MAT 205: Calculus III

Spring 2008
Department of Mathematics
SUNY at Stony Brook

This course is the third semester in the Calculus sequence. We will develop the theory and practice of differentiation and integration of functions of several variables. This builds strongly on the calculus of 1 variable that you've had in previous calculus classes. There will be a balance of theory and computation, although formal proofs will not be emphasized.

Instructor: Dr. Corbett Redden
Math Tower 3-114. Phone: 632-8261. email: *redden at math dot sunysb dot edu*
Office Hours: M 4-5p, Tu 10a-12p, or drop-in, or by appointment.

Teaching Assistant: Ki Song
Math. Tower 2-104, e-mail: *kiwisquash at math dot sunysb dot edu*
Office hours: Wed 10-11a MLC, Wed 4-5p office, Wed 6:30-7:30p office

Homework: Working homework problems is the only way to really learn the material. While you are encouraged to work with others, you must write up all solutions on your own. Homework sets will usually be collected in class on Thursdays. Homeworks turned in late will only receive half-credit. Also, you are encouraged to read the corresponding section of the text book *before* attending each lecture.

Exams:

- Midterm 1: Thursday, March 6 (in class) Review Sheet, Midterm 1 Solutions
- Midterm 2: Thursday, April 10 (in class) Review Sheet, Review Answers, Exam Solutions.
- Final Exam: Thursday, May 15, 5:00-7:30p, SBUnion 236, Final Exam Review, Review Solutions (in progress).

  These dates are firm, and make-ups will only be given in the case of unforseeable circumstances beyond the student's control. In such a case, the student should contact the instructor as soon as possible.

Class schedule:

Lecture	TuTh	5:20-6:40p	S B Union 231	Corbett Redden
Recitation	W	5:20-6:40p	S B Union 231	Ki Song

Textbook:: *Vector Calculus* (3rd Edition), by Colley. Pearson Prentice Hall, 2006.

Course Grade: Midterm 1: 20%, Midterm 2: 20%, Final Exam: 40%, Homework/Quiz/Participation: 20%

MLC: The Math Learning Center is located in Math Tower S-240A and offers free help to any student requesting it. It also provides a locale for students wishing to form study groups.

Disabilities: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.www.ehs.stonybrook.edu/fire/disabilities.shtml

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Starred (*) problems are optional problems (not to be turned in) for those students wishing to challenge themselves.

Week	Section	Notes	Homework
1/28-2/1	1.1-1.4	Due Thursday 2/7	**§1.1**: 3, 6 **§1.2**: 7, 13, 17, 23, 29, 37a **§1.3**: 3, 7, 11, 13, 20, (25*) **§1.4**: 10, 17, 18, 23, 26a-d, 37
2/4-2/8	1.5, 3.1, 3.2	Due Thursday 2/14	**§1.5**: 5, 8, 13, 18, 21, 30 **§3.1**: 5, 9, 16, 23, 25 **§3.2**: 3, 10ac
2/11-2/15	1.7, 2.1, 2.2, 2.3	Due Thursday 2/21	**§1.7**: 11, 15, 24, 27, 33 **§2.1**: 12, 15, 16, 36, 43 **§2.2**: 11, 12, 15, 29, 36 **§2.3**: 2, 3
2/18-2/22	2.3	Due Thursday 2/28	**§2.3**: 5, 10, 14, 25, 27, 31, 35
2/25-2/29	2.4, 2.5, 2.6	Due Thursday 3/13 (after midterm). It is suggested you work on these problems before the midterm.	**§2.4**: 2, 6, 9 **§2.5**: 2, 4, 10, 15, 16, 20 **§2.6**: 3, 5, 12, 13, 19, 28
3/3-3/7	Review, **Midterm 1 3/6 (Review Sheet)** Midterm 1 and Solutions		
3/10-3/14	4.1, 4.2	Due Thursday 3/27	**§4.1**: 4, 8, 9, 14, 27a **§4.2**: 3, 6, 8, 22a, 36
3/17-3/21	Spring Break		
3/24-3/28	4.3, 4.4	Due Thursday 4/3	**§4.2**: 32, 33 **§4.3**: 3, 21, 22, 27 **§4.4**: 3a, 7, 10
3/31-4/4	5.1, 5.2, 5.3	Due Tuesday 4/8	**§5.1**: 3, 7 **§5.2**: 5, 9, 13 **§5.3**: 4, 18

4/7-4/11	Review, **Midterm 2 4/10** Review Sheet, Answers Midterm 2 and Solutions	No HW due 4/17	
4/14-4/18	5.4, 5.5	Due 4/24	**§5.4**: 1, 7, 9, 11, 14 **§5.5**: 1, 9, 11, 13, 16
4/21-4/25	5.6, 6.1	Due Tuesday 5/6	**§5.6**: 3, 4, 22 **§6.1**: 4, 5, 7, 11, 16, 28
4/28-5/2	6.1, 6.2, 6.3	Due Thursday 5/8	**§6.2**: 1, 7, 8, 14, 19, 21 **§6.3**:3, 6, 11, 18, 23, 25
5/5-5/9	6.2, Review	Final next week!	
5/12-5/16	No classes, **Final 5/15** **Review Sheet**, Review Solutions Office hours: T W 5-7p Final Exam		

Name: ID:

Math 205: Midterm 1

March 6, 2008

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

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Total	

1. (20 points) Let $\mathbf{a} = (0, 3, -1), \mathbf{b} = (2, -1, 1), \mathbf{c} = (1, 1, 2)$.

a. Calculate $(3\mathbf{a} + \mathbf{b}) - \mathbf{c}$.

b. What is $\mathbf{a} \cdot \mathbf{b}$?

c. Find the angle between the two vectors $\mathbf{b}$ and $\mathbf{c}$.

d. Find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{c}$.

