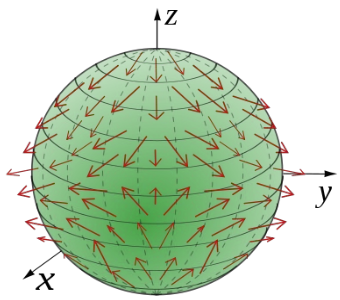
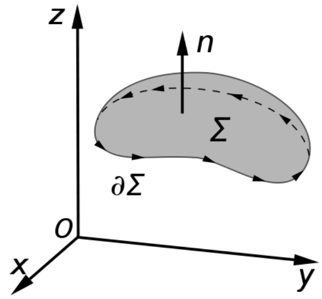


MAT 203
Calculus III with applications

Spring 2014



It is the student's responsability to check this page frequently for changes and updates. Changes will be announced in class and, if appropriate, on the web page. Students are responsible for announcements made in class and/or on the web-page. Academic Calendar.

Click here for the content and prerequisites for the course.

We will be following ***Multivariable Calculus, 10th Edition***, by **Ron Larson and Bruce Edwards**.

Instructors

	Name	Time and location	Email
Lecture	Claudio Meneses	MWF 11:00-11:53 Javits 103	claudio.meneses AT stonybrook.edu
R 01	Xin Zhang	Tu 2:30-3:23 Light Engineering 152	xzhang AT math.sunysb.edu
R 02	Xin Zhang	W 10:00-10:53 Light Engineering 152	xzhang AT math.sunysb.edu
R 03	Shaosai Huang	M 12:00-12:53 Javits 101	ahuang AT math.sunysb.edu

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Office Hours
(all in the Math Tower)

Claudio Meneses	W 1:00-2:00 pm 2-115 F 8:00-10:00 am 2-115
Xin Zhang	Tu 4:15-5:15 pm 2-104 W 4:00-6:00 pm MLC
Shaosai Huang	Tu 9:00-10:00 am S240-G W 2:00-4:00 pm MLC

Syllabus and Homework

Starting from the second week of classes, you should submit your homework to your TA during recitation. Late homework will not be accepted. A collection of 5 problems per assignment will be graded every week, chosen randomly by your TA.

Week	Sections	Homework
January 27	11.1 11.2 11.3	4, 8, 28, 50 6, 26, 44, 58 4,10,14, 42

February 3	11.4	1-6 ,16, 28, 38 (due on week 4 recitation)
February10	11.5 11.6 11.7	14, 26, 42, 82 10, 14, 20, 24 6, 12, 20, 28, 44
February 17	12.1 12.2 12.3	12, 26, 32,50 10, 22, 40, 56 6,18, 28, 34
February 24	12.4 12.5 (Arc length only)	2, 6, 14, 16, 34, 50 4, 6, 14, 18
February 26	Review for midterm 1	You may find useful to practice with the following problems, taken from the review sections of each chapter: Chapter 11: 9, 10, 12, 16, 18, 22, 26, 30, 36, 40, 47, 52, 54, 74. Chapter 12: 10, 12, 16, 24, 30, 36, 40, 46, 48, 50, 58.
February 28	**Midterm 1 (in class)**	solutions.
March 3	13.1 13.2 13.3	16, 18, 38, 40 18, 20, 42, 50 20, 26, 36, 50, 52
March 10	13.4 13. 5 13.6	10, 25, 30, 33 4, 7, 9, 22, 23 2, 12, 15, 34 (due after spring break)
March 17	Spring break	
March 24	13.7 13.8 13.9 13.10	8, 16, 29, 46 4, 10, 16, 38 2, 4, 10, 12 4, 6, 14, 26
March 31	14.1 14.2 14.3 14.4	8, 10, 20, 30 6, 10, 14, 32 7, 14, 20, 26, 4, 6, 18, 36
April 7	14.5 14.6 14.7	10, 16, 18, 36 6, 8, 16, 26 6, 10, 14, 22
April 14	14.7 14.8 (Review for midterm 2)	24, 26, 34, 42 2, 4, 8, 12, 16, 18, 22 You may find useful to practice with the following problems, taken from the review sections of each chapter: Chapter 13: 3, 5,9, 12, 13, 15, 18, 25, 30, 33, 35, 37, 41, 44, 45, 49, 52, 56, 75, 80. Chapter 14: 2, 3, 5, 8, 9, 13, 16, 20, 23, 25, 28, 32, 34, 37, 43, 47, 56, 60, 66, 70, 72.
April 21	**Midterm 2 (in class)**	solutions.
April 23	15.1 15.2	18, 23, 42, 46, 63, 69 7, 9, 13, 17, 35, 45
April 28	15.3 15.4 15.5	4, 8, 12, 20, 28 2, 8, 12, 20 8, 10, 26, 28, 40
May 5	15.6 15.7 15.8	6, 18, 26,30 2, 8, 18, 24 6, 10, 16
May 15	**Final Exam**	(Thursday, May 15, Javits 103, 11:15 am-1:45 pm) The final is cumulative. You may try to practice solving the following review exercises from chapter 15, together with all the previous ones: 4, 5, 6, 8, 9, 12, 14, 16, 19, 22, 26, 28, 32, 35, 36, 39, 42, 46, 48, 50, 52, 54, 56, 57, 60.

