MAT 118: Introduction to Mathematical Thinking

Syllabus

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MAT 118, Spring 2011
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COURSE DESCRIPTION

This course is intended for those students who want a broad introduction to interesting mathematics, but who only intend to take a single semester of college math. We will focus on topics outside the traditional high school curriculum and college calculus sequence, with an emphasis on problem solving. The course gives students an appreciation for the intellectual scope of mathematics.

TEXTBOOK

The official textbook for this class is *Excursions in Modern Mathematics*, by Peter Tannenbaum, seventh edition. While this textbook will be helpful as a reference, it is **not required**. We will follow several chapters of the book closely, but we will also make occasional departures from the text material.

HOMEWORK

Homework is an essential part of this class. A problem set will be assigned each week to be turned in on Friday at the beginning of class. These problems will be posted on the class website.

GRADING

There will be two midterm exams and a final exam. In addition to the exams, weekly homework will be assigned and collected. Final course grades will be based on this breakdown:

30 % final exam, 20 % each midterm exam, 30 % homework

The exam dates are as follows:

First Midterm: Friday, February 25, in class
Second Midterm: Friday, April 1, in class
Final Exam: Wednesday, May 18, 2:15 PM-4:45 PM

DISABILITIES

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services at http://studentaffairs.stonybrook.edu/dss/ or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

http://www.stonybrook.edu/ehs/fire/disabilities.shtml

ACADEMIC INTEGRITY

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

http://www.stonybrook.edu/uaa/academicjudiciary/

CRITICAL INCIDENCE MANAGEMENT

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

Name: Solutions

1. (a) (15 pts) Consider the election given by the following preference schedule:

Number of voters	5	4	2
First choice	A	B	C
Second choice	C	C	B
Third choice	B	A	A

Find the winner of the election using the run-off method. Show your work.

First-place votes
A - 5
B - 4
C - 2

C has the fewest first-place votes, so eliminate C. Then you have:

First-place votes
A - 5
B - 6

Now, B has the most first-place votes, so $\boxed{\text{B wins}}$.

(b) (15 pts) Rank the candidates in the election above using the recursive plurality method. Show your work.

First-place votes
A - 5
B - 4
C - 2

A has the most first-place votes, so A wins. Now, eliminate A. Then you have:

First-place votes
B - 4
C - 7

Now C has the most first-place votes, so C comes in second. It follows that the ranking is

$\boxed{\text{First - A} \quad \text{Second - C} \quad \text{Third - B}}$

(c) (15 pts) Consider the election given by the following preference schedule:

Number of voters	3	2
First choice	A	B
Second choice	B	C
Third choice	C	A

Use this example to show that the Borda Count method does not satisfy the Independence of Irrelevant Alternatives Criterion.

A receives $3(3)+2(1) = 11$ points.
B receives $3(2)+2(3) = 12$ points.
C receives $3(1)+2(2) = 7$ points.

So B wins.

Now, suppose that C withdraws from the race. Then we have:

A receives $3(2)+2(1) = 8$ points.
B receives $3(1)+2(2) = 7$ pts.

So A wins.

Since C's withdrawal affects the outcome of the race, the Borda Count does not satisfy IIA.

2. (a) (15 pts) Ten legislative seats are to be apportioned to the 4 states whose populations are listed below.

State	A	B	C	D
Population	150	235	255	460

Find the apportionment of seats given by Hamilton's method. Show your work.

The standard divisor is $\frac{1100}{10} = 110$, so the standard quotas are:

A	B	C	D
$1\frac{5}{11}$	$2\frac{1.5}{11}$	$2\frac{3.5}{11}$	$4\frac{2}{11}$

The state with the highest fractional part is A, so the final apportionment is:

A	B	C	D
2	2	2	4

(b) (15 pts) Suppose now that the number of seats available increases to 11. Find the new apportionment given by Hamilton's method. Use this example to prove that Hamilton's method is subject to the Alabama paradox.

The standard divisor is $\frac{1100}{11} = 100$, so the standard quotas are:

A	B	C	D
1.5	2.35	2.55	4.6

The 2 states with the highest fractional parts are C and D, so the final is:

A	B	C	D
1	2	3	5

Notice that A lost a seat, simply because the total of seats was increased. This is an example of the Alabama paradox.