2. (15 points) Find the distance between the point $(5, -1, 2)$ and the plane $2x - 2y + z = 7$.

3. (15 points) Let $\mathbf{r}(t) = t^2\mathbf{i} - (t+2)\mathbf{j} + e^t\mathbf{k}$ be a curve in $\mathbb{R}^3$. Write an equation for the tangent line to the curve at $t = 3$.

4. (10 points) Consider the surface given by the equation

$$z = y - 2x^2.$$

a. Find the level curve at an arbitrary height and then sketch several sample level curves (e.g. curves for $c = -1, 0, 1, 2$ is sufficient).
b. Then, use this information to give a *rough* sketch of the surface.

5. (10 points) Evaluate the following limits. If they fail to exist, explain why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 1} =?, \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2} =?$$

6. (15 points) Consider the surface given by the function

$$z = f(x, y) = (x + y)e^{2y}.$$

a. Find ∇f at the point $(2, 2)$.
b. Write an equation for the tangent plane at the point $(2, 2)$.
c. Using linear approximation, estimate $f(1.8, 2.1)$.

7. (15 points) Let $\mathbf{P}(r,\theta) = (r\cos\theta, r\sin\theta)$ be a function $\mathbb{R}^2 \to \mathbb{R}^2$. Notice that $\mathbf{P}$ converts polar coordinates to rectangular coordinates.
a. Calculate $D\mathbf{P}$.
b. Suppose $f(x,y)$ is a function $\mathbb{R}^2 \to \mathbb{R}$, and at the point $\mathbf{x_0} = (0,2)$,

$$\frac{\partial f}{\partial x}(\mathbf{x_0}) = -3, \qquad \frac{\partial f}{\partial y}(\mathbf{x_0}) = 4.$$

Then, what is $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ at this same point? (Hint: consider $D(f \circ \mathbf{P})$, and convert the point $\mathbf{x_0}$ to polar coordinates).

Math 205 Midterm 1 Review

The exam will take place in class on Thursday, March 6, 2008. No books, notes, or calculators are allowed. You will be responsible for the material we have covered so far. While we have roughly covered 1.1-3.2, there are a number of subtopics the book discusses that we have ignored. If you have not seen any homework problems on these, you will not be accountable for them (e.g. Newton's method, delta-epsilon proofs, inverse function theorem, curvature formulas are all things we skipped over). The following is a list of useful formulas/equations and sample problems.

Vector equations and properties:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = a_1b_1+\cdots+a_nb_n \\
\|\mathbf{a}\| &= \sqrt{\mathbf{a}\cdot\mathbf{a}} \\
\mathbf{a}\times\mathbf{b} &= det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\
\|\mathbf{a}\times\mathbf{b}\| &= \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta \\
\mathrm{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}
\end{aligned}$$

Parametric equation for line passing through the point $\mathbf{x_0}$ with direction $\mathbf{b}$

$$\mathbf{x}(t) = \mathbf{x_0} + t\mathbf{b}$$

Equation for plane with normal vector $\mathbf{n}$ through the point $\mathbf{x_0}$

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{x_0}) = 0$$

Parametric equation for the plane spanned by the two vectors $\mathbf{a}, \mathbf{b}$ and containing the point $\mathbf{x_0}$

$$\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{x_0}$$

Differentiation formulas:

$$f:\mathbb{R}^n\to\mathbb{R}, \qquad \nabla f = \left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)$$

$$f:\mathbb{R}^n\to\mathbb{R}, \qquad Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m, \qquad D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Chain Rule: $\quad D(\mathbf{f}\circ\mathbf{u}) = D\mathbf{f}(\mathbf{u})D\mathbf{u}$

Specific case: $\quad \dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$

Sample problems:

1. Let $\mathbf{a} = (1, 0, -4), \mathbf{b} = (3, 2, 4), \mathbf{c} = (-2, -3, 1)$. Compute $\mathbf{a} - 2\mathbf{b}$, the length of $\mathbf{a}$, $(\mathbf{a} - 2\mathbf{b}) \cdot \mathbf{c}$, and $\mathbf{b} \times \mathbf{c}$.

2. What is the distance between the two points $(1, 2, 4)$ and $(-1, 1, 6)$? Write an equation for a line passing through those two points.

3. Find the angle between the two vectors $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (0, 1, -1)$. Find a vector of length 1 which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Write an equation for a plane passing through the origin and spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.

4. Find the distance between the parallel planes $2x - 2y + z = 5$ and $2x - 2y + z = 20$.

5. Determine/describe/sketch level curves for the function $f(x, y) = 3x^2 + 2y^2$.

6. Determine whether the following limits exist. If they do, calculate the limit.

$$\lim_{(x,y)\to(1,2)} \frac{y^2}{x^2 + y^2}, \qquad \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$$

7. Calculate ∇f at the point $(1, 2)$ for $f(x, y)) = \ln(x^2 + y^2)$. Write an equation for the tangent plane to f at the point $(1, 2)$.

8. Compute the matrix Df where $f(x, y, z) = (xz - y^2, \sin x - \cos y)$.

9. Suppose that an insect flies along the helical curve

$$x = \cos(2\pi t), y = \sin(2\pi t), z = 3t,$$

and that the temperature (in some nice but nonstandard units) at any point (x, y, z) is given by the function

$$f(x, y, z) = (x^2 + y^2)z.$$

What is the instantaneous rate of change of temperature felt by the insect at time $t = \frac{1}{2}$? Also, what is the derivative of $f(x, y, z)$ in the direction $(0, -2\pi, \frac{1}{2})$?

