Alexander Kuznetsov. Exercises on Exceptional Collections. Let $P$ be a $k$-point of $\mathbb{P}^2_k$. Let $\pi: X \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2_k$ at $P$. Let $i: E \to X$ be the inclusion of the exceptional divisor, $E \cong \mathbb{P}^1_k$. Let $f_P: X \to \mathbb{P}^1$ be the linear projection away from $P$.

Associated to the blowing up $\pi$ there is a full exceptional collection in $D^b(X)$,

$$\mathcal{E}_1 = \langle i_*\mathcal{O}_E(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{O}_{\mathbb{P}^2}(2) \rangle.$$ 

Associated to the $\mathbb{P}^1$-bundle $f$ there is another full exceptional collection in $D^b(X)$,

$$\mathcal{E}_2 = \langle f^*\mathcal{O}_{\mathbb{P}^1}, f^*\mathcal{O}_{\mathbb{P}^1}(1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_X f^*\mathcal{O}_{\mathbb{P}^1}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_X f^*\mathcal{O}_{\mathbb{P}^1}(1) \rangle.$$ 

**Problem 0.1.** Find a sequence of mutations from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Next, let $P_1$ and $P_2$ be two distinct $k$-points of $\mathbb{P}^2_k$. Let $\rho: Y \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2_k$ at $P_1$ and $P_2$. The exceptional locus $F$ has two disjoint connected components, $F_1$ and $F_2$, each isomorphic to $\mathbb{P}^1_k$. Denote by $j: F \to X$ the inclusion. The linear projections $f_{P_1}$ and $f_{P_2}$ induce a morphism,

$$(f_{P_1}, f_{P_2}): Y \to \mathbb{P}^1_k \times \text{Spec}(k) \mathbb{P}^1_k.$$

This morphism is a blowing up at the point $Q = (f_{P_1}(P_2), f_{P_2}(P_1))$. Denote by $h: G \to Y$ the exceptional divisor of this blowing up, $G \cong \mathbb{P}^1_k$.

Associated to the blowing up $\rho$ there is a full exceptional collection in $D^b(Y)$,

$$\mathcal{F}_1 = \langle j_*\mathcal{O}_{F_1}(-1), j_*\mathcal{O}_{F_2}(-1), \rho^*\mathcal{O}_{\mathbb{P}^2}, \rho^*\mathcal{O}_{\mathbb{P}^2}(1), \rho^*\mathcal{O}_{\mathbb{P}^2}(2) \rangle.$$ 

Associated to the blowing up $(f_{P_1}, f_{P_2})$ there is a full exceptional collection in $D^b(Y)$,

$$\mathcal{F}_2 = \langle h_*\mathcal{O}_G(-1), f_{P_1}^*\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_V, f_{P_2}^*\mathcal{O}_{\mathbb{P}^1}, f_{P_1}^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_V, f_{P_2}^*\mathcal{O}_{\mathbb{P}^1}, f_{P_1}^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_V f_{P_2}^*\mathcal{O}_{\mathbb{P}^1}(1) \rangle.$$

**Problem 0.2.** Find a sequence of mutations from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Burt Totaro. Exercises on Base Change Homomorphisms. Let $k$ be a field. For every $k$-variety $X_k$ and for every integer $q$, there is the free Abelian group of all $q$-cycles,

$$Z_q(X_k) = \langle [V_k] | V_k \subset X_k, \text{ closed, integral, dim}(V_k) = q \rangle.$$ 

Inside $Z_q(X_k)$ there is the subgroup $\text{Rat}_q(X_k)$ generated by all $q$-cycles of the form $u_*\text{div}(f)$, where $u: W \to X_k$ is any proper morphism from any normal $k$-variety of
dimension $q + 1$, where $f$ is any nonzero rational function on $W$, and where $\text{div}(f)$ is the principal (Weil) divisor on $W$ of $f$. The quotient group is the **Chow group**,

$$\text{CH}_q(X_k) = Z_q(X_k)/\text{Rat}_q(X_k).$$

For every field extension $F/k$, denote $X_k \times_{\text{Spec}(k)} \text{Spec}(F)$ by $X_F$. For every $V_k \subset X_k$ as above, the base change $V_F \subset X_F$ is a closed subscheme of pure dimension $q$ that gives a cycle $[V_F] \in Z_q(X_F)$ (note that $V_F$ may not be integral). The induced base change homomorphism,

$$u_{F/k,X_k,\text{CH}_q}: \text{CH}_q(X_k) \to \text{CH}_q(X_F), \quad [V_k] \mapsto [V_F],$$

maps $\text{Rat}_q(X_k)$ to $\text{Rat}_q(X_F)$. Thus, there is a well-defined homomorphism of Chow groups,

$$u_{F/k,X_k,\text{CH}_q}: \text{CH}_q(X_k) \to \text{CH}_q(X_F).$$

There is also an induced homomorphism of $\mathbb{Q}$-vector spaces,

$$u_{F/k,X_k,\text{CH}_q} \otimes \mathbb{Q}: \text{CH}_q(X_k) \otimes \mathbb{Q} \to \text{CH}_q(X_F) \otimes \mathbb{Q}.$$

**Problem 0.3.** Find an example of a field $k$, a smooth, projective $k$-variety $X_k$, field extension $F/k$, and an integer $q$ such that the induced homomorphism $u_{F/k,X_k,\text{CH}_q}$ is not surjective. In fact, find an example such that $u_{F/k,X_k,\text{CH}_q} \otimes \mathbb{Q}$ is not surjective.

**Problem 0.4.** Find an example of a field $k$, a smooth, quasi-projective $k$-variety $X_k$, field extension $F/k$, and an integer $q$ such that the induced homomorphism $u_{F/k,X_k,\text{CH}_q}$ is not injective. For a challenge, find an example where $X_k$ is projective.

**Problem 0.5.** Prove that for every field $k$, for every quasi-projective $k$-variety $X_k$, for every field extension $F/k$, and for every integer $q$ the induced homomorphism $u_{F/k,X_k,\text{CH}_q} \otimes \mathbb{Q}$ is injective.

**Claire Voisin. Exercises on Torsion Cohomology, Griffiths Groups and Decompositions of the Diagonal.**

**Problem 0.6.** For every (second countable, Hausdorff) topological manifold $M$, prove that the singular cohomology group $H^1(M; \mathbb{Z})$ is torsion-free.

**Problem 0.7.** For every smooth, projective, complex variety $X$, for the subgroups $\text{CH}_1(X)_{\text{alg}} \subset \text{CH}_1(X)_{\text{hom}} \subset \text{CH}_1(X)$ of cycles that are algebraically equivalent to zero, resp. homologically equivalent to zero, prove that the quotient **Griffiths group**, $\text{CH}_1(X)_{\text{hom}}/\text{CH}_1(X)_{\text{alg}}$ is a birational invariant.

**Problem 0.8.** For every smooth, projective, complex variety $X$, for every element $\alpha \in H^0(X, \Omega^1_X)$, if there exists a dense, Zariski open subset $U \subset X$ such that $\alpha|_U$ is exact, then prove that $\alpha$ equals 0. Please do this without using mixed Hodge structures.
Problem 0.9. For every (second countable, Hausdorff) topological manifold $M$ that is connected and oriented, prove that the Künneth homomorphism,
\[ \bigoplus_{0 \leq p, q} H^p(M; \mathbb{Z}) \otimes H^q(M; \mathbb{Z}) \rightarrow \bigoplus_{0 \leq r} H^r(M \times M; \mathbb{Z}), \]
is an isomorphism if and only if $\bigoplus_r H^r(M; \mathbb{Z})$ is torsion-free.

Problem 0.10. For every smooth, projective, complex variety $X$ of dimension $n$, for the diagonal class $[\Delta_X] \in CH_n(X \times X)$, there exists a decomposition
\[ [\Delta_X] = a_1[Z_1 \times T_1] + \cdots + a_r[Z_r \times T_r] \]
for integers $a_1, \ldots, a_r$, and for integral subvarieties of $X$, $Z_1, \ldots, Z_r, T_1, \ldots, T_r$ if and only if numerical equivalence of cycles in $X$ equals rational equivalence.

Problem 0.11. For every smooth, projective, complex surface $X$, for the cohomological diagonal class $[\Delta_X]$ in $H^*(X^{an} \times X^{an}; \mathbb{Q})$, resp. in $H^*(X^{an} \times X^{an}; \mathbb{Z})$, there exists a decomposition
\[ [\Delta_X] = a_1[Z_1 \times T_1] + \cdots + a_r[Z_r \times T_r] \]
for integers $a_1, \ldots, a_r$, and for integral subvarieties of $X$, $Z_1, \ldots, Z_r, T_1, \ldots, T_r$ if and only if $q(X) = p_g(X) = 0$, resp. if and only if both $q(X) = p_g(X) = 0$ and $H^*(X^{an}, \mathbb{Z})$ is torsion-free.

Lev Borisov. Degrees of Calabi-Yaus. Recall that a smooth, projective variety $X$ is a Calabi-Yau variety if $\omega_X \cong \mathcal{O}_X$, if $X$ is simply connected, and if $h^0(X, \Omega_X^q)$ vanishes for $0 < q < \dim(X)$.

Problem 0.12. Using computer code, using (skew-symmetric) Thom-Porteous, and using Schubert calculus, compute the degrees of Pfaffian Calabi-Yau varieties, respectively Grassmannian Calabi-Yau varieties.

Alena Pirutka. Problems on Rationality.

Problem 0.13. For a smooth quadric hypersurface $X_k \subset \mathbb{P}^n_k$, prove that $X$ is rational if and only if $X$ has a $k$-point.

Problem 0.14. Let $k$ be an algebraically closed field. For every $k$-variety $X_k$ and for every field extension $K/k$, prove that $X_k$ is $k$-rational if and only if the base change $X_K$ is $K$-rational.

For the next sequence of exercises, let $k$ be a field (not necessarily algebraically closed nor even infinite). Let $X_k$ be a $k$-variety of dimension $m$. Let $\phi : \mathbb{A}^n_k \rightarrow X_k$ be a dominant rational transformation. Necessarily $n \geq m$, and these exercises investigate whether there exists $\phi$ with $n = m$.

Problem 0.15. Prove that there exists a dense, Zariski open $U \subset \mathbb{A}^n_k$ such that $\phi|_U$ is a morphism whose (nonempty) fibers are pure-dimensional of dimension $d = n - m$. 

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Problem 0.16. Assume now that $k$ is infinite. Prove that there exists a $k$-point $u$ of $U$ and a hyperplane $H \subset \mathbb{A}_k^n$ containing $u$ such that the restriction of $\phi$ to $U \cap H$ is dominant. Use induction on $n$ to prove that there exists a dominant rational transformation from $\mathbb{A}_k^m$ to $X_k$.

Problem 0.17. Finally, assume that $k$ is a finite field. Let $\ell$ be an integer different from the characteristic. Let $K \subset \overline{k}$ be the union of all extension fields of $k$ of degree $\ell^s$, $s > 0$.

(a) First prove that $U(K)$ is not empty.
(b) For an arbitrary point $u = (u_1, \ldots, u_n)$ in $\mathbb{A}_k^n(K)$, use the Primitive Element Theorem to prove that, up to a permutation, $k(u_n) \subset k(u_{n-1}) \subset \cdots \subset k(u_1)$. Use this to prove that the ideal $\mathfrak{m}_u \subset k[x_1, \ldots, x_{n-1}, x_n]$ is generated by elements in $k[x_1, \ldots, x_{n-1}]$ and elements of the form $x_n - P(x_1, \ldots, x_{n-1})$.
(c) Finally, prove that there exists an affine hypersurface $Z = \text{Zero}(x_n - P(x_1, \ldots, x_{n-1}))$ in $\mathbb{A}_k^n$ containing $u$ such that the restriction of $\phi$ to $Z \cap U$ is dominant. Again use induction to prove that there exists a dominant rational transformation from $\mathbb{A}_k^m$ to $X_k$. 