Quizzes

Every week, you will get a quiz at the end of your recitation with one problem of the same nature and difficulty as your homework.

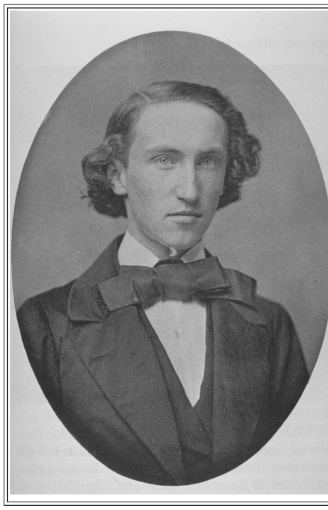
Exams and grading policy

You should bring your Stony Brook ID to all exams. Books, notes, calculators, cell phones, etc. are not allowed.

Important: there will be **no make-ups for missed exams**, unless an acceptable and documented reason is given. In such situation the corresponding grade will be dropped in computing your course grade.

A course numerical grade will be calculated according to the following rule: Midterm 1 = 20%, Midterm 2 =20%, Final = 40%, Homework = 10%, Quizzes 10%. The numerical grade will be converted to a final letter grade only **after** the final test has been graded.

In order to do well in this class you are strongly encouraged to: read the section to be covered before class, do the homework, attend recitations, and start preparing for tests well in advance. You may find it useful to visit the Math Learning Center. You are specially encouraged to attend your instructor's or TA's office hours.



Josiah Willard Gibbs, considered by many the greatest american mathematical physicist of the XIX century, is credited as being the creator of vector calculus. You can find a free online copy of his first treatise here.

- **Disability support services (DSS) statement**:
  If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities/asp.
- **Academic integrity statement:**
  Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/ .
- **Critical incident management:**
  Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

MAT 203 Spring 2014 Midterm 1

Name: ________________________________ SB ID number: ______________

Please circle the number of your recitation:

R01	Tu 2:30pm- 13:23pm	Lgt Engr Lab 152	Xin Zhang
R02	W 10:00am-10:53am	Lgt Engr Lab 152	Xin Zhang
R03	M 12:00pm- 12:53pm	Javits Lectr 101	Shaosai Huang

**************DO NOT WRITE BELOW THIS LINE****************

1	3	3	4	5	TOTAL

Instructions: The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., then **please raise your hand**.

1.- Given $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

(a) (*5 points*): Calculate $\mathbf{u} \times \mathbf{v}$.

By definition,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 2 \\ -4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -5 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 6 & 2 \\ -4 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 6 & -5 \\ -4 & 2 \end{vmatrix} \mathbf{k} = -19\mathbf{i} - 26\mathbf{j} - 8\mathbf{k}.$$

(b) (*5 points*): Calculate $(\mathbf{u} + \mathbf{v}) \times \mathbf{u}$

Here we can use some properties of the vector product:

$$\begin{aligned} &(\mathbf{u} + \mathbf{v}) \times \mathbf{u} \\ &\qquad = \mathbf{u} \times \mathbf{u} + \mathbf{v} \times \mathbf{u} \quad \text{(distributivity)} \\ &\qquad = \mathbf{v} \times \mathbf{u} \qquad \text{(since} \quad \mathbf{u} \times \mathbf{u} = \mathbf{0}) \\ &\qquad = -\mathbf{u} \times \mathbf{v} \qquad \text{(anticommutativity)} \end{aligned}$$

therefore, from (a) we conclude that $(\mathbf{u} + \mathbf{v}) \times \mathbf{u} = 19\mathbf{i} + 26\mathbf{j} + 8\mathbf{k}$

(c) (*10 points*): Find a unit vector that is orthogonal to both $\mathbf{w}_1 = \langle 2, -10, 8 \rangle$ and $\mathbf{w}_2 = \langle 4, 6, -8 \rangle$

