrix}$.

Indeed, $\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} 2-x \\ -y-4 \end{pmatrix} = \begin{pmatrix} x+(2-x)-1 \\ y+(-y-4)+2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

- The real number 1 is the multiplicative identity: $1 \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.x-1+1 \\ 1.y+2.1-2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

- Associativity of scalar multiplication:

$$c_1 \odot \left(c_2 \odot \begin{pmatrix} x \\ y \end{pmatrix}\right) = c_1 \odot \begin{pmatrix} c_2 x - c_2 + 1 \\ c_2 y + 2c_2 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1(c_2x - c_2 + 1) - c_1 + 1 \\ c_1(c_2y + 2c_2 - 2) + 2c_1 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1c_2x - c_1c_2 + c_1 - c_1 + 1 \\ c_1c_2y + 2c_1c_2 - 2c_1 + 2c_1 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1c_2x - c_1c_2 + 1 \\ c_1c_2y + 2c_1c_2 - 2 \end{pmatrix} = (c_1c_2) \odot \begin{pmatrix} x \\ y \end{pmatrix}$$

- Distributivity

(i)

$$(c_1 + c_2) \odot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} (c_1+c_2)x - (c_1+c_2) + 1 \\ (c_1+c_2)y + 2(c_1+c_2) - 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1x + c_2x - c_1 - c_2 + 1 \\ c_1y + c_2y + 2c_1 + 2c_2 - 2 \end{pmatrix}$$

$$c_1 \odot \begin{pmatrix} x \\ y \end{pmatrix} \oplus c_2 \odot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} c_1x - c_1 + 1 \\ c_1y + 2c_1 - 2 \end{pmatrix} \oplus \begin{pmatrix} c_2x - c_2 + 1 \\ c_2y + 2c_2 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1x - c_1 + 1 + c_2x - c_2 + 1 - 1 \\ c_1y + 2c_1 - 2 + c_2y + 2c_2 - 2 + 2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1x + c_2x - c_1 - c_2 + 1 \\ c_1y + c_2y + 2c_1 + 2c_2 - 2 \end{pmatrix}$$

So, $(c_1 + c_2) \odot \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \odot \begin{pmatrix} x \\ y \end{pmatrix} \oplus c_2 \odot \begin{pmatrix} x \\ y \end{pmatrix}$

(ii) $c \odot \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)$

$= c \odot \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 + 2 \end{pmatrix}$

$= \begin{pmatrix} c(x_1 + x_2 - 1) - c + 1 \\ c(y_1 + y_2 + 2) + 2c - 2 \end{pmatrix}$

$= \begin{pmatrix} cx_1 + cx_2 - 2c + 1 \\ cy_1 + cy_2 + 4c - 2 \end{pmatrix}$

③

$c \odot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus c \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$= \begin{pmatrix} cx_1 - c + 1 \\ cy_1 + 2c - 2 \end{pmatrix} \oplus \begin{pmatrix} cx_2 - c + 1 \\ cy_2 + 2c - 2 \end{pmatrix}$

$= \begin{pmatrix} cx_1 - c + 1 + cx_2 - c + 1 - 1 \\ cy_1 + 2c - 2 + cy_2 + 2c - 2 + 2 \end{pmatrix}$

$= \begin{pmatrix} cx_1 + cx_2 - 2c + 1 \\ cy_1 + cy_2 + 4c - 2 \end{pmatrix}$

So, $c \odot \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = c \odot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus c \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Hence, $(\mathbb{R}^2, \mathbb{R}, \oplus, \odot)$ is a vector space.

---

2) Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a L.I. Set in V.

a) Suppose that there exist scalars $a, b, c \in \mathbb{R}$ such that

$$(*) \rightarrow a\alpha_1 + b(\alpha_1 + \alpha_2) + c(\alpha_1 + \alpha_2 + \alpha_3) = 0.$$

Then,

$$a\alpha_1 + b\alpha_1 + b\alpha_2 + c\alpha_1 + c\alpha_2 + c\alpha_3 = 0$$

$$\Rightarrow (a+b+c)\alpha_1 + (b+c)\alpha_2 + c\alpha_3 = 0.$$

Since, $\{\alpha_1, \alpha_2, \alpha_3\}$ are L.I., we must have

$$\begin{cases} a+b+c = 0 \\ b+c = 0 \\ c = 0 \end{cases}$$

But this implies that $a = b = c = 0$.

Thus, $(*)$ holds if and only if $\underline{a = b = c = 0}$.

Hence, $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ is also, a L.I. Set. $\boxtimes$

b) The set $\{\alpha_1-\alpha_2, \alpha_2-\alpha_3, \alpha_3-\alpha_1\}$ is not L.I. because we have the following non-trivial relation:

$$1\cdot(\alpha_1-\alpha_2)+1\cdot(\alpha_2-\alpha_3)+1\cdot(\alpha_3-\alpha_1)=0.$$

$\otimes$

3) $\dim P_m = (m+1)$.

To prove that $\{P_0, \ldots, P_m\}$ is a linearly dependent set, we will assume the contrary and arrive at a contradiction.

So let us assume that $\{P_0, \ldots, P_m\}$ is L.I. We know that in an $(m+1)$-dimensional vector space, a collection of $(m+1)$ linearly independent vectors is a basis for the vector space.

Hence, $\{P_0, \ldots, P_m\}$ is a basis for $\mathcal{P}_m$.

In particular, they span all of $\mathcal{P}_m$.

Note that the polynomial $q(x) = x$ lies in $\mathcal{P}_m$. Hence, there exists scalars $a_0, \ldots, a_m \in \mathbb{R}$ with

$$q(x) = a_0 P_0(x) + \cdots + a_m P_m(x), \quad \forall x \in \mathbb{R}$$

But then,

$$q'(x) = a_0 P_0'(x) + \cdots + a_m P_m'(x), \quad \forall x \in \mathbb{R}$$

$$\Rightarrow q'(1) = a_0 P_0'(1) + \cdots + a_m P_m'(1)$$

$$\Rightarrow \underline{q'(1) = 0}. \quad (\text{Since, } P_i'(1) = 0, \ \forall i = 0, \ldots, m)$$

However, $q(x) = x$

$$\Rightarrow q'(x) = 1 \Rightarrow \underline{q'(1) = 1}.$$

But this a contradiction which proves that $\{P_0, \ldots, P_m\}$ is not linearly independent. $\boxtimes$

~~4 $\{\alpha_1, \ldots, \alpha$~~

4 $\{v_1, \ldots, v_n\}$ is a L.I. set in V.

Suppose that $\{v_1 + w, \ldots, v_n + w\}$ is linearly dependent. Then, there exist scalars $a_1, \ldots, a_n$; not all equal to 0; such that

$$a_1(v_1 + w) + \ldots + a_n(v_n + w) = 0$$

$$\Rightarrow (a_1 v_1 + \ldots + a_n v_n) + (a_1 + \ldots + a_n)\, w = 0 \rightarrow (*)$$

Now, if $(a_1 + \ldots + a_n) = 0$, then

$(*)$ reduces to $a_1 v_1 + \ldots + a_n v_n = 0$.

Since $\{v_1, \ldots, v_n\}$ is a L.I. set, we must have $a_1 = \ldots = a_n = 0$, which contradicts the fact that not all a_i s are 0.

Therefore, we must have $(a_1 + \ldots + a_n) \neq 0$.

Now, $(*)$ implies that

$$(a_1 + \cdots + a_n)w = -(a_1 v_1 + \cdots + a_n v_n)$$

$$\Rightarrow w = \frac{-a_1}{a_1 + \cdots + a_n} v_1 - \cdots - \frac{a_n}{a_1 + \cdots + a_n} v_n .$$

$$\Rightarrow w \in \operatorname{span}\{v_1, \ldots, v_n\}. \quad \boxtimes$$

---

5) $V = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}.$

Note that any matrix $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in V$ can be written as:

$$a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There fore,

$$\text{Span}\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\} = V.$$

Moreover, the set

$$\left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$$

is <u>L. I.</u> because if

$$a_1\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + a_2\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} + a_3\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} + a_4\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + a_5\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} + a_6\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

then

$$\begin{pmatrix}a_1&a_2&a_3\\0&a_4&a_5\\0&0&a_6\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\Rightarrow \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0.$$

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Hence, $\left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} \right\}$

is a linearly independent spanning set for V, and thus a basis for V.

It follows that $\dim(V) = 6$. $\boxtimes$

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6) $V = \{ f \in P_3 : f(0) = f(1),\ f''(0) = f''(1) \}$.