(c) (15 pts) Ten legislative seats are to be apportioned to the 3 states whose populations are listed below.

State	A	B	C
Population	860	74	66

Find the apportionment of seats given by Jefferson's method. Use this example to show that Jefferson's method violates the quota rule. Show your work. (Hint: try using the divisor $D = 80$.)

The standard divisor is $\frac{1000}{10} = 100$, so the standard quotas are:

A	B	C
8.6	.74	.66

Using the divisor $D = 80$, however, the modified quotas are:

A	B	C
$10\frac{3}{4}$	$\frac{74}{80}$	$\frac{66}{80}$

Rounding each of these down, we obtain the apportionment:

A	B	C
10	0	0

Notice that A receives 10 seats, which is more than its upper quota of 9. It follows that Jefferson's method violates the quota rule.

3. (10 pts) Do **ONE** of the following two problems. Make sure to explain your answer as best as possible.

(a) Is Adams' method subject to the Alabama paradox? Prove your answer.

Claim: Adams' method is not subject to the Alabama paradox.

Proof: Suppose that we increase the number of available seats. We want to show that no state loses a seat. Let A be a state with population P. Let D_1 be our original "suitable" divisor, and D_2 the "suitable" divisor after we increase the number of seats. Notice that $D_1 \geq D_2$, so $\frac{P}{D_1} \leq \frac{P}{D_2}$. Rounding both sides of the inequality up, we see that $\left\lceil \frac{P}{D_1} \right\rceil \leq \left\lceil \frac{P}{D_2} \right\rceil$. But $\left\lceil \frac{P}{D_1} \right\rceil$ is the number of seats originally apportioned to state A by Adams' rule, and $\left\lceil \frac{P}{D_2} \right\rceil$ is the number of seats apportioned to state A by Adams' rule after we increase the number of available seats. It follows that state A does not lose a seat.

(b) For two days of the week A and B, we write $A \leq B$ if day B occurs anywhere from 0 to 3 days after day A. So, for example, Friday $\leq$ Monday, since Monday is 3 days after Friday, but Friday $\not\leq$ Tuesday, since Tuesday is 4 days after Friday. Is this a total ordering on the days of the week? Prove your answer.

The transitive property says that, for any 3 days A, B, and C, if $A \leq B$ and $B \leq C$, then $A \leq C$. This means that if day B occurs within 3 days after day A, and day C occurs within 3 days after day B, then day C occurs within 3 days after day A.

Note, however, that

Monday $\leq$ Wednesday, and Wednesday $\leq$ Friday, but

Monday $\not\leq$ Friday. So this is not a total ordering on the days of the week.

MAT118 Homework 1

Due Friday, February 11

1. Consider the election given by the following preference schedule:

Number of voters	14	10	8	4	1
First choice	A	C	D	B	C
Second choice	B	B	C	D	D
Third choice	C	D	B	C	B
Fourth choice	D	A	A	A	A

   Find the winner of the election for each of the methods we discussed in class (Plurality, Run-Off, Sequential Run-Off, Borda Count). Show your work.

2. Show that the Borda Count method does not satisfy the Independence of Irrelevant Alternatives Criterion by constructing a counterexample. (You must explain why your example is a counterexample in order to receive credit.)

3. Does a dictatorship satisfy the Unanimity Criterion? How about the Independence of Irrelevant Alternatives Criterion? Explain why or why not.

MAT118 Homework 1

Due Friday, February 11

1. Consider the election given by the following preference schedule:

Number of voters	14	10	8	4	1
First choice	A	C	D	B	C
Second choice	B	B	C	D	D
Third choice	D	D	B	C	B
Fourth choice	C	A	A	A	A

   Rank the candidates using both the extended plurality and recursive plurality methods. Show your work.

2. Explain, in your own words, how the recursive dictatorship ranking system works.

3. For each of the following true/false statements about two candidates, determine whether or not it is a total ordering. If it is not a total ordering, which of the four properties does it satisfy, and which does it fail to satisfy?