A vector orthogonal to a given pair can be obtained by means of the cross product. Thus consider

$$\mathbf{w}_1 \times \mathbf{w}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -10 & 8 \\ 4 & 6 & -8 \end{vmatrix} = \begin{vmatrix} -10 & 8 \\ 6 & -8 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 8 \\ 4 & -8 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -10 \\ 4 & 6 \end{vmatrix} \mathbf{k} = 32\mathbf{i} + 48\mathbf{j} + 52\mathbf{k}.$$

Since $||\mathbf{w}_1 \times \mathbf{w}_2|| = \sqrt{(32)^2 + (48)^2 + (52)^2} = 4\sqrt{377}$, a unit vector perpendicular to $\mathbf{w}_1$ and $\mathbf{w}_2$ would be $\frac{1}{\sqrt{377}} \langle 8, 12, 13 \rangle$.

2.- (a) (*10 points*): Use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -\mathbf{j} + 2\mathbf{k}$.

Recall that the triple scalar product of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is defined as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} = 10.$$

In general, the triple scalar product measures *oriented volume*, but in this case, since the triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forms a right-handed triple, the volume of the parallelepiped spanned by these vector is 10.

(b) (*20 points*): Find the equation of the plane containing the lines given by

$$\frac{x-1}{-2} = y = z + 1 \qquad \text{and} \qquad \frac{x+1}{-2} = y - 1 = z - 2$$

.

In principle, it is *false* that two random lines in space will lie on a plane, however, in this case, the lines have parametric equations

$$\langle x, y, z \rangle = \langle 1, 0, -1 \rangle + t\langle -2, 1, 1 \rangle \quad \text{and} \quad \langle x, y, z \rangle = \langle -1, 1, 2 \rangle + t\langle -2, 1, 1 \rangle$$

in particular, *these lines are parallel.* One tentative, simple option would have been to find the normal vector to the plane by taking the cross product of the direction vectors of the two lines, but this would be $\mathbf{0}$. Instead, we can find 3 points in the prospective plane. Setting $t = 0$, we find $(1, 0, -1)$ and $(-1, 1, 2)$. A third point is for instance, with $t = 1$ in the first line, $(-1, 1, 0)$. Thus

$$\langle 1, 0, -1 \rangle - \langle -1, 1, 0 \rangle = \langle 2, -1, -1 \rangle \quad \text{and} \quad \langle -1, 1, 2 \rangle - \langle -1, 1, 0 \rangle = \langle 0, 0, 2 \rangle$$

are vectors parallel to the plane. Therefore

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 0 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 4\mathbf{j}$$

is a vector normal to the plane, and

$$-2(x+1) - 4(y-1) = 0, \quad \text{or} \quad x + 2y = 1$$

is the equation for the plane.

3.-(*20 points*): Convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.

$$x^2 - y^2 = 2z.$$

(a) In cylindrical coordinates,

$$x^2 - y^2 = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta),$$

thus the ecuation would be

$$r^2\cos(2\theta) = 2z.$$

(b) In spherical coordinates,

$$x^2 - y^2 = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta) = \rho^2\sin^2\phi\cos(2\theta),$$

thus, since $z = \rho\cos\phi$, we get

$$\rho^2\sin^2\phi\cos(2\theta) = 2\rho\cos\phi$$

or

$$\rho\sin^2\phi\cos(2\theta) = 2\cos\phi.$$

4.- Consider the vector-valued function

$$\mathbf{r}(t) = e^t \cos(t)\mathbf{i} + e^t \sin(t)\mathbf{j}$$

(a) (*15 points*) Find the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$ at $t = \dfrac{\pi}{2}$.

Notice that the trace of this vector-valued function is *planar*. Since $\mathbf{r}'(t) = e^t(\cos(t) - \sin(t))\,\mathbf{i} + e^t(\sin(t) + \cos(t))\,\mathbf{j}$, then

$$\begin{aligned}
||\mathbf{r}'(t)|| =& e^t \sqrt{(\cos(t) - \sin(t))^2 + (\cos(t) + \sin(t))^2} \\
=& e^t \sqrt{2\left(\cos^2(t) + \sin^2(t)\right)} = \sqrt{2}e^t,
\end{aligned}$$

therefore $\mathbf{T}(t) = \dfrac{1}{\sqrt{2}}((\cos(t) - \sin(t))\mathbf{i} + (\sin(t) + \cos(t))\mathbf{j})$.

