MAT 118: Mathematical Thinking
Spring 2016

MAT 118: Mathematical Thinking Spring 2016

General Information

In the course we will explore various applications of mathematics. The main objective is to develop your mathematical thinking and problem solving abilities. During the semester we will work on different real-life mathematical problems such as: determining a winner in elections, finding efficient route, studying population growth etc..

Instructor:

Artem Dudko, artem.dudko@stonybrook.edu
Lectures: MWF 10:00-10:53am, Harriman Hall 137
Office hours: W 11:00-1:00pm, Math Tower 3114, and W 1:00-2:00pm, Math Learning Center, Math Tower S-240A

Recitations:

R01, W 9-9:53am, Harriman Hall 112, Santai Qu
R02, M 1:00-1:53pm, Library W4535, Santai Qu
R03, Th 1:00-1:53pm, Library E4310, Harrison Pugh

Textbook: Excursions in Modern Mathematics, by Peter Tannenbaum (8th edition, preferably)

Topics: Mathematics behind elections, power and sharing (Chapters 1-3); mathematics of getting around and touring (Chapters 5 and 6); population growth models and financial mathematics (Chapters 9 and 10); Fibonacci numbers and the golden ratio (Chapter 13); probabilities and expectations (Chapter 16).

Assignments: There will be weekly homework assignments (with a few exceptions) posted on the course web page due on Friday. You should hand in your assignments to the instructor at the end of Friday classes. Each homework will consist of several problems two or three of which will be graded (but you don't know which, so expected to do all of them). Also, there will be recommended problem sets. The focus of the course is on learning how to recognise, formulate and solve mathematical problems, therefore it is highly recommended that you work on recommended problems as well (even though it is not for grading).

Tests:

Final Exam: Monday, May 16, 8:00am-10:45am, Harriman 137 (our regular classroom). There will be 10 problems and 4 multiple choice questions. For the final exam you need to know everything we learn after midterm 2 (6 problems + 4 multiple choice questions for this part): population models (Chapter 9), financial mathematics (Chapter 10) and Fibonacci numbers (Chapter 13); and the following topics from the material covered by the midterms:
pairwise comparison method, Banzhaf power and lone-chooser method (2 problems);
method of markers, eulerization and nearest neighbor algorithm (2 problems).
Notice, some questions will require a **calculator**.
Review session will be on Wednesday, May 11, 12-2pm in Melville Library E-4315.
Grade improvement possibility: if your final exam grade is better than the average of the midterms it will replace the midterms (and so your final grade will improve).

Midterm II: Friday, April 8, in class. It covers the following material: method of sealed bids (Section 3.5), method of markers (Section 3.6); the mathematics of getting around (Chapter 5), which includes street-routing problems, graphs, Euler's theorems and Fleury's algorithm, eulerization and semi-eulerization; the mathematics of touring (Chapter 6), which includes traveling salesman problem, Hamilton paths and circuits, brute-force algorithm, nearest-neighbor and repetitive nearest neighbor algorithms, cheapest-link algorithm. The format is the same as for midtem I. There will be 4 problems of the same type as homework problems and a few multiple choice type questions. There will be a **review session** on Friday, April 1, 5-7pm in ESS 131.

Midterm I: Friday, February 26, in class. It covers all material we learned before today: Mathematics of elections, Mathematics of power and Mathematics of Sharing (introduction and the lone chooser method). There will be 4 problems of the same type as homework problems and a few multiple choice type questions. There will be a **review session** on Wednesday, February 24, 6-8pm in Melville Library E4320.

Last day of classes: Friday, May 7.

Course grade is computed by the following scheme:
Homework: 20%
Midterms: 40%
Final Exam: 40%

Letter grade cutoffs:
85-100 A
80-85 A-
75-80 B+
65-75 B
60-65 B-

55-60 C+
45-55 C
35-45 D
0-35 F

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

MAT 118 Homework assignments.

The exercises (unless stated otherwise) are from the course book "Excursions in modern mathematics", 8th edition, by Peter Tannenbaum. They can be found at the end of the corresponding chapter. Only 2-3 problems from each assignment will be graded, but you don't know which and are expected to do all of them. It is recommended that you read the corresponding chapters before doing the problems. Recommended problems are not for grading, but for practicing purposes.

HW1 (due on Friday, February 5):
Chapter 1, problems 1, 4, 14, 24, 28.
Recommended problems: 7, 10, 17, 27.

HW2 (due on Friday, February 12):
Chapter 1, problems 34, 37, 44, 49, 53.
Recommended problems: 35, 39, 41, 47.

HW3 (due on Friday, February 19):
Chapter 2, problems 2, 4, 6, 12, 14, 19 (a) and (d).
Recommended problems: 7, 9, 15, 17, 20 (a) and (c).

HW4 (due on Friday, March 4):
Chapter 3, problems 39, 41, 44, 48.
Recommended problems: 38, 43, 45.

HW5 (due on Friday, March 11):
Chapter 3, problems 52, 55. Chapter 5, problems 1, 8, 10.
Recommended problems from Chapter 3: 57; Chapter 5: 2, 9, 11.

HW6 (due on Friday, March 25):
Chapter 5, problems 14, 19, 32, 35, 43, 47.
Recommended problems: 22, 25, 27, 30, 54.

HW7 (due on Friday, April 1):
Chapter 6, problems 4, 13, 28, 31, 39.
Recommended problems: 7, 19, 30, 33, 38.

HW8 (due on Friday, April 15):
Chapter 9, problems 9, 11, 20, 26, 34.

Recommended problems: 3, 6, 13, 35.

HW9 (due on Friday, April 22):
Chapter 9, problems 37, 41, 50, 55, 61.
Recommended problems: 39, 43, 51, 59, 62.

HW10 (due on Friday, April 29):
Chapter 10, problems 2, 13, 22, 26, 33, 38.
Recommended problems: 6, 18, 23, 31, 36.

HW11 (due on Friday, May 6):
Chapter 10, problems 47, 51, 54. Chapter 13, problems 3, 8, 17.
Recommended problems from Chapter 10: 40, 52, 55; Chapter 13: 2, 13, 15.

MAT 118: Mathematical Thinking Spring 2016

General Information
Homework Assignments
Solutions

Solutions

Midterm 1 Solutions
Midterm 2 Solutions
Final Review
List of formulas
Final exam solutions

1. The following table shows the preference schedule for an election with four candidates (A, B, C and D). Use the Borda method to find the complete ranking of the candidates.

Number of voters	7	8	4	3	1
1st	B	A	A	D	B
2nd	C	B	B	C	A
3rd	A	D	C	B	D
4th	D	C	D	A	C

Solution :

	7	8	4	3	1
1st (4)	B(28)	A(32)	A(16)	D(12)	B(4)
2nd (3)	C(21)	B(24)	B(12)	C(9)	A(3)
3rd (2)	A(14)	D(16)	C(8)	B(6)	D(2)
4th (1)	D(7)	C(8)	D(4)	A(3)	C(1)

A: $14+32+16+3+3 = 68$

B: $28+24+12+6+4 = 74$

C: $21+8+8+9+1 = 47$

D: $7+16+4+12+2 = 41$

1st B, 2nd A, 3rd C, 4th D

Answer: B, A, C, D

2. Find the complete ranking of the candidates from the election of problem 1 using the Plurality method. Explain using problem 1 that the Borda method violates the Majority fairness criterion.

Solution: Number of first place votes

A: $8+4=12$, B: $7+1=8$,

C: 0, D: 3

1st A, 2nd B, 3rd D, 4th C

Answer: A, B, D, C.

Notice, there were $7+8+4+3+1=23$ voters. A has $12 > \frac{23}{2}$ first place votes (majority). Majority criterion says A should win. However, in Borda count B is the winner. This violates the Majority criterion.

3. Find the Shapley-Shubik power indexes of the weighted voting system [8 : 7, 5, 2]. You can leave the answer in the form of a simple fraction (like $\frac{2}{7}$).

Solution

List all sequential coalitions, underline pivotal players.