   (a) $A \clubsuit B$ – Candidate A is precisely as smelly as candidate B.

   (b) $A \bigtriangledown B$ – Candidate A is at least as old as candidate B. (Assume that no two distinct candidates are the exact same age.)

   (c) $A \star B$ – Candidate A is a rock star.

MAT118 Homework 3

Due Friday, February 25

1. A transportation company operates 5 different bus routes, each one traveling to a different New York City borough. The company owns exactly 40 buses. The number of buses apportioned to each route is based on the number of passengers riding that route, which are as follows:

Route	Manhattan	Brooklyn	Bronx	Queens	Staten Island
Riders (in 100s)	1408	1048	648	548	348

   Find the apportionment of buses to each route using the following methods:

   (a) Hamilton's Method

   (b) Jefferson's Method (use $D = 93$)

   (c) Adams' Method (use $D = 108$)

   (d) Webster's Method (use $D = 99.65$)

   (e) Huntington-Hill Method (use $D = 100$)

2. One hundred legislative seats are to be apportioned to the 3 states whose populations are listed below.

State	A	B	C
Population	9010	580	410

   Use this example to show that Adams' method violates the quota rule.

3. Prove that Adams' method cannot produce upper-quota violations. If you were a member of congress, would you prefer Jefferson's method or Adams' method? Why?

MAT118 Homework 4

Due Friday, March 4

1. Four friends each own different portions of their internet start-up company. Mark owns 5000 shares of stock, Steve owns 4000, Henry owns 3000, and Al owns 1000. Each of the friends receives as many votes as shares of stock he owns, and a majority of the 13000 votes are required for a resolution to pass.

   (a) Compute the Banzhaf power index of each of the 4 friends.

   (b) Compute the Shapley-Shubik power index of each of the 4 friends.

2. In this simplified version of the US federal system, there are 5 voters: 2 representatives, 2 senators, and the president. In order for a resolution to pass, it requires the support of at least one representative, at least one senator, and the president.

   (a) Compute the Banzhaf power index of each of the 5 voters.

   (b) Compute the Shapley-Shubik power index of each of the 5 voters.

   (c) Show that this system is not swap robust. Conclude that it is not a weighted voting system.

3. In the system to amend the Canadian constitution, each of the 10 provinces gets a vote. In order for an amendment to pass, it requires the support of at least 7 of the 10 provinces, and the total population of the supporting provinces must be at least 50 % of the total population of Canada.

   (a) Show that this system is swap robust.

   (b) Use the chart below to show that this system is not trade robust. Conclude that it is not a weighted voting system.

Province	Population (%)	Province	Population (%)
Ontario	38	Saskatchewan	3
Quebec	23	Nova Scotia	3
British Columbia	13	New Brunswick	2
Alberta	11	Newfoundland	1
Manitoba	4	Prince Edward Island	0

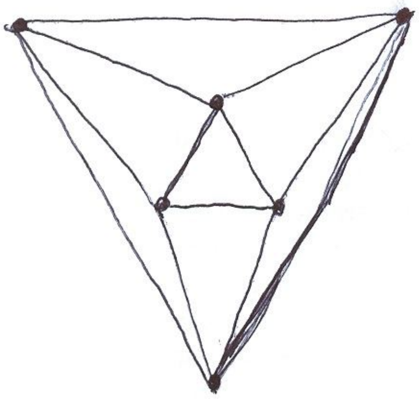
MAT118 Homework 5

Due Friday, March 11

1. For each of the items below, determine whether a graph exists with the described properties. If such a graph exists, draw an example. If not, explain why not.
   (a) A graph with 3 vertices, all of degree 3.
   (b) A graph with 4 vertices, all of degree 3.
   (c) A connected graph with 4 vertices, all of degree 1.
2. Draw an example of a connected graph such that every edge is a disconnecting edge.
3. For each of the graphs on the following page, determine whether the graph has:
   (a) An Euler walk.
   (b) An Euler circuit.

