**General Information**

Differential equation is an equation relating an unknown function and its derivatives. Various scientific laws can be translated into differential equations. The course is dedicated to standard techniques for solving ordinary differential equations, including numerical methods, and their applications in different branches of science such as physics, biology, chemistry, economics and social sciences.

**Instructor:**

Artem Dudko, artem.dudko@stonybrook.edu
Lectures: TuTh 10:00-11:20am, Engineering 143
Office hours: TuTh 12:00-1:00pm, Math Tower 3114, and F 11:00-12:00pm, Math Learning Center, Math Tower S-240A

**Teaching assistants:**

Shaosai Huang
R01, W 10:00-10:53am, Earth&Space 069
R02, F 1:00-1:53pm, Library N3063
Office hours: W 9:00-10:00am, Math Tower 3-103, and Tu 2:00-4:00pm, Math Learning Center, Math Tower S-240A

Oleksandr Tsymbaliuk
R03, Tu 5:30-6:23pm, Physics P127
R04, W 7:00-7:53pm, Light Eng. Lab. 152
Office hours: Tu 12:30-1:30pm and W 9:00-10:00am, SCGP 302, and F 12:00-1:00pm, Math Learning Center, Math Tower S-240A


**Topics:** an introduction to first order differential equations; phase plane analysis; numerical methods; higher order linear equations and systems; nonlinear phenomena.

**Prerequisite** is completion of one of the standard calculus sequences (either MAT 125-127 or MAT 131-132 or MAT 141-142) with a grade C or higher in MAT 127, 132 or 142 or AMS 161. Also,
MAT 203/205 (Calculus III) and AMS 261/MAT 211 (Linear Algebra) are recommended. Informally, students should know integration and differentiation techniques and, desirably, be familiar with complex numbers and basic aspects of linear algebra.

**Tests, quizzes, assignments:**
There will be homework assignments and quizzes alternating weekly. The first quiz is on Thursday, September 4. Starting the week of September 23 the quizzes will be written on Tuesdays during the last 20 minutes of the class. You should hand in your assignments to the instructor during Tuesday class. The first homework assignment will be due on Tuesday, September 16. No late assignments will be accepted.
Midterm Test I: Thursday, Oct 2, in class.
Midterm Test II: Tuesday, Nov 4, in class.
Final Exam: Friday, Dec. 12, 11:15am-1:45pm, location TBA.
Last day of classes: Thursday, December 4.

**Course grade** is computed by the following scheme:
- Homework and Quizzes: 20%
- Midterm Test I: 20%
- Midterm Test II: 20%
- Final Exam: 40%

**Information for students with disabilities**
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:
http://www.sunysb.edu/ehs/fire/disabilities.shtml
## Syllabus

The following is a tentative schedule for MAT 303.

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<th>Week of</th>
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<td>Aug 26</td>
<td>1.1 Differential equations and mathematical models</td>
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<td>Sep 2</td>
<td>1.2 Integrals as general and particular solutions</td>
<td>No class on Tue, Sep 2</td>
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<td>1.3 Slope fields and solution curves</td>
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<td>1.6 Substitution methods and exact equations</td>
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<td>Sep 23</td>
<td>2.1 Population models</td>
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<td>Sep 30</td>
<td>2.3 Acceleration-Velocity models</td>
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<td>2.4 Numerical approximation: Euler's method</td>
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<td>Oct 7</td>
<td>3.1 Introduction: second-order linear equations</td>
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<td>3.2 General solutions of linear equations</td>
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<td>Oct 14</td>
<td>3.3 Homogeneous equations with constant coefficients</td>
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<td>3.4 Mechanical vibrations</td>
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<td>Oct 21</td>
<td>3.5 Nonhomogeneous equations</td>
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<td>3.6 Forced oscillations and resonance</td>
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<td>Nov 4</td>
<td>5.1 Matrices and linear systems</td>
<td>Midterm 2 on Tue, Nov 4</td>
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<td>Nov 11</td>
<td>5.2 The eigenvalue method</td>
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<td>5.3 Second-order systems and applications</td>
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<td>Nov 18</td>
<td>5.4 Multiple eigenvalue solutions</td>
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<td>5.5 Matrix exponentials and linear systems</td>
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<td>Nov 25</td>
<td>6.1 Stability and the phase plane</td>
<td>No class on Th, Nov 27</td>
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<tr>
<td>Dec 2</td>
<td>6.2 Linear and almost linear systems</td>
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# MAT 303: Calculus IV with Applications
## Fall 2014

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Copyright 2008 Stony Brook University
Recommended problems from the course book.

Section 1.1: 5, 7, 9, 27, 40, 45.
Section 1.2: 6, 7, 8, 24, 32, 38.
Section 1.3: 15, 16, 17.
Section 1.4: 6, 9, 17, 23, 25, 27, 31, 34, 44, 61.
Section 1.5: 7, 13, 16, 20, 36.
Section 1.6: 10, 15, 16, 30, 32, 37, 40.
Section 2.1: 4, 9, 17, 26.
Section 2.2: 4, 7.
Section 2.3: 1, 4, 10, 17, 20.
Section 2.4: 2, 4, 10.
Section 3.1: 4, 7, 14, 18, 20, 22, 25, 30, 31, 34, 41, 47, 48.
Section 3.2: 1, 3, 5, 8, 10, 13, 17, 27, 30, 33.
Section 3.3: 4, 10, 16, 19, 22, 23, 24, 27, 31, 39, 42.

Nonhomogeneous equations: in each of the following question find by inspection a particular solution of the given differential equation, then find the general solution of the associated homogeneous equation and compose the general solution of the nonhomogeneous differential equation. Solve the initial value problem, if the initial conditions are given.

1. $y'' - 9y = \sin(2x)$, $y(0) = 2$, $y'(0) = -2/13$.
2. $y'' + 4y = 3x - 1$, $y(0) = -1/4$, $y'(0) = 3/4$.
3. $y''' + 3y'' + 3y' + y = \exp(x)$.

Section 3.4: 1, 2, 15, 17, 20.
Section 3.5: 1, 3, 9, 17, 25, 30.
Section 3.6: 1, 3, 5.
Section 4.1: 5, 7, 10, 16, 19, 24.
Section 4.2: 3, 5, 7, 9, 12, 14.
Section 5.1: 1, 2, 5, 12, 20, 22, 26.
Section 5.2: 4, 8, 11, 19, 22, 26.
Section 5.4: 2, 6, 7.
Section 6.2: solve the system, determine the type of the critical point, sketch the phase portrait for the questions 5, 7, 8, 9.
Examples

- Final exam from Fall 2012 semester
- Final exam from Fall 2012 semester, solutions
- Practice midterm II (1)
- Practice midterm II (2)
- Practice midterm II (1) solutions
- Practice midterm II (2) solutions
- Midterm II review
MAT 303 Assignment 1.
Hand in to the instructor in class on Tuesday, September 16.

**Problem 1.** In each case verify by substitution that the function is a solution of the corresponding differential equation

1) \( y(x) = \sin\left(\frac{x}{2}\right) - 2 \cos\left(\frac{x}{2}\right), \quad 4y'' + y = 0, \)
2) \( y(x) = e^{x^2}, \quad y' = 2xy, \)
3) \( y(x) = \sqrt{x^2 + 1}, \quad (y')^2 = 1 - \frac{1}{y^2}. \)

**Problem 2.** Find the general solutions of the following differential equations:

1) \( \frac{dx}{dt} = 3t^2 + 2t - \cos(2t), \quad 2) \ y' = x^2 \sin(x^3). \)

**Problem 3.** Solve the initial value problems:

1) \( \frac{dy}{dt} = \frac{t}{t^2 + 1}, \quad y(0) = 5, \quad 2) \ xy' = x^2 - 2, \quad y(-1) = 0. \)

**Problem 4.** A car starting from rest reached the velocity 30 mi/h (44 ft/s) after traveling the distance of 44 ft. Assuming that the car had constant acceleration find this acceleration and the time which took the car to reach 60 mi/h.

**Problem 6.** Solve the following first order separable differential equations:

1) \( y' = x^2y, \quad y(2) = 1, \quad 2) \ \frac{dx}{dt} = x + \frac{1}{x}, \)

**Problem 6.** Among the following differential equations solve the one which is first order and separable

1) \( \frac{d^2x}{dt^2} = x^2t^2, \quad 2) \ \frac{dy}{dt} = t^2 + y \sin t, \)
3) \( y' - 1 = xy + x + y, \quad 4) \ (y')^2 = x^2 + y^2. \)
Problem 7. Show by substitution that the formula

\[ y(x) = \frac{2}{1 + Ce^x} - 1, \]  

where \( C \) is a constant, gives a general solution of the differential equation

\[ 2y' = y^2 - 1. \]

Show that formula (1) is not the general solution of the given equation by finding a solution which is not described by (1).
MAT 303 Assignment 2.
Hand in to the instructor in class on Tuesday, September 30.

Problem 1. A tank contains 800 liters (L) of a solution consisting of 50 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 2L/s, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 20 kg of salt remains in the tank?

Problem 2. Solve the differential equation
\[
\frac{dx}{dt} = (2x - 4t)^3 + 2.
\]

Problem 3. For each of the following equations determine its type (separable, linear, substitution applicable, homogeneous, Bernoulli, reducible second order). If several methods are applicable, list all of them. Don’t solve the equations.

1) \(y' = x^2 + xy\), 2) \(\frac{dx}{dt} + x^2t = t\sin x + t\), 3) \(x''x'x = t^2 + 1\),
4) \(y^3 + 2yx^2 = 3x^3\frac{dy}{dx}\), 5) \((x')^2 = x^2 + (x'')^2\), 6) \(\sin \frac{dy}{dt} = \cos y + t^2\).