$\langle P_1, \underline{P_2}, P_3 \rangle$

$\langle P_1, \underline{P_3}, P_2 \rangle$

$\langle P_2, \underline{P_1}, P_3 \rangle$

$\langle P_2, P_3, \underline{P_1} \rangle$

$\langle P_3, \underline{P_1}, P_2 \rangle$

$\langle P_3, P_2, \underline{P_1} \rangle$

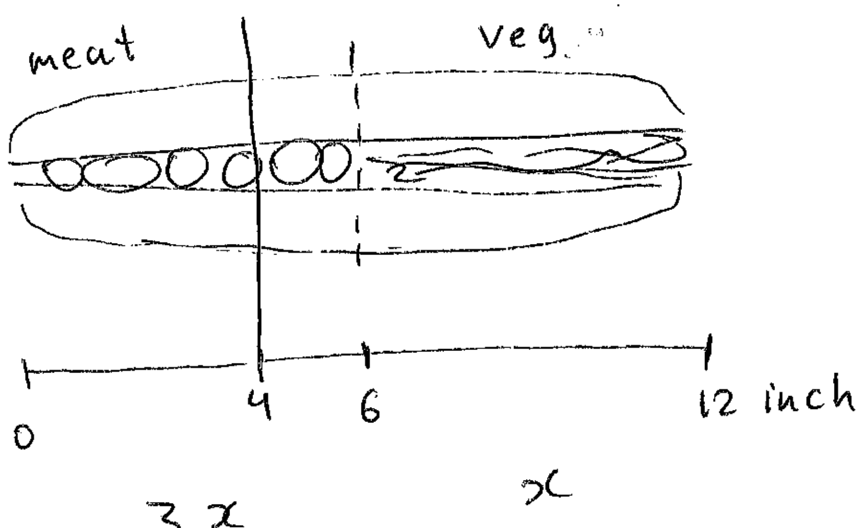
Pivotal counts: $S_1 = 4$, $S_2 = S_3 = 1$

Indexes: $\sigma_1 = \frac{4}{6} = \frac{2}{3}$, $\sigma_2 = \sigma_3 = \frac{1}{6}$

Answer: $\sigma_1 = \frac{2}{3}$, $\sigma_2 = \sigma_3 = \frac{1}{6}$

4. A friend treated Layla and Steve with the half meatball - half vegetarian foot-long sandwich for \$9. They plan to divide it using the divider-chooser method. Layla likes the meatball part three times more than the vegetarian, Steve likes meatball part two times more than the vegetarian. Layla divides the sandwich by one vertical cut and then Steve chooses the part he likes more. Describe the outcome (where does Layla makes the cut and which part Steve chooses) and give the value of the shares to Layla and Steve.

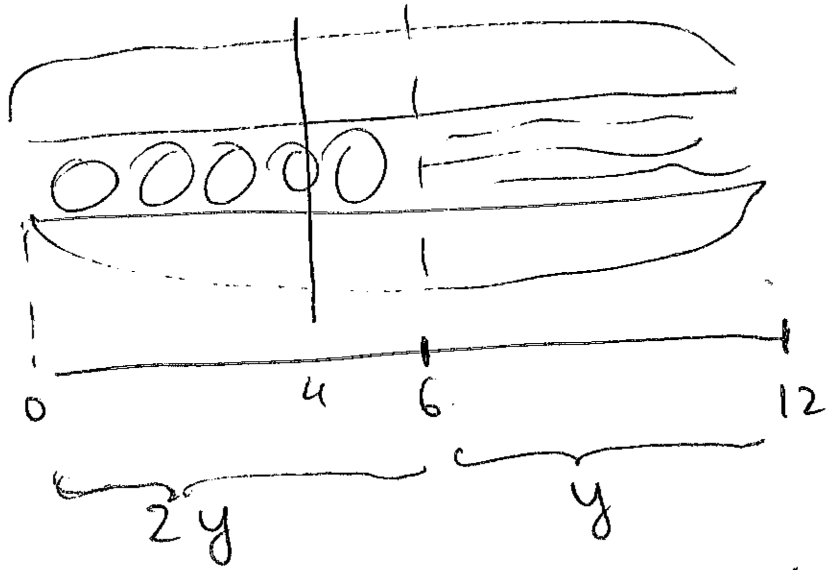
Solution To get the division find out the values of parts for Layla.



If \$x is the value of veg. part for her then \$3x is for meat. In total $x + 3x = 4x = 9$. Thus, $x = \frac{9}{4} = 2.25$.

She needs to cut into equal value $\frac{9}{2} = \$4.5 = 2x$ parts. This is $\frac{2}{3}$ of the meat value, so the cut is at $\frac{2}{3} \cdot 6 = \frac{12}{3} = 4$ inch

To find out what Steve chooses calculate the values of parts for him.



If \$y for veg. then \$2y for meat
In total $y + 2y = 3y = \$9$. Thus,
$y = 3$, $2y = 6$. The $\frac{2}{3}$ of the meat ball
part is worth $\frac{2}{3} \cdot 6 = \$4$ for him,
the rest's worth $9 - 4 = \$5$.
Therefore, Steve chooses the part
containing veg. The rest goes to Layla.
It's worth 4.5\$ to her

Multiple choice answers:

1. d, 2. C, 3. C, 4. b

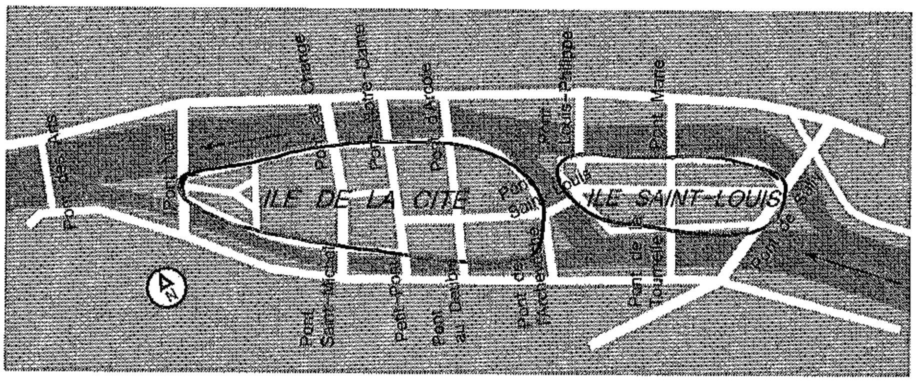
1. Four family members, Anna, Alex, Victor and Lisa, want to split an antique statue they possess using the method of sealed bids. Anna bids 800, Alex bids 1400, Victor bids 800 and Lisa bids 1000. Describe the outcome.

	Anna	Alex	Victor	Lisa
Bids	800	(1400)	800	1000
Fair Shares	200	350	200	250
To(from)	(200)	1050	(200)	(250)
Final settlement	gets 300	Gets statue pays 950	gets 300	gets 350

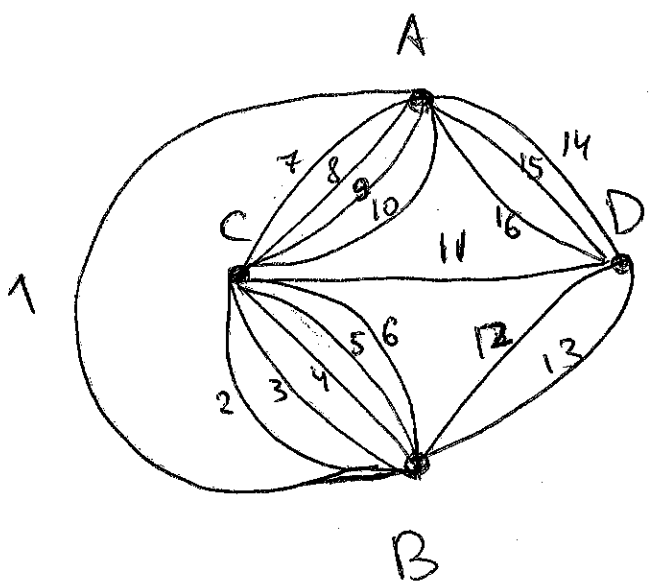
Surplus: 1050 − 200 − 200 − 250 = 400

Fair share of surplus $\frac{400}{4} = 100$

2. A plan of central Paris bridges is shown below. A tourist wants to make a tour passing through each bridge exactly once. If there exists such a tour find it. Model the corresponding street-routing problem using a graph. You don't need to draw the route on the plan, a path or a circuit on the graph is sufficient.