10. Suppose a particle moving in 3-dimensional space with constant acceleration (due to gravity with a nicer constant)

$$\mathbf{a}(t) = (0, 0, -10).$$

Given an initial position $\mathbf{x_0} = (0, 0, 0)$ and initial velocity $\mathbf{v_0} = (2, 4, 10)$, what is the position function for the particle? Write the equation for the tangent line to the particle at $t = \frac{1}{2}$.

Math 205: Midterm 1 Solutions

1. Let $\mathbf{a} = (0,3,-1), \mathbf{b} = (2,-1,1), \mathbf{c} = (1,1,2)$.
a. $(3\mathbf{a}+\mathbf{b}) - \mathbf{c} = 3(0,3,-1) + (2,-1,1) - (1,1,2) = (1,7,4)$.
b. $\mathbf{a}\cdot\mathbf{b} = (0,3,-1)\cdot(2,-1,1) = 0*2+3*-1+-1*1 = -4$
c. Find the angle between the two vectors $\mathbf{b}$ and $\mathbf{c}$.

$$\begin{aligned}
\mathbf{b}\cdot\mathbf{c} &= \|\mathbf{b}\|\|\mathbf{c}\|\cos\theta \\
\cos\theta &= \frac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|} \\
\cos\theta &= \frac{(2,-1,1)\cdot(1,1,2)}{\sqrt{2^2+1^2+1^2}\sqrt{1^2+1^2+2^2}} = \frac{1}{2} \\
\theta &= \frac{\pi}{3}
\end{aligned}$$

d. Find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{c}$.

$$\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 7\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

2. Find the distance between the point $(5,-1,2)$ and the plane $2x-2y+z=7$.
Let $\mathbf{v}$ be a vector from the point to the plane, and let $\mathbf{n}$ be the normal vector to the plane. From the coefficients of the equation for the plane, we see that

$$\mathbf{n} = (2,-2,1),$$

and if we choose $(0,0,7)$ as our point on the plane, then

$$\mathbf{v} = (5,-1,2) - (0,0,7) = (5,-1,-5).$$

The distance between the point and the plane will then be the distance of the vector $\mathbf{v}$ projected onto the normal vector $\mathbf{n}$.

$$\begin{aligned}
Proj_{\mathbf{n}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} \\
&= \frac{(5,-1,5)\cdot(2,-2,1)}{(2,-2,1)\cdot(2,-2,1)}(2,-2,1) = \frac{7}{9}(2,-2,1) \\
\|Proj_{\mathbf{n}}\mathbf{v}\| &= \frac{7}{9}\sqrt{9} = \frac{7}{3}
\end{aligned}$$

The distance between the point and plane is 7/3.

3. Let $\mathbf{r}(t) = t^2\mathbf{i} - (t+2)\mathbf{j} + e^t\mathbf{k}$ be a curve in $\mathbb{R}^3$. Write an equation for the tangent line to the curve at $t = 3$.
The tangent line $\mathbf{l}(t)$ is given by the equation

$$\mathbf{l}(t) = \mathbf{r}(3) + \mathbf{r}'(3)t.$$

(It goes through the point $\mathbf{r}(3)$ and has direction $\mathbf{r}'(3)$.) We then calculate

$$\begin{aligned}
\mathbf{r}(3) &= 9\mathbf{i} - 5\mathbf{j} + e^3\mathbf{k} \\
\mathbf{r}'(t) &= 2t\mathbf{i} - \mathbf{j} + e^t\mathbf{k} \\
\mathbf{r}'(t) &= 6\mathbf{i} - \mathbf{j} + e^e\mathbf{k}
\end{aligned}$$

$$\mathbf{l}(t) = (9 + 6t)\mathbf{i} + (-5 - t)\mathbf{j} + (e^3 + e^3 t)\mathbf{k}$$

4. Consider the surface given by the equation

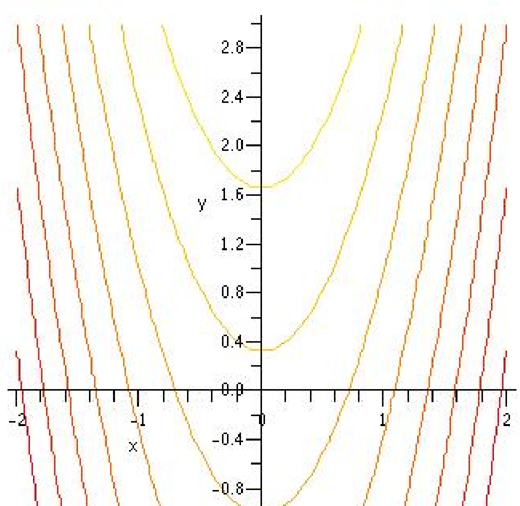
$$z = y - 2x^2.$$

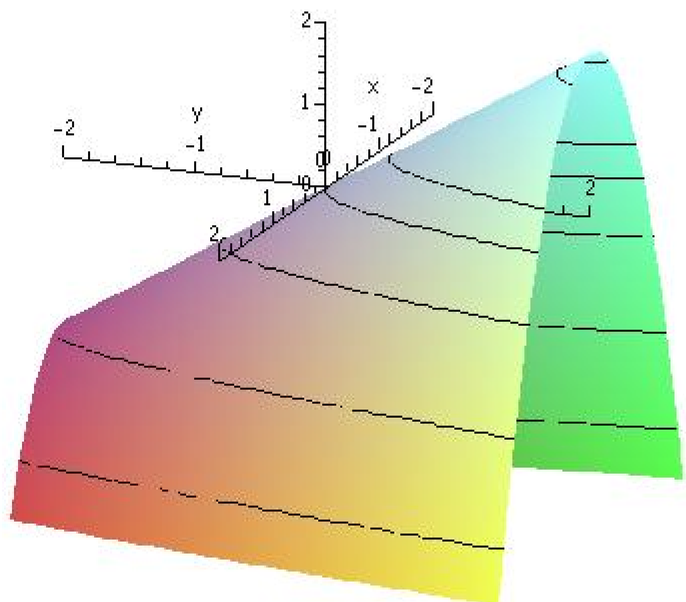
a. Find the level curve at an arbitrary height and then sketch several sample level curves (e.g. curves for $c = -1, 0, 1, 2$ is sufficient).
b. Then, use this information to give a *rough* sketch of the surface.