Problem 4. Solve the initial value problem
\[2x + 2yy' = 3(x^2 + y^2), \quad y(0) = 0.\]

Problem 5. Solve the differential equation
\[x^2\frac{dy}{dx} = y^2 - yx + x^2.\]

Problem 6. Solve the initial value problem
\[y' = \frac{y}{4} - \frac{1}{2y}, \quad y(2) = -\sqrt{2}.\]
Problem 7. A particle is moving on a straight line in such a way that at each moment of time the product of its coordinate on the line and its acceleration is equal to the square of its velocity. Initially the particle positioned at the point with coordinate 1 on the line and has velocity 0.1. Find the position function of the particle.
MAT 303 Assignment 3.
Hand in to the instructor in class on Tuesday, October 14.

**Problem 1.** Let $P(t)$ be a rabbit population satisfying the extinction-explosion equation

$$\frac{dP}{dt} = aP^2 - bP.$$ 

The threshold population is $M = 540$ rabbits. In 1975 the population was 360 rabbits. In 1985 the population was 180 rabbits. Find the size of the population in 1995.

**Problem 2.** A fish population of a lake was attacked by a disease and the fishes cannot reproduce. The death rate per week per fish is proportional to $\frac{1}{\sqrt{P}}$. Initially there were 900 fishes. After 6 weeks the number of fishes decreased to 441. When the population will extinct?

**Problem 3.** A population of a certain country satisfying to the logistic equation was equal to 20 mln people and growing with the rate 0.2 mln per year in 1900. Also, this population was equal to 30 mln people and growing with the rate 0.15 mln per year in 1955. Find the limiting population and the predicted population for the year 2000.

**Problem 4.** Solve the differential equations:

1) $y' = 3y - 4y^2$,  
2) $\frac{dx}{dt} = x^2 - 1$,  
$x(0) = 4$.

**Problem 5.** Solve the differential equation

$$\frac{dx}{dt} = x(x^2 - 1).$$
Hint: you can either use partial fractions, or simplify the equation first by using an appropriate substitution.
Problem 1. Verify that functions $y_1 = e^x \sin(2x)$, $y_2 = e^x \cos(2x)$ are solutions of the differential equation
$$y'' - 2y' + 5y = 0.$$ 
Find the general solution. Solve the initial value problem
$$y(\pi) = 0, \; y'(\pi) = 2.$$

Problem 2. Show that the functions $y_1 = x^2$, $y_2 = x^{-1}$ are solutions of
$$x^2y'' - 2y = 0.$$ 
Find the general solution. Find a solution $y$ of this differential equation such that
$$y(-1) = 0, \; y'(-1) = 2.$$

Problem 3. Show that the functions $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

Problem 4. Find constants $b, c$ such that the quadratic function $y(x) = x^2 + bx + c$ is a solution of the second order differential equation
$$y'' - y' + 2y = 2x^2.$$

Problem 5. Find the Wronskian of the pairs of functions
1) $f(x) = x^2 - 1$, $g(x) = x^2 + 1$,
2) $f(x) = e^x \sin(2x)$, $g(x) = e^x \cos(2x)$.

Problem 6. Show that there are no everywhere continuous functions $p(x), q(x)$ such that the equation
$$y'' + p(x)y' + q(x)y = 0$$
has solutions

\[ a) \ y_1 = x^2 - 1, \ y_2 = x^2 + 1, \ b) \ y_1 = \sin x, \ y_2 = x \sin x. \]

**Problem 7.** Solve

\[ 1) \ y'' + y = 0, \ 2) \ yy'' + (y')^2 = 0, \ y(1) = -1 \]

as reducible second order differential equations.
MAT 303 Assignment 5.
Hand in to the instructor in class on Tuesday, November 18.

**Problem 1.** Describe the motion of a body of mass $m$ with initial position $x_0$ and initial velocity $v_0$ in a mass-spring-dashpot system with a spring constant $k$ and damping constant $c$ if

a) $m = 3$, $c = 10$, $k = 7$, $x_0 = 6$, $v_0 = 2$;
b) $m = 2$, $c = 8$, $k = 10$, $x_0 = 10$, $v_0 = 2$.

**Problem 2.** Solve the initial value problem

$$x'' + 4x = 2 \cos 3t, \quad x(0) = x'(0) = 0.$$ 

Sketch the graph of the solution.

**Problem 3.** Transform the differential equation into a system of first-order differential equations:

$$x^{(4)} \sin t - t^2 x'' + x^2 = 0.$$ 

**Problem 4.** Transform the system of differential equations into a system of first-order differential equations:

$$x'' + x^2 y' = \sin y, \quad y'' - y = 2x + x'.$$

**Problem 5.** Solve the initial value problem:

$$x' = -2y, \quad y' = 2x, \quad x(0) = 1, \quad y(0) = -2.$$ 

**Problem 6.** Solve the initial value problem:

$$x' = y - 3x, \quad y' = 5y - 16x, \quad x(1) = 2, \quad y(1) = 9,$$

by reducing it to a second order differential equation.
with(DEtools): with(plots):

Example 1

> eqn1 := diff(y(t), t) = 2*y(t);

\[
\text{eqn1} := \frac{d}{dt} y(t) = 2y(t)
\]

> DEplot(eqn1, [y(t)], -1..1, [[0, 1], [0, 3], [0, 7], [0, -1], [0, -2], [0, -3], [1, 1], [-1, -1]], y=-8..8, stepsize=0.03, linecolor=blue);

> dsolve(eqn1);

\[
y(t) = \text{-C1 e}^{2t}
\]

Example 2

> eqn2 := diff(y(x), x)*x - 3*y(x) = x^4*cos(x);

\[
eqn2 := \left( \frac{d}{dx} y(x) \right) x - 3y(x) = x^4 \cos(x)
\]

> DEplot(eqn2, [y(x)], -4..0, [[-1, 1], [-1, 0.5], [-1, 0], [-1, -0.5]], y=-10..10, stepsize=0.03, linecolor=blue);
Example 3

dsolve(eqn2);

\[ y(x) = x^3 \sin(x) + x^3 \_C1 \]

\[ \text{eqn3:=diff(y(t),t)=-3*t*y(t)}; \]

\[ \text{eqn3 := } \frac{d}{dt} y(t) = -3 \, t \, y(t) \]

\[ \text{DEplot(eqn3, [y(t)], -2..2, \{[0,1], [0,3], [0,7], [0,-1], [0,-2], [0,-3], [1,1], [-1,-1]\}, y=-8..8, stepsize=0.03, linecolor=blue);} \]
\[ y(t) = C_1 e^{-\frac{3t^2}{2}} \]
with(DEtools): with(plots):

eqn1:=diff(y(t),t)=2*y(t);

DEplot(eqn1,[y(t)],-1..1,
{[0,1],[0,3],[0,7],[0,-1],[0,-2],[0,-3],[1,1],[-1,-1]},
y=-8..8,stepsize=0.03,linecolor=blue);

Example 1

C020AD89A70A9BDDBF698B860884BC02029FE7190DA0BX0FF(F)
C020AD89A70A9BDDBF698B860884BC02029FE7190DA0BX0FF(F)

CBFB50D32BCC01FB8B503DC9CBFBE3060945F9A88C01A8B319C7C3886BF61D947AC
DB03C001978C53E3B81AX0FF(F)

ABC039AF48D7BFEF1B0C78F2A377C0117E5AB8C5E45BEF55DA70076D93DC0161ECAE6
249943X0FF(F)

EF025E333F11F3C013AC6A814475ECBF698501ED38F1C012B31AF96124AAX0FF(F)

EE7E170AEBO81EBFDD7C7ADF752E7BFE68366CC2B5DF6DF6D93D06A3D0XX0FF(F)

3FDD7C7A0E1AC176BFEF7C880DDEC13A3E0F4587364EBDXX0FF(F)

BABAFF656A10ED543F5750E53D566X0FF(F)

3FDD7C7A0E1AC176BFEF7C880DDEC13A3E0F4587364EBDXX0FF(F)

2EE64CBF2C465B4010334B807B4543BFED6B6EBA3421004FA1922322BDCSEX0FF(F)

EE64CBF2C465B4010334B807B4543BFED6B6EBA3421004FA1922322BDCSEX0FF(F)

BABAFF656A10ED543F5750E53D566X0FF(F)

EE64CBF2C465B4010334B807B4543BFED6B6EBA3421004FA1922322BDCSEX0FF(F)

BABAFF656A10ED543F5750E53D566X0FF(F)