Draw a vertex for each bank and each island, draw an edge for each bridge

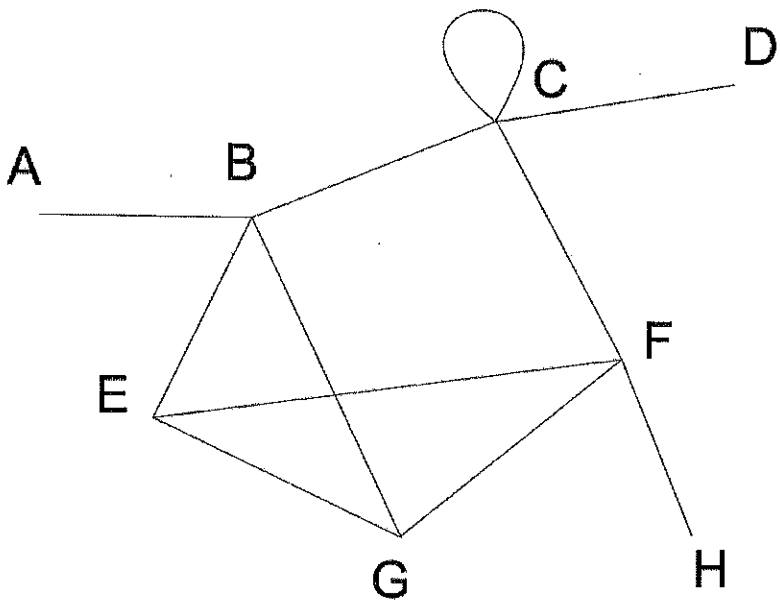


There are no odd vertices => has an Euler circuit

Find it using Fleury's algorithm.

Start from any vertex say A

3. Does the following graph has a Hamilton path? If yes, find one. If no, explain why.

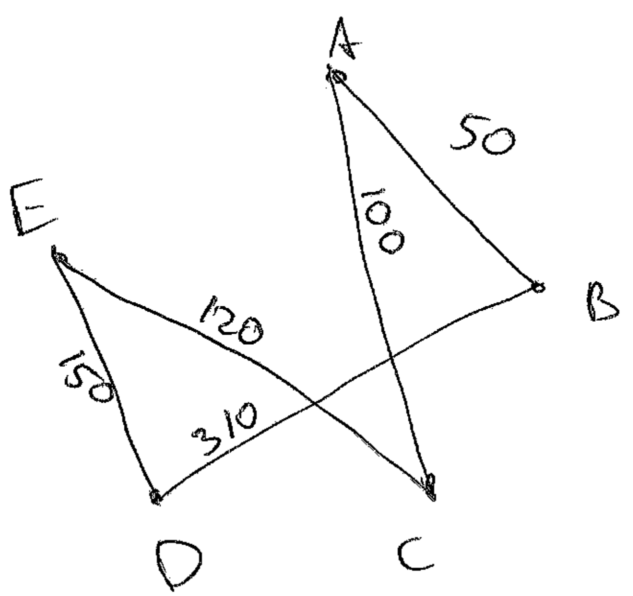


There are 3 vertices which has only one edges A, D and H. Each vertex in a Hamilton path (except the ends) should be connected by at least two edges (one to come in, another to leave the vertex). So A, D, H can't be in the middle of a Hamilton path. They should be the ends. But there are only two ends of a path. Thus, it's impossible to construct a Hamilton path.

4. Distances between 5 cities are given in a table. A rock band plans a tour visiting all the cities starting and ending at A. Find an effective route for the band using the Cheapest Link Algorithm. Find the total length of this route.

	A	B	C	D	E
A	x	50	100	300	220
B	50	x	110	310	200
C	100	110	x	210	120
D	300	310	210	x	150
E	220	200	120	150	x

Construct the path from cheapest links avoiding
1) 3-edges from one vertex
2) partial circuits



Cheapest : AB 50
2nd : AC 100
3rd : ~~BC~~ 110
can't use since create a partial cycle.

~~next~~ next cheapest : EC 120
4th : ED 150
Finish by closing the circuit BD 310

Total: 50+100+120+150+310 = 730

Write starting at A: A→B→D→E→C→A.

5. In each of the following multiple choice questions circle the correct answer.

1) Which of the following is TRUE about the Method of Markers:

a) this is a method for solving the Traveling Salesman Problem;

b) this method guarantees that each of the participants gets a fair share;

c) this method works equally well for any type of fair division games;

d) this method always involves cash.

2) An Euler path is:

a) a path on a graph passing through each vertex exactly ones;

b) the total number of edges starting at a given vertex;

c) a path on a graph passing through each edge exactly ones;

d) a solution of a Traveling Salesman Problem.

3) Which of the following is sufficient for having an Euler's path:

a) the graph is connected;

b) degree of each vertex is even;

c) the graph is connected and there are at most two odd vertices;

d) the graph has no bridges.

4) Which of the following is an approximate method for solving the Traveling Salesman Problem:

a) Method of Sealed Bids;

b) Fleury's Algorithm;

c) Brute Force Algorithm;

d) Repetitive Nearest Neighbor Algorithm.

Spring 2016 MAT 118 Final review

Section 9: Population growth

Sequences

Can be described by:

- words
- several terms $a_1, a_2, a_3, a_4, \dots$
- recursive relation
- general (explicit) formula

Examples

1) given the firs four terms $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ write down the next two terms of the sequence, find the general formula.

Solution Since $2 = \frac{2}{1}$ we see that n'th number in the sequence is $\frac{n+1}{n}$.
The general formula: $\boxed{a_n = \frac{n+1}{n}}$.
After $a_4 = \frac{5}{4}$ we have $a_5 = \frac{6}{5}$, $a_6 = \frac{7}{6}$.

Answer $\frac{6}{5}, \frac{7}{6}$. $a_n = \frac{n+1}{n}$.

2) a sequence starts with $a_1 = 2$ and $a_2 = 0.5$ Each next term is a half difference of the previous and second previous. Write the recursive formula and find a_3, a_4, a_5.

Solution $a_{n+1} = \frac{1}{2}(a_n - a_{n-1})$.

2.

$a_3 = \frac{1}{2}(a_2 - a_1) = \frac{1}{2}(0.5 - 2) = -0.75$

$a_4 = \frac{1}{2}(a_3 - a_2) = \frac{1}{2}(-0.75 - 0.5) = -0.625$

$a_5 = \frac{1}{2}(a_4 - a_3) = \frac{1}{2}(-0.625 - (-0.75)) = 0.0625$

Answer: $a_{n+1} = \frac{1}{2}(a_n - a_{n-1})$

$a_3 = -0.75$, $a_4 = -0.625$, $a_5 = 0.0625$

Linear growth model means that the population changes by the same amount d in equal time periods.

Population sequence is of the form:

- $P_0, P_0 + d, P_0 + 2d, P_0 + 3d, \ldots$
- Recursive formula: $\boxed{P_{n+1} = P_n + d}$
- General formula: $\boxed{P_n = P_0 + nd}$

d is the common difference.

The sequence P_n is an arithmetic sequence

Example The population of deers in a forest was 100 in 2010 and 130 in 2016. Assuming linear growth in which year it will reach 200?

Solution $P_n = P_0 + nd$. Since the question is "which year" the equal time periods are one year. $P_0 = 100$ in 2010. Let P_n be the population n years after 2010.

Then $P_6 = 130$ in 2016.

$P_n = P_0 + n \cdot d$ $\quad n=6$:

$P_6 = P_0 + 6d$

$130 = 100 + 6d$, $6d = 30$, $d = 5$.

$$\boxed{P_n = 100 + 5n}$$

When $P_n = 200$:

$200 = 100 + 5n$ $\quad 5n = 100$, $n = 20$.

Answer in 2030.

Arithmetic sum formula:

$$\boxed{P_0 + P_1 + \dots + P_{n-1} = \frac{P_0 + P_{n-1}}{2} \cdot n}$$

Example Find $3 + 5 + 7 + \dots + 37$.

Solution $P_0 = 3$, $d = 2$

$P_n = P_0 + n \cdot d = 3 + 2n$

We want: $P_{n-1} = 37$

$3 + 2(n-1) = 37$, $\quad 2n = 36$, $n = 18$.

Thus, $3 + 5 + 7 + \dots + 37 = P_0 + P_1 + \dots + P_{17} =$

$\frac{P_0 + P_{17}}{2} \cdot 18 = \frac{3 + 37}{2} \cdot 18 = 360.$

4. **Exponential growth model** means that in equal time periods the population grows by the same constant factor R.
The population sequence has the form:

- $P_0, RP_0, R^2P_0, R^3P_0, \ldots$
- Recursive formula: $\boxed{P_{n+1} = RP_n}$
- General formula: $\boxed{P_n = R^n P_0}$

R is called the common ratio

The sequence P_n is a geometric sequence.

Example In 2010 the population of Manhattan was ≈ 1,585,873, in 2013 ≈ 1,626,159. Assuming exponential growth what will be the population in 2022?

Solution 2013 is 3 years after 2010
2022 is 12 years after 2010.