   Show your work.

a)



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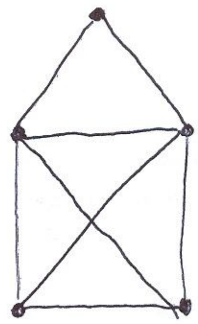

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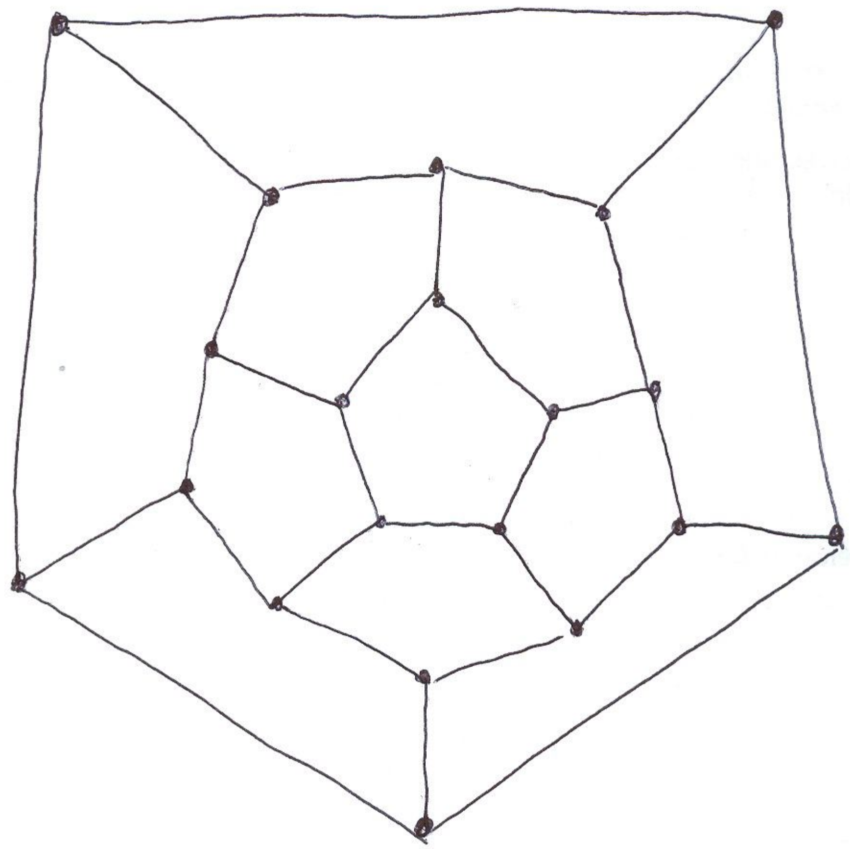
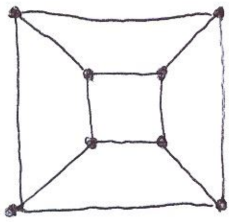


MAT118 Homework 6

Due Friday, March 18

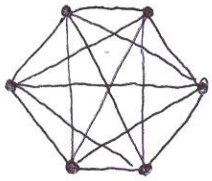
1. For each of the connected planar graphs below, count the number of vertices, edges, and faces. Verify that the Euler characteristic of each one is 2.

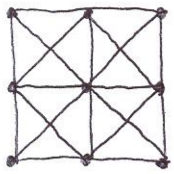
a) 

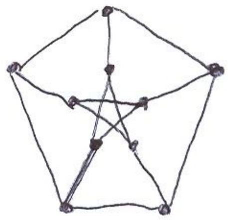
b) 

c) 

2. For each of the graphs below, determine whether or not it is planar. If the graph is planar, draw a planar realization of it. If not, use one of our theorems from class to show that it is not.

a) 

b) 

c) 

3. Last year, a major city attempted to hire former SBU professor and mathematical sculptor George Hart to build sculptures of "new" Platonic graphs. Explain why Dr. Hart had to turn them down.

MAT118 Homework 7

Due Friday, March 25

1. Draw an example of:

   (a) A planar graph with no loops or multiple edges that is not 3-colorable

   (b) A graph with no loops or multiple edges that is not 4-colorable (Note: your example can't be planar!)