So, $f(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3 \in V$,

if and only if

$$\begin{cases} f(0) = a_3 = a_0 + a_1 + a_2 + a_3 = f(1), \\ \text{and} \\ f''(0) = 2a_1 = 6a_0 + 2a_1 = f''(1) \end{cases}$$

i.e. $\begin{cases} a_0 + a_1 + a_2 = 0, \text{ and} \\ 6a_0 = 0 \end{cases}$

i.e. $\begin{cases} a_0 = 0, & \text{and} \\ a_1 + a_2 = 0 & . \end{cases}$

i.e. $\begin{cases} a_0 = 0, & \text{and} \\ a_2 = -a_1 & . \end{cases}$

Hence, $V = \{ f \in P_3 : f(x) = a_1 x^2 - a_1 x + a_3,$
$a_1, a_3 \in \mathbb{R} \}$

$= \{ f \in P_3 : f(x) = a_1 (x^2 - x) + a_3 . 1,$
$a_1, a_3 \in \mathbb{R} \}.$

Now set $f_1(x) = x^2 - x$, $f_2(x) = 1$.

Then, $V = \text{span}\{f_1, f_2\}$, as every element in V is a linear combination of f_1 and f_2.

Moreover, if

$$\lambda_1 (x^2 - x) + \lambda_2 \cdot 1 = 0,$$

Here, 0 is the 0 polynomial, which is the additive identity in V

then,

$$\lambda_1 (x^2 - x) + \lambda_2 = 0, \quad \forall x \in \mathbb{R}.$$

So, $\lambda_1 (1^2 - 1) + \lambda_2 = 0 \Rightarrow \underline{\lambda_2 = 0}$, and

$$\lambda_1 (2^2 - 2) + \lambda_2 = 0 \Rightarrow 2\lambda_1 + \lambda_2 = 0$$

$$\Rightarrow 2\lambda_1 = 0 \quad [\text{as } \lambda_2 = 0]$$

$$\Rightarrow \underline{\lambda_1 = 0}.$$

Thus, $\underline{\lambda_1 (x^2 - x) + \lambda_2 \cdot 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0}$.

This proves that $\{f_1, f_2\}$ are L.I.,

and they span V.

So, $\{f_1, f_2\}$ is a basis for V.

$$\Rightarrow \dim(V) = 2.$$

⊠

7) a) $T: V \to W$ is a linear map, and $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V.

Then, $\text{Range}(T)$

$$= \{T(\alpha) : \alpha \in V\}.$$

But every $\alpha \in V$ is a linear combination of the basis vectors $\alpha_1, \ldots, \alpha_n$.

~~Let us pick $\alpha \in V$.~~
~~Then, $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$~~

Pick <u>any</u> $T(\alpha) \in \text{Range}(T)$.

Then, $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$, for some $c_1, \ldots, c_n \in \mathbb{R}$

$$\Rightarrow T(\alpha) = T(c_1\alpha_1 + \cdots + c_n\alpha_n)$$

$$\Rightarrow T(\alpha) = c_1 T(\alpha_1) + \cdots + c_n T(\alpha_n) \quad \text{(By linearity of } T\text{)}$$

$$\Rightarrow \underline{T(\alpha) \in \text{Span}\{T(\alpha_1), \ldots, T(\alpha_n)\}}.$$

Since $T(\alpha)$ was an arbitrary element in

Range (T), it follows that every vector in Range(T) lies in $\underline{\text{span}\{T(\alpha_1), \ldots, T(\alpha_n)\}}$.

But the vectors $T(\alpha_1), \ldots, T(\alpha_n)$ themselves lie in Range(T), and so do their linear combinations (as Range(T) is a subspace).

Therefore, $\text{span}\{T(\alpha_1), \ldots, T(\alpha_n)\} \subseteq \text{Range}(T)$,

and $\underline{\text{Range}(T) \subseteq \text{span}\{T(\alpha_1), \ldots, T(\alpha_n)\}}$.

So, $\text{Range}(T) = \text{span}\{T(\alpha_1), \ldots, T(\alpha_n)\}$. ⊠

b) Let us fix a basis

$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ of $\mathbb{R}^3$.

We define a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$

such that $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$,

$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$, and

$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Then, by part (a), Range(T) = $\text{Span}\left\{T\begin{pmatrix}1\\0\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix}, T\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$= \text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

$= \text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$. ⊠

8) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with

$$T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},\quad T\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix},\quad T\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Then,

$$T\begin{pmatrix}1\\-1\end{pmatrix} + T\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow T\left(\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix} \quad \left[\text{By linearity of } T\right]$$

$$\Rightarrow T\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow -T\begin{pmatrix}3\\-2\end{pmatrix} = -\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow T\left(-\begin{pmatrix}3\\-2\end{pmatrix}\right) = \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}.$$

But this contradicts the fact that $T\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

Hence, Such a linear map T does not exist. $\boxtimes$

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9) Suppose that

$$c_1 T(\alpha_1) + \cdots + c_n T(\alpha_n) = 0_W.$$

for $c_1, \ldots, c_n \in \mathbb{R}$.

Then, $T(c_1\alpha_1 + \cdots + c_n\alpha_n) = 0_W$ [By linearity of T]

$\Rightarrow c_1\alpha_1 + \cdots + c_n\alpha_n \in$ Null Space (T)

But as T is injective, we have that Null Space $(T) = \{0_V\}$.

Hence, $c_1\alpha_1 + \cdots + c_n\alpha_n = 0_V$.

Finally, since $\{\alpha_1, \ldots, \alpha_n\}$ is L.I. in V, we have that

$$c_1 = \cdots = c_n = 0.$$

Therefore,

$$c_1 T(\alpha_1) + \cdots + c_n T(\alpha_n) = 0_W$$

$$\Rightarrow c_1 = \cdots = c_n = 0.$$

This proves that $\{T(\alpha_1), \ldots, T(\alpha_n)\}$ is a L.I. set in W.

10) Let T be a linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$\text{Null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,/\, x_1 = 5x_2,\ x_3 = 7x_4\}$$

$$= \left\{\left(x_1, \frac{x_1}{5}, x_3, \frac{x_3}{7}\right) \Big| \ x_1, x_3 \in \mathbb{R}\right\}$$

$$= \left\{\left(x_1, \frac{x_1}{5}, 0, 0\right) + \left(0, 0, x_3, \frac{x_3}{7}\right) \Big| \ x_1, x_3 \in \mathbb{R}\right\}$$

$$= \left\{x_1\left(1, \tfrac{1}{5}, 0, 0\right) + x_3\left(0, 0, 1, \tfrac{1}{7}\right) \Big| \ x_1, x_3 \in \mathbb{R}\right\}$$

$$= \text{span}\left\{\left(1, \tfrac{1}{5}, 0, 0\right), \left(0, 0, 1, \tfrac{1}{7}\right)\right\}.$$

Thus, Null(T) is the span of the L.I. set of vectors $\left\{\left(1, \tfrac{1}{5}, 0, 0\right), \left(0, 0, 1, \tfrac{1}{7}\right)\right\}$.

Hence, the above two vectors form a basis for Null(T).

$\Rightarrow$ dim Null$(T) = 2$.

By the rank-nullity theorem, we've that

$$\dim(\mathbb{R}^4) = \dim(\text{range}(T)) + \dim(\text{Null}(T))$$

$$\Rightarrow 4 = \dim(\text{range}(T)) + 2$$

$$\Rightarrow \underline{\dim(\text{range}(T)) = 2}.$$

Thus, $\text{Range}(T) \subseteq \mathbb{R}^2$ is a 2-dimensional subspace of the 2-dimensional vector space $\mathbb{R}^2$.

Hence, $\text{Range}(T) = \mathbb{R}^2$

$\Rightarrow$ T is surjective. $\boxtimes$

1. *(10 pts)*

Suppose that the vectors u_1, u_2 and u_3 in a vector space V are linearly independent. Show that the vectors $u_1 + u_2$, $u_2 + u_3$ and $u_3 + u_1$ are also linearly independent.

Solution: Let $c_1, c_2, c_3 \in \mathbb{R}$ be scalars such that

$$c_1(u_1 + u_2) + c_2(u_2 + u_3) + c_3(u_3 + u_1) = 0.$$

Then,

$$(c_1 + c_3)u_1 + (c_1 + c_2)u_2 + (c_2 + c_3)u_3 = 0.$$

As u_1, u_2 and u_3 are linearly dependent, it follows that

$$c_1 + c_3 = c_1 + c_2 = c_2 + c_3 = 0.$$

Solving the above system of equations in c_1, c_2, c_3, we conclude that

$$c_1 = c_2 = c_3 = 0.$$

Therefore,

$$c_1(u_1 + u_2) + c_2(u_2 + u_3) + c_3(u_3 + u_1) = 0 \implies c_1 = c_2 = c_3 = 0.$$

Hence, the vectors $u_1 + u_2$, $u_2 + u_3$ and $u_3 + u_1$ are also linearly independent.

2. *(10 pts)*
Let $\mathcal{P}_3$ be the vector space of real polynomials of degree at most 3 (with respect to usual addition of polynomials and multiplication of scalars with polynomials). Let V be the subspace of $\mathcal{P}_3$ defined as:

$$V = \{f \in \mathcal{P}_3 : f(0) + f(1) = 0,\ f'(0) = f'(1)\}.$$

Find a basis for V.

Solution: An arbitrary element of P_3 is of the form $f(x) = a + bx + cx^2 + dx^3$, $a, b, c, d \in \mathbb{R}$.