3 divides 12, so we can chose time period 3 years. P_n = the population $3n$ years after 2010.

In 2010: $P_0 = 1585873$ $\qquad 12:3=4$.

In 2013: $P_1 = 1626159$

In 2022: $P_4 = ?$

We have: $P_1 = RP_0$, so $R = \frac{P_1}{P_0} = \frac{1626159}{1585873} \approx 1.025403.$

$P_4 = R^4 P_0 \approx 1.025403^4 \cdot 1585873 \approx 1{,}753{,}262$.

Answer: about 1,753,262

5. Geometric sum formula:

$$P_0 + RP_0 + R^2P_0 + \dots + R^{n-1}P_0 = \frac{R^n - 1}{R-1} P_0.$$

Example Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{10}}$

Solution $P_0 = 1$, $R = \frac{1}{2}$, $P_n = 1 \cdot \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$

We want: $R^{n-1} P_0 = \frac{1}{2^{10}}$

$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{10}}$, so $n-1 = 10$, $n = 11$.

By the formula,

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{10}} = \frac{R^{11} - 1}{R - 1} P_0 = \frac{\left(\frac{1}{2}\right)^{11} - 1}{\frac{1}{2} - 1} =$$

$$2 - \frac{1}{2^{10}}$$

Answer: $2 - \frac{1}{2^{10}}$

Logistic growth model

Elements: maximal carrying capacity C, p-value of the population $p_n = \frac{P_n}{C}$, natural growth parameter R.

Population satisfies the recurrent formula

$$\boxed{p_{n+1} = R(1 - p_n)p_n}$$

Example $R = 0.8$, $P_0 = 9000$, $C = 10000$. Find P_1, P_2, P_3, P_4, P_5. Predict the long term behavior.

6. <u>Solution</u> p-value $p_0 = \frac{P_0}{C} = \frac{9000}{10000} = 0.9.$

$p_1 = R(1-p_0)p_0 = 0.8 \cdot 0.1 \cdot 0.9 = 0.072$, $P_1 = Cp_1 = 720$

$p_2 = R(1-p_1)p_1 = 0.8 \cdot (1-0.072) \cdot 0.072 \approx 0.0535$, $P_2 = Cp_2 = 535$

$p_3 = R(1-p_2)p_2 \approx 0.0405$, $P_3 = Cp_3 = 405$

$p_4 \approx 0.0311$, $P_4 = 311$

$p_5 \approx 0.0241$. $P_5 = 241$.

We see that the population constantly decreases. Prediction: the population will decrease to extinction.

<u>Answers</u>: 720, 535, 405, 311, 241. Decreases to extinction.

7. **Section 10: Financial mathematics**

Percentages

- $P\%$ as decimal is $p = \frac{P}{100}$
- $P\%$ of B is $p \cdot B = \frac{P \cdot B}{100}$
- Starting with baseline value B adding $P\%$ you get $F = (1+p)B = \left(1 + \frac{P}{100}\right)B$

Example Kevin's Salary was $60,000. First in increased by 5%, then by 8%. But later it decreased by 3%. What's the final salary?

Solution $p_1 = 0.05$, $p_2 = 0.08$, $p_3 = -0.03$.

$B = 60{,}000$.

$F_1 = B(1+p_1) = 60000 \cdot 1.05 = 63000$

$F_2 = F_1(1+p_2) = 63000 \cdot 1.08 = 68040$

$F_3 = F_2(1+p_3) = 68040 \cdot 0.97 \approx 65999 \approx 66000$

Answer: about 66,000

Interest

Basic elements

- Principal P, final value F, total interest $I = F - P$
- Interest rate r
- Term t

Simple interest means that the interest rate is applied to the principal value only and does not accumulate

Simple interest formula:

$$\boxed{F = P(1+tr)}$$

- F — final
- P — principal
- t — term (number of times interest rate applied).
- r — periodic interest rate

Example: A government bond has price \$5000 and the future value after 4 years is \$6000. What is the annual percentage rate?

Solution: $P=5000$, $F=6000$, $t=4$.

$$6000 = 5000\cdot(1+r\cdot 4)$$

$$1000 = 5000\cdot r\cdot 4$$

$$r = \frac{1000}{5000\cdot 4} = 0.05$$

The percentage rate is $100\cdot 0.05 = 5\%$.

Answer: APR is 5%.

9. *Compound interest* means that interest is applied to both the principal value and the previously accumulated interest.

Compound interest formula

$$\boxed{F = P(1+r)^t}$$

Example Borrowing \$5000 for two years with monthly compounding, APR = 3.6%. How much to return?

Solution $P = 5000$, decimal value of APR is $\frac{3.6}{100} = 0.036$. Monthly interest rate is $r = \frac{0.036}{12} = 0.003$. The term is $t = 2 \cdot 12 = 24$ months. Thus

$$F = P(1+r)^t = 5000 \cdot (1.003)^{24} \approx 5373$$

Answer 5373.

Annual percentage yield is the actual annual interest rate. If APR decimal value is r then APY decimal value is $\boxed{\left(1+\frac{r}{n}\right)^n - 1}$ if compounded n times per year.

10. <u>Example</u> Which rate gives more interest:

a) 10% compounded annually or

b) 9.8% compounded monthly

<u>Solution</u> Let's find the effective interest rate APY in b) to compare it with a).

APY decimal value is

$\left(1+\frac{r}{n}\right)^n-1$ with $r=\frac{9.8}{100}=0.098$, $n=12$.

$\left(1+\frac{0.098}{12}\right)^{12}-1 \approx 0.1025$

Thus, in b) APY is $\approx 10.25\% > 10\%$.

So, even thought in a) APR is higher, the accumulated interest in a) is lower than in b).

<u>Answer</u> 9.8% compounded monthly is higher

11. Installment loans

Amortization formula for monthly payment M on a loan with principle P, annual percentage rate r paid after T monthly installments

$$M = P \cdot \frac{p(1+p)^T}{(1+p)^T - 1}$$

where $p = \frac{r}{12}$ is the monthly interest rate

Example Buying a house for $500,000 financing for 27 years at 4% APR. How much will be paid in total?

Solution $r = 4\% = 0.04$

$P = 500{,}000$, $p = \frac{r}{12} = \frac{0.04}{12} \approx 0.00333$

$T = 27 \cdot 12 = 324$.

$$M = 500000 \cdot \frac{0.00333 \cdot 1.00333^{324}}{1.00333^{324} - 1} \approx 2562.38$$

Total payment

$T \cdot M = 324 \cdot 2562.38 \approx 830211$

12. **Section 13: Fibonacci numbers**

Basic facts:

- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
- $F_1 = F_2 = 1$, recursive formula $\boxed{F_{n+1} = F_n + F_{n-1}}$
- general formula (Binet) $F_n = [[(\frac{\sqrt{5}+1}{2})^n / \sqrt{5}]]$
- $\frac{F_{n+1}}{F_n}$ approach the golden ratio $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.6$

Examples 1. Simplify: $2F_{n+1} - F_n + F_{n-2}$

Solution $2F_{n+1} - F_n + F_{n-2} = F_{n+1} + \underbrace{F_{n+1} - F_n}_{F_{n-1}} + F_{n-2}$

$F_{n+1} + \underbrace{F_{n-1} + F_{n-2}}_{F_n} = F_{n+1} + F_n = F_{n+2}$.

Answer F_{n+2}.

2. Find approximate value of $\frac{F_{31}}{F_{30}}$ up to three decimal digits.

Solution $\frac{F_{n+1}}{F_n}$ approaches $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618$ rapidly, so $\frac{F_{31}}{F_{30}} \approx 1.618$

Answer 1.618

13. 3. Find F_{18}

Solution $F_{18} = [[\left(\frac{\sqrt{5}+1}{2}\right)^{18} / \sqrt{5}]] = [[2584.00008...]] =$

2584.

Answer 2584.

golden ratio $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618$ is a positive solution of $\varphi^2 = \varphi + 1$.

$$\boxed{\varphi^n = F_n \cdot \varphi + F_{n-1}}$$

Example Find φ^7 without calculating powers of numbers. Write three decimal digits.