A level curve at height $z = c$ is given by the equation

$$\begin{aligned}
c &= y - 2x^2 \\
y &= 2x^2 + c
\end{aligned}$$

Thus, the level curves will all be parabolas shifted vertically by the constant c.





5. Evaluate the following limits. If they fail to exist, explain why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 1} = \frac{0^2 - 0^2}{0^2 + 1} = \frac{0}{1} = 0.$$

Since the numerator and denominator of the above function are polynomial, and the denominator is not 0 at the point $(0, 0)$, then the limit is obtained by plugging in the point $(x, y) = (0, 0)$.

Consider approaching $(0, 0)$ through the curve $y = mx$. Then

$$\lim_{(x,mx)\to(0,0)} \frac{x^2 - y^2}{x^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2}(1 - m^2) = 1 - m^2$$

The above limit gives a different value for different curves approaching $(0, 0)$, so the overall limit does not exist.

6. (15 points) Consider the surface given by the function

$$z = f(x,y) = (x+y)e^{2y}.$$

a. Find ∇f at the point $(2,2)$.
b. Write an equation for the tangent plane at the point $(2,2)$.
c. Using linear approximation, estimate $f(1.8, 2.1)$.

a. $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (e^{2y}, (2x+2y+1)e^{2y})$.

$$\nabla f(2,2) = (e^4, 9e^4) = e^4(1,9).$$

b. $f(2,2) = 4e^4$ will be our point on the tangent plane. The normal vector to the tangent plane is given by

$$\mathbf{n} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1)_{|(2,2)} = (e^4, 9e^4, -1).$$

(This can be memorized or seen from the formula (up to a negative sign) for

$$\mathbf{n} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}).$$

Therefore, our equation for the tangent plane is

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x_0}) &= 0 \\ (e^4, 9e^4, -1) \cdot (x-2, y-2, z-4e^4) &= 0 \\ e^4(x-2) + 9e^4(y-2) - z + 4e^4 &= 0 \\ e^4\left[(x-2) + 9(y-2) + 4\right] &= z \end{aligned}$$

c. The linear approximation $L(x,y)$ to the function f at (2,2) is given by the equation of the tangent plane at $(2,2)$.

$$\begin{aligned} L(x,y) &= e^4\left[(x-2) + 9(y-2) + 4\right] \\ L(1.8, 2.1) &= e^4\left[(1.8-2) + 9(2.1-2) + 4\right] \\ &= e^4\left(-.2 + .9 + 4\right) = 4.7e^4 \end{aligned}$$

7. (15 points) Let $\mathbf{P}(r,\theta) = (r\cos\theta, r\sin\theta)$ be a function $\mathbb{R}^2 \to \mathbb{R}^2$. Notice that $\mathbf{P}$ converts polar coordinates to rectangular coordinates.

a. Calculate $D\mathbf{P}$.

b. Suppose $f(x,y)$ is a function $\mathbb{R}^2 \to \mathbb{R}$, and at the point $\mathbf{x_0} = (0,2)$,

$$\frac{\partial f}{\partial x}(\mathbf{x_0}) = -3, \qquad \frac{\partial f}{\partial y}(\mathbf{x_0}) = 4.$$

Then, what is $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ at this same point? (Hint: consider $D(f \circ \mathbf{P})$, and convert the point $\mathbf{x_0}$ to polar coordinates).

$$DP = \begin{bmatrix} \dfrac{\partial P_1}{\partial r} & \dfrac{\partial P_1}{\partial \theta} \\ \\ \dfrac{\partial P_2}{\partial r} & \dfrac{\partial P_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Notice that the point $(x,y) = (0,2)$ is equal to $P(2,\pi/2)$. In other words, $P(2,\pi/2) = (0,2)$. This can be seen by converting to polar coordinates. Therefore, we have

$$\begin{aligned} D(f \circ P)_{(r,\theta)} &= Df_{P(r,\theta)} DP_{(r,\theta)} \\ D(f \circ P)_{(2,\pi/2)} &= Df_{(0,2)} DP_{(2,\pi/2)} \\ &= \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix}_{(0,2)} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}_{(2,\pi/2)} \\ &= \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \dfrac{\partial f}{\partial r}|_{\mathbf{x_0}} & \dfrac{\partial f}{\partial \theta}|_{\mathbf{x_0}} \end{bmatrix} = D(f \circ P)_{(2,\pi/2)} = \begin{bmatrix} 4 & 6 \end{bmatrix}$$

Hence, $\dfrac{\partial f}{\partial r} = 4, \dfrac{\partial f}{\partial \theta} = 6$ at the point $(x,y) = (0,2)$.

Name: ID:

Math 205: Midterm 2
April 10, 2008

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 100 possible points. Good luck!