The condition $f(0) + f(1) = 0$ implies that

$$(a) + (a + b + c + d) = 0 \Rightarrow \underline{a = -\tfrac{1}{2}(b + c + d)} \to (1)$$

Since $f'(x) = b + 2cx + 3dx^2$, the condition

$$f'(0) = f'(1)$$

$$\Rightarrow b = b + 2c + 3d \Rightarrow \underline{c = -\frac{3d}{2}} \to (2)$$

Therefore, $V = \{f \in P_3 \mid f(0) + f(1) = 0,\ f'(0) = f'(1)\}$

$$= \left\{a + bx + cx^2 + dx^3 \,\middle|\, a = -\tfrac{1}{2}(b + c + d),\ c = -\frac{3d}{2}\right\}$$

$$= \left\{a + bx + cx^2 + dx^3 \,\middle|\, c = -\frac{3d}{2},\ a = -\frac{1}{2}\left(b - \frac{3d}{2} + d\right)\right\}$$

$$= \left\{a + bx + cx^2 + dx^3 \,\middle|\, c = -\frac{3d}{2},\ a = -\frac{1}{2}b + \frac{d}{4}\right\}$$

$$= \left\{\left(\frac{d}{4} - \frac{b}{2}\right) + bx - \frac{3d}{2}x^2 + dx^3 \,\middle|\, b, d \in \mathbb{R}\right\}$$

$$= \left\{b\left(x - \frac{1}{2}\right) + d\left(x^3 - \frac{3x^2}{2} + \frac{1}{4}\right) : b, d \in \mathbb{R}\right\} \longrightarrow$$

$$= \operatorname{span}\left\{ \left(x - \frac{1}{2}\right), \left(x^3 - \frac{3x^2}{2} + \frac{1}{4}\right) \right\}.$$

Hence, a basis for V is given by

$$\left\{ \left(x - \frac{1}{2}\right), \left(x^3 - \frac{3x^2}{2} + \frac{1}{4}\right) \right\}.$$

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3. *(10 pts)*
Let $\mathrm{Mat}_2(\mathbb{R})$ be the vector space of all 2×2 real matrices (with respect to usual matrix addition and multiplication of scalars with matrices) over the scalar field $\mathbb{R}$. Further, let

$$V = \{A \in \mathrm{Mat}_2(\mathbb{R}) : A^{tr} = A\};$$

i.e. V is the set of all symmetric matrices in $\mathrm{Mat}_2(\mathbb{R})$. Show whether or not V is a subspace of $\mathrm{Mat}_2(\mathbb{R})$. If it is a subspace, furnish a basis for V, and give its dimension.

Solution: An arbitrary element of $\mathrm{Mat}_2(\mathbb{R})$ is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$.

Now,

$$V = \{A \in \mathrm{Mat}_2(\mathbb{R}) \mid A^{tr} = A\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{tr} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} b = c \in \mathbb{R} \\ a, d \in \mathbb{R} \end{matrix} \right\} = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$$
$$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$$
$$= \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Since V is the span of three vectors it is necessarily a subspace.

Clearly, $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ are linearly independent. Hence, they form a basis for V, and $\dim(V) = 3$.

4. *(10 pts)*
If
$$\alpha_1 = (1, -1),\ \alpha_2 = (2, -1),\ \alpha_3 = (3, -2),$$
and
$$\beta_1 = (1, 0),\ \beta_2 = (0, 1),\ \beta_3 = (1, 1),$$
is there a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\beta_i) = \alpha_i$, for $i = 1, 2, 3$? If yes, what is the null-space of such a linear map?

Solution: Let us first note that $\{\beta_1 = (1, 0), \beta_2 = (0, 1)\}$ is a basis for $\mathbb{R}^2$. We define a linear map $S : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $S(\beta_1) = \alpha_1 = (1, -1)$, $S(\beta_2) = \alpha_2 = (2, -1)$, and extending it linearly to all of $\mathbb{R}^2$ (we know that the linear map S is uniquely determined by its action on the basis $\{\beta_1, \beta_2\}$).

Now, $S(\beta_3) = S(1, 1) = S(1, 0) + S(0, 1) = S(\beta_1) + S(\beta_2) = \alpha_1 + \alpha_2 = (3, -2) = \alpha_3$.

Therefore, $S(\beta_j) = \alpha_j$, for $j = 1, 2, 3$.

Hence, a linear map T with the desired properties exists, and it is given by $T := S$ as above.

We now note that the image of T is equal to $\text{span}(\alpha_1, \alpha_2) = \mathbb{R}^2$. Thus, the dimension of image(T) is 2. By the rank-nullity theorem, we have that
$$\dim(\text{null}(T)) = \dim(\mathbb{R}^2) - \dim(\text{image}(T)) = 2 - 2 = 0.$$
Thus, dim(null(T))=0; i.e. $\text{null}(T) = \{(0, 0)\}$.

5. *(10 pts)*
Let W_1, W_2 and W_3 be subspaces of a vector space V such that W_1 is contained in $W_2 \cup W_3$. Show that W_1 is either contained in W_2, or contained in W_3.

Solution: W_1, W_2 and W_3 are subspaces of a vector space V such that W_1 is contained in $W_2 \cup W_3$.

Let us suppose that W_1 is neither contained in W_2, nor contained in W_3 (which is the negation of what we are required to prove). Then, we can pick $\alpha \in W_1 \setminus W_2$, and $\beta \in W_1 \setminus W_3$. Since $W_1 \subset W_2 \cup W_3$, we must have that $\alpha \in W_3$ and $\beta \in W_2$.

Moreover, since $\alpha, \beta \in W_1$, and W_1 is a subspace, we conclude that $\alpha + \beta \in W_1$. As $W_1 \subset W_2 \cup W_3$, we must have $(\alpha + \beta) \in W_2$ or $(\alpha + \beta) \in W_3$.

Case 1. Let $(\alpha + \beta) \in W_2$. We also know that $\beta \in W_2$. As W_2 is a subspace, we have that $\alpha = (\alpha + \beta) - \beta \in W_2$. But this contradicts our selection of α from $W_1 \setminus W_2$.

Case 2. Let $(\alpha + \beta) \in W_3$. We also know that $\alpha \in W_3$. As W_3 is a subspace, we have that $\beta = (\alpha + \beta) - \alpha \in W_3$. But this contradicts our selection of β from $W_1 \setminus W_3$.

Since we arrived at a contradiction in both cases, our assumption that W_1 is neither contained in W_2, nor contained in W_3 was wrong. This proves that W_1 is either contained in W_2, or contained in W_3.

6. *(10 pts)*
Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, \text{ and } x_3 = x_4 = x_5\}$.

Solution: Let us set $V := \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, \text{ and } x_3 = x_4 = x_5\}$. Clearly, we can rewrite

$$\begin{aligned} V &= \{(3x_2, x_2, x_3, x_3, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2 \cdot (3,1,0,0,0) + x_3 \cdot (0,0,1,1,1) : x_2, x_3 \in \mathbb{R}\} \\ &= \text{Span}\{(3,1,0,0,0), (0,0,1,1,1)\}. \end{aligned}$$

Thus, $\{(3,1,0,0,0), (0,0,1,1,1)\}$ is a basis for V, and hence $\dim(V) = 2$.

Now suppose that $T : \mathbb{R}^5 \to \mathbb{R}^2$ is a linear map with $\text{null}(T) = V$. Then, $\dim(\text{null}(T)) = \dim(V) = 2$.

By the rank-nullity theorem, we have that

$$\begin{aligned} \dim(\mathbb{R}^5) &= \dim(\text{image}(T)) + \dim(\text{null}(T)) \\ &\implies 5 = \dim(\text{image}(T)) + 2 \\ &\implies \dim(\text{image}(T)) = 3. \end{aligned}$$

However, $\text{image}(T) \subset \mathbb{R}^2$, and hence, $\dim(\text{image}(T)) \leq \dim(\mathbb{R}^2) = 2$.

But this implies that $3 = \dim(\text{image}(T)) \leq 2$; i.e. $3 \leq 2$, a contradiction. This contradiction proves that there cannot exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ whose null space equals V.

(1) Suppose that V and W are both finite dimensional vector spaces. Prove that there exists a surjective linear map from V onto W if and only if $\mathrm{Dim}(W) \leq \mathrm{Dim}(V)$.

(2) Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

(3) Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenspaces of T. Is T diagonalizable?

(4) Define $T \in \mathcal{L}(\mathbb{R}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenspaces of T. Is T diagonalizable?

(5) Suppose $T \in \mathcal{L}(V)$ and $\mathrm{Rank}(T) = k$. Prove that T has at most $k+1$ distinct eigenvalues.

(6) Suppose $P \in \mathcal{L}(V)$ and $P^2 = I$. Find all eigenvalues of P. Prove that P is diagonalizable. (Hint: for every $v \in V$, we have that $v = (v + P(v))/2 + (v - P(v))/2$.

(7) Prove or disprove: there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$||(x_1, x_2)|| = \mathrm{Max}(|x_1|, |x_2|),$$

for all $(x_1, x_2) \in \mathbb{R}^2$.