BABAFF656A10ED543F5750E53D566X0FF(F)
Example 2

```maple
deq := diff(y(x), x) = 3*y(x) + x^4*cos(x); 
DEplot(deq, [y(x)], x = -4 .. 0, [[-1, 1], [-1, 0.5], [-1, 0], [-1, -0.5]], y = -10 .. 10, stepsize = 0.03, linecolor = blue);```

Example 3

\[ \frac{dy}{dt} = -3ty \]

DEplot(eqn3, [y(t)], -2..2, [...], y = -8..8, stepsize = 0.03, linecolor = blue);
\text{dsolve(eqn3);}

\text{Curve 1} \text{ Curve 2} \text{ Curve 3} \text{ Curve 4} \text{ Curve 5} \text{ Curve 6} \text{ Curve 7} \text{ Curve 8} \text{ Curve 9}
MAT 303 FALL 2012 Final Exam

NAME: ID:

There are ten problems. Each problem has the same value 10 points.

Show your work

Do not tear-off any page

No calculators No cells etc.

On your desk: only test, pen, pencil, eraser, and student ID

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1. Solve the initial value problem
\[ x e^y y' - 2 e^y = 3x^3, \quad y(1) = 0. \]
2. Show that the following differential equation is exact, then solve it:

\[(3y - \sin x)dy + (1 - y\cos x)dx = 0.\]
3.

Suppose that a motorboat is moving at $4m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $2m/s$. Assume that the resistance motorboat encounters is proportional to its velocity. Find the velocity of the boat in 20 seconds after the motor has quit.
4.

In a population $P(t)$ of rabbits both the birth and the death rates are proportional to $P^2(t)$. Initially, there were one thousand of rabbits. After 9 years only 800 of rabbits left. Find $P(t)$. 
5.

Solve the non-homogeneous equation

\[ y'' - 3y' + 2y = \sin x. \]
Using the elimination method, solve the initial value problem:

$$X' = \begin{bmatrix} -t & 1 \\ -t^2 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$
7.

Show that the following three functions are linearly independent on the real line.

\[ X_1(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} t \\ t^2 + t \\ 3t \end{bmatrix}, \quad X_3(t) = \begin{bmatrix} t + 1 \\ 0 \\ 1 \end{bmatrix}. \]
8.

Solve the system of differential equations

\[ X' = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} X. \]
9.

Solve the initial value problem

\[ X' = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}. \]
10.

Solve the system of homogeneous equations

\[ X' = \begin{bmatrix} 3 & -4 \\ 3 & -5 \end{bmatrix} X. \]

Determine the type of the critical point at 0. Sketch the phase portrait.
THERE ARE TEN PROBLEMS. EACH PROBLEM HAS THE SAME VALUE 10 POINTS.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser and student ID

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2.

1.

Solve the initial value problem

\[ xe^y y' - 2e^y = 3x^3, \quad y(1) = 0. \]

**Solution.** The term \( e^y \) appears twice, so it is natural to try the substitution \( u = e^y \). We have \( \frac{du}{dx} = e^y \frac{dy}{dx} \), so the equation can be rewritten in terms of \( u \) as:

\[ x \frac{du}{dx} - 2u = 3x^3, \quad \text{or} \quad \frac{du}{dx} - \frac{2}{x} u = 3x^2. \]

This is a linear equation in terms of \( u \). We have:

\[ \rho(x) = \exp\left(- \int \frac{2}{x} dx\right) = \exp(-2 \ln |x|) = x^{-2}. \]

Multiplying our equation by \( x^{-2} \) we obtain:

\[ x^{-2} \left( \frac{du}{dx} - \frac{2}{x} u \right) = \frac{d}{dx} (x^{-2} u) = 3. \]

It follows that \( x^{-2} u = \int 3 dx = 3x + C, \quad u = 3x^3 + Cx^2. \) Since \( u = e^y \), plugging \( y(1) = 0 \) we obtain: \( 1 = 3 + C, \quad C = -2. \)

**Answer:** \( y = \ln(3x^3 - 2x^2). \)
2.

Show that the following differential equation is exact, then solve it:

\[(3y - \sin x)dy + (1 - y \cos x)dx = 0.\]

**Solution.** \(\frac{\partial}{\partial x}(3y - \sin x) = -\cos x = \frac{\partial}{\partial y}(1 - y \cos x),\) therefore, the equation is exact. We have:

\[
\frac{\partial F}{\partial x} = 1 - y \cos x, \quad F = \int (1 - y \cos x)\,dx = x - y \sin x + C(y),
\]

\[
\frac{\partial F}{\partial y} = -\sin x + C'(y) = 3y - \sin x, \quad C'(y) = 3y, \quad C(y) = \frac{3y^2}{2}.
\]

Thus, \(F(x, y) = x - y \sin x + \frac{3y^2}{2}.\)

**Answer:** \(x - y \sin x + \frac{3y^2}{2} = C,\) where \(C\) is any constant.
3.

Suppose that a motorboat is moving at $4m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $2m/s$. Assume that the resistance motorboat encounters is proportional to its velocity. Find the velocity of the boat in 20 seconds after the motor has quit.

See Practice midterm II(2), problem 3.
4.

In a population $P(t)$ of rabbits both the birth and the death rates are proportional to $P^2(t)$. Initially, there were one thousand of rabbits. After 9 years only 800 of rabbits left. Find $P(t)$.

See Practice midterm II(1), problem 1.
5.

Solve the non-homogeneous equation

\[ y'' - 3y' + 2y = \sin x. \]

**Solution.** First, let us find a particular solution \( y_p \) of this equation. The derivatives of \( f(x) = \sin x \) involve only \( \sin x \) and \( \cos x \) (with \( \pm \) sign). Therefore, try \( y_p = a \sin x + b \cos x \). We have:

\[
(-a \sin x - b \cos x) - 3(a \cos x - b \sin x) + 2(a \sin x + b \cos x) =
\]

\[
(a + 3b) \sin x + (b - 3a) \cos x = \sin x.
\]

It follows that \( a + 3b = 1 \), \( b = 3a \), \( 10a = 1 \), \( a = 0.1 \), \( b = 0.3 \). Thus, \( y_p = 0.1 \sin x + 0.3 \cos x \).

Next, let us find the general solution \( y_c \) of the associated homogeneous equation: \( y'' - 3y' + 2y = 0 \). We have: \( r^2 - 3r + 2 = 0 \), \( r_1 = 1 \), \( r_2 = 2 \). Thus, \( y_c = C_1 e^x + C_2 e^{2x} \).

The general solution of the initial equation is \( y = y_p + y_c \).

**Answer:** \( y(x) = C_1 e^x + C_2 e^{2x} + 0.1 \sin x + 0.3 \cos x \).
6.

Using the elimination method, solve the initial value problem:

\[ X' = \begin{bmatrix} -t & 1 \\ -t^2 & t \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}. \]

**Solution.** Let \( X = \begin{bmatrix} x \\ y \end{bmatrix} \). Then the system can be written in coordinate form as:

\[ x' = -tx + y, \quad y' = -t^2 x + ty. \]

From the first equation we get: \( y = x' + tx \). Then \( y' = x'' + tx' + x \).

Plugging this formulas into the second equation we obtain:

\[ x'' + tx' + x = -t^2 x + t(x' + tx), \quad x'' + x = 0, \quad x = C_1 \cos t + C_2 \sin t. \]

Finally, \( y = x' + tx = -C_1 \sin t + C_2 \cos t + t(C_1 \cos t + C_2 \sin t) \). Substitution the initial condition \( x(0) = 5, y(0) = -2 \) we get: \( C_1 = 5, C_2 = -2 \).

**Answer:** \( X(t) = \begin{bmatrix} 5 \cos t - 2 \sin t \\ -5 \sin t - 2 \cos t + t(5 \cos t - 2 \sin t) \end{bmatrix} \).
7. Show that the following three functions are linearly independent on the real line.

\[ X_1(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} t \\ t^2 + t \\ 3t \end{bmatrix}, \quad X_3(t) = \begin{bmatrix} t + 1 \\ 0 \\ 1 \end{bmatrix}. \]

**Solution.** One has

\[ W(t) = \det \begin{bmatrix} 1 & t & t + 1 \\ 2 & t^2 + t & 0 \\ 3 & 3t & 1 \end{bmatrix}. \]

To show that the vector functions \( X_1, X_2, X_3 \) are linearly independent on the real line it is sufficient to find a point \( a \) such that \( W(a) \neq 0 \). It is easy to see that the first two natural guesses \( a = 0 \) and \( a = 1 \) do not work: \( W(0) = W(1) = 0 \). However,

\[ W(-1) = \det \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 3 & -3 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} 2 \neq 0 \]

(here we expanded the determinant by the last column). This means that the vector functions \( X_1, X_2 \) and \( X_3 \) are linearly independent.
Solve the system of differential equations

\[ X' = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} X. \]

**Solution.** First, calculate the characteristic polynomial and find the eigenvalues.

\[
p(\lambda) = \det \begin{bmatrix} 3 - \lambda & -1 & 0 \\ -2 & 2 - \lambda & 0 \\ 0 & 2 & -1 - \lambda \end{bmatrix} = ((3 - \lambda)(2 - \lambda) - 2)(-1 - \lambda) = -(\lambda + 1)(\lambda^2 - 5\lambda + 4) = -(\lambda + 1)(\lambda - 1)(\lambda - 4).