Solution $\varphi^7 = F_7 \cdot \varphi + F_6 = 13 \cdot \frac{\sqrt{5}+1}{2} + 8 \approx$

29.034

answer 29.034

Sum of the first n Fibonacci numbers

$$\boxed{F_1 + F_2 + \dots + F_n = F_{n+2} - 1}$$

Sum of the squares

$$\boxed{F_1^2 + F_2^2 + \dots + F_n^2 = F_n \cdot F_{n+1}}$$

14. **Pairwise-comparison method**

Example Find the complete ranking in the following electionsusing pairwise comparison method

Number of voters	2	6	3	8
1st	A	B	B	D
2nd	B	D	D	C
3rd	D	A	C	A
4th	C	C	A	B

Solution

	Count	Winner
A v B	10 : 9	A
A v C	8 : 11	C
A v D	2 : 17	D
B v C	11 : 8	B
B v D	11 : 8	B
C v D	0 : 19	D

A : 1, B : 2, C : 1, D : 2

Answer B, D share first-second, A, C share third-fourth

Banzhaf power

Basic elements: winning coalition, critical player, critical counts $B_1, B_2, \ldots B_N$, total critical count $T = B_1 + B_2 + \ldots + B_N$, Banzhaf power indexes

$$\beta_1 = \frac{B_1}{T}, \quad \beta_2 = \frac{B_2}{T}, \quad \ldots \quad \beta_N = \frac{B_N}{T}.$$

Example Find Banzhaf power in the weighted voting system

$$[8 : 5, 5, 3, 1]$$

Solution

Winning coalitions	Weights
$\{\underline{P_1}, \underline{P_2}\}$	10
$\{\underline{P_1}, \underline{P_3}\}$	8
$\{\underline{P_2}, \underline{P_3}\}$	8
$\{P_1, P_2, P_3\}$	13
$\{\underline{P_1}, \underline{P_2}, P_4\}$	11
$\{\underline{P_1}, \underline{P_3}, P_4\}$	9
$\{\underline{P_2}, \underline{P_3}, P_4\}$	9
$\{P_1, P_2, P_3, P_4\}$	14

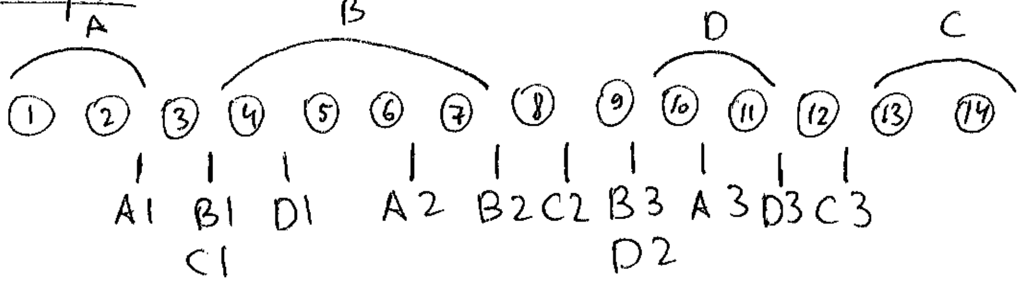
$B_1 = 4, \quad B_2 = 4, \quad B_3 = 4, \quad B_4 = 0, \quad T = 12$

Answer: $\beta_1 = \beta_2 = \beta_3 = \frac{4}{12} = \frac{1}{3}, \ \beta_4 = 0.$

16. <u>Method of markers</u>

Fair division method for fine grained or continuous asset.

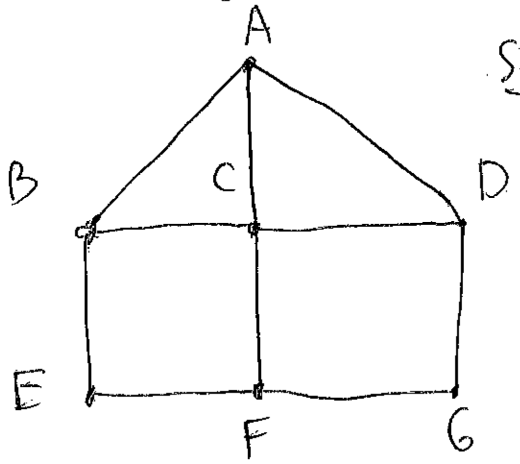
<u>Example</u> Describe the division and the leftover.



<u>Answer</u> A gets 1-2, B gets 4-7, C gets 13-14, D gets 10-11. Leftover: 3, 8, 9, 12.

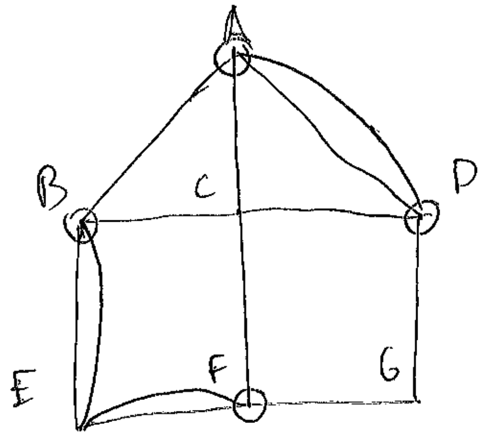
<u>Eulerization</u> is doubling some edges of the graph to make all vertices even degree. After Eulerization in a connected graph it is possible to find a route covering each edge once and returning to the initial vertex. (Euler circuit)

<u>Example</u> Find an effective route for a security starting and ending at A.

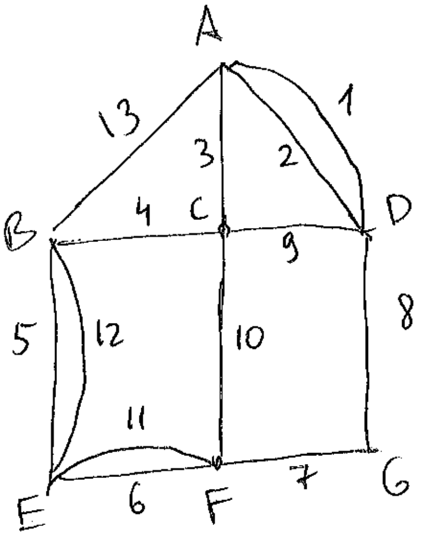


<u>Solution</u> There are 4 odd vertices: A, B, D, F. Need to double some edges to make them even

17.



On the new graph we can find an Euler's path using Fleury's algorithm.

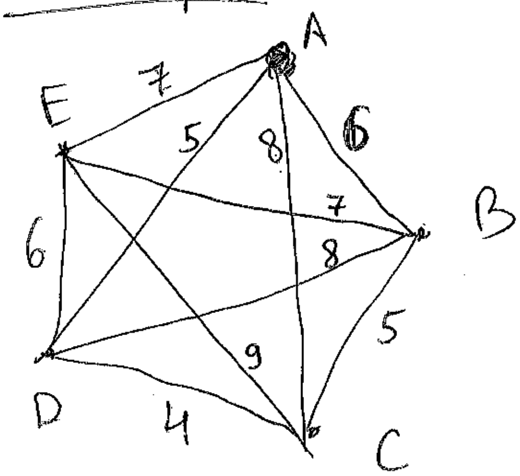


Double edges mean security will need to cover corresponding segments twice

Nearest Neighbor algorithm

An approximate algorithm for solving Traveling Salesman Problem. Each step go to the "closest" among remaining vertexes.

Example Find an effective route visiting each city starting and ending at C. Find the cost



Solution

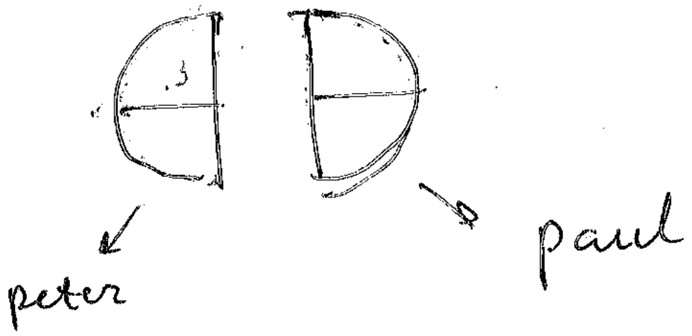
$C \xrightarrow{4} D \xrightarrow{5} A \xrightarrow{6} B \xrightarrow{7} E \xrightarrow{9} C$