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Total	

1. (15 points) Let $f(x, y) = x^{3/2} \sin y$.

a. Find the 2nd-order Taylor polynomial for $f(x, y)$ at the point $(1, 0)$.

b. Use part (a) to approximate $\sqrt{1.1^3} \sin(-.1)$

2. (20 points) Let $f(x, y) = x^3y + 12x^2 - 8y$. Find all critical points of f and classify their behavior (i.e. determine if a critical point is a local minimum, maximum, saddle, or if it cannot be determined).

3. (20 points) Consider the function $f(x, y) = x^2y$. Find the absolute maximum and minimum values of f on the bounded region

$$x^2 + 2y^2 \leq 16.$$

4. (10 points) *In the following problem, you do not have to solve for a final answer. You only have to translate the question into solving a system of algebraic equations.*

The base of an open-top aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, set up a problem to find the dimensions of the aquarium that minimize the cost of the materials. Be sure to

a. Draw a picture.

b. Identify and label your variable names.

c. Take any necessary derivatives.

d. Note what your system of equations will tell you.

5. (10 points) Find the volume of the 3-dimensional region bounded by the planes

$$x = 0, x = 1, y = 0, y = 2, z = 0, z = 3x + 4y.$$

6. (10 points) Compute the integral

$$\int\int_D x^3y^2 dA$$

where $D = \{(x, y) \mid 0 \leq x \leq 2, -x \leq y \leq x.\}$.

7. (15 points) Compute the integral

$$\int_0^4 \int_{y/2}^2 \sin(x^2) dx dy$$

by

a. Drawing the region being integrated over.

b. Switching the order of integration and then integrating.

Math 205 Midterm 2 Review

The exam will take place in class on Thursday, April 10, 2008. No books, notes, or calculators are allowed. You will be held explicitly responsible for the material we have covered since the last midterm, though much of that depends implicitly on earlier material. Roughly speaking, the midterm covers information from sections 4.1-4.4, 5.1-5.3. The following is a rough summary of the concepts covered, followed by sample problems.

- Differentials and Taylor's Theorem: Creating linear and quadratic approximations to multivariable functions.
- Local extrema of functions (finding critical points, classifying them local mins/maxes, and saddles).
- Global extrema of functions on bounded regions.
- Extrema of functions with (1 or more) constraints. In particular, using Lagrange multipliers.
- Extrema in physical situations. You should be able to translate a real-life problem into a mathematical question (e.g. find equilibrium points in a physical situation with a conservative vector field.)
- Double integrals over general regions $D \subset \mathbb{R}^2$.
- Interpret double integrals in terms of volume.
- Change order of integration on double integrals.

1. a. Find the first-order Taylor polynomial for $f(x, y) = e^{2x} \cos y$ at the point $(0, 0)$.

b. Find the second-order Taylor polynomial for $f(x, y) = e^{2x} \cos y$ at the point $(0, 0)$.

c. Use a second-order Taylor approximation to estimate $e^{.2} \cos .2$.

2. Let $f(x, y) = x^2 + xy + y^2 + 2x - 2y + 5$. Find all critical points of f and determine their nature (local min, max, saddle).

3. Suppose the temperature of a space is given by the function

$$T(x, y, z) = e^{-y}(x^2 - y^2).$$

Find the hottest point on the ball $x^2 + y^2 + z^2 \leq 4$.

4. Suppose a particle is constrained to live on the surface

$$z = -xye^{-x^2-y^2}$$

and moves under the influence of a gravitational force (given by the potential energy function $V(x, y, z) = mgz$. Find any equilibrium points.

5. Find the minimum distance from the origin to the surface $x^2 - (y - z)^2 = 1$.

6. Suppose a coffee shop sells beans from Arabia and Hawaii. If the Arabian beans are priced at x dollars per pound and the Hawaiian beans are priced at y dollars per pound, then research estimates that each week approximately $80 - 100x + 40y$ pounds of Arabian beans will be sold and $20 + 60x - 35y$ pounds of Hawaiian beans will be sold. The beans cost the coffee house \$2/lb for the Arabian and \$4/lb for the Hawaiian beans. How should the owners price the coffee beans in order to maximize their profits?

7. Find the volume of the region bounded by the planes

$$x = 0, x = \pi, y = 1, y = 2, z = 0, z = y \sin x.$$

8. Evaluate $\int\int_D 3y dA$, where D is the region bounded by $y = x^2 + 2$ and $y = 2x^2 - 2$.

9. Evaluate the following integral by switching the order of integration:

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy.$$

Math 205 Midterm 2 Review Solutions

1. $p_1 = 1 + 2x$, $p_2 = 1 + 2x + 2x^2 - \frac{1}{2}y^2$, $e^{.2}\cos .2 \approx 1.2$.
2. $f(-2, 2)$ is a minimum.
3. (I meant for this to be 2-d problem). Max temp is $4(\sqrt{3} - 1)e^{-4+2\sqrt{3}}$.
4. There are equilibrium points at $(0, 0, 0), (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2e}), (\mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2e})$.
5. Minimum distance is 1
6. Owners should price the Arabian at $x =$ \$2.70 and the Hawaiian at $y =$ \$4 per pound.
7. 3
8. $\frac{-2^7}{5}$
9. 2

Math 205: Midterm 2 Solutions

1. (15 points) Let $f(x,y) = x^{3/2}\sin y$.
   a. Find the 2nd-order Taylor polynomial for $f(x,y)$ at the point $(1,0)$.
   b. Use part (a) to approximate $\sqrt{1.1^3}\sin(-.1)$

$$\begin{aligned}
p_2(x,y) =& f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \frac{1}{2}f_{xx}(1,0)(x-1)^2 \\
&+ f_{xy}(1,0)(x-1)(y-0) + \frac{1}{2}f_{yy}(1,0)(y-0)^2 \\
=& y + \frac{3}{2}(x-1)y \\
\sqrt{1.1^3}\sin(-.1) =& f(1.1,-.1) \approx p_2(1.1,-.1) \\
\sqrt{1.1^3}\sin(-.1) \approx& -.1 + \frac{3}{2}(1.1-1)(-.1) = -.115
\end{aligned}$$