\]

Thus, the eigenvalues are \( \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 4 \). Now, let us find the corresponding eigenvectors

1) \( \lambda_1 = -1 \).

\[
\begin{bmatrix} 4 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.
\]

We obtain that \( v_1 = v_2 = 0 \) and \( v_3 \) can be any. Set \( v_3 = 1 \). Then \( V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

2) \( \lambda_2 = 1 \).

\[
\begin{bmatrix} 2 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.
\]

We obtain that \( v_2 = 2v_1 \) and \( v_3 = v_2 \). Set \( v_1 = 1 \). Then \( V_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \).
3) $\lambda_3 = 4$. 

$$
\begin{bmatrix}
-1 & -1 & 0 \\
-2 & -2 & 0 \\
0 & 2 & -5
\end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.
$$

We obtain that $v_2 = -v_1$ and $v_3 = \frac{2}{5}v_2$ can be any. Set $v_1 = 1$. Then

$$V_3 = \begin{bmatrix} 1 \\ -1 \\ -\frac{2}{5} \end{bmatrix}.$$  

**Answer:** $X(t) = C_1 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + C_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ -\frac{2}{5} \end{bmatrix}.$
9.

Solve the initial value problem

\[ X' = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}. \]

**Solution.**

\[ p(\lambda) = \det \begin{bmatrix} 1 - \lambda & -5 \\ 1 & 3 - \lambda \end{bmatrix} = \lambda^2 - 4\lambda + 8. \]

The eigenvalues are \( \lambda = 2 \pm 2i \). To find the corresponding solutions of the linear system it is sufficient to consider only one of the conjugate eigenvalues. For \( \lambda = 2 + 2i \), \( p = q = 2 \), we have:

\[ \begin{bmatrix} -1 - 2i & -5 \\ 1 & 1 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \]

Thus, \( v_1 = (2i - 1)v_2 \). Set \( v_2 = 1 \). Then the eigenvector is

\[ V = \begin{bmatrix} 2i - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \]

So, \( a = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \) The corresponding real solutions are:

\[ X_1 = e^{pt}(a \cos qt - b \sin qt) = e^{2t}(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t), \]

\[ X_2 = e^{pt}(b \cos qt + a \sin qt) = e^{2t}(\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin 2t). \]

The general solution is: \( X = C_1X_1 + C_2X_2 \). For \( t = 0 \) we have:

\[ \begin{bmatrix} 0 \\ 6 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \]

Thus, \(-C_1 + 2C_2 = 0, C_1 = 6, C_2 = 3. \) We obtain: \( X = 6X_1 + 3X_2. \)

**Answer:** \( X = e^{2t}(\cos 2t \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \sin 2t \begin{bmatrix} -15 \\ 3 \end{bmatrix}). \)
10.

Solve the system of homogeneous equations

\[ X' = \begin{bmatrix} 3 & -4 \\ 3 & -5 \end{bmatrix} X. \]

Determine the type of the critical point at 0. Sketch the phase portrait.

\[ p(\lambda) = (3 - \lambda)(-5 - \lambda) + 12 = \lambda^2 + 2\lambda - 3. \] The eigenvalues are: \( \lambda_1 = -3, \lambda_2 = 1. \) Real of opposite sign, therefore, the origin is a saddle point.

1) \( \lambda_1 = -3. \)

\[ \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \]
Thus, \( v_2 = \frac{3}{2} v_1. \) Set \( v_1 = 1. \) Then \( V_1 = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}. \)

2) \( \lambda_2 = 1. \)

\[ \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \]
Thus, \( v_1 = 2v_2. \) Set \( v_2 = 1. \) Then \( V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \)

The general solution is \( X(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} + C_2 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \)
In a population $P(t)$ of rabbits both the birth and the death rates are proportional to $P^2(t)$. Initially, there were one thousand of rabbits. After 9 years only 800 of rabbits left. Find $P(t)$. 
2.

Consider the differential equation

\[
\frac{dy}{dx} = (x - 2)y^2, \quad y(2) = 1.
\]

Using Euler’s method with step size \( h = 0.2 \) find approximate value of \( y(2.4) \).
3.

Verify that the functions \( y_1 = x^2, \ y_2 = \frac{1}{x} \) are solutions of the differential equation

\[ x^2 y'' - 2y = 0. \]

Solve the initial value problem

\[ y(1) = 5, \ y'(1) = -3. \]
4.

Using Wronskian show that the functions $y_1 = x^2, y_2 = \sin x, y_3 = \cos x$ are linearly independent on $\mathbb{R}$. 
5.

Find the general solution of the differential equation

\[ y^{(4)} - y = 2x. \]
6.

Solve the initial value problem

\[ y^{(3)} - 3y'' + 3y' - y = 0, \quad y(0) = 1, y'(0) = 1, y''(0) = 2. \]
1.

In a population of rabbits the birth rate is $0.0005P$ (births per month per rabbit) and the death rate is constant equal to 0.05 (deaths per month per rabbit). Initially, there are $P_0$ rabbits. Write down the population function $P(t)$. Describe the behavior of the population with time depending on $P_0$. 
2.

Find all critical points of the autonomous differential equation

\[
\frac{dx}{dt} = (e^x - 1)^2(x^2 - 4).
\]

determine their types (stable, unstable, or semistable). Draw the phase diagram.
3.

Suppose that a motorboat is moving at $4m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $2m/s$. Assume that the resistance motorboat encounters is proportional to its velocity. Find the velocity of the boat in 30 seconds after the motor has quit.
4.

Verify that the functions \( y_1 = x, y_2 = x \ln |x| \) are solutions of the differential equation
\[
x^2 y'' - xy' + y = 0.
\]

Solve the initial value problem
\[
y(-e) = e, y'(-e) = -3.
\]
Find the general solution of the differential equation

\[ y^{(3)} - 2y'' + y' = \sin x. \]
6.

Show directly (without using Wronskian) that the functions $x^3 + 2x^2$, $x^2 - 1$ and $x^3 - 5x^2 - 2$ are linearly independent on $\mathbb{R}$. 
1. In a population $P(t)$ of rabbits both the birth and the death rates are proportional to $P^2(t)$. Initially, there were one thousand of rabbits. After 9 years only 800 of rabbits left. Find $P(t)$.

Solution. We have: $\frac{dP}{dt} = kP^2 \cdot P = kP^3$ (a priori, the rates are per time unit per population unit). Solving this separable equation we get:

$$\frac{dP}{P^3} = kdt, \quad \int \frac{dP}{P^3} = \int kdt, \quad -\frac{1}{2P^2} = kt + C, \quad P = \frac{1}{\sqrt{At} + B},$$

where $A = -2k, B = -2C$. For simplicity, for the unit of population we will use thousand of rabbits. Initially, for $t = 0$, we have one thousand of rabbits, therefore,

$$B = \frac{1}{P^2(0)} = 1.$$

For $t = 9$ we have:

$$9A + 1 = \frac{1}{P^2(9)} = \frac{1}{(0.8)^2} = \frac{25}{16}, \quad A = \frac{1}{9}\left(\frac{25}{16} - 1\right) = \frac{1}{16}.$$

Thus,

$$P(t) = \frac{1}{\sqrt{t/16} + 1} = \frac{4}{\sqrt{t + 16}}.$$

Answer: $P(t) = \frac{4}{\sqrt{t + 16}}$ thousand of rabbits.

2. Consider the differential equation

$$\frac{dy}{dx} = (x - 2)y^2, \quad y(2) = 1.$$

Using Euler’s method with step size $h = 0.2$ find approximate value of $y(2.4)$. 

1
Solution. We have: \( x_0 = 2, \ y_0 = 1, \ x_n = x_0 + nh = 2 + 0.2n, \ 2.4 = x_2. \) The approximations \( y_n \) of \( y(x_n) \) are defined by:
\[
y_{n+1} = y_n + h(x_n - 2)y_n^2 = y_n + 0.04ny_n^2.
\]
Thus, \( y_1 = y_0 + 0 = 1, \ y_2 = y_1 + 0.04y_1^2 = 1 + 0.04 = 1.04. \)

Answer: \( y(2.4) \approx 1.04. \)

3. Verify that the functions \( y_1 = x^2, \ y_2 = \frac{1}{x} \) are solutions of the differential equation
\[
x^2y'' - 2y = 0.
\]
Solve the initial value problem
\[
y(1) = 5, \ y'(1) = -3.
\]
Solution. We have: \( y_1'' = 2, \ x^2y_1'' - 2y_1 = 2x^2 - 2x^2 = 0, \ y_2'' = 2x^{-3}, \ x^2y_2'' - 2y_2 = 2x^{-1} - 2x^{-1} = 0. \) Thus, \( x^2 \) and \( x^{-1} \) are solutions. By Principle of Superposition, for any \( c_1, c_2, \ y(x) = c_1x^2 + c_2x^{-1} \) is a solution. We have: \( y'(x) = 2c_1x - c_2x^{-2}. \) Substituting \( x = 1 \) we get:
\[