Cost $4+5+6+7+9=31$

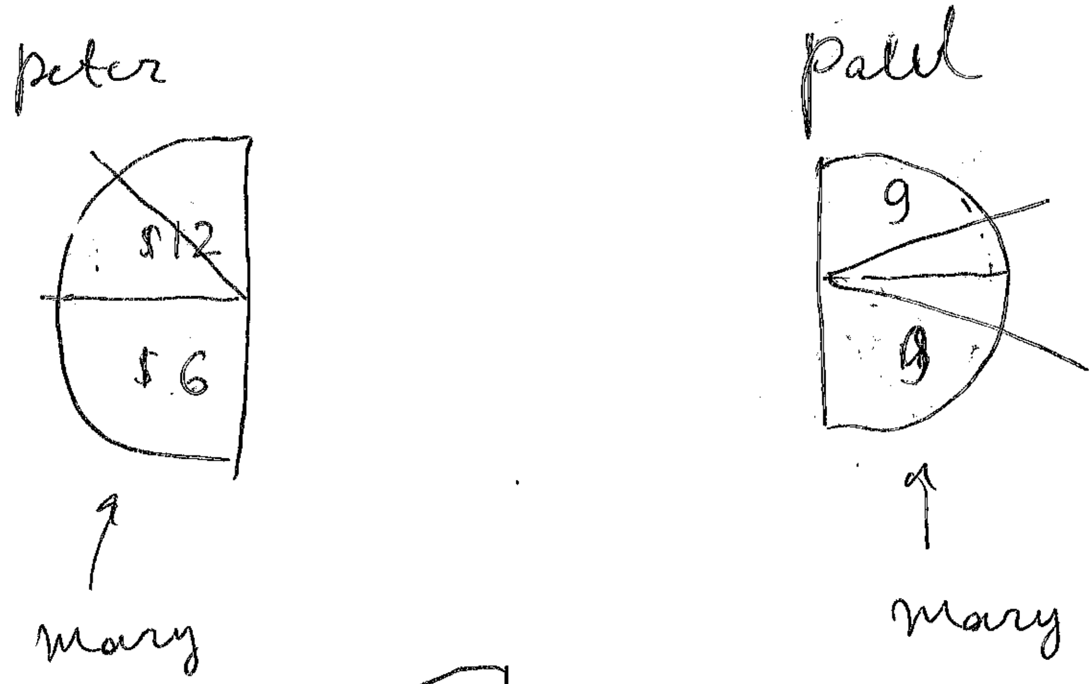
7)


peter
paul
mary
Chocolate
Cheese
$ 24
$ 12
$ 18
$ 18
0
$36

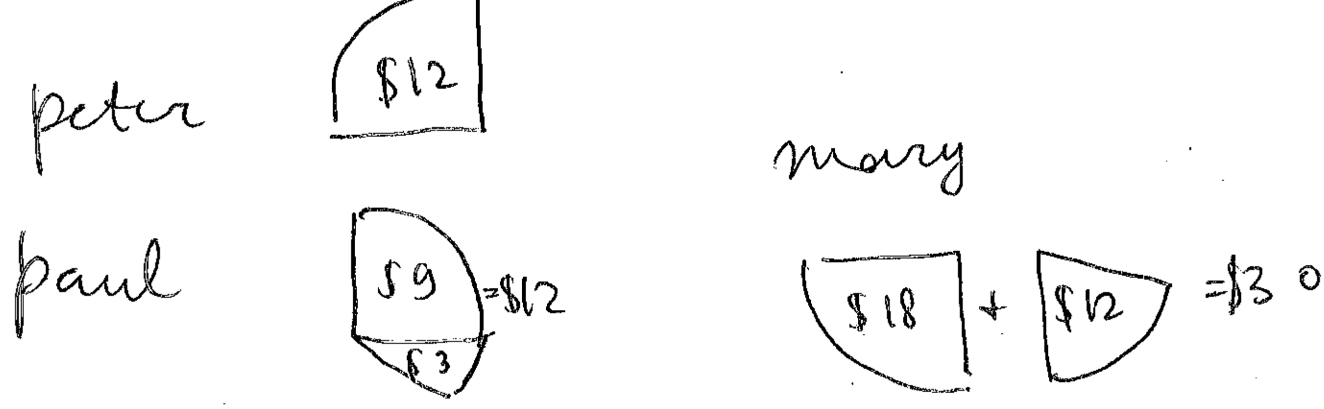
1) peter


peter
paul

2)


peter
$12
$6
mary
paul
9
9
mary

3)