Similarly, $\mathbf{T}'(t) = -\dfrac{1}{\sqrt{2}}((\sin(t) + \cos(t))\mathbf{i} + (\sin(t) - \cos(t))\mathbf{j})$. Thus

$$||\mathbf{T}'(t)|| = \frac{1}{\sqrt{2}}\sqrt{(\sin(t) + \cos(t))^2 + (\sin(t) - \cos(t))^2} = 1.$$

Therefore $\mathbf{N}(t) = \mathbf{T}'(t)$. At $t = \dfrac{\pi}{2}$ we find

$$\mathbf{T}(\pi/2) = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle, \qquad \mathbf{N}(\pi/2) = \langle -1/\sqrt{2}, -1/\sqrt{2}\rangle.$$

(b) (*15 points*) Find the tangential and normal componens of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = \dfrac{\pi}{2}$.

A direct computation shows that $\mathbf{a}(t) = \mathbf{r}''(t) = -2e^t \sin(t)\mathbf{i} + 2e^t \cos(t)\mathbf{j}$. Without any further computations, we can observe that

$$\mathbf{a}(t) = \sqrt{2}e^t\mathbf{T}(t) + \sqrt{2}e^t\mathbf{N}(t)$$

Therefore, $a_{\mathbf{T}}(t) = a_{\mathbf{N}}(t) = \sqrt{2}e^t$. In particular, $a_{\mathbf{T}}(\pi/2) = a_{\mathbf{N}}(\pi/2) = \sqrt{2}e^{\pi/2}$.

5.- Extra credit (*15 points*): Using Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, and Newton's Law of Gravitation

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r}$$

show that $\mathbf{a}$ and $\mathbf{r}$ are parallel, and that $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{L}/m$ is a constant vector. So, $\mathbf{r}(t)$ moves in a *fixed plane*, orthogonal to $\mathbf{L}$ (In physics, $\mathbf{L}$ is know as *angular momentum.* As this problem shows, an intrinsic property of any central force problem is that motion always occurs in a plane. Probably the first documented manifestation of the conservation of angular momentum is *Kepler's second law of planetary motion*).

(a) As a consequence of Newton's second law, we find that $\mathbf{a} = -\dfrac{GM}{r^r}\mathbf{r} = f(r)\cdot\mathbf{r}$, where $f(r)$ is a *scalar function.* Even though this function is not constant, for every value of r we find that the vectors $\mathbf{r}$ and $\mathbf{a}$ are parallel.

(b) Let $\mathbf{L}/m = \mathbf{r} \times \mathbf{r}'$ (notice that here m is just a constant, representing the mass of a moving particle). Then

$$\begin{aligned}
\frac{d(\mathbf{L}/m)}{dt} = \frac{1}{m}\frac{d\mathbf{L}}{dt} = \frac{1}{m}\frac{d}{dt}(\mathbf{r} \times \mathbf{r}') &= \frac{1}{m}(\mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'') \\
&= \frac{1}{m}(\mathbf{r} \times \mathbf{a}) \\
&= \frac{f(r)}{m}(\mathbf{r} \times \mathbf{r}) = 0.
\end{aligned}$$

Therefore, $\mathbf{L}$ is constant.

MAT 203 Spring 2014 Midterm 2

Name: ________________________ SB ID number: ____________

Please circle the number of your recitation:

R01	Tu 2:30pm- 13:23pm	Lgt Engr Lab 152	Xin Zhang
R02	W 10:00am-10:53am	Lgt Engr Lab 152	Xin Zhang
R03	M 12:00pm- 12:53pm	Javits Lectr 101	Shaosai Huang

***************DO NOT WRITE BELOW THIS LINE*****************

1	3	3	4	5	TOTAL

Instructions: The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., then **please raise your hand**.

1.- (a) (*10 points*): Find the gradient of the function $f(x,y) = 9x^2 - 4y^2$ and the maximum value of the directional derivative at the point $P(3,2)$.

Solution: We calculate

$$\nabla f = 18x\mathbf{i} - 8y\mathbf{j}.$$

Recall that the minimum value of the directional derivative for f is attained in the opposite direction of the gradient; thus, since

$$\| \nabla f(3,2) \| = \sqrt{(18 \cdot 3)^2 + (-8 \cdot 2)^2} = \sqrt{3172}$$

the miminum value of the directional derivative at $P(3,2)$ is $-\sqrt{3172}$.

(b) (*5 points*): Find a unit normal vector to the level curve $f(x,y) = 65$ at $P(3,2)$.

Solution: We only need to normalize $\nabla f(3,2) = 54\mathbf{i} - 16\mathbf{j}$. From (a), we found that $\| \nabla f(3,2) \| = \sqrt{3172}$. Therefore

$$\mathbf{u} = \frac{1}{\sqrt{3172}} (54\mathbf{i} - 16\mathbf{j}).$$

(c) (*5 points*): Find the tangent line to the previous level curve at P.