c_1 + c_2 = 5, \ 2c_1 - c_2 = -3, \ 3c_1 = 5 - 3 = 2, \ c_1 = \frac{2}{3}, \ c_2 = 5 - \frac{2}{3} = 4\frac{1}{3}.
\]
Answer: \( y(x) = \frac{2}{3}x^2 + 4\frac{1}{3}x^{-1}. \)

4. Using Wronskian show that the functions \( y_1 = x^2, \ y_2 = \sin x, \ y_3 = \cos x \) are linearly independent on \( \mathbb{R}. \)

Solution:
\[
W(x) = \det \begin{bmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{bmatrix}.
\]
To show that the function are linearly independent it is sufficient to find one point \( a \) such that \( W(a) \neq 0. \) Take point \( 0. \)
\[
W(0) = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.
\]
Expanding by the first row we get

\[ W(0) = 1 \cdot \det \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = -2. \]

Therefore, the functions are linearly independent.

5. Find the general solution of the differential equation

\[ y^{(4)} - y = 2x. \]

**Solution.** First, find the general solution of the associated homogeneous equation: \( y^{(4)} - y = 0 \). The characteristic equation is \( r^4 - 1 = 0 \). Equivalently, \( (r^2 - 1)(r^2 + 1) = 0 \). Thus, the roots are \( r_{1,2} = \pm 1 \), \( r_{3,4} = \pm i \). We have: \( y_1 = e^x, y_2 = e^{-x}, y_3 = \sin x, y_4 = \cos x \). The general solution of the homogeneous equation is \( y_h = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x \).

Now, find a particular solution of the initial equation \( y^{(4)} - y = 0 \). According to the general theory, we should try to find \( y_p \) as a combination of \( f(x) = 2x \) and its derivatives. \( f'(x) = 2, f''(x) = 0 \). Thus, we can take \( y_p(x) = a_1 x + a_2 \). Substituting \( y_p \) in the equation we get \(-a_1 x - a_2 = 2x\). Therefore, \( y_p = -2x \).

The general solution of the initial equation is \( y(x) = y_h(x) + y_p(x) \).

**Answer:** \( y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x - 2x \).

6. Solve the initial value problem

\[ y^{(3)} - 3y'' + 3y' - y = 0, \quad y(0) = 1, y'(0) = 1, y''(0) = 2. \]

**Solution.** First, find the general solution. The characteristic equation is \( r^3 - 3r^2 + 3r - 1 = 0 \). Equivalently, \( (r - 1)^3 = 0 \). Thus, \( r_1 = 1 \) is a root repeated 3 times. Therefore, the general solution is \( y(x) = (c_1 + c_2 x + c_3 x^2) e^x \). We have:

\[
\begin{align*}
y'(x) &= (c_1 + c_2 + (c_2 + 2c_3)x + c_3 x^2) e^x, \\
y''(x) &= (c_1 + 2c_2 + 2c_3 + (c_2 + 4c_3)x + c_3 x^2) e^x.
\end{align*}
\]
Substituting the initial condition, we obtain a system:

\[ c_1 = 1, \ c_1 + c_2 = 1, \ c_1 + 2c_2 + 2c_3 = 2. \]

We get: \( c_2 = 0, c_3 = \frac{1}{2}. \)

**Answer:** \( y(x) = (1 + \frac{x^2}{2})e^x. \)
1. We have: \( \frac{dP}{dt} = (0.005P - 0.05)P = 0.0005P(P - 100) \). This is an extinction/explosion equation with \( a = 0.0005 \) and the threshold \( M = 100 \). The general solution is:

\[
P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{aMt}} = \frac{100P_0}{P_0 + (100 - P_0)e^{0.05t}}.
\]

When \( 0 < P_0 < 100 \) the population decreases in time and converges to 0 when \( t \) goes to \( \infty \). \( P_0 = 100 \) is the equilibrium population. When \( P_0 > 100 \) the population increases in time and converges to \( \infty \) in finite time.

2. The critical points are solutions of \( f(x) = 0 \). \((e^x - 1)^2(x^2 - 4) = 0 \) when \( x = 0 \) or \( \pm 2 \). When \( x < -2 \) \( x' = (e^x - 1)^2(x^2 - 4) > 0 \), \( x \) increases; when \( -2 < x < 0 \) \( x' < 0 \), \( x \) decreases; when \( 0 < x < 2 \) \( x' < 0 \), \( x \) decreases; when \( x > 2 \) \( x' > 0 \), \( x \) increases. From the phase diagram we see that \(-2\) is a stable critical point, \( 0 \) and \( 2 \) are unstable (0 is semistable).

3. From the second Newton’s law we get

\[
\frac{dv}{dt} = a = F/m = kv,
\]
where \( k \) is some constant. Solution of this differential equation is
\[ v(t) = v_0 e^{kt}. \]
We have \( v_0 = 4 \) and \( v(10) = 2 \). Thus
\[ e^{10k} = \frac{2}{4} = \frac{1}{2}, \quad k = -\frac{\ln 2}{10}. \]
We obtain:
\[ v(30) = 4e^{30k} = 4e^{-3\ln 2} = 4/2^3 = 0.5. \]
Answer: 0.5 m/s.

4. We have:
\[ x^2 y'' - xy' + y = -x + x = 0, \quad x^2 y'' - xy_2 + y_2 = x^2 \left( \frac{1}{x} \right) - x(\ln |x| + 1) + x \ln |x| = 0, \]
thus, \( x \) and \( x \ln |x| \) are solutions. By the Principle of Superposition, \( y(x) = c_1 x + c_2 x \ln |x| \) is a solution for any \( c_1, c_2 \). We have: \( y'(x) = c_1 + c_2 (\ln |x| + 1) \). Substituting the initial condition, we get:
\[ -c_1 e + c_2 e = e, \quad c_1 + 2c_2 = -3 \Rightarrow c_1 - c_2 = -1, \quad 3c_2 = -2, \quad c_2 = -\frac{2}{3}, c_1 = -\frac{5}{3}. \]
Answer: \( y(x) = -\frac{5}{3}x - \frac{2}{3}x \ln |x| \).

5. First, find the general solution of the associated homogenous equation \( y''' - 2y'' + y' = 0 \). The characteristic equation \( r^3 - 2r^2 + r = 0 \), \( r(r - 1)^2 \) has a root \( r_1 = 0 \) and a repeated root \( r_2 = 1 \) of order 2. Therefore, \( y_c = c_1 + (c_2 + c_3 x)e^x \).

Now, find a particular solution of \( y''' - 2y'' + y' = \sin x \). One has \( f'(x) = \cos x, f''(x) = -\sin x, \ldots \). Thus, \( f(x) \) and its derivatives involve only \( \sin x \) and \( \cos x \). We can try \( y_p(x) = a \sin x + b \cos x \). Substituting it in the equation, we get:
\[ y_p''' - 2y_p'' + y_p' = (-a \cos x + b \sin x) - 2(-a \sin x - b \cos x) + (a \cos x - b \sin x) = 2a \sin x + 2b \cos x = \sin x, \]
therefore, \( a = 1/2, b = 0 \). \( y_p(x) = 0.5 \sin x \).

The general solution of the initial equation is of the form \( y(x) = y_c(x) + y_p(x) \).
**Answer:** \( y(x) = c_1 + (c_2 + c_3x)e^x + 0.5\sin x. \)

6. Assume that these functions are linearly dependent. Then there exists constants \( c_1, c_2, c_3 \) (not all equal to zero) such that

\[
c_1(x^3 + 2x^2) + c_2(x^2 - 1) + c_3(x^3 - 5x^2 - 2) = 0
\]

for all \( x \). Gather coefficients in front of each power of \( x \) together. We get:

\[
(c_1 + c_3)x^3 + (2c_1 + c_2 - 5c_3)x^2 + (-c_2 - 2c_3) = 0.
\]

A polynomial is equal to zero everywhere only if all of its coefficients are zeros. Thus,

\[
c_1 + c_3 = 0, \ 2c_1 + c_2 - 5c_3 = 0, \ -c_2 - 2c_3 = 0.
\]

From the first and the last equations we obtain: \( c_2 = -2c_3, c_1 = -c_3 \). Substituting these formulas into the second equation we get: \(-9c_3 = 0, c_3 = 0\), and so \( c_1 = c_2 = 0 \) as well. Thus, all the coefficients are zeros. There are no \( c_1, c_2, c_3 \) not all equal to zero satisfying the conditions, therefore, the functions are linearly independent.
population models

If \( \beta, \delta \) are birth and death rates per unit of time per unit of population, then

\[
\frac{dP}{dt} = (\beta - \delta) \cdot P
\]

(less common; \( \beta, \delta \) are time rates, then \( \frac{dP}{dt} = \beta - \delta \))

Important models:

1) Exponential: \( \frac{dP}{dt} = kP \), \( k \) const \( \Rightarrow \)

\[
P(t) = P_0 e^{kt}
\]

2) Logistic: \( \frac{dP}{dt} = kP(M-P) \)

\[
P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}
\]

3) Extinction/explosion: \( \frac{dP}{dt} = kP(P-M) \)

\[
P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}
\]
Example. A population satisfies the exponential equation. Initially there were one million people. After 100 years only half a million left. After another 100 years only 0.2 million people left.

Find the threshold population.

Solution. For convenience measure population in millions (so, \( P_0 = 1 \)). Then

\[
P(t) = \frac{M}{1 + (M-1)e^{kt}}.
\]

We have:

\[
P(1) = \frac{M}{1 + (M-1)e^k} = 0.5 \Rightarrow M = 0.5 + 0.5(M-1)e^k
\]

\[
P(2) = \frac{M}{1 + (M-1)e^{2k}} = 0.2 \Rightarrow M = 0.2 + 0.2(M-1)e^{2k}
\]

To simplify, set \( x = e^k \). Then \( e^{2k} = x^2 \)

\[
M = 0.5 + 0.5(M-1)x \Rightarrow x = \frac{M - 0.5}{0.5(M-1)} = \frac{2M-1}{M-1}
\]

\[