peter
$12
paul
$9
$3
=$12
mary
$18
+
$12
=$30

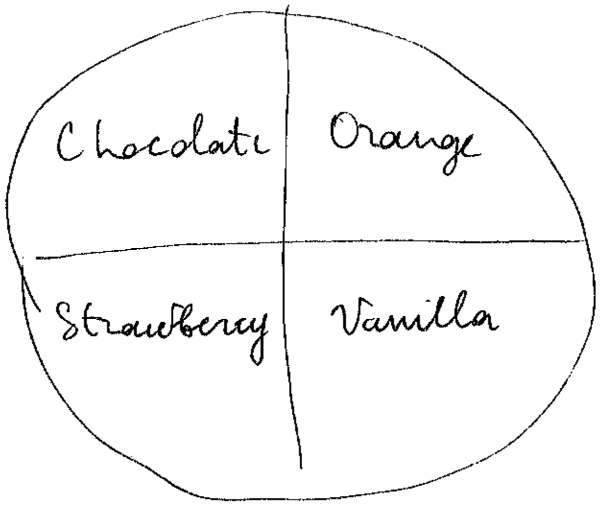
9) N40 p. 98

Arthur, Brian, Carl

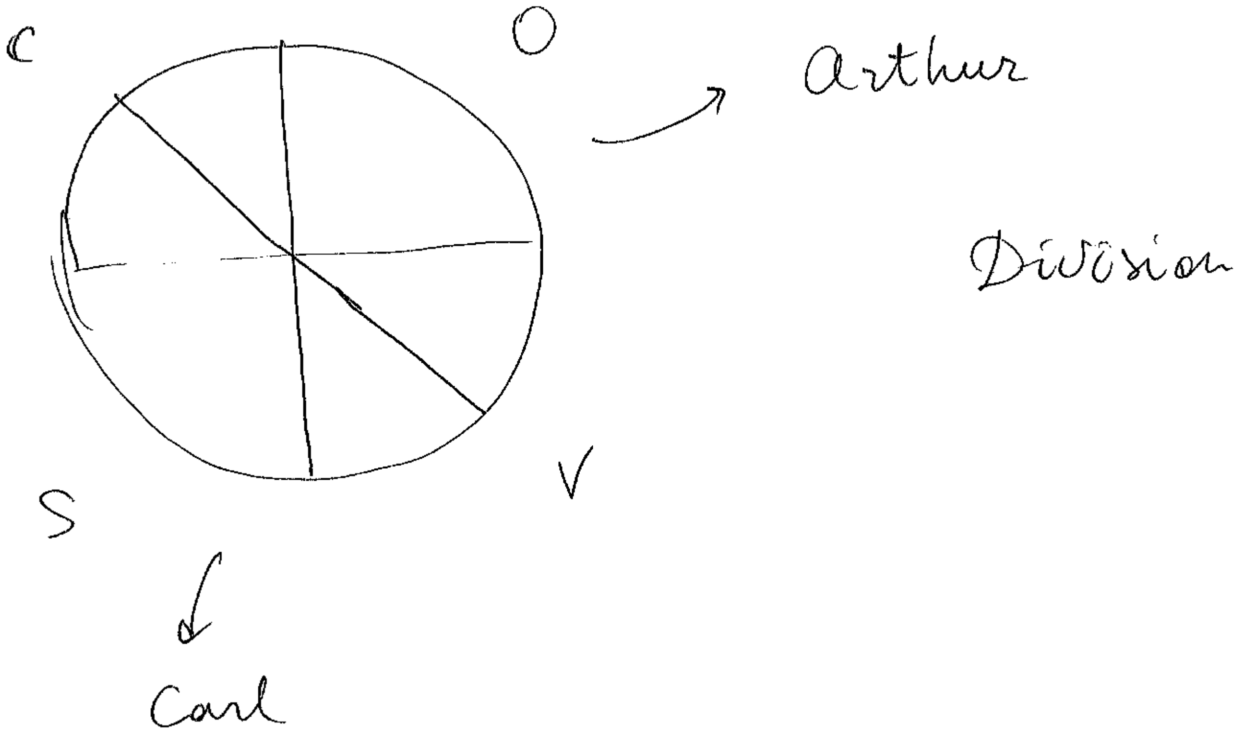
Arthur: loves C = O, hates S, V

Brian: loves C = S, hates O, V

Carl: loves C = V, hates O, S

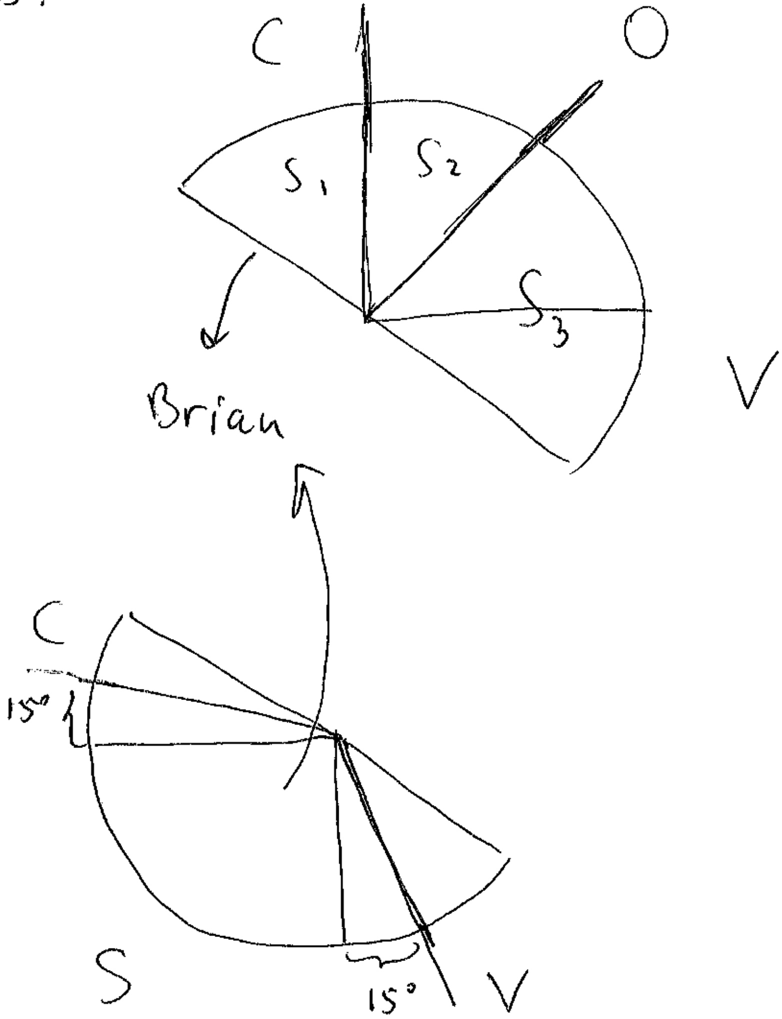


Carl, Arthur are dividers, Carl divides:



10) Subdivisions:

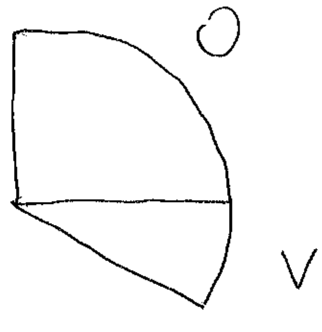
Arthur



Carl

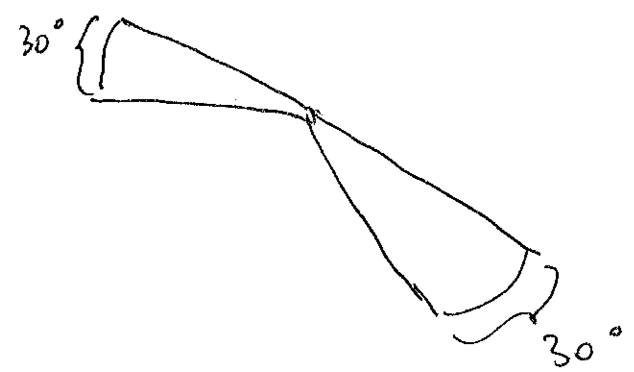
Values:

Arthur



50%

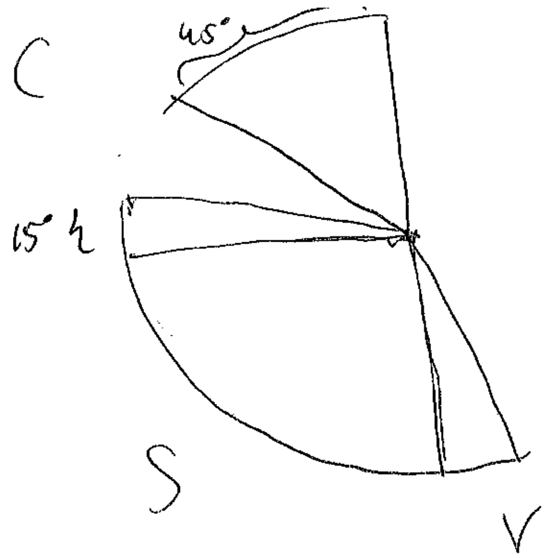
Carl



$\frac{1}{3}$ of C +
$\frac{1}{3}$ of V
$\frac{1}{3}$ of the total value
≈ 33,3%

11)

Brian



Whole S + 60° of C = 150° of the total 180°.

$$\frac{150°}{180°} \cdot 100\% \approx 83,3\%.$$

Formulas to memorize for MAT118 Final exam

Linear growth $P_{n+1} = P_n + d$ $P_n = P_0 + nd$

Exponential growth $P_{n+1} = R P_n$ $P_n = R^n P_0$

Logistic growth $p_n = \frac{P_n}{C}$ $p_{n+1} = R p_n (1 - p_n)$

P % of B is $\frac{P \cdot B}{100} = pB$, where $p = \frac{P}{100}$

Simple interest formula $F = P \cdot (1 + tr)$

t: number of times interest is applied

r: periodic interest rate

Compound interest formula $F = P \cdot (1+r)^t$

Fibonacci numbers: $F_0 = F_1 = 1, \ F_{n+1} = F_n + F_{n-1}$

Binet's formula $F_n = \left[\left[\left(\frac{1+\sqrt{5}}{2}\right)^n / \sqrt{5} \right]\right]$

Sum of Fibonacci numbers $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$,

Sum of squares of Fibonacci numbers $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n \cdot F_{n+1}$

MAT 118 SPRING 2016 FINAL EXAM

NAME : ID :

RECITATION : (M, W or Th)

THERE ARE 10 PROBLEMS, 16 POINTS EACH
AND 4 MULTIPLE CHOICE QUESTIONS, 10 POINTS EACH
SHOW YOUR WORK
DO NOT TEAR-OFF ANY PAGE

1		16pts
2		16pts
3		16pts
4		16pts
5		16pts
6		16pts
7		16pts
8		16pts
9		16pts
10		16pts
11		40pts
Total		200pts

1. The following table shows a preference schedule for an election with four candidates (A, B, C and D). Use the pairwise comparison method to find the complete ranking of the candidates.

Number of voters	3	5	4	1	6
1st	C	A	D	D	A
2nd	A	B	B	C	D
3rd	D	D	C	A	B
4th	B	C	A	B	C

Solution:

Pair to compare	Count	Winner of the pair
A v B	15 : 4	A
A v C	11 : 8	A
A v D	14 : 5	A
B v C	15 : 4	B
B v D	5 : 14	D
C v D	3 : 16	D

A: 3pts, B: 1pt, C: 0 pt, D: 2pt.

Answer A is 1st, D is 2nd, B is 3rd, C is 4th

2. Find the Banzhaf power indexes of the weighted voting system [10 : 8, 6, 2, 1]. You can leave the answer in the form of a simple fraction (like $\frac{2}{7}$).

Solution

Winning coalition	Weight
$\{P_1, P_2\}$	14
$\{P_1, P_3\}$	10
$\{P_1, P_2, P_3\}$	16
$\{P_1, P_2, P_4\}$	15
$\{P_1, P_3, P_4\}$	11
$\{P_1, P_2, P_3, P_4\}$	17

Critical counts:

$B_1 = 6$, $B_2 = 2$, $B_3 = 2$, $B_4 = 0$

Total count $T = 6 + 2 + 2 = 10$.

Banzhaf powers:

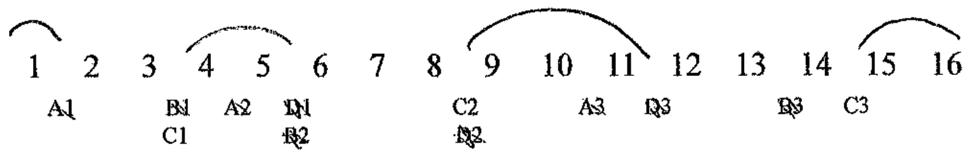
$\beta_1 = \frac{B_1}{T} = \frac{6}{10} = 0.6$

$\beta_2 = \beta_3 = \frac{2}{10} = 0.2$

$\beta_4 = 0$.