(8) Suppose $\{e_1, \cdots, e_m\}$ is an orthonormal list of vectors in V, and $v \in V$. Prove that $||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$ if and only if $v \in \mathrm{Span}(e_1, \cdots, e_m)$.

(9) On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \text{ for all } f, g \in \mathcal{P}_2(\mathbb{R}).$$

Apply the Gram-Schmidt procedure to the basis $\{1, x, x^2\}$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

(10) On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \text{ for all } f, g \in \mathcal{P}_2(\mathbb{R}).$$

(a) Prove that $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ defined as

$$T(p) = p(2)$$

is a linear functional.

(b) Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p(2) = \int_0^1 p(x)q(x)dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

(11) In $\mathbb{R}^4$ (equipped with the standard dot product of vectors), let

$$U = \mathrm{Span}((1, 0, 0, 1), (1, 2, 1, 2)).$$

Find $u \in U$ such that $||u - (2, 1, 2, 1)||$ is as small as possible.

(12) Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

1) V and W are finite dimensional vector spaces.

$\boxed{\Rightarrow}$ Suppose that $T: V \to W$ is a surjective linear map. By the rank-nullity theorem, we have that

$$\circledast \to \dim(V) = \dim(\text{range}(T)) + \dim(\text{null}(T)).$$

Since T is surjective, we have that
$\text{range}(T) = W$.

so, $\circledast$ reduces to:

$$\dim(V) = \dim(W) + \dim(\text{null}(T))$$
$$\geq \dim(W) + 0$$

as the dimension of any subspace is non-negative

$\Rightarrow \dim(V) \geq \dim(W)$.

$\boxed{\Leftarrow}$ Now let $\dim(V) \geq \dim(W)$.

Let us pick a basis $\{\alpha_1, \ldots, \alpha_k\}$ for W (where $k = \dim(W)$).

and a basis $\{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n\}$ for V (where $n = \dim(V) \geq \dim(W) = k$)

Now define a linear map

$T: V \to W$ such that

$$\begin{cases} T(\beta_i) = \alpha_i, & i = 1, \ldots, k \\ T(\beta_i) = 0_W, & i = k+1, \ldots, n \end{cases}$$

Then, Range(T)

$= \text{span}\{T(\beta_1), \ldots, T(\beta_k), T(\beta_{k+1}), \ldots, T(\beta_n)\}$

$= \text{span}\{\alpha_1, \ldots, \alpha_k\}$

$= W$ [as $\{\alpha_1, \ldots, \alpha_k\}$ is a basis of W]

$\Rightarrow$ Range$(T) = W$

$\Rightarrow$ $T: V \to W$ is the desired surjective linear map. $\boxtimes$

2) Suppose that W is finite dimensional, and $T \in \mathcal{L}(V, W)$.

$\boxed{\Leftarrow}$ Suppose that there exists $S \in \mathcal{L}(W, V)$ such $ST = Id_V$.

Let $\alpha_1, \alpha_2 \in V$ s.t.

$$T(\alpha_1) = T(\alpha_2).$$

Then,
$$S(T(\alpha_1)) = S(T(\alpha_2))$$
$$\Rightarrow ST(\alpha_1) = ST(\alpha_2)$$
$$\Rightarrow Id_V(\alpha_1) = Id_V(\alpha_2)$$
$$\Rightarrow \alpha_1 = \alpha_2.$$

Therefore,
$$T(\alpha_1) = T(\alpha_2)$$
$$\Rightarrow \alpha_1 = \alpha_2; \text{ i.e.}$$

T is injective.

$\boxed{\Rightarrow}$ Conversely let $T : V \to W$ be injective.

Let us choose a basis $\{\beta_1, \ldots, \beta_k\}$ of Range$(T) \subseteq W$, and extend it to a basis $\{\beta_1, \ldots, \beta_k, \beta_{k+1} \ldots \beta_n\}$ of W.

Since $\beta_1, \ldots, \beta_K \in \text{Range}(T)$, there exist $\alpha_1, \ldots, \alpha_K \in V$ such that

$$T(\alpha_i) = \beta_i, \quad i = 1, \ldots, K. \rightarrow \text{A}$$

Let us define a linear map

$S : W \rightarrow V$ such that

$$\begin{cases} S(\beta_i) = \alpha_i, & i = 1, \ldots, K \\ S(\beta_i) = 0_V, & i = K+1, \ldots, n. \end{cases}$$

We'll now show that $ST = Id_V$.

To this end, pick any $\alpha \in V$, and write $T(\alpha) = \sum_{i=1}^{K} c_i \beta_i$, for some $c_1, \ldots, c_K \in \mathbb{C}$.

[This possible because $T(\alpha) \in \text{Range}(T) = \text{Span}\{\beta_1, \ldots, \beta_K\}$]

So, $T(\alpha) = \sum_{i=1}^{K} c_i T(\alpha_i)$ [By equation A]

$\Rightarrow T\left(\alpha - \sum_{i=1}^{K} c_i \alpha_i\right) = 0_W$

$\Rightarrow \alpha - \sum_{i=1}^{K} c_i \alpha_i = 0_V$ [As T is injective, $\text{Null}(T) = \{0_V\}$]

$\Rightarrow \alpha = \sum_{i=1}^{K} c_i \alpha_i \rightarrow \text{B}$

(5)

$$\Rightarrow T(\alpha) = \sum_{i=1}^{K} c_i \beta_i \quad \text{[Again by (A)]}$$

$$\Rightarrow ST(\alpha) = \sum_{i=1}^{K} c_i S(\beta_i)$$

$$\Rightarrow ST(\alpha) = \sum_{i=1}^{K} c_i \alpha_i \quad \text{[by definition of } S\text{]}$$

$$\Rightarrow ST(\alpha) = \alpha \quad \text{[by equation (B)]}$$

Hence, $ST(\alpha) = \alpha$, for any $\alpha \in V$.

$$\Rightarrow ST = Id_V.$$

$\boxtimes$

3) $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$T(w, z) = (z, w).$$

Suppose that $\lambda \in \mathbb{C}$ be an eigenvalue of T with associated eigenvector $(w_0, z_0) \neq (0,0)$.

Then, $T(w_0, z_0) = \lambda (w_0, z_0)$

$$\Rightarrow (z_0, w_0) = (\lambda w_0, \lambda z_0)$$

$$\Rightarrow \begin{cases} z_0 = \lambda w_0 \\ w_0 = \lambda z_0. \end{cases}$$

Hence, $z_0 = \lambda w_0 = \lambda(\lambda z_0)$

$$\Rightarrow \lambda^2 z_0 = z_0$$

$$\Rightarrow z_0(\lambda^2 - 1) = 0. \rightarrow (*)$$

<u>Case-I: ($z_0 = 0$).</u>

In this case, $w_0 = \lambda z_0 = 0$; i.e. $(z_0, w_0) = (0,0)$, a contradiction to our choice of (w_0, z_0).

Case-II ($z_0 \neq 0$).

Then, by ⊛, we've that

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

So, the eisenvalues of T are ± 1.

$\boxed{\text{Eigenspace of } 1:}$:

Suppose that (w_0, z_0) is an e.vector of T associated to the e.value 1.

Then, ~~T(w0~~ $T(w_0, z_0) = 1 \cdot (w_0, z_0)$.

$$\Rightarrow (z_0, w_0) = (w_0, z_0)$$

$$\Rightarrow z_0 = w_0.$$

Thus, every e.vector in the e.space of 1 is of the form $\begin{pmatrix} z_0 \\ z_0 \end{pmatrix}$, where $z_0 \in \mathbb{R}$.

So, $\mathcal{E}$.space of 1 = span$\{(1,1)\}$.

$$\Rightarrow \dim(\mathcal{E}.\text{space of } 1) = 1.$$

$\boxed{\text{E. space of } -1 :}$ (8)

Let $\begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$ be an e.vector of T associated with the e.value -1.

Then $T\begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = -1\begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$

$\Rightarrow (z_0, w_0) = (-w_0, -z_0)$

$\Rightarrow \underline{w_0 = -z_0}$.

So, every e.vector of T associated with -1 is of the form $(-z_0, +z_0)$, where $z_0 \in \mathbb{R}$.

So, eigenspace of $-1 = \text{Span}\{(-1, 1)\}$

$\Rightarrow \underline{\dim(\text{E. space of } -1) = 1}$.

Now, the sum of the dimensions of the eigenspaces of T is

$$\underline{1 + 1 = 2 = \dim(\mathbb{R}^2)}.$$

Hence, T is diagonalizable. ⊠

4) Solved in class. (9)

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5) $T : V \to V$ is a linear map, and

$\text{Rank}(T) = K$.

Suppose that $\lambda_1, \dots, \lambda_n$ are n distinct e.values of T with associated non-zero e.vectors $\alpha_1, \dots, \alpha_n$.