M = 0.2 + 0.2(M-1)x^2 = 0.2 + 0.2(M-1) \cdot \frac{(2M-1)^2}{(M-1)^2} = 0.2 + 0.2 \cdot \left(1 + \frac{(2M-1)^2}{M-1}\right) \Rightarrow
\]

\[
M^2 - M = 0.2 + 0.2 \cdot (2M-1)^2,
\]

\[
5(M^2 - M) = 1 + 4M^2 - 4M + 1
\]

\[
M^2 - M - 2 = 0, \ (M+1)(M-2) = 0 \Rightarrow M = -1, \text{ or } M = 2.
\]

Answer. 2 million.
Acceleration - Velocity models

Movement in a resisting medium.
Typically, resistance is proportional to \( v^b \). If no other forces are involved, then
\[
\frac{dv}{dt} = k v^b
\]

Example: A body moves in a resisting medium with resistance proportional to \( v^3 \) without any external forces involved. Initial velocity was 1 m/s. After 10 seconds the body slowed down to 0.5 m/s. Find the distance travelled during these 10 seconds.

Solution:
\[
\frac{dv}{v^3} = k \, dt
\]
\[
-\frac{1}{2} v^{-2} = k \, t + C
\]
\[
\frac{1}{v^2} = -2k t - 2C = k_t + C
\]
\[
v = \sqrt{\frac{1}{k_t + C}}, \quad v_0 = 1 \Rightarrow \frac{1}{k + C} = 1 \Rightarrow C = 1, \quad v(t) = 0.5 \Rightarrow \frac{1}{10k + 1} = 0.5 \Rightarrow 10k + 1 = 4 \Rightarrow k = 0.3.
\]
Thus,
\[
v(t) = \sqrt{0.3t + 1}
\]
\[
\Delta \text{distance} = \int_0^{10} \frac{1}{\sqrt{0.3t + 1}} dt = \frac{2}{0.3} \left[ \sqrt{0.3t + 1} \right]_0^{10} = \frac{20(\sqrt{7} - 1)}{3} \approx 6.7
\]

Answer: 6.7 meters.
Wronskian and linear dependence

\[
W(f_1, f_2, \ldots, f_n) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}
\]

If \( f_1, \ldots, f_n \) are linearly dependent on \( I \) (that is, there exist \( c_1, \ldots, c_n \) not all zero:
\[
c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0 \text{ on } I
\]
then \( W(x) \equiv 0 \) on \( I \). To show linear independence of \( f_1, \ldots, f_n \) on \( I \) enough to find an \( a \in I \) such that \( W(a) \neq 0 \).

Example: Show that \( f_1(x) = e^{x} \), \( f_2(x) = x^2 - x \), \( f_3(x) = x^2 + 1 \) are linearly independent on \( (-3, 3) \).

Solution: \( W(x) = \det \begin{pmatrix} e^{x} & x^2 - x & x^2 + 1 \\ -e^{x} & 2x - 1 & 2x \\ e^{x} & 2 & 2 \end{pmatrix} \)

Convenient to consider \( x = 1 \in (-3, 3) \)

\[
W(1) = \det \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - 0 + 2 \cdot \det \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} = 1 \cdot (2 - 4) + 2 \cdot (-2 - 1) = -8 \neq 0
\]

\( \Rightarrow f_1, f_2, f_3 \) are linearly independent on \((-3, 3)\).
and particular solutions of homogeneous equations

Let \( y_1, y_2 \) be linearly independent solutions of
\[ y'' + p_3(x)y' + \ldots + p_n(x)y = 0, \text{ where } p_i(x) \text{ are constant.} \]

Then the general solution is
\[ y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n. \]

For any \( a \in \mathbb{R}\) and any \( b_0, b_1, \ldots, b_{n-1} \in \mathbb{R} \)

\[ \exists! \text{ solution } y(x), \text{ such that } \]
\[ y(a) = b_0, y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}. \]

Example: Show that \( y_1 = x^{-1}, y_2 = 1, y_3 = x \)
are linearly independent solutions of
\[ x y''' + 3 y'' = 0 \text{ on } (0, +\infty). \]

Solve the initial value problem: \( y(1) = 2, y'(1) = 1, y''(1) = 0 \).

Solution: \( y''' = y'' = y' = y'' = 0 \Rightarrow y_2 \) and \( y_3 \) are solutions,
\[ x y''' + 3 y_1'' = -6 x^{-1} x + 3 \Rightarrow \]
\[ y_1 \text{ is a solution,} \]
\[ W(x) = \begin{pmatrix} x^{-1} & 1 & x \\ -x^{-2} & 0 & 1 \\ x^{-3} & 0 & 0 \end{pmatrix} = x^{-1} \text{det} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 1 \text{det} \begin{pmatrix} x^{-2} & 0 \\ 0 & 0 \end{pmatrix} + x \text{det} \begin{pmatrix} 0 & 0 \\ 2x^{-3} & 0 \end{pmatrix} = 2 x^{-3} \neq 0 \Rightarrow \]
\( y_1, y_2, y_3 \) are linearly independent.
The general solution is
\[ y(x) = C_1y_1 + C_2y_2 + C_3y_3 = \frac{c_1}{x} + C_2 + C_3x. \]

Substitute the initial conditions:
\[ y(1) = C_1 + C_2 + C_3 = 2, \]
\[ y'(1) = -C_1 + C_3 = 1, \]
\[ y''(1) = 2C_1 = 0 \Rightarrow C_1 = 0, \quad C_3 = 1, \quad C_2 = 1. \]

Answer: \[ y(x) = x + 1. \]

Homogeneous equations with constant coefficients
\[ y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = 0. \] (* *)

Char. equation:
\[ r^n + a_1r^{n-1} + \ldots + a_{n-1}r + a_n = 0. \] (*)

Fundamental Theorem of Algebra:
(*) has \( n \) solutions in total (real and complex, counting multiplicity).

Thus 1) If \( r \) is a real root of multiplicity \( k \) then \[ y_1 = e^{rx}, \quad y_2 = xe^{rx}, \quad \ldots \quad y_k = x^{k-1}e^{rx} \] are solutions of (* *)

2) If \( r \) is a complex root of multiplicity \( k \)
If \( r = a + bi \), then \( r = a - bi \) is also a root of multiplicity \( k \), \( y_1 = e^{ax} \cos bx \), 
\( y_2 = e^{ax} \sin bx \), \( y_3 = xe^{ax} \cos bx \), \( y_4 = xe^{ax} \sin bx \), 
\( y_{k+1} = e^{ax} \cos bx \), \( y_{k+2} = xe^{ax} \sin bx \), 
\( y_{k+3} = e^{ax} \sin bx \), \( y_{k+4} = xe^{ax} \cos bx \). These are solutions.

3) The \( n \) solutions corresponding to the \( n \) roots of \( (*) \) are linearly independent. The general solution is their linear combination: \( y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \).

**Example:** The characteristic equation of a homogeneous eq. has the form: 

\[
(r-2)^3 (r^2+2)^2 = 0.
\]

Find the general solution.

**Solution:** The roots (counting multiplicity) are: \( 2, 2, 2, 0, -\sqrt{2}i, -\sqrt{2i}, \sqrt{2i}, \sqrt{2i} \). The corresponding solutions are: \( e^{2x}, xe^{2x}, x^2 e^{2x}, \cos \sqrt{2}x, x \cos \sqrt{2}x, x^2 \cos \sqrt{2}x, \sin \sqrt{2}x, x \sin \sqrt{2}x, x^2 \sin \sqrt{2}x \). The general solution is

\[
y(x) = c_1 e^{2x} + c_2 xe^{2x} + c_3 x^2 e^{2x} + c_4 +
\]
\[
c_5 \cos \sqrt{2}x + c_6 x \cos \sqrt{2}x + c_7 \sin \sqrt{2}x + c_8 x \sin \sqrt{2}x.
\]
Solutions of non-homogeneous equations

\[ y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = f(x). \quad (1) \]

The general solution is of the form:

\[ y(x) = y_c + y_p, \] where \( y_c \) is the general solution of

\[ y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = 0 \]
and \( y_p \) is a particular solution of \((1)\).

Case of constant coefficients

\[ y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(x). \]

If \( f(x) \) and its derivatives are linear combinations of some functions \( f_1, f_2, \ldots, f_k \), then try \( y_p(x) = c_1 f_1 + c_2 f_2 + \ldots + c_k f_k \).

Example: Solve

\[ y^{(n)} + 2y'' + y = xe^x. \]

**Solution:** Consider \( y^{(n)} + 2y'' + y = 0 \).

\[ r^4 + 2r^2 + 1 = 0, \quad (r^2 + 1)^2 = 0. \] Roots are

\[ \pm i, \text{ mult. 2} \Rightarrow \]

\[ y_c(x) = c_1 \cos x + c_2 x \cos x + c_3 \sin x + c_4 x \sin x. \]
2) Find \( y_p \).

We have \( f(x) = xe^x \), \( f'(x) = (x+1)e^x \),
\( f''(x) = (x+2)e^x \), \( \ldots \) \( f^n(x) \) and its
derivatives are linear combinations of
\( xe^x \) and \( e^x \). Try \( y_p = a_1xe^x + a_2e^x \).

We have:
\[
y_p = a_1(x+1)e^x + a_2e^x,
\]
\[
y_1'' = a_1(x+1)e^x + a_2e^x \]
\[
y_1'' = a_1(x+1)e^x + a_2e^x.
\]
\[
a_1(x+1)e^x + a_2e^x + 2(a_1(x+2)e^x + a_2e^x) + \]
\[
a_1xe^x + a_2e^x = xe^x.
\]
\[
y_1xe^x + (8a_1 + 4a_2)e^x = xe^x \Rightarrow \]
\[
y_1xe^x + 8a_1e^x + 4a_2e^x = xe^x \Rightarrow \]
\[
y_1 = 1, \quad a_1 = \frac{1}{4}, \quad a_2 = -2a_1 = -\frac{1}{2}.
\]

Answer:
\[
y(x) = y_c + y_p = \]
\[
c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x + \]
\[
c_5xe^x - \frac{1}{2}e^x.
\]
1. (6 pts)

Solve differential equations:

\[ 1) \frac{dy}{dx} = \sqrt{x} + x\sqrt{x}. \]

\[ y(x) = \int (\sqrt{x} + x\sqrt{x}) \, dx = \]

\[ \frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} + C \]
2) \( \frac{dy}{dx} = \sqrt{x} + y\sqrt{x} \). (9 pts)

\[
\text{Assuming } y \neq -1 \\
\Rightarrow \frac{dy}{y+1} = \sqrt{x}
\]

\[
\ln|y+1| = \frac{2}{3} x^{\frac{3}{2}} + C
\]

\[
|y+1| = e^{\frac{2}{3} x^{\frac{3}{2}} + C}
\]

\[
y+1 = \pm e^{\frac{2}{3} x^{\frac{3}{2}}} = Ae^{\frac{2}{3} x^{\frac{3}{2}}}
\]