   (c) A non-planar graph with no loops or multiple edges that is 2-colorable

2. 25 students are sitting at their desks, which are arranged in a 5×5 grid. Their teacher asks them to all stand up and move to an adjacent desk. Each student may move to the desk immediately to her right, immediately to her left, immediately in front of her, or immediately behind her.

   (a) Draw a graph in which each desk is represented by a vertex, and two vertices are connected by an edge if the two corresponding desks are adjacent.

   (b) Show that this graph is 2-colorable.

   (c) Use your answer to part (b) to show that it is impossible for the students to do what their teacher asked.

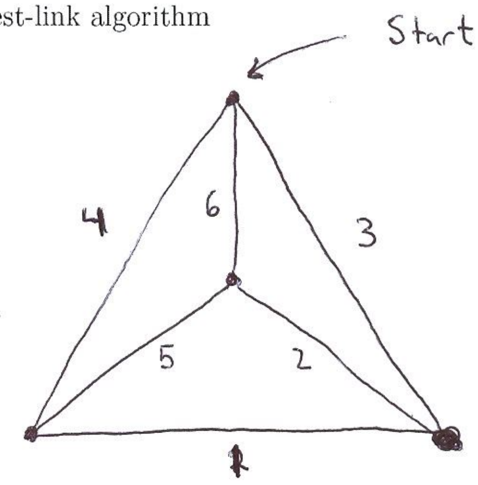
3. Show that, if a graph has no loops and each of its vertices has degree at most $k - 1$, then it is k-colorable.

MAT118 Homework 8

Due Friday, April 1

1. Draw an example of:
   (a) A graph with both an Euler circuit and a Hamiltonian circuit
   (b) A graph with an Euler circuit but no Hamiltonian circuit
   (c) A graph with a Hamiltonian circuit but no Euler circuit
   (d) A graph with neither a Hamiltonian circuit nor an Euler circuit

2. For the graph below, compute the Hamiltonian circuit produced by:
   (a) The nearest-neighbor algorithm
   (b) The repetitive nearest-neighbor algorithm
   (c) The cheapest-link algorithm



3. Explain why the cheapest edge in any graph is always part of the Hamiltonian circuit obtained using the nearest-neighbor algorithm.

MAT118 Homework 9

Due Friday, April 8

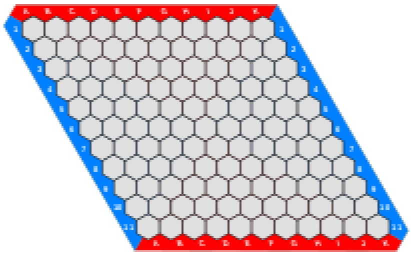
1. In the game of Divisor Chomp, two players alternate turns. On a player's turn, she must name a divisor of 4000. She cannot name a number that has already been mentioned, or any multiple of a number that has already been mentioned. The player who names the number 1 loses. Show that the first player has a winning strategy.

2. This less interesting version of the game Nim is played with two piles of matchsticks. One of the piles contains 20 matchsticks, and the other contains 21. The two players alternate turns. On a player's turn, she may remove as many matchsticks as she likes, as long as they are all from the same pile. The player who takes the last matchstick wins. Show that the first player has a winning strategy. What is this winning strategy?

3. The game of Hex is played on a hexagonal grid. (See the attached illustration.) Players take turns placing a stone of their color on a single cell within the overall playing board. The goal is to form a connected path of your stones linking the opposing sides of the board marked by your colors, before your opponent connects his or her sides in a similar fashion. The first player to complete his or her connection wins the game. The four corner hexagons each belong to both adjacent sides.

   (a) Show that one of the two players must win the game (there are no draws).

   (b) Show that the first player in Hex has a winning strategy.



MAT118 Homework 10

Due Friday, April 15

1. (a) Suppose that, in an effort to make soccer more exciting, FIFA decides that goals are now worth 5 points each. Show that this new game is equivalent to soccer.

   (b) Suppose that, in an effort to make baseball more exciting, MLB decides to implement the following scoring system: each run is worth 1 point, and, at the end of each inning, a team is awarded a number of bonus points equal to the number of runs they scored in that inning. Show that this new game is equivalent to baseball.