Then, $T(\alpha_i) = \lambda_i \alpha_i$, $i = 1, \dots, n$ $(*)$

$\Rightarrow \alpha_i = \lambda_i^{-1} T(\alpha_i) = T(\lambda_i^{-1} \alpha_i)$, if $\lambda_i \neq 0$.

Note that since all λ_is are distinct, at most one of them can be equal to 0. By equation $(*)$, if $\lambda_i \neq 0$, then $\alpha_i \in \text{Range}(T)$.

It follows that at least $(n-1)$ vectors out of $\{\alpha_1, \dots, \alpha_n\}$ lie in Range (T).

But we know that [eigenvectors of T] distinct e.values form a linearly independent set.

If $(n-1)$ is ~~not~~ larger than (10)
k, then we would obtain $(n-1)$
linearly independent vectors in the
subspace Range (T) of dimension k,
which is a contradiction.

Hence, $(n-1)$ cannot be larger than k.

$\Rightarrow \quad (n-1) \leq k$

$\Rightarrow \quad n \leq k+1.$

Therefore, the number of distinct
e-values of T is at most $(k+1)$.

6) $P: V \to V$ is a linear map such that $P^2 = I$.

Let λ be an e-value of P with an associated non-zero eigenvector α.

Then, $P(\alpha) = \lambda\alpha$

$$\Rightarrow P(P(\alpha)) = P(\lambda\alpha) = \lambda P(\alpha)$$

$$\Rightarrow P^2(\alpha) = \lambda \cdot \lambda\alpha$$

$$\Rightarrow I(\alpha) = \lambda^2\alpha$$

$$\Rightarrow \lambda^2\alpha = \alpha$$

$$\Rightarrow (\lambda^2 - 1)\alpha = 0_V$$

$$\Rightarrow \lambda^2 - 1 = 0 \quad [\text{as } \alpha \neq 0_V]$$

$$\Rightarrow \underline{\lambda = \pm 1}.$$

So, the e.values of P are ± 1.

~~E space of 1:~~

~~Let $P(\alpha) = 1.\alpha$~~

~~$\Rightarrow P^2(\alpha) =$~~

Note that

$$V = \frac{(V+P(V))}{2} + \frac{(V-P(V))}{2},$$

for all $v \in V$.

But, $P\left(\frac{V+P(V)}{2}\right) = \frac{1}{2}\left(P(V)+P^2(V)\right)$

$= \frac{1}{2}\left(P(V)+V\right)$ [as $P^2 = I$]

$\Rightarrow$ $\frac{V+P(V)}{2}$ is an e.vector of P associated to 1.

Again, $P\left(\frac{V-P(V)}{2}\right) = \frac{1}{2}\left(P(V)-P^2(V)\right)$

$= \frac{1}{2}\left(P(V)-V\right)$ [as $P^2 = I$]

$= -\left(\frac{V-P(V)}{2}\right).$

$\Rightarrow$ $\frac{V-P(V)}{2}$ is an e.vector of P associated to -1.

Therefore, every vector of V can be written as the sum of an e.vector of 1 and an e.vector of -1.

$\Rightarrow$ The sum of the eigenspaces of P is equal to V.

$\Rightarrow$ P is diagonalizable.

$\boxtimes$

---

7) Suppose that there is an inner product $\langle , \rangle$ on $\mathbb{R}^2$ such that

$$\|(x_1, x_2)\| = \sqrt{\langle (x_1, x_2), (x_1, x_2) \rangle}$$
$$= \max\{|x_1|, |x_2|\}.$$

We know that on any inner product space, the parallelogram law holds.

Applying the parallelogram law on the vectors $(1,0)$ and $(0,1)$ yields:

$$2\|(1,0)\|^2 + 2\|(0,1)\|^2 = \|(1,0)+(0,1)\|^2 + \|(1,0)-(0,1)\|^2$$

$$\Rightarrow 2(\max\{1,0\})^2 + 2(\max\{0,1\})^2$$
$$= (\max\{1,1\})^2 + (\max\{1,1\} \quad)^2$$

$\Rightarrow 2.1^2 + 2.1^2 = 1^2 + 1^2$

$\Rightarrow \underline{4 = 2, \text{ a Contradiction}}.$

This contradiction proves that the norm $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$ ~~does not in~~ is not induced by an inner product.

$\boxtimes$

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8) Suppose that $\{e_1, .., e_m\}$ is an orthonormal set in V, and $v \in V$.

Let us assume that

$$v \in \text{Span}(e_1, .., e_m).$$

Then, $v = \sum_{i=1}^{m} c_i e_i$, for some $c_1, .., c_k \in \mathbb{C}$

$$\Rightarrow \langle v, e_k \rangle = \left\langle \sum_{i=1}^{m} c_i e_i, e_k \right\rangle,$$

for $k = 1, .., m$

$$\Rightarrow \langle v, e_k \rangle = \sum_{i=1}^{m} c_i \langle e_i, e_k \rangle$$

$$\Rightarrow \langle v, e_k \rangle = c_k \quad \left[\begin{array}{l} \text{as } \langle e_i, e_k \rangle = 0, k \neq i \\ \langle e_k, e_k \rangle = 1 \end{array}\right]$$

Hence, $$V = \sum_{i=1}^{m} \langle v, e_i \rangle e_i$$

$$\Rightarrow \|v\|^2 = \sum_{i=1}^{m} |\langle v, e_i \rangle|^2,$$ by Pythagorean theorem.

Conversely, let

$$\|v\|^2 = \sum_{i=1}^{m} |\langle v, e_i \rangle|^2 \rightarrow (*)$$

Consider $U = \text{span}\{e_1, \ldots, e_m\}$.

~~Since~~ $\cancel{V = U \oplus U^{\perp}}$

We know that

$$V = P_U(v) + (v - P_U(v)), \rightarrow (**)$$

where P_U is orthogonal projection onto U.

Moreover,

$$P_U(v) = \sum_{i=1}^{m} \langle v, e_i \rangle e_i \rightarrow (***)$$

Again, Pythagorean theorem applied to (**) yields that

$$\|v\|^2 = \|P_U(v)\|^2 + \|v - P_U(v)\|^2$$

[Recall that $P_U(v) \in U$, and $(v - P_U(v)) \in U^\perp$]

$$\Rightarrow \|v\|^2 = \sum_{i=1}^{m} |\langle v, e_i \rangle|^2 + \|v - P_U(v)\|^2$$

[From (***)]

$$\Rightarrow \sum_{i=1}^{m} |\langle v, e_i \rangle|^2 = \sum_{i=1}^{m} |\langle v, e_i \rangle|^2 + \|v - P_U(v)\|^2$$

[by (*)]

$$\Rightarrow \|v - P_U(v)\| = 0$$

$$\Rightarrow v - P_U(v) = 0_V \Rightarrow P_U(v) = v$$

$$\Rightarrow v = \sum_{i=1}^{m} \langle v, e_i \rangle e_i \in \operatorname{span}\{e_1, \dots, e_m\}$$

$$\Rightarrow v \in \operatorname{span}\{e_1, \dots, e_m\}. \quad \boxtimes$$

12) $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$.

Note that

$$(T-\lambda I)^* = T^* - \bar{\lambda} I^* = T^* - \bar{\lambda} I.$$

Now,

$$\begin{aligned}
&\operatorname{Null}(T-\lambda I) \\
&= \left(\operatorname{Range}(T-\lambda I)^*\right)^{\perp} \\
&= \left(\operatorname{Range}(T^*-\bar{\lambda} I)\right)^{\perp}
\end{aligned}$$

Finally, λ is an eigenvalue of T

$$\begin{aligned}
&\iff \operatorname{Null}(T-\lambda I) \neq \{0_V\} \\
&\iff \left(\operatorname{Range}(T^*-\bar{\lambda} I)\right)^{\perp} \neq \{0_V\} \\
&\iff \operatorname{Range}(T^*-\bar{\lambda} I) \neq V
\end{aligned}$$

Since, $V = \operatorname{Range}(T^*-\bar{\lambda} I) \oplus \left(\operatorname{Range}(T^*-\bar{\lambda} I)\right)^{\perp}$

$$\begin{aligned}
&\cancel{\iff \dim(\operatorname{Range}(T^*-\bar{\lambda} I)) < \dim V} \\
&\iff \dim\left(\operatorname{Range}(T^*-\bar{\lambda} I)\right) < \dim V \\
&\iff \dim\left(\ker(T^*-\bar{\lambda} I)\right) > 0 \\
&\iff \ker(T^*-\bar{\lambda} I) \neq \{0_V\} \\
&\iff \bar{\lambda} \text{ is an e.vector of } T^*. \quad \boxtimes
\end{aligned}$$

By rank-nullity theorem applied on $(T^*-\bar{\lambda} I)$

11) $U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ (18)

Let's apply Gram-Schmidt on $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ to find an orthonormal basis of U.