Solution: Since a point in the line is already given to us, all we need is to find a vector parallel to the level curve at $P(3,2)$. This vector will be perpendicular to the gradient there, and therefore we can choose $\mathbf{v} = 16\mathbf{i} + 54\mathbf{j}$ (or any nonzero scalar multiple of it). A parametric equation of the line is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (3 + 16t)\mathbf{i} + (2 + 54t)\mathbf{j}.$$

2.- (*20 points*): Use Lagrange multipliers to maximize $f(x, y) = 2xy$ subject to the constraint $2x + y = 12$, assuming that x and y are positive.

Solution: Letting $g(x, y) = 2x+y$, the constraint is read as the level set $g(x, y) = 12$. We calculate:

$$\nabla f = 2y\mathbf{i} + 2x\mathbf{j}, \qquad \nabla g = 2\mathbf{i} + \mathbf{j}.$$

Thus the equation $\nabla f = \lambda \nabla g$ becomes

$$2y\mathbf{i} + 2x\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j},$$

or equivalently,

$$2y = 2\lambda, \qquad 2x = \lambda.$$

If we replace these in $2x + y = 12$, we find

$$\lambda = 6.$$

Since x and y are assumed to be positive, we finally obtain

$$x = 3, \qquad y = 6.$$

At this point, the constrained maximum is found to be $f(x, y) = f(3, 6) = 36$.

3.-(*20 points*): Find the area of the surface given by $z = 25 - x^2 - y^2$ over the region $R = \{(x,y) : x^2 + y^2 \leq 25\}$. Feel free to use the coordinate system of your choice.

Solution: The region of integration is a polar sector that can be parametrized as

$$R = \{(r,\theta) : 0 \leq r \leq 5,\ 0 \leq \theta \leq 2\pi\}.$$

Thus, it will be convenient to use polar coordinates. We calculate

$$f_x = -2x, \qquad f_y = -2y.$$

Hence, the element of surface area is given as

$$\sqrt{1 + f_x^2 + f_y^2}\,dA = \sqrt{1 + 4x^2 + 4y^2}\,dA = \sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

and the surface area is equal to

$$S = \int_0^{2\pi} \left(\int_0^5 \sqrt{1 + 4r^2}\,r\,dr \right) d\theta$$

Letting $u = 1 + 4r^2$, $du = 8rdr$, we get

$$S = 2\pi \left(\int_1^{101} \sqrt{u}\frac{du}{8} \right) = \frac{\pi}{4} \left. \frac{u^{3/2}}{3/2} \right|_1^{101} = \frac{\pi}{6} \left(\sqrt{(101)^3} - 1 \right).$$

4.- (*20 points*):Use cylindrical coordinates to find the volume of the solid bounded above by $z = 8 - x^2 - y^2$ and below by $z = x^2 + y^2$.

Solution: Notice that both surfaces are circular paraboloids, and thus have z-axial symmetry. They intersect in circle, whose projection to the xy-plane is given by the equation

$$8 - x^2 - y^2 = x^2 + y^2, \quad \text{or equivalently,} \quad x^2 + y^2 = 4.$$

In cylindrical coordinates, the surfaces have equations $z = 8 - r^2$ and $z = r^2$. Therefore, a parametrization of the solid region Q is given in cylindrical coordinates as

$$Q = \{(r, \theta, z) : 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi,\ r^2 \leq z \leq 8 - r^2\}$$

and

$$V = \int_0^2 \left(\int_{r^2}^{8-r^2} \left(\int_0^{2\pi} d\theta \right) dz \right) r dr$$

$$= 2\pi \int_0^2 \left((8 - r^2) - r^2 \right) r dr = 4\pi \int_0^2 (4r - r^3) dr = 4\pi \left(2(2)^2 - \frac{2^4}{4} \right) = 16\pi.$$

5.- EXTRA CREDIT (*20 points*): The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called *Laplace's equation*. Rewrite it in cylindrical coordinates. You must show ALL your work.

Solution: Recall the cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The problem is just a computation based on the chain rule. A trick that simplifies a bit the tons of computations we should perform is to notice that

$$\frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial \theta},$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial \theta},$$

and if we invert this 2×2 system, we obtain

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta},$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$$

similarly,

$$\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta},$$

thus

$$\frac{\partial^2 u}{\partial y^2} = \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta},$$

and miraculously, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r},$$

which finally gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$