On Antipode Preserving Cubic Maps, (Draft of Feb. 15, 2015)

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Abstract. This note will study a family of cubic rational maps which carry antipodal points of the Riemann sphere to antipodal points.

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1. Introduction.

Recall the following classical result (see [**Bo**]).

Theorem of Borsuk. *There exists a map $f : S^n \to S^n$ of degree d which carries antipodal points to antipodal points if and only if d is odd.*

For the ***Riemann sphere***, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$, the ***antipodal map*** is defined to be the fixed point free map $\mathcal{A}(z) = -1/\overline{z}$. We are interested in rational maps from the Riemann sphere to itself which carry antipodal points to antipodal points[1], so that $\mathcal{A} \circ f = f \circ \mathcal{A}$. If all of the zeros q_j of f lie in the finite plane,

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[1]The proof of Borsuk's Theorem in this special case is quite easy. A rational map of degree d has $d+1$ fixed points, counted with multiplicity. If these fixed points occur in antipodal pairs, then $d+1$ must be even.

then f can be written uniquely as

$$f(z) = u \prod_{j=1}^{d} \frac{z - q_j}{1 + \overline{q}_j z} \qquad \text{with} \qquad |u| = 1 .$$

The simplest interesting case is in degree $d = 3$. To fix ideas we will discuss only the special case[2] where f has a critical fixed point. Putting this fixed point at the origin, we can take $q_1 = q_2 = 0$, and write q_3 briefly as q. It will be convenient to take $u = -1$, so that the map takes the form

$$f(z) = f_q(z) = z^2 \frac{q - z}{1 + \overline{q} z} . \tag{1}$$

(Note that there is no loss of generality in choosing one particular value for u, since we can always change to any other value of u by rotating the z-plane appropriately.)

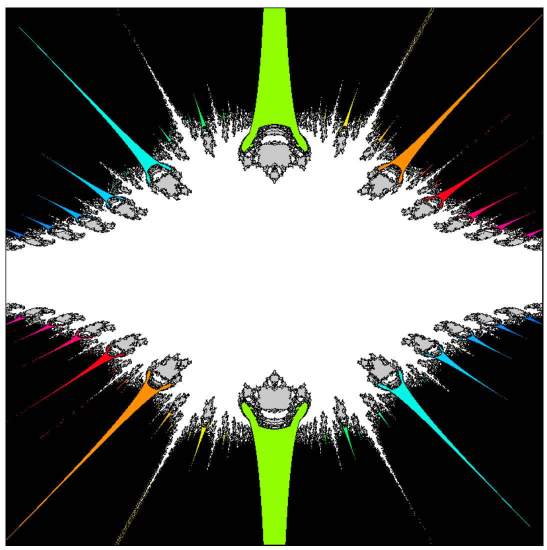

FIGURE 1. *The q-parameter plane.*

Figure 1 illustrates the q-parameter plane for this family of maps (1). In this figure:

• The central white region, resembling a porcupine, is the hyperbolic component centered at $f_0(z) = -z^3$. It consists of all q for which the Julia set is a Jordan curve separating the basins of zero and infinity.

• The black region is the ***Herman ring locus***. For the overwhelming majority of parameters q in this region, the map f_q has a Herman ring which separates the basins of zero and infinity.

• The colored regions will be called ***tongues***, in analogy with Arnold tongues. Each of these is a hyperbolic component, stretching out to infinity, which represent maps with a self-antipodal attracting periodic orbit having a well defined rotation number around the origin, always with even denominator. The colors vary from red (for rotation number close to zero) to blue (for rotation number close to one).

[2] For a different family, consisting of maps with two critical points, see [**Mi1**, §7] or [**GH**].

• For rotation numbers with odd denominator, there are no such tongues. Instead there are channels leading out to infinity, which are part of the central hyperbolic component. These will be called ***fjords***.

• The grey regions and the nearby small white regions in Figure 1 represent ***capture components***, such that both of the critical points in $\mathbb{C}\smallsetminus\{0\}$ have orbits which eventually land in the immediate basin of zero or infinity.