$$e_1 = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\|} = \frac{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{1^2+0^2+0^2+1^2}} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

$$e_2 = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}}{\| \quad - \quad - \quad \|}$$

$$= \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \left(1 \cdot \frac{1}{\sqrt{2}} + 2 \cdot 0 + 1 \cdot 0 + 2 \cdot \frac{1}{\sqrt{2}} \right) \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}}{\| \quad - \quad - \quad - \quad \|}$$

$$= \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{3}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}}{\| \quad - \quad \|} = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 0 \\ 0 \\ 3/2 \end{pmatrix}}{\| \quad - \quad - \quad \|}$$

$$= \frac{\begin{pmatrix} -1/2 \\ 2 \\ 1 \\ 1/2 \end{pmatrix}}{\sqrt{\left(-\frac{1}{2}\right)^2 + 2^2 + 1^2 + \left(\frac{1}{2}\right)^2}} = \begin{pmatrix} -1/2 \\ 2 \\ 1 \\ 1/2 \end{pmatrix} \Big/ \sqrt{\frac{1}{4} + 4 + 1 + \frac{1}{4}}$$

$$= \frac{\begin{pmatrix} -1/2 \\ 2 \\ 1 \\ 1/2 \end{pmatrix}}{\sqrt{\frac{11}{2}}} = \sqrt{\frac{2}{11}} \begin{pmatrix} -1/2 \\ 2 \\ 1 \\ 1/2 \end{pmatrix}.$$

Now, the vector $u \in U$ for which $\|u - (2,1,2,1)\|$ is minimum is given by:

$$u = P_U \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \left(\left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\rangle \right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$+ \left(\left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \sqrt{\frac{2}{11}} \begin{pmatrix} -1/2 \\ 2 \\ 1 \\ \frac{1}{2} \end{pmatrix} \right\rangle \right) \cdot \sqrt{\frac{2}{11}} \begin{pmatrix} -1/2 \\ 2 \\ 1 \\ \frac{1}{2} \end{pmatrix}.$$

$\circledast$

9) and 10)

Done in class.

1. *(10 pts)*
Suppose that V is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

($\Leftarrow$) Assume there exists $S: W \to V$ such that $TS = I_W$. For any vector $w \in W$, we get $T(Sw) = w$. This mean w is in the range of T.
$\therefore$ T is surjective

($\Rightarrow$) Assume $T: V \to W$ is surjective.
Fix any basis $\{f_1, \dots, f_m\}$ of W. Since T is surjective, there exist vectors $e_1, \dots, e_m \in V$ with $Te_1 = f_1, Te_2 = f_2, \dots, Te_m = f_m$
Now simply define a linear map $S: W \to V$ by $S(a_1f_1 + \dots + a_mf_m) = a_1e_1 + \dots + a_me_m$.

We can see

$$\begin{aligned} T(S(a_1f_1 + \dots + a_mf_m)) &= T(a_1e_1 + \dots + a_me_m) \\ &= a_1T(e_1) + \dots + a_mT(e_m) \\ &= a_1f_1 + \dots + a_mf_m \end{aligned}$$

$\therefore$ $T \circ S$ is the identity map on W.
We have explicitly constructed a linear map S with the desired property, so there exists such a map S.

2. *(5+5+5 pts)*
Suppose $P \in \mathcal{L}(V, V)$, and $P^2 = P$.

i) Prove that the only eigenvalues of P are 0 and 1.

Let λ be an eigenvalue of P.
That is, there exists a nonzero vector $v \in V$ (eigenvector) s.t. $Pv = \lambda v$. ①
We want to compute $P^2 v$ in two different ways:
On the one hand, $P^2 v = Pv$ ($\because P^2 = P$)
$= \lambda v$ ($\because$ ①)
On the other hand, $P^2 v = P(\lambda v)$ ($\because$ ①)
$= \lambda P(v) = \lambda \cdot \lambda v$ ($\because$ ①)
$= \lambda^2 v$.
Hence we have $\lambda v = \lambda^2 v$. $\Rightarrow (\lambda^2 - \lambda) v = 0$
Since $v \neq 0$, we must have $\lambda^2 - \lambda = 0$ $\therefore \lambda = 0$ or 1

ii) (Contd.) Prove that the Eigenspace of 0 is equal to Ker(T), and the Eigenspace of 1 is equal to Range(T). (here $T=P$)

The eigenspace of 0 is by definition

$\ker(T-0\cdot I) = \ker T$.

Let us prove the eigenspace of 1 is range(T)

That is, we want to prove $\ker(T-I) = \text{range}(T)$

($\supset$) Choose any element $Tv \in \text{range}(T)$

We have $(T-I)(Tv) = T^2v - Tv = 0$ since $T^2=T$

$\therefore Tv \in \ker(T-I)$ $\therefore \ker(T-I) \supset \text{range}(T)$

($\subset$) Conversely, let $v \in \ker(T-I)$, i.e. $(T-I)v = 0$.

We have $Tv = v$. But then $v = Tv \in \text{range}(T)$

$\therefore \ker(T-I) \subset \text{range}(T)$

This proves $\ker(T-I) = \text{range}(T)$

iii) (Contd.) Prove that P is diagonalizable.
(Hint: Use the fact that a linear map $T : V \to V$ is diagonalizable if and only if the sum of the dimensions of its eigenspaces equals $\mathrm{Dim}(V)$.)

Recall the property

T: diagonalizable $\Leftrightarrow$ $\dim V$ = sum of $\dim(\ker(T-\lambda_i I))$ for all λ_i: eigenvalues.

By (i), we have only two eigenvalues of $T(=P)$.

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$

Hence, T: diagonalizable $\Leftrightarrow$ $\dim V = \dim(\ker T) + \dim(\ker(T-I))$

From (ii), we have $\ker(T-I) = \mathrm{range}(T)$.

But then we can use the rank-nullity theorem

$$\dim V = \dim(\ker T) + \dim(\mathrm{range}\, T) \quad (\because \text{r/k-nullity})$$

$$= \dim(\ker T) + \dim(\ker(T-I))$$

$\therefore$ T is diagonalizable

$(T=P)$

3. *(10 pts)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined as

$$T(x, y, z) = (0, x, y).$$

Is T diagonalizable? Justify your answer.

Sol 1 Fix a standard basis $e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

The matrix form of T, with respect to the std basis, is

$$[T] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ This is already lower-triangular

but not diagonal. This is enough to conclude T is not diagonalizable

Sol 2 We compute the eigenspaces of T.

Need to solve: $T(x,y,z) = \lambda(x,y,z)$ to compute eivalues.

This is $\begin{cases} 0 = \lambda x \\ x = \lambda y \\ y = \lambda z \end{cases}$

① If $\lambda = 0$, then we get $x = y = 0$.
This means the eigenspace of 0 is $\text{span}\{(0,0,1)\}$
(equivalently, only one eigenvector $(0,0,1)$)

② If $\lambda \neq 0$, then from $0 = \lambda x$, we have $x = 0$.
From $x = \lambda y$, we have $y = 0$, and similarly from $y = \lambda z$, we have $z = 0$. $\therefore (x,y,z) = (0,0,0)$ is the only vector in $\mathbb{R}^3$ satisfying $T(x,y,z) = \lambda(x,y,z)$, if $\lambda \neq 0$. But this cannot be an eigenvector, since eigenvectors have to be nonzero
$\therefore$ Any $\lambda \neq 0$ cannot be an eigenvalue of T.

Hence the only eigenspace is the eigenspace of 0, and is

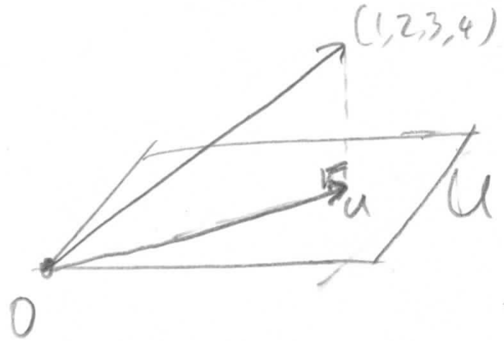
$\text{span}\{(0,0,1)\}$.

Hence T does not satisfy the criterion.

" $\dim V =$ sum of $\dim(\ker(T-\lambda_i I))$ for $\forall \lambda_i$: eigenvalues"

This means T is not diagonalizable

4. *(10 pts)*
In $\mathbb{R}^4$, let
$$U = \mathrm{Span}((1,1,0,0),(1,1,1,2)).$$
Find $u \in U$ such that $||u-(1,2,3,4)||$ is as small as possible. (Here $\mathbb{R}^4$ is viewed as an inner product space equipped with the standard dot product of vectors.)



Such $u \in U$ is precisely the projection vector of $(1,2,3,4)$ on the space U.

To compute $\mathrm{proj}_U(1,2,3,4)$, need to compute at least one possible orthonormal basis. This can be computed by applying G-S process to the given (non orthonormal) basis of U.