Answer: $\beta_1 = 0.6$, $\beta_2 = \beta_3 = 0.2$, $\beta_4 = 0$.

3. Four kids Andrew, Bobby, Clare and Daisy are dividing a stack of 16 candies using the method of markers. They place the markers as shown on the picture (candies are numbered from 1 to 16). Describe a possible outcome (the shares of each child and the leftover). You don't need to describe the division of the leftover between the kids.

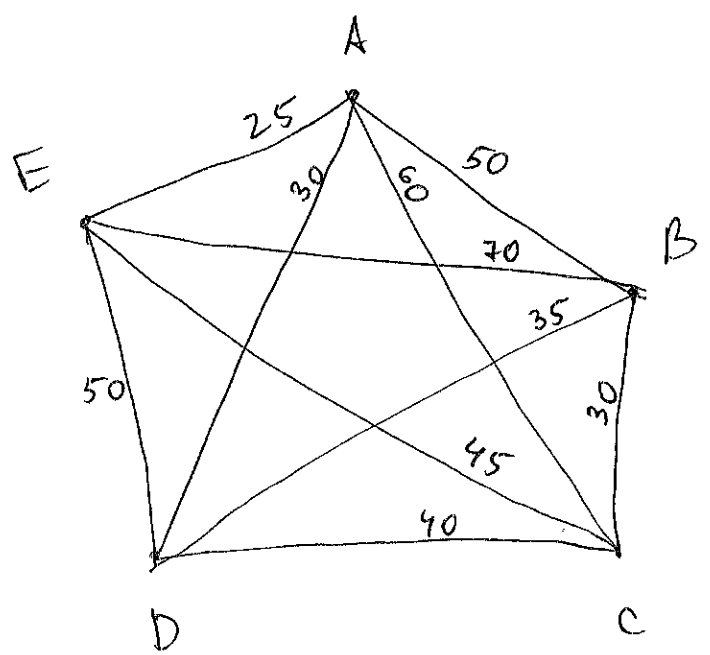


<u>Answer</u> Andrew gets 1,
Bobby gets 4-5,
Clare gets 15-16,
Daisy gets 9-11.

The leftover is: 2-3, 6-8, 12-14.

4. Prices of traveling by train between 5 cities (Asmond, Brown, Chatter, Doorville and Eagletown) are given in a table. A salesman wants to visit all the cities starting at Chatter and returning to Chatter at the end. Find an effective route for the salesman using the nearest neighbor algorithm. Find the total cost of this route.

	A	B	C	D	E
A	x	50	60	30	25
B	50	x	30	35	70
C	60	30	x	40	45
D	30	35	40	x	50
E	25	70	45	50	x



$C \xrightarrow{30} B \xrightarrow{35} D \xrightarrow{30} A \xrightarrow{25} E \xrightarrow{45} C$

Total cost:

$30+35+30+25+45=165$

answer. C, B, D, A, E, C, total cost 165.

5. A scientist investigates a colony of bacteria that grows according to the linear growth model. The colony started with 40 bacteria. After 3 hours it became 100 bacteria. What will be the size of the colony after another 10 hours?

Linear growth: $P_n = P_0 + nd$.

Given: $P_0 = 40$, $P_3 = 100$.

Take $n = 3$ in the formula:

$P_3 = P_0 + 3d$,

$100 = 40 + 3d$,

$60 = 3d$, $d = 20$.

Thus, $P_n = P_0 + nd = 40 + n \cdot 20$.

Aften another 10 hours (that is 13 hour after start):

$P_{13} = 40 + 13 \cdot 20 = 300$.

Answer: 300 bacteria.

6. A population of fish in a pond grows according to the logistic model with the natural growth parameter $r = 2.5$, carrying capacity 1000 fish and initial population 350 fish. Compute how many fish there will be in a pond after one, two, three and four years. Predict what will be the long term behavior of the population.

Logistic model: $p_{n+1} = r\cdot(1-p_n)\cdot p_n$,

where $p_n = \frac{P_n}{C}$ (and so $P_n = p_n \cdot C$)

We have: $P_0 = 350$, $C = 1000$, $r = 2.5$.

Thus, $p_0 = \frac{350}{1000} = 0.35$

$p_1 = r(1-p_0)\cdot p_0 = 2.5\cdot(1-0.35)\cdot 0.35 = 0.56875$,

$P_1 = p_1 \cdot C \approx 569$

$p_2 = r(1-p_1)\cdot p_1 = 2.5\cdot(1-0.56875)\cdot 0.56875 \approx 0.61318$

$P_2 = p_2 \cdot C \approx 613$

$p_3 = r(1-p_2)\cdot p_2 \approx 2.5\cdot(1-0.61318)\cdot 0.61318 \approx 0.59298$

$P_3 = p_3 \cdot C \approx 593$

$p_4 = r(1-p_3)\cdot p_3 \approx 2.5(1-0.59298)\cdot 0.59298 \approx 0.60339$

$P_4 = p_4 \cdot C \approx 603$.

We see that the p-value approach 0.6 and the population approach 600.

Answer: 569, 613, 593, 603. Approach 600.

7. The price of a certain product was $120. First the price increased by 5%, then after some time it decreased by 15%, then later it again increased by 5%. In percents, how much in total the price changed?

Solution 1:

$B = 120$

after increase by 5%:

$F_1 = 120 \cdot (1 + \frac{5}{100}) = 126.$

after decrease by 15%:

$F_2 = 126 \cdot (1 - \frac{15}{100}) = 107.1$

after increase by 5%:

$F_3 = 107.1 \cdot (1 + \frac{5}{100}) = 112.455 \approx 112.46$

In percents: $\frac{F_3}{B} \cdot 100 = \frac{112.46}{120} \cdot 100 \approx 93.7\%$

Final price is 93.7 % of the initial.

Thus, the price decreased by

$100 - 93.7 = 6.3\%$.

Solution 2 $p_1 = \frac{5}{100} = 0.05$, $p_2 = -\frac{15}{100} = -0.15$,

$p_3 = \frac{5}{100} = 0.05$.

The price changes by factor

$(1+p_1)(1+p_2)(1+p_3) = (1+0.05)(1-0.15)(1+0.05) \approx 0.937$

The final price is 93.7 % of the initial.

Decreased by $\approx 100 - 93.7 = 6.3\%$

Answer decreased by 6.3%

8. Jim wants to put some money in a trust fund for his newborn grandson Michael under 5% APR compounded annually. He wants the amount on the fund to be $10,000 when Michael turns 20. How much money should Jim put in the trust fund?

Compounded interest:

$$F = P \cdot (1+r)^t$$

$t = 20$, $r = \frac{5}{100} = 0.05$, $F = 10,000$

$P = ?$

We have:

$$10000 = P \cdot (1+0.05)^{20} \approx 2.653 \cdot P$$

$$P \approx \frac{10000}{2.653} \approx 3769.32.$$

Answer: Jim needs to put $3769.32 in the trust fund.

9. *a*) Write down the first ten Fibonacci numbers.
b) Find F_{22} using Binet's formula.

a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

b) $F_n = \left[\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right]\right]$

$F_{22} = \left[\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{22}}{\sqrt{5}} \right]\right] = [[17711.0000113...]] =$

17711

10. Find the sum of the first 20 Fibonacci numbers.

$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$

$F_1 + F_2 + F_3 + \ldots + F_{20} = F_{22} - 1 =$

$17711 - 1 = 17710.$

Answer: 17710

11. In each of the following multiple choice questions circle the correct answer.

1) The exponential growth population model is characterized by the following:

a) population sequence is always increasing;

b) the difference between consecutive sizes of population stays the same;

c) the ratio of consecutive sizes of population stays the same;

d) the population lives in a bounded habitat.

2) Which of the following is TRUE about the logistic population model:

a) the population sequence may admit a random behavior;

b) the population sequence is always decreasing;

c) the population sequence admits a general (explicit) formula;

d) this model describes a population living in an unbounded habitat.

3) Which of the following statements is FALSE:

d) the amount of money borrowed from a lender is called the principal value;

b) simple interest is applied only to the principle value;

c) compound interest is applied both to the principle value and previously accumulated interest;

(d) annual percentage yield is a type of the interest compounded once per year.

4) Which of the following statements is TRUE:

a) all Fibonacci numbers are even;

b) 17 is a Fibonacci number;

(c) the ratio of consecutive Fibonacci numbers approaches the golden ratio;

d) Fibonacci sequence is an example of a linear growth model.