2. (20 points) Let $f(x,y) = x^3y + 12x^2 - 8y$. Find all critical points of f and classify their behavior (i.e. determine if a critical point is a local minimum, maximum, saddle, or if it cannot be determined).

$$\begin{cases} 0 = \frac{\partial f}{\partial x} = 3x^2y + 24x \\ 0 = \frac{\partial f}{\partial y} = 12y + 48 \end{cases} \quad \Rightarrow x = 2, y = -4.$$

$$H(x,y) = \begin{bmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{bmatrix}$$

$$H(2,-4) = \begin{bmatrix} -24 & 12 \\ 12 & 0 \end{bmatrix}$$

$$Det(H(2,-4)) < 0 \Rightarrow \text{ (2,-4) is a saddle point.}$$

3. (20 points) Consider the function $f(x,y) = x^2y$. Find the absolute maximum and minimum values of f on the bounded region

$$x^2 + 2y^2 \leq 16.$$

First, we find critical points in the interior region:

$$\nabla f = \mathbf{0}$$

$$\begin{cases} 0 = f_x = 2xy \\ 0 = f_y = x^2 \end{cases} \quad \Rightarrow x = 0.$$

$$f(0,y) = 0^2y = 0$$

Therefore, any point $(0, y)$ (for $y^2 < 8$) will be a critical point, and $f(0, y) = 0$ at any such point.

We now check for extremal points on the boundary $g(x, y) = x^2 + 2y^2 = 16$ using Lagrange multipliers. Note that we may assume $x \neq 0$ since we have already checked those points.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 16 \end{cases}$$

$$\begin{cases} 2xy = \lambda 2x \\ x^2 = \lambda 4y \\ x^2 + 2y^2 = 16 \end{cases}$$

$$\Rightarrow y^2 = \frac{8}{3}, y = \pm\sqrt{8/3}, x^2 = \frac{32}{3}, x = \pm 4\sqrt{\frac{2}{3}}.$$

The above calculation gives us 4 points, and we must check the value of f at each of these and compare with the other values. Doing so shows us that

$$f(\pm 4\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}) = \frac{64}{3}\sqrt{\frac{2}{3}} \text{ is the maximum.}$$

$$f(\pm 4\sqrt{\frac{2}{3}}, -2\sqrt{\frac{2}{3}} = -\frac{64}{3}\sqrt{\frac{2}{3}} \text{ is the minimum.}$$

4. (10 points) *In the following problem, you do not have to solve for a final answer. You only have to translate the question into solving a system of algebraic equations.*

The base of an open-top aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, set up a problem to find the dimensions of the aquarium that minimize the cost of the materials. Be sure to

a. Draw a picture.
b. Identify and label your variable names.
c. Take any necessary derivatives.
d. Note what your system of equations will tell you.

The setup for this problem depends on the picture drawn. In here, assume we have a rectangular tank with width and depth x and y and height z. Then, we wish to maximize the total cost of material subject to the constraint of a fixed volume. The cost function will be

$$C(x, y, z) = 5xy + 2xz + 2yz,$$

(technically, C should be some constant times the above equation. However, that constant is irrelevant when maximizing.) subject to the constraint

$$xyz = V$$

(where V is a constant). To optimize such a constrained system, we use Lagrange multipliers, giving us the system of equations

$$\begin{cases} 5y + 2z = \lambda yz \\ 5x + 2z = \lambda xz \\ 2x + 2y = \lambda xy \\ xyz = V \end{cases}$$

This is a system of 4 equations with 4 unknowns. The solution(s) (x, y, z, λ) will give us the dimensions (x, y, z) which optimize C subject to the constraint $xyz = V$.

5. (10 points) Find the volume of the 3-dimensional region bounded by the planes

$$x = 0, x = 1, y = 0, y = 2, z = 0, z = 3x + 4y.$$

$$\int_0^1 \int_0^2 (3x + 4y) dy dx = \int_0^1 3xy + 2y^2|_0^2 dx = \int_0^1 (6x + 8) dx$$
$$= 3x^2 + 8x|_0^1 = 11$$

6. (10 points) Compute the integral

$$\int\int_D x^3 y^2 dA$$

where $D = \{(x, y) \mid 0 \leq x \leq 2, -x \leq y \leq x.\}$.

$$\int\int_D x^3 y^2 dA = \int_{x=0}^{x=2} \int_{y=-x}^{y=x} x^3 y^2 dy dx = \int_0^2 \frac{2}{3} x^6 dx = \frac{2^8}{21}$$

7. (15 points) Compute the integral

$$\int_0^4 \int_{y/2}^2 \sin(x^2) dx dy$$

by
a. Drawing the region being integrated over.
b. Switching the order of integration and then integrating.

The region drawn should be the region bounded by the lines

$$x = 0, y = 0, y = 4, y = 2x.$$

This gives

$$\int_0^4 \int_{y/2}^2 \sin(x^2) dx dy = \int_{x=0}^{x=2} \int_{y=0}^{y=2x} \sin(x^2) dy dx = \int_0^2 2x \sin(x^2) dx = 1 - \cos 4.$$

Name: ID:

Math 205: Final Exam

May 15, 2008

The exam is all partial credit. Please write neatly and clearly, showing all of your work. No calculators, cell phones, books, or notes may be used. The test contains 130 possible points, and you have 2.5 hours to work. Good luck!