$\|(1,1,0,0)\| = \sqrt{2} \Rightarrow e_1 = \frac{1}{\sqrt{2}}(1,1,0,0)$

$e_2' = (1,1,1,2) - \langle (1,1,1,2), e_1 \rangle \cdot e_1$

$= (1,1,1,2) - \langle (1,1,1,2), \frac{1}{\sqrt{2}}(1,1,0,0) \rangle \cdot \frac{1}{\sqrt{2}}(1,1,0,0)$

$= (1,1,1,2) - (1,1,0,0) = (0,0,1,2)$

$\|e_2'\| = \sqrt{5} \Rightarrow e_2 = \frac{1}{\sqrt{5}}(0,0,1,2)$

$\therefore \{e_1 = \frac{1}{\sqrt{2}}(1,1,0,0),\ e_2 = \frac{1}{\sqrt{5}}(0,0,1,2)\}$ is (one possible) orthonormal basis of U.

Now $\mathrm{proj}_U(1,2,3,4) = \langle (1,2,3,4), e_1 \rangle e_1 + \langle (1,2,3,4), e_2 \rangle e_2$

$= \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,1,0,0) + \frac{11}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}(0,0,1,2) = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right)$.

5. *(5+10 pts)*

Consider the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ defined as

$$T(p) = p(1).$$

(Here $\mathcal{P}_2(\mathbb{R})$ stands for the vector space of all real polynomials of degree at most two.)

i) Show that T is a linear functional.

Need to show T respects addition & scalar multiplication

For $p, q \in P_2(\mathbb{R})$, we have

$$T(p+q) = (p+q)(1) = p(1) + q(1)$$
$$= T(p) + T(q)$$

$\therefore$ T respects addition

For $p \in P_2(\mathbb{R})$ and $a \in \mathbb{R}$, we have

$$T(ap) = (ap)(1) = a \cdot p(1) = a \cdot T(p)$$

$\therefore$ T respects scalar multiplication

$\therefore$ T is a linear map (linear functional)

ii) $\mathcal{P}_2(\mathbb{R})$ turns into an inner product space if we define an inner product by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \text{ for all } f, g \in \mathcal{P}_2(\mathbb{R}).$$

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$p(1) = \int_0^1 p(x)q(x)dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Step1 Find an orthonormal basis of $P_2(\mathbb{R})$

There is an obvious choice of basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$.

We apply G-S process to this basis to produce an orthonormal basis of $P_2(\mathbb{R})$.

$\|1\| = \sqrt{\int_0^1 1\cdot 1\cdot dx} = \sqrt{\int_0^1 dx} = 1$. $\therefore e_1 = 1. (\in P_2(\mathbb{R}))$

$e_2' = x - \langle x, e_1\rangle e_1$

$= x - \left(\int_0^1 x\,dx\right)\cdot 1 = x - \frac{1}{2}.$

$\|e_2'\| = \sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}$

$= \frac{1}{\sqrt{12}}.$

$\therefore e_2 = \frac{e_2'}{\|e_2'\|} = \sqrt{12}(x - \frac{1}{2}) = \sqrt{3}(2x-1)$

$e_3' = x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2$

$= x^2 - \left(\int_0^1 x^2 dx\right)\cdot 1 - \left(\int_0^1 x^2\sqrt{3}(2x-1)\,dx\right)\cdot\sqrt{3}(2x-1)$

$= x^2 - \frac{1}{3} - \frac{1}{2}(2x-1) = x^2 - x + \frac{1}{6}$

$\|e_3'\| = \sqrt{\int_0^1 (x^2-x+\frac{1}{6})^2 dx} = \cdots = \frac{1}{\sqrt{180}}.$

$\therefore e_3 = \frac{e_3'}{\|e_3'\|} = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right) = \sqrt{5}\,(6x^2 - 6x + 1)$

Step 2 Finding the polynomial $q \in P_2(\mathbb{R})$

Note that the given condition is precisely $p(1) = \langle q, p \rangle$ (for all $p \in P_2(\mathbb{R})$)

Use the identity $q = \langle q, e_1 \rangle \cdot e_1 + \langle q, e_2 \rangle e_2 + \langle q, e_3 \rangle e_3$.

Since $\langle q, e_1 \rangle = e_1(1) = 1$,

$\langle q, e_2 \rangle = e_2(1) = \sqrt{3}\,(2 \times 1 - 1) = \sqrt{3}$,

$\langle q, e_3 \rangle = e_3(1) = \sqrt{5}\,(6 \times 1^2 - 6 \times 1 + 1) = \sqrt{5}$,

we get $q = e_1 + \sqrt{3}\, e_2 + \sqrt{5}\, e_3$

$= 1 + \sqrt{3} \cdot \sqrt{3}\,(2x - 1) + \sqrt{5} \cdot \sqrt{5}\,(6x^2 - 6x + 1)$

$= 30x^2 - 24x + 3$.

1. Let $\mathcal{P}_5(\mathbb{R})$ be the vector space of real polynomials of degree at most 4, and

$$U := \{p(z) = az^3 + bz^5 : a, b \in \mathbb{R}\}.$$

Find a subspace W of $\mathcal{P}_5(\mathbb{R})$ such that $\mathcal{P}_5(\mathbb{R}) = U \oplus W$.

2. Let V be finite-dimensional, $T \in \mathcal{L}(V)$, and $M = [T]_{\mathcal{B}}$ be the matrix of T with respect to some basis $\mathcal{B}$ of V. Assume that the matrix M is lower-triangular. Prove that T is surjective if and only if every entry on the principal diagonal of M is different from 0.

3. Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of real polynomials of degree at most 3, and the linear map $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be defined as $T(p) = p''$.

(a) Find the eigenvalues and eigenspaces of T. Is T diagonalizable?
(b) Find the generalized eigenspace for each eigenvalue of T.

4. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = \mathrm{Id}_V$ if and only if T is bijective and S is the inverse of T.

5. Let V be an n-dimensional inner product space, and $T \in \mathcal{L}(V)$. Further suppose that U is a subspace of V, $\{\beta_1, \cdots, \beta_k\}$ is a basis for U, $\{\beta_{k+1}, \cdots, \beta_n\}$ is a basis for $U^\perp$, and P_U is the orthogonal projection operator to U.

(a) Show that $\mathcal{B} := \{\beta_1, \cdots, \beta_k, \beta_{k+1}, \cdots, \beta_n\}$ is a basis for V.
(b) Prove that $P_U T = T P_U$ if and only if the matrix $[T]_{\mathcal{B}}$ is of the form

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

where M_1 is a $k \times k$ matrix and M_2 is an $(n-k) \times (n-k)$ matrix.

6. Let $\mathcal{P}_4(\mathbb{R})$ be the inner product space of real polynomials of degree at most 4 equipped with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx,$$

for all $p, q \in \mathcal{P}_4(\mathbb{R})$, and consider its subspace $U = \mathrm{Span}\{x, x^3\}$. Find $U^\perp$.

7. Let T be a diagonalizable operator on an n-dimensional complex vector space V.

(a) Show that Null $(T^2) =$ Null (T).
(b) Assume further that T^{n+1} is the zero operator on V; i.e. $T^{n+1}(\alpha) = 0_V$ for all $\alpha \in V$. Show that T itself is the zero operator on V.

8. Suppose V is an n-dimensional complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that $1, 2$, and 3 are the only distinct eigenvalues of T.

(a) Prove that the dimension of each generalized eigenspace of T is at most $(n-2)$.
(b) Show that $(T-I)^{n-2}(T-2I)^{n-2}(T-3I)^{n-2}(\alpha) = 0_V$, for all $\alpha \in V$.

6) $P_4(\mathbb{R})$ is the inner product space of real polynomials of degree ≤ 4 with the inner product:

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\,q(x)\,dx, \quad p, q \in P_4(\mathbb{R}).$$

~~Let~~ Then, $\dim(P_4(\mathbb{R})) = 5$.

Let $U = \text{span}\{x, x^3\}$

$\Rightarrow \dim U = 2$.

We know that $P_4(\mathbb{R}) = U \oplus U^\perp$

$\Rightarrow \dim P_4(\mathbb{R}) = \dim U + \dim U^\perp$

$\Rightarrow \dim U^\perp = 5-2 = 3$

$\Rightarrow \underline{\dim U^\perp = 3.} \rightarrow$ A

We'll now show that $1, x^2, x^4 \in U^\perp$.