2. Where does the game of Divisor Chomp get its name? Show that the game of Divisor Chomp from the previous homework is equivalent to the game of Chomp we played in class. Conclude that the first player has a winning strategy.

3. (a) Recall the game of 24 from class. Suppose that you are now allowed to take 1,2,3, or 4 stones. Which player is in a better position at the start of the game? Describe a winning strategy for this player.

   (b) Suppose that you are now allowed to take 1,2, or 7 stones. Which player is in a better position at the start of the game? Describe a winning strategy for this player.

MAT118 Homework 11

Due Friday, April 29

1. Determine whether each of the following positions in Nim is balanced or unbalanced. What is the best possible next move?

   (a) Three piles – one with 2 stones, one with 6 stones, and one with 7 stones.

   (b) Five piles, each containing 5 stones.

   (c) Four piles – one with 1 stone, one with 2 stones, one with 4 stones, and one with 8 stones.

2. The payoff matrix for the game of Chicken is depicted below. Find all of the Nash equilibria in this game. Explain in your own words why each is a Nash equilibrium.

	Swerve	Don't Swerve
Swerve	2,2	1,3
Don't Swerve	3,1	4,4

3. Two employees at a small company are upset with the new policy regarding vacation time. Each is considering sending a memo to their boss outlining their complaints. Each employee knows that, if the boss receives two memos, he will have to take the complaint seriously – this is the best possible outcome. If the boss receives only one memo, however, he may simply fire the person who complained. This is the worst outcome for the person who complained, but the second-best for the person who kept quiet; at least he knows that the boss is aware of the problem. If both employees keep quiet, of course, then nothing will happen.

   (a) Draw the payoff matrix for this game.

   (b) This game is equivalent to one of those that was described in class. Which one?

   (c) What are the Nash equilibria in this game? Why?

MAT118 Homework 12

Due Friday, May 6

1. Simplify each of the expressions below.

   (a) $(7 + 3i) - (3 - 2i)$

   (b) $(2 + i)(2 - i)$

   (c) $(1 + i)^2 + (1 + i)$

2. Plot each of the complex numbers below in the complex plane.

   (a) $1 + 2i$

   (b) -4

   (c) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

3. Suppose you plot an arbitrary complex number $a+bi$ as a vector in the complex plane. Describe what would happen to that vector geometrically if you were to then multiply by each of the following numbers.

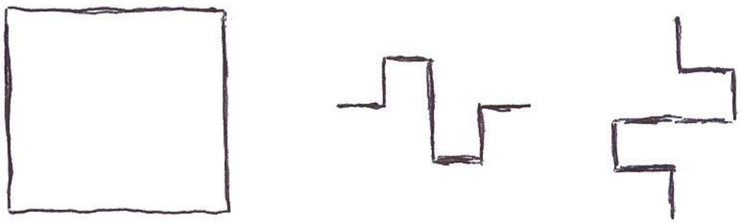
   (a) 3

   (b) $-i$

   (c) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

MAT118 Homework 13
Due Friday, May 13

1. The *quadratic Koch island* begins with a 16×16 square, as shown below. At each step, we replace each horizontal line with the "sawtooth" shown on the left, and each vertical line with the "sawtooth" shown on the right.



   (a) Draw the next two steps of the quadratic Koch island.
   (b) What is the area of the figure after the first step?
   (c) What is the area of the figure after the second step?
   (d) Make a conjecture about the area of the quadratic Koch island.

2. In the plane, label the origin with the letter A, the point $(32, 0)$ with the letter B and the point $(0, 32)$ with the letter C. The three points form the vertices of a right triangle, on which we will play the chaos game. Suppose that we start at the point B, and then roll a die six times with the following outcomes: 1,6,4,5, and 5. Carefully plot the points corresponding to these outcomes.

3. For each of the following complex numbers, determine the first four terms of the Mandelbrot sequence. Is the Mandelbrot sequence escaping, periodic, or attracting? Explain.

   (a) $-\frac{1}{2}$
   (b) $2i$
   (c) $1 + i$