Here \( A \neq 0 \) any. But \( y = -1 \) is also a solution \( \Rightarrow A = 0 \) works.

Answer: \( y = Ae^{\frac{2}{3} x^{\frac{3}{2}}} - 1, \ A \) any
2. (15 pts)

Solve the initial value problem

\[(x^2 + 1) \frac{dx}{dt} = tx, \quad x(-1) = -1.\]

2. separable

\[(x + \frac{1}{x}) dx = t dt \quad \leftarrow \text{Assuming } x \neq 0 \quad x = 0 \text{ is a solution.}\]

\[
\frac{x^2}{2} + \ln |x| = \frac{t^2}{2} + C
\]

\[
|x| e^{\frac{x^2}{2}} = e^{\frac{t^2}{2} + C}
\]

3. \(x e^{\frac{x^2}{2}} = t e^{\frac{t^2}{2}} = A e^{\frac{t^2}{2}}, \quad A \neq 0.\)

Since \(x = 0\) is a solution, \(A\) can be 0.

\(x(-1) = -1.\)

3. \(-1 e^{\frac{1}{2}} = A e^{\frac{1}{2}} \Rightarrow A = -1.\)

Answer: \(x e^{\frac{x^2}{2}} = -e^{\frac{t^2}{2}}\) (implicit solution).
A cup of tea at temperature 80°C was left on a table in the kitchen, where the temperature is 26°C. After 20 min the tea cooled down to 62°C. What will be the temperature of the tea after another 20 min?

\[ \frac{dT}{dt} = -k(T - A) \]

\[ T(t) = A + (T_0 - A) e^{-kt} = 26 + (80 - 26) e^{-kt} = 26 + 54 e^{-kt} \]

\[ T(20) = 62 \]

\[ 62 = 26 + 54 e^{-k \cdot 20} \]

\[ 36 = 54 e^{-k \cdot 20} \]

\[ e^{-k \cdot 20} = \frac{2}{3} \]

\[ k = -\frac{\ln \frac{3}{2}}{20} = \frac{\ln \frac{3}{2}}{20} \]

\[ T(40) = 26 + 54 e^{-k \cdot 40} = 26 + 54 e^{-2 \ln \frac{3}{2}} = 26 + 54 \left( \frac{3}{2} \right)^{-2} = \]

\[ = 26 + \frac{4}{9} \cdot 54 = 26 + 24 = 50 \]

Answer: 50°C
4. (20pts)

A tank initially contains 100 lb of salt dissolved in 200 gal of water. A brine containing 1 lb/gal of salt flows into the tank at the rate 2 gal/min and the well stirred mixture flows out of the tank at the rate 1 gal/min. How much salt will be inside the tank after 50 min?

Let $x(t)$ be the amount of salt at time $t$.

$$\frac{dx}{dt} = 1.2 - \frac{1}{V(t)} \cdot x(t),$$

where $V(t)$ is the amount of liquid in the tank at time $t$, $V(t) = 200 + (2-1)t = 200 + t$.

$$\frac{dx}{dt} = 2 - \frac{x}{200 + t}; \quad \frac{dx}{dt} + \frac{x}{200 + t} = 2 \quad \text{Linear}$$

$$\int \frac{1}{200 + t} \, dt = \ln(200 + t) = 200 + t$$

$$\frac{d}{dt} \left((200 + t)x\right) = (200 + t) \left(\frac{dx}{dt} + \frac{x}{200 + t}\right) = 2 \cdot (200 + t)$$

$$(200 + t)x = (200 + t)^2 + C, \quad x = 200 + t + \frac{C}{200 + t}$$

$x(0) = 100 \Rightarrow 100 = 200 + \frac{C}{200}, \quad C = -20000$

$x(t) = 200 + t - \frac{2 \cdot 10^4}{200 + t}$,
\[ x(50) = 250 - \frac{2 \cdot 10^4}{250} = 250 - 80 = 170. \]

Answer 170 lb
5. (12 pts)

Solve differential equations by using appropriate substitution:

1) \( 2yy' = (x - y^2)^2 + 1, \ y(1) = 1. \)

2. \( u = x - y^2 \)

\[
\frac{du}{dx} = 1 - 2y \frac{dy}{dx} = 1 - (x - y^2)^2 + 1 = -u^2
\]

\[
\frac{du}{u^2} = -dx \quad \text{(assuming} \ u \neq 0) \]

\[
2 \left( \frac{1}{u} \right) = -x + C, \quad u = \frac{1}{x - C}
\]

2 or \( u = 0 \) (singular solution).

\( x - y^2 = u \Rightarrow a) y = \pm \sqrt{x - u^2} = \pm \sqrt{x - \frac{1}{x - C}} \)

or \( b) y = \pm \sqrt{x} \) (when \( u = 0 \))

2 \( y(1) = 1 : \ a) 1 = \pm \sqrt{1 - \frac{1}{1 - C}} \Rightarrow " + " \)

\( 1 = 1 - \frac{1}{1 - C}, \quad \frac{1}{1 - C} = 0 \quad \text{impossible, no C} \)

2 \( b) y = \pm \sqrt{1} \Rightarrow " + " \cdot \quad C = 0 \)

Thus, \( y = \sqrt{x} \) is the unique solution.

Answer \( y = \sqrt{x} \)
2) \( xt \frac{dx}{dt} = x^2 - 2t^2 \).

\[
\frac{dx}{dt} = \frac{x}{t} - \frac{2t}{x}, \quad \text{Homogeneous.}
\]

2 substitution \( v = \frac{x}{t} \)

\[ x = vt, \quad \frac{dx}{dt} = v \frac{dv}{dt} + t \frac{dv}{dt} \]. Thus

\[ vt + t \frac{dv}{dt} = v - \frac{2}{v} \]

2 \( t \frac{dv}{dt} = -\frac{2}{v} \), \( \int v \, dv = -\int \frac{2 \, dt}{t} \)

\[
\frac{v^2}{2} = -2 \ln |t| + C
\]

2 \( v = t \sqrt{-4 \ln |t| + A} \), \( A = 2C \)

2 \( x = tvt = t \sqrt{-4 \ln |t| + A} \), \( A \) any number.

Answer \( x = t t \sqrt{-4 \ln |t| + A} \), \( A \) any.
Show that the following differential equation is exact; then solve it.

\[(e^x + y)dx + (\sin y + x)dy = 0.\]

\[M, N \text{ continuous everywhere}\]

\[\text{Exact } \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\]

3. \[\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \implies \text{the equation is exact.}\]

3. Need to find \(F\) such that \[\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N\]

3. \[F = \int M \, dx = \int (e^x + y) \, dx = e^x + xy + C(y)\]

3. \[\frac{\partial F}{\partial y} = N \implies x + c'(y) = \sin y + x,\]

\[c'(y) = \sin y, \quad c(y) = -\cos y + C.\]

3. Thus, \[F(x, y) = e^x + xy - \cos y + C.\]

\[\text{Answer } e^x + xy - \cos y + C = 0.\]
THERE ARE TEN PROBLEMS. EACH PROBLEM HAS THE SAME VALUE 10 POINTS.

SHOW YOUR WORK

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser and student ID

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1. Find all solutions of the initial value problem
\[ y' = (2x - 3y)^3 + \frac{2}{3}, \quad y(3) = 2. \]

**Solution:** Let us use a substitution \( u(x) = 2x - 3y(x) \). We have:
\[ u' = 2 - 3y' = 2 - 3(u^3 + \frac{2}{3}) = -3u^3. \]
Observe that \( u = 0 \) is a solution. It corresponds to \( y = \frac{2}{3}x \). This function satisfies the initial condition, and so is a solution of the initial value problem.

Let \( u \neq 0 \). Then we can divide by \( u^3 \):
\[ \frac{du}{u^3} = -3dx, \quad -\frac{1}{2u^2} = -3x + C, \quad u = \pm \frac{1}{\sqrt{6x + A}}. \]
Thus,
\[ y = \frac{1}{3}(2x - u) = \frac{1}{3}(2x \pm \frac{1}{\sqrt{6x + A}}). \]
Substituting the initial condition \( y(3) = 2 \) we obtain:
\[ 2 = 2 \pm \frac{1}{3\sqrt{18 + A}}. \]
This equation does not have any solutions, and so \( u \neq 0 \) does not give any solutions of the initial value problem.

**Answer:** \( y = \frac{2}{3}x \).
2. A tank contains 500 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 2L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How much salt will there be in the tank after 50 seconds?

Solution: Let $x(t)$ be the amount of salt in the tank at time $t$. Notice that the amount of solution is 500 L and does not change with time. And so at time $t$ the solution contains $\frac{x}{500}$ kg of salt per L. The salt leaves the tank with the rate $2 \frac{x}{500} = \frac{x}{250}$ kg/s. We obtain:

$$\frac{dx}{dt} = -\frac{x}{250}.$$  

This is an exponential model with $k = -\frac{1}{250}$. The solution is:

$$x(t) = x_0 \exp(kt) = 100 \exp\left(-\frac{t}{250}\right).$$

Thus, at time $t = 50$ s, there is

$$100 \exp\left(-\frac{50}{250}\right) = 100 \exp\left(-\frac{1}{5}\right)$$

L of salt.

Answer: $100 \exp\left(-\frac{1}{5}\right)$ L of salt.
3. In a population $P(t)$ of some species the birth rate is constant equal to 0.1 (births per month per individual) and the death rate is equal to 0.002$P$ (deaths per month per individual). Initially, there were $P_0$ species. Write down the population function $P(t)$. Describe the behavior of the population with time depending on $P_0$.

**Solution:** We have:

$$\frac{dP}{dt} = P(0.1 - 0.002P) = 0.002(50 - P).$$

This is a logistic equation with $k = 0.002$ and $M = 50$. The solution is:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} = \frac{50P_0}{P_0 + (50 - P_0)e^{-0.1t}}.$$

From general theory of the logistic equation, $P_0 = 0$ and $P_0 = 50$ give constant (equilibrium) populations $P(t) = 0$ and $P(t) = 50$. If $0 < P_0 < 50$, the population is increasing and converges to 50 when $t \to \infty$. If $P_0 > 50$, the population is decreasing and converges to 50 when $t \to \infty$. 
4. Solve the non-homogeneous equation \( x'' - 3x' + 2x = t + 2 \).