Note that Figure 1 is invariant under 180° rotation. In fact each f_q is linearly conjugate to $f_{-q}(z) = -f_q(-z)$. In order to eliminate this duplication, it will be convenient to make a simple change of parameter, setting $s = q^2$. The s-plane will be referred to as ***moduli space***, since each $s \in \mathbb{C}$ corresponds to a unique holomorphic conjugacy class of mappings of the form (1) with a marked critical fixed point at the origin.[3]

Definition 1.1. It will be useful to add a circle of points at infinity to the s-plane. By the ***circled s-plane***

$$\circledC = \mathbb{C} \cup (\text{circle at infinity})$$

we will mean the compactification of the s-plane which is obtained by adding one point at infinity

$$\infty_{\mathbf{t}} = \lim_{r\to+\infty} r\, e^{2\pi i\,\mathbf{t}}$$

corresponding to each point $e^{2\pi i\,\mathbf{t}}$ on the unit circle. This circled plane is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ under a homeomorphism $\eta : \circledC \xrightarrow{\cong} \overline{\mathbb{D}}$ which maps each radial line $e^{2\pi i\,\mathbf{t}}[0,\,\infty]$ onto its subset $e^{2\pi i\,\mathbf{t}}[0,\,1]$. The precise choice of η will not be important; however, for illustrative purposes, it is often useful to show the image of the circled plane under such a homeomorphism η, since we can then visualize the entire circled plane in one figure.

This circle at infinity has an important dynamical interpretation:

Lemma 1.2. *Suppose that $s = q^2$ converges to the point $\infty_{\mathbf{t}}$ on the circle at infinity. Then throughout any annulus $\epsilon \le |z| \le 1/\epsilon$ the associated maps $f_{\pm q}$ converge uniformly to the rotation*

$$z \mapsto e^{2\pi i\,\mathbf{t}}\, z\ .$$

Hence all of the interesting dynamics must be concentrated within the small disk $|z| < \epsilon$ and its antipodal image. The proof is a straightforward exercise. □

2. Fatou Components and Hyperbolic Components

Theorem 2.1. *Every Fatou component U for a map $f = f_q$ in our family* (1) *is either an annulus or is simply connected. Furthermore, in the annulus case, either U or some forward image $f^{\circ k}(U)$ is a Herman ring.*[4]

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[3]Thus we do not allow conjugacies which interchange the roles of zero and infinity, and hence replace s by $\overline{s}$. The punctured s-plane, with the origin removed, can actually be considered as a parameter space, setting for example $F_s(w) = q\, f_q(w/q) = w^2(s-w)/(s+|s|w)$.

[4]For the moment, this Herman ring may be periodic of period $m \ge 1$. However §7 will show that such a ring is necessarily fixed ($m = 1$).

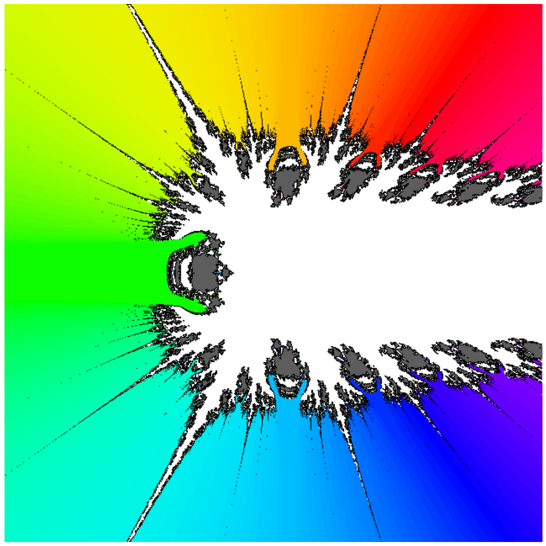

FIGURE 2. *The s-moduli plane. Here the colors code the rotation number, not distinguishing between tongues and Herman rings.*