To ~~their~~ this end, note that:

$$\left.\begin{aligned} \langle 1, x\rangle &= \int_{-1}^{1} x\,dx = \left(\frac{x^2}{2}\right)_{-1}^{1} = 0 \\ \langle 1, x^3\rangle &= \int_{-1}^{1} x^3\,dx = \left(\frac{x^4}{4}\right)_{-1}^{1} = 0 \end{aligned}\right\} \Rightarrow \langle 1, \alpha\rangle = 0 \ \forall \alpha \in U \Rightarrow \underline{1 \in U^\perp}.$$

$$\langle x^2, x\rangle = \int_{-1}^{1} x^3\,dx = \left(\frac{x^4}{4}\right)_{-1}^{1} = 0$$

$$\langle x^2, x^3\rangle = \int_{-1}^{1} x^5\,dx = \left(\frac{x^6}{6}\right)_{-1}^{1} = 0$$

$\Rightarrow \langle x^2, \alpha\rangle = 0 \quad \forall \alpha \in U$

$\Rightarrow x^2 \in U^\perp.$

$$\langle x^4, x\rangle = \int_{-1}^{1} x^5\,dx = \left(\frac{x^6}{6}\right)_{-1}^{1} = 0$$

$$\langle x^4, x^3\rangle = \int_{-1}^{1} x^7\,dx = \left(\frac{x^8}{8}\right)_{-1}^{1} = 0$$

$\Rightarrow \langle x^4, \alpha\rangle = 0 \quad \forall \alpha \in U$

$\Rightarrow x^4 \in U^\perp.$

Thus, $1, x^2, x^4 \in U^\perp$

$\Rightarrow \operatorname{span}\{1, x^2, x^4\} \subseteq U^\perp \rightarrow$ (B) (Since $U^\perp$ is a subspace)

But, $\dim \operatorname{span}\{1, x^2, x^4\} = 3$

& $\dim U^\perp = 3$ (by (A))

$\Rightarrow \dim \operatorname{span}\{1, x^2, x^4\} = \dim U^\perp \rightarrow$ (C)

(B) & (C) together imply that

$U^\perp = \operatorname{span}\{1, x^2, x^4\}.$

$\boxtimes$

3) $P_3(\mathbb{R})$ is the vector space of all real polynomials of degree ≤ 3.

$\dim P_3(\mathbb{R}) = 4$.

$T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$, $T(p) = p''$.

a) Suppose that λ is an eigenvalue of T with associated eigenvector

$p(x) = a + bx + cx^2 + dx^3 \neq 0$

Now, $\circledast \Rightarrow T(p) = \lambda p$

$\Rightarrow 2c + 6dx = \lambda a + \lambda bx + \lambda c x^2 + \lambda d x^3$

$\Rightarrow 2c = \lambda a, \; 6d = \lambda b, \; \lambda c = 0, \; \lambda d = 0$

Case. I: $(\lambda \neq 0)$.

Then, $\lambda c = 0 = \lambda d \Rightarrow c = d = 0$.

Again, $2c = \lambda a$

$\Rightarrow \lambda a = 0$

$\Rightarrow a = 0 \; (\text{as } \lambda \neq 0)$.

Similarly. $6d = \lambda b$

$\Rightarrow \lambda b = 0 \Rightarrow b = 0 \; (\text{as } \lambda \neq 0)$.

So, $a = b = c = d = 0$

$\Rightarrow P(x) \equiv 0$, a contradiction.

So, T has no non-zero eigenvalue.

Case-II: $(\lambda = 0)$.

$2c = \lambda a$ $\quad$ $6d = \lambda b$

$\Rightarrow 2c = 0$ $\quad$ $\Rightarrow 6d = 0$

$\Rightarrow c = 0$ $\quad$ $\Rightarrow d = 0$

So, $p(x) = a + bx$

Therefore, 0 is an eigenvalue of T.

and the associated eigenspace is

$\{a + bx \mid a, b \in \mathbb{R}\}$

$= \text{Span}\{1, x\}$

Hence, $\dim E.\text{Space}(0) = 2$

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Hence, 0 is the only eigenvalue of T, and

$$\dim E.space(0) = 2 < 4 = \dim P_3(\mathbb{R}).$$

$\Rightarrow$ T is not diagonalizable. $\boxtimes$

b) We only need to compute the generalized eigenspace of T corresponding to the eigenvalue 0, and this is

$$\text{Null}\left(T - 0.I\right)^4$$

$$= \text{Null}\left(T^4\right)$$

$$= \{a + bx + cx^2 + dx^3 \mid T^4(a + bx + cx^2 + dx^3) = 0\}$$

$$= \{a + bx + cx^2 + dx^3 \mid T^3(2c + 6dx) = 0\}$$

$$= \{a + bx + cx^2 + dx^3 \mid T^2(0) = 0\} = P_3(\mathbb{R}).$$

Note that $T(p) = p''$

⑥

Hence, $\text{Null}(T^4) = \mathcal{P}_3(\mathbb{R})$; i.e. the generalized e.space of 0 is $\mathcal{P}_3(\mathbb{R})$. ☒

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1) $U = \{az^3 + bz^5 : a, b \in \mathbb{R}\}$ in $\mathcal{P}_5(\mathbb{R})$

So, $U = \text{span}\{z^3, z^5\}$.

Let us consider the subspace

~~$V = \text{span}\{1, z^2\}$~~

$V = \text{span}\{1, z, z^2, z^4\}$ of $\mathcal{P}_5(\mathbb{R})$.

Note that $\dim V = 4$.

Let $p(z) \in U \cap V$. So

$p(z) \in \text{span}\{z^3, z^5\} \cap \text{span}\{1, z, z^2, z^4\}$.

Hence, $p(z) = az^3 + bz^5 = c + dz + ez^2 + fz^4$,

for some $a, b, c, d, e, f \in \mathbb{R}$.

But this implies that

$$c + dz + ez^2 - az^3 + fz^4 + bz^5 = 0$$

in $P_5(\mathbb{R})$.

$\Rightarrow a = b = c = d = e = f = 0$.

$\Rightarrow p(z) = 0$ in $P_5(\mathbb{R})$.

Hence, $\underline{U \cap V = \{0\}}$. $\rightarrow$ A

~~Therefore~~ Moreover, any element in $P_5(\mathbb{R})$ is of the form

$$a_1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + a_6 z^5$$

$$= (a_4 z^3 + a_6 z^5) + (a_1 + a_2 z + a_3 z^2 + a_5 z^4).$$

$\Rightarrow$ Any element of $P_5(\mathbb{R})$ can be written as the sum of some element of U & some element of V.

$\Rightarrow \underline{U + V = P_5(\mathbb{R})} \rightarrow$ B

A & B together imply that

$$U \oplus V = P_3(\mathbb{R}).$$ $\boxtimes$

2) $M = [T]_\beta = \begin{pmatrix} m_{11} & 0 & 0 & \cdots & 0 & 0 \\ m_{21} & m_{22} & 0 & \cdots & 0 & 0 \\ m_{31} & m_{32} & m_{33} & \cdots & 0 & 0 \\ \vdots & & & & & 0 \\ m_{n1} & m_{n2} & \cdots & & & m_{nn} \end{pmatrix}$

is the matrix of $T: V \to V$

Note first that T is surjective

$\iff T$ is injective

$\iff \ker(T) = \{0_V\}$

$\iff \text{Nullity}(M) = 0.$

Let us first assume that

$\underline{m_{KK} \neq 0} \quad \forall K = 1, \ldots, n.$

Now, $\text{Null}(M)$

$= \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : M \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$

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$$= \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \begin{pmatrix} m_{11} a_1 \\ m_{21} a_1 + m_{22} a_2 \\ \cdots \\ m_{n1} a_1 + m_{n2} a_2 + \cdots + m_{nn} a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Now, $m_{11} a_1 = 0$ and $m_{11} \neq 0$ implies that $\underline{a_1 = 0}$.

Putting $a_1 = 0$ in the second equation, we see that $m_{22} a_2 = 0$ and $m_{22} \neq 0$

Hence, $\underline{a_2 = 0}$

Putting $a_2 = a_1 = 0$ in the the third equation, and using the fact that $m_{33} \neq 0$, we conclude that $\underline{a_3 = 0}$.

Thus, we can inductively carry on the above procedure to conclude that

$$M \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence, $\mathrm{Null}(M) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\} \Rightarrow \mathrm{Ker}(T) = \{0_V\}$

$\Rightarrow$ $\underline{T \text{ is bijective.}}$

Conversely, let T be surjective;
i.e. $\underline{\text{Nullity}\ (M) = 0}$.

We'll prove that $m_{kk} \neq 0 \ \forall k = 1, \ldots, n$
by ~~cont~~ contradiction.
To this end, assume that $\underline{m_{ii} = 0}$,
for some $i \in \{1, \ldots, n\}$.
We'll now solve the system of linear equations

$$M \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ m_{i+1,i} X_i + m_{i+1,i+1} X_{i+1} \\ \vdots \\ m_{n,i} X_i + \cdots + m_{n,n} X_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \rightarrow (*)$$

Since the first i entries on the left column vector above are 0, $(*)$ reduces to a system of $(n-i)$ equations in $(n-i+1)$ variables.

By the rank-nullity theorem, there exists a non-zero vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{x}_i \\ \vdots \\ \tilde{x}_n \end{pmatrix}$ satisfying $(*)$

Set $\alpha = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{x}_i \\ \vdots \\ \tilde{x}_n \end{pmatrix}$.

So, $M\alpha = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \alpha \in \text{Null}(M)$, but $\alpha \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

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$\Rightarrow$ Nullity $(M) \geq 1$ [Since Null(M) Contains a non-zero vector, Null (M) has Positive dimension.]

But this Contradicts our assumption that (T is surjective $\Leftrightarrow$ Nullity$(M) = 0$.)

Hence, our assumption that
$m_{ii} = 0$, for some $i \in \{1, \ldots, n\}$
was wrong.

$\Rightarrow$ $m_{kk} \neq 0$ $\forall k = 1, \ldots, n$. ⊠