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

1. a. (5 points) Find a vector perpendicular to both $(1, 2, -1)$ and $(0, 1, 0)$.

b. (5 points) What is the angle between $\mathbf{i} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j} - \mathbf{k}$?

2. (7 points) Suppose a retail company's profit depends on 3 variables, x, y, and z. In turn, each of these 3 variables depends on the 2 variables u and t. For instance, let u be a measure of population density and t be time. Then, we could have $x(u, t)$ be a labor cost, $y(u, t)$ some economic indicator, and $z(u, t)$ a product cost.

Suppose that the profit is given by the function

$$P(x, y, z) = y^{\frac{1}{2}} z^{\frac{-1}{2}} - 2(x^2 + z).$$

Then, find the rate of change of profit with respect to time (i.e. find $\frac{\partial P}{\partial t}$). Your answer should be expressed in the variables x, y, z and the partial derivatives of x, y, z with respect to u, t.

3. (13 points total)

a. Find the equation of the tangent plane to

$$z = \cos x \sin y$$

at the point $(0, \frac{\pi}{3})$. (In case you forgot, $\cos\frac{\pi}{3} = \frac{1}{2}$, and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.)

b. Use linear approximation to estimate $\cos(.2)\sin(\frac{\pi}{3} - .1)$.

4. a. (10 points) Suppose a particle (in $\mathbb{R}^2$) moves subject to a conservative force with potential

$$f(x, y) = x^2 - xy + \frac{3}{2}y^2 + 5x + 4.$$

At what point(s) is the potential energy minimized?

b. (10 points) Suppose the particle from part (a) is constrained to live on the curve

$$x + 2y = 6.$$

Find any equilibrium points (i.e. critical points of the potential energy subject to our constraint).

5. (10 points) Calculate the following integral (Hint: change the order of integration)

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx.$$

6. (10 points) Let D be the parallelogram in $\mathbb{R}^2$ bounded by the four lines

$$x + y = 0, \quad x + y = \frac{\pi}{2}, \quad 2x - y = 0, \quad 2x - y = \pi.$$

Calculate the integral

$$\iint_D \cos(x + y) \sin(2x - y) dA.$$

(Hint: Use a change of variables.)

7. (5 points) Consider the integral

$$\iiint_E (x^2 y + z)\, dV$$

where E is the region bounded by the planes

$$z = 2 - x - y, \quad x = 0, \quad y = 0, \quad z = 0.$$

Convert this integral into an iterated integral (i.e. properly set up bounds of integration for all three integrals). You do not need to evaluate.

8. Let $C : [0, 2\pi] \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ given by

$$C(t) = (\cos t, \sin t, 2t).$$

a. (8 points) Find the length of the curve C.

b. (6 points) Suppose a force is given by the vector field

$$\mathbf{F} = (x, y, z).$$

What is the *average magnitude* of the force $\mathbf{F}$ along the curve C? You do not need to solve completely, but instead, you may stop once you have the answer expressed in terms of an ordinary integral.

8 (cont'd). We are still considering the curve C traced out by

$$C(t) = (\cos t, \sin t, 2t), \qquad 0 \leq t \leq 2\pi$$

and the force

$$\mathbf{F} = (x, y, z).$$

c. (8 points) Calculate the work done by $\mathbf{F}$ in transporting a particle along the curve C by computing the line integral

$$\int_C \mathbf{F} \cdot \mathbf{ds}.$$

d. (8 points) Is $\mathbf{F}$ conservative? If so, find a potential and use this to calculate part (c) in a different way.

9. a. (7 points) Show that

$$\int_C y\,dx + (2x + \sin(e^y))dy = 0$$

for any curve C contained in the $x-$axis.

b. (8 points) Let γ be a curve in $\mathbb{R}^2$ given by

$$\gamma(t) = (\cos t, \sin t), \qquad 0 \leq t \leq \pi.$$

Use part (a) to relate γ to a closed curve and then use **Green's Theorem** to calculate

$$\int_\gamma y\,dx + (2x + \sin(e^y))\,dy.$$

10. (10 points) Let D be the triangle formed by the 3 points $(0,0),(1,0),(1,2)$, and let C be the (counter-clockwise) curve along the boundary (tracing out the above triangle). Compute

$$\int_C (y+1)e^{x^2}\,dx.$$

(Hint: Use Green's Theorem, and be careful about how you set up your new integral.)

Math 205 Final Exam Review

The final exam will take place on Thursday, May 15, 2008 from 5:00 - 7:00 p.m. No books, notes, or calculators are allowed. You will be held responsible for all the material we have covered this semester. The exam will have a slight emphasis on the material covered since the last midterm. All previous exams, review sheets, and homework will provide useful study material.

Office hours during finals week will be Tuesday and Wednesday from 5-7p (or later).

A complete set of practice problems can be obtained from your previous two review sheets plus these additional problems (from the book):

2.5: 19
5.3: 16
5.4: 12
5.5: 23, 25
5.6: 5
5.8: 10
6.5: 3, 21, 22, 23, 24, 35

Math 205 Final Exam Review Solutions

2.5:19. The chain rule essentially says that $D(f\circ g) = Df \cdot Dg$, where the right hand side indicates matrix multiplication. We use the notation that $x = st, y = tu, z = su$ (obtained from g). Therefore,

$$\begin{aligned}
D(f\circ g) = Df \cdot Dg &= \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 3x^2 & -zeyz & -ye^{yz} \end{bmatrix} \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 3(st)^2 & -suestu^2 & -tue^{stu^2} \end{bmatrix} \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \\
&= \begin{bmatrix} t+u & s+u & t+s \\ 3(st)^2 t - tu^2 e^{stu^2} & 3(st)^2 s - su^2 e^{stu^2} & -2stue^{stu^2} \end{bmatrix}
\end{aligned}$$

5.3: 16

$$\begin{aligned}
\int_{y=0}^{y=\pi} \int_{x=y}^{x=\pi} \frac{\sin x}{x} dx dy &= \int_{x=0}^{x=\pi} \int_{y=0}^{y=\pi} \frac{\sin x}{x} dy dx \\
&= \int_0^\pi \sin x dx = 2
\end{aligned}$$

5.4: 12. The region W, when projected to the $x-z$ axis, is the unit circle D, given by $x^2 + z^2 = 1$ (with $y = 0$). Over, the region D, the y-values range from 0 to $2 - x - z$. Therefore,

$$\begin{aligned}
\iiint_W y\, dV = \iint_D \int_{y=0}^{y=2-x-z} y\, dy\, dA &= \iint_D \frac{1}{2}(2-x-z)^2 dA \\
&= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (2 - r\cos\theta - r\sin\theta)^2 r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (4r - 4r^3\cos\theta - 4r^2\sin\theta + r^3 + 2r^3\sin\theta\cos\theta)\, dr\, d\theta = \frac{9}{4}
\end{aligned}$$

5.5: 23. Using polar coordinates, we get that

$$\begin{aligned}
\iint_D \cos(x^2+y^2) dA = \int_0^1 \int_{\pi/3}^{\pi} \cos(r^2) r\, d\theta\, dr &= (\pi - \frac{\pi}{3}) \int_0^1 \cos(r^2) r\, dr \\
&= \frac{\pi}{3} \sin 1
\end{aligned}$$

5.5: 25. Let D be the disc of radius 2 in the $x-y$ plane.

$$\iiint_w (x^2+y^2+2z^2)\,dV = \iint_D \int_{z=-1}^{z=2} (x^2+y^2+2z^2)\,dz\,dA = \iint_D 3(x^2+y^2)+6\,dA$$
$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (3r^2+6)r\,dr\,d\theta = 48\pi$$

5.6: 5. The temperature function $f(x,y,z)$ is given by

$$f(x,y,z) = k(x^2+y^2+z^2)$$

for some constant k of proportinality. Then,

$$\iiint_W f\,dV = k\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x^2+y^2+z^2)\,dz\,dy\,dx = 8k$$

and the volume of W is 8. Therefore,

$$f_{avg} = \frac{\iiint_W f\,dV}{\iiint_W\,dV} = \frac{8k}{8} = k$$

The set of points of average temperature is then

$$\begin{aligned} k(x^2+y^2+z^2) &= k \\ x^2+y^2+z^2 &= 1 \end{aligned}$$

which is just the sphere of radius 1 centered at the origin.

5.8: 10 We use the change of variables

$$u = x+2y, v = y.$$

Then,

$$du\,dv = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx\,dy = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} dx\,dy$$

After substituting, we have the integral

$$\int_{y=0}^{y=6} \int_{x+2y=0}^{x+2y=1} y^3(x+2y)^2 e^{(x+2y)^3}\,dx\,dy = \int_{y=0}^{y=6} y^3u^2e^{u^3}\,du\,dy$$
$$= \int_0^6 y^3\frac{1}{3}(e-1)\,dy = \frac{6^4}{12}(e-1)$$

6.5: 3. Parameterize the curve by $(x,y) = (a\cos t, a\sin t)$ with $0 \le t \le \pi$.

$$\int_C y\,ds = \int_{t=0}^{t=\pi} y\sqrt{(dx/dt)^2+(dy/dt)^2}dt = \int_0^\pi a\sin ta\,dt = 2a^2$$
$$\int_C ds = \text{Length of } C = \pi a$$
$$[y]_{avg} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

6.5: 21. $\mathbf{x}(t) = (t^3, -t^2, t)$. $\frac{d\mathbf{x}}{dt} = (3t^2, -2t, 1)$.

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} dt = \int_{t=0}^{t=1} (\sin t^3, \cos(-t^2), t^4) \cdot (3t^2, -2t, 1) dt \\ &= \int_0^1 3t^2 \sin t^3 - 2t \cos(-t^2) + t^4 \, dt = -\cos 1 - \sin 1 + \frac{6}{5} \end{aligned}$$

6.5: 22. Let C be the curve traced out as described. Notice that C is traveling in a clockwise direction (as opposed to counterclockwise), so to use Green's theorem properly, we should consider the curve $-C$ going in the opposite direction. Let T be the interior of the triangle formed by the curve C.

$$\begin{aligned} \int_C x^2 y \, dx + (x+y) y \, dy &= -\int_{-C} x^2 y \, dx + (xy + y^2) \, dy = -\iint_T (y - x^2) \, dA \\ &= -\int_{x=0}^{x=1} \int_{y=0}^{y=-x+1} (y - x^2) \, dy \, dx \\ &= -\int_0^1 \frac{1}{2}(-x+1)^2 + x^3 - x^2 \, dx = \frac{11}{12} \end{aligned}$$

6.5: 23. This is easy to do without Green's Theorem.

$$\iint_D dA = \int_{\theta=a}^{\theta=b} \int_{r=0}^{r=f(\theta)} r \, dr \, d\theta = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$$

6.5: 24 Since C is closed (and without any intersection points), we can apply Green's Theorem. Let D be the region bounding C.

$$\int_C f(x) dx + g(y) dy = \iint_D \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y} \right) = \iint_D 0 \, dA = 0$$