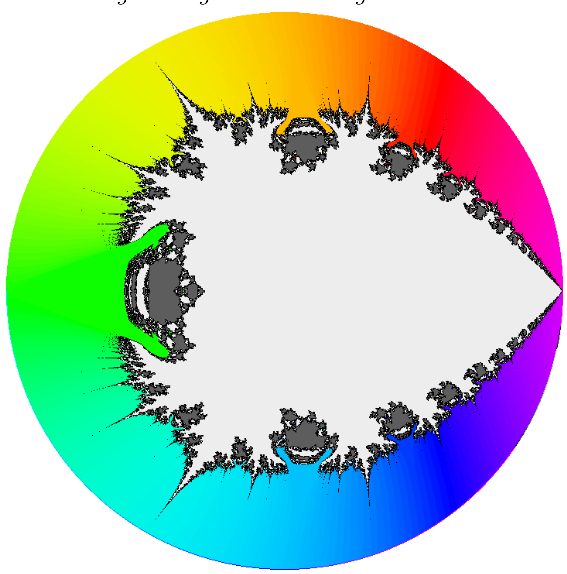

FIGURE 3. *Image of the circled s-plane under the homeomorphism $\eta : \copyright \to \overline{\mathbb{D}}$ which shrinks the circled plane to the unit disk.*

Proof. First consider the special case where U is a periodic Fatou component.

Case 1. A Rotation Domain. If there are no critical points in this cycle of Fatou components, then U must be either a Herman ring, or a Siegel disk

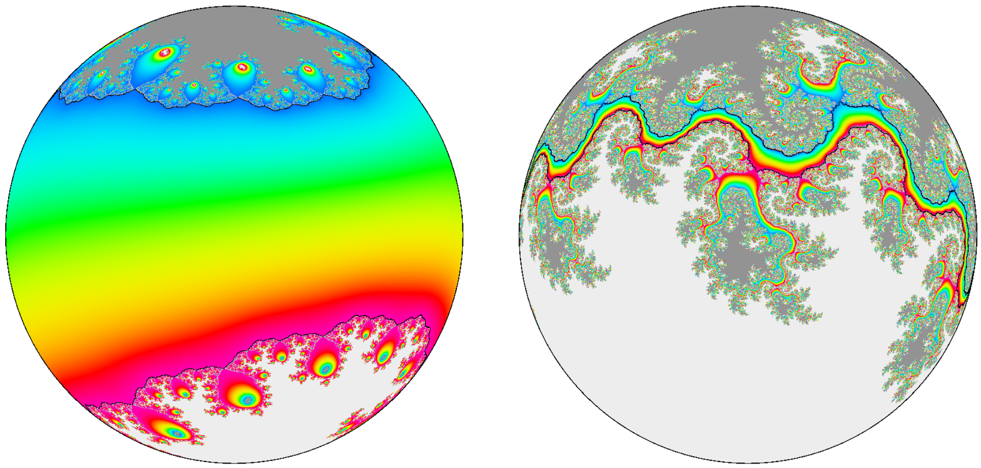

FIGURE 4. *Irrational rotation number: two Herman rings on the Riemann sphere, projected orthonormally. The front (visible) half of the sphere corresponds to the right half-plane, with* $z = 0$ *at the bottom and* $z = \infty$ *at the top. The boundaries of the rings have been outlined. Otherwise, every point* z *maps to a point* $f_q(z)$ *of the same color. Left example:* $q \approx 6 + i$, *modulus* ≈ 0.2081. *Right example:* $q \approx 1.8 + 3.21\,i$, *modulus* ≈ 0.0063.

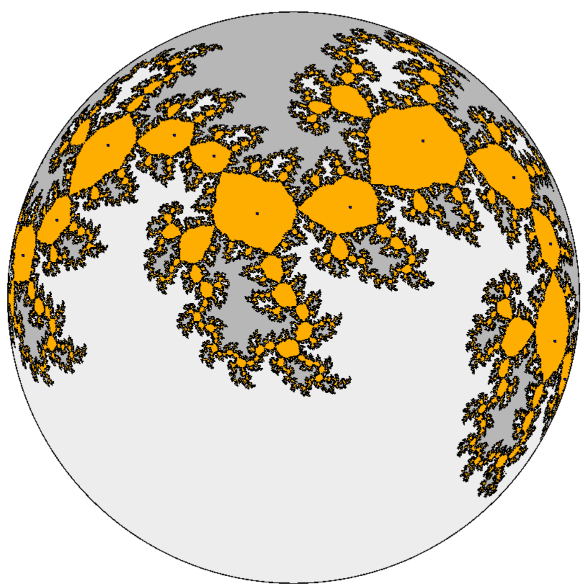

FIGURE 5. *Rational rotation number: Julia set for a point in the* $(21/26)$*-tongue. Note the self-antipodal attracting orbit of period* 26*, which has been marked. The associated ring of attracting Fatou components separates the basins of zero and infinity.*