**Solution:** The general solution is \( x = x_c + x_p \), where \( x_c \) is the general solution of the associated homogeneous equation \( x'' - 3x' + 2x = 0 \) and \( x_p \) is a particular solution of the initial equation \( x'' - 3x' + 2x = t + 2 \).

We have: \( r^2 - 3r + 2 = 0 \), \( r_1 = 1 \), \( r_2 = 2 \), so \( x_c = C_1 e^t + C_2 e^{2t} \). Try \( x_p = at + b \). We have:

\[
x_p'' - 3x_p' + 2x_p = -3a + 2(at + b) = t + 2,
\]

therefore, \( 2a = 1 \), \( a = \frac{1}{2} \), \( -3a + 2b = 2 \), \( b = \frac{7}{4} \).

**Answer:** \( x = C_1 e^t + C_2 e^{2t} + \frac{1}{2}t + \frac{7}{4} \).
5. In a mass-spring-dashpot system a mass $m = 1$ is attached to a spring with spring constant $k = 25$ and a dashpot with damping constant $c = 6$. The initial position of the mass is at the equilibrium point of the spring and the initial velocity is $-4$. Find the position function of the mass.

Solution: We have:

$$x'' + 6x' + 25x = 0, \quad x_0 = 0, \quad v_0 = -4.$$ 

Solve the characteristic equation:

$$r^2 + 6r + 25 = 0, \quad r = \frac{1}{2}(-6 \pm \sqrt{36 - 100}) = -3 \pm 4i.$$ 

Thus, the general solution is:

$$x(t) = e^{-3t}(C_1 \cos 4t + C_2 \sin 4t).$$ 

Substituting the initial condition we get:

$x(0) = C_1 = 0 \Rightarrow x(t) = C_2 e^t \sin 4t \Rightarrow x'(0) = 4C_2 = -4, \quad C_2 = -1.$ 

Answer: $x(t) = -e^t \sin 4t.$
6. Using the elimination method, solve the system:

\[ tx' = 2x + y, \quad ty' = (t^2 - 2)x - y. \]

**Solution:** From the first equation we obtain:

\[ y = tx' - 2x \implies y' = tx'' + x' - 2x' = tx'' - x'. \]

Substituting these identities in the second equation we get:

\[ t(tx'' - x') = (t^2 - 2)x - (tx' - 2x). \]

Simplifying we obtain \( t^2 x'' - t^2 x = 0, \ x'' - x = 0 \). Solution of this equation is \( x = C_1 e^t + C_2 e^{-t} \). Finally,

\[ y = tx' - 2x = C_1 e^t (t - 2) - C_2 e^{-t} (t + 2). \]

**Answer:** \( x = C_1 e^t + C_2 e^{-t}, \ y = tx' - 2x = C_1 e^t (t - 2) - C_2 e^{-t} (t + 2). \)
7. Show that there is no everywhere continuous $2 \times 2$ matrix function $P(t)$ such that both vector functions

$$X_1(t) = \begin{bmatrix} t + 1 \\ 2t \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} t \\ t + 2 \end{bmatrix}$$

are solutions of the system $X' = P(t)X$.

**Solution:** Assume that there exists such $P(t)$. By theorem on Wronskian, either $W(t) = 0$ for all $t$ or for each $t$ $W(t) \neq 0$. We have:

$$W(t) = (t + 1)(t + 2) - 2t^2 = -t^2 + 3t + 2.$$  

This is a quadratic function. The discriminant is $D = (-3)^2 - 4 \cdot (-1) \cdot 2 = 17 > 0$. Therefore, there exist exactly two points

$$t = \frac{1}{2}(-3 \pm \sqrt{17})$$

at which $W(t) = 0$ and $W(t) \neq 0$ for all other points. Thus, neither one of the two options from Theorem on Wronskian are satisfied. This contradiction shows that there is no $P(t)$ satisfying the conditions of the problem.
8. Solve the system of differential equations

\[
X' = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -7 & 2 & -1 \end{bmatrix} X.
\]

**Solution:** First, let us find the eigenvalues of the matrix \( A \) of the system.

\[
p(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ -4 & 4 - \lambda & 0 \\ -7 & 2 & -1 - \lambda \end{bmatrix} = (\lambda^2 - 4\lambda + 4)(-\lambda - 1) = -(\lambda + 1)(\lambda - 2)^2.
\]

Thus, the eigenvalues are \( \lambda_1 = -1 \) and \( \lambda_{2,3} = 2 \). Let us find the corresponding eigenvectors.

1) \( \lambda = -1 \):

\[
\begin{bmatrix} 1 & 1 & 0 \\ -4 & 5 & 0 \\ -7 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.
\]

We get \( v_1 = v_2 = 0, v_3 \) is any. Set \( v_3 = 1 \). Then

\[
V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_1(t) = e^{-t}V_1.
\]

2) \( \lambda = 2 \):

\[
\begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.
\]

We get \( v_2 = 2v_1, v_3 = \frac{1}{3}(-7v_1 + 2v_2) \). Set \( v_1 = 1 \). Then \( v_2 = 2, v_3 = -1, \)

\[
V_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad X_2(t) = e^{2t}V_2.
\]

We see that there are no 2 linearly independent eigenvectors corresponding to the repeated eigenvalue \( \lambda = 2 \), therefore, this eigenvalue
is defective. We need to find a generalized eigenvector $V_3$ such that $(A - \lambda I)V_3 = V_2$:

$$
\begin{bmatrix}
-2 & 1 & 0 \\
-4 & 2 & 0 \\
-7 & 2 & -3
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}.
$$

We see that $v_2 = 2v_1 + 1, v_3 = \frac{1}{3}(-7v_1 + 2v_2 + 1)$. Set $v_1 = 0$. Then $v_2 = 1, v_3 = 1$,

$$V_3 = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad X_3(t) = e^{2t}(tV_2 + V_3).$$

The general solution is $X = C_1 X_1 + C_2 X_2 + C_3 X_3$.

**Answer:** $X(t) = C_1 e^{-t} \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} + C_2 e^{2t} \begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix} + C_3 e^{2t}(t \begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix}).$
9. Solve the initial value problem

\[ X' = \begin{bmatrix} 1 & -2 \\ 5 & -5 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \]

**Solution:** First, find the eigenvalues: \( p(\lambda) = (1 - \lambda)(-5 - \lambda) - (-2) \cdot 5 = \lambda^2 + 4\lambda + 5 = 0, \quad \lambda = \frac{1}{2}(-4 \pm \sqrt{-4}) = -2 \pm i. \) In the case of conjugate complex eigenvalues to find the solutions it is sufficient to consider only one of the eigenvalues.

\( \lambda = -2 + i. \) Let us find the complex eigenvector by solving \((A - \lambda I)V = 0:\)

\[ \begin{bmatrix} 3 - i & -2 \\ 5 & -3 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \]

We obtain: \( v_2 = \frac{1}{2}(3 - i)v_1. \) Take \( v_1 = 3. \) Then \( v_2 = 3 - i, \)

\[ V = \begin{bmatrix} 2 \\ 3 - i \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \]

In this case, the two real solutions corresponding to \( \lambda \) and \( \bar{\lambda} \) are:

\[ X_1 = e^{pt}(a \cos qt - b \sin qt) = e^{-2t}(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t) \]

\[ X_2 = e^{pt}(b \cos qt + a \sin qt) = e^{-2t}(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \sin t). \]

The general solution is \( X(t) = C_1X_1 + C_2X_2. \) Substituting the initial condition we obtain:

\[ X(0) = C_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \]

Thus, \( 2C_1 = 2, \ C_1 = 1, \ 3C_1 - C_2 = 4, \ C_2 = -1. \) Finally,

\[ X(t) = e^{-2t}(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t) - e^{-2t}(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \sin t) \]

**Answer:** \( X(t) = e^{-2t}(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cos t - \begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin t). \)
10. Solve the system of homogeneous equations

\[ X' = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix} X. \]

Determine the type of the critical point at 0. Sketch the phase portrait.

**Solution:** We have:

\[ p(\lambda) = -\lambda(4 - \lambda) + 3 = \lambda^2 - 4\lambda + 3. \]

So the eigenvalues are \( \lambda_1 = 1, \lambda_2 = 3 \). Distinct real eigenvalue of positive sign, therefore, the critical point is an improper source.

1) \( \lambda_1 = 1 \). The corresponding eigenvector:

\[
\begin{bmatrix}
-1 & -1 \\
3 & 3
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = 0, \quad v_1 = -v_2, \quad V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

2) \( \lambda_2 = 3 \). The corresponding eigenvector:

\[
\begin{bmatrix}
-3 & -1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = 0, \quad v_2 = -3v_1, \quad V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.
\]

The general solution is

\[ X(t) = C_1 e^{t} V_1 + C_2 e^{3t} V_2 = C_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \]

When \( t \to -\infty \) \( e^t \) is dominating \( e^{3t} \), therefore, all non-straight trajectories are tangent to the line in direction of \( \pm V_1 \) near the origin. When \( t \to \infty \) \( e^{3t} \) is dominating \( e^t \), therefore, all non-straight trajectories approach the direction of \( \pm V_2 \) at infinity.