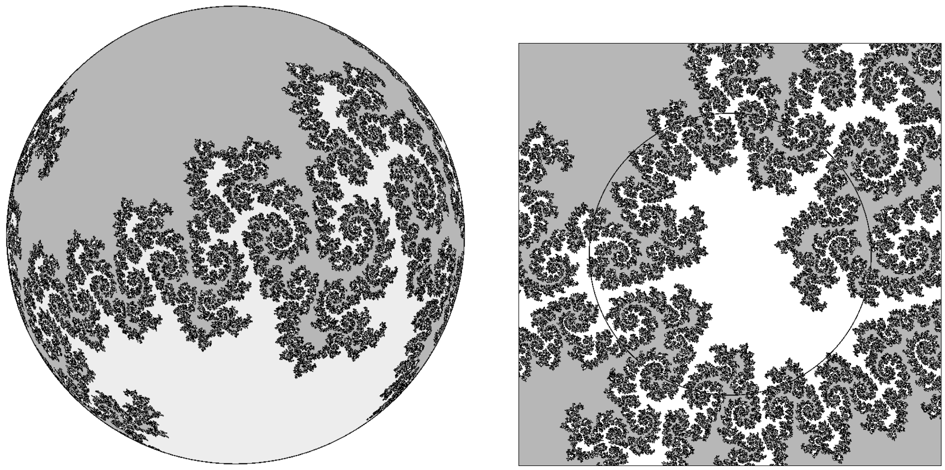

FIGURE 6. *Julia set for a point in the* $(5/11)$*-fjord. This is a Jordan curve, separating the basins of zero and infinity. The left side illustrates the Riemann sphere version, and the right side shows the view in the* z*-plane, with the unit circle drawn in.*

(necessarily simply connected). In either case there is nothing to prove. Thus we are reduced to the attracting and parabolic cases.

Case 2. An Attracting Domain of period $m \geq 2$**.** Then U contains an attracting periodic point z_0 of period m. Let $\mathscr{O}$ be the orbit of z_0, and let U_0 be a small disk about z_0 which is chosen so that $f^{\circ m}(U_0) \subset U_0$, and so that the boundary ∂U_0 does not contain any postcritical point. For each $j \geq 0$, let U_j be the connected component of $f^{-j}(U_0)$ which contains a point of $\mathscr{O}$, thus $f(U_j) = U_{j-1}$. Assuming inductively that U_j is simply connected, we will prove that U_{j+1} is also simply connected. If U_{j+1} contains no critical point, then it is an unbranched covering of U_j, and hence maps diffeomorphically onto U_j. If there is only one critical point, mapping to the critical value $v \in U_j$, then $U_{j+1} \smallsetminus f^{-1}(v)$ is an unbranched covering space of $U_j \smallsetminus \{v\}$, and hence is a punctured disk. It follows again that U_{j+1} is simply connected. There cannot be two critical points in U_{j+1}, since then one of them would have to be the critical fixed point 0 or ∞. In this case the period would be $m = 1$, contradicting the hypothesis. Thus each U_j is simply connected. Since the entire Fatou component U is the nested union of the U_{jm}, it is also simply connected.

Case 3. A Parabolic Domain of period $m \geq 2$**.** The argument is the same, except that in place of a small disk centered at the parabolic point, we take a small attracting petal which intersects all orbits in its Fatou component.

Case 4. Period One: The Basin of Zero or Infinity. Since the two free fixed points are antipodal, they are necessarily distinct—There cannot be a fixed point of multiplier $+1$. Hence the holomorphic index formula implies easily that the two free fixed points are strictly repelling. (Compare [**Mi2**, Corollary 12.7].)

Thus there cannot be a parabolic fixed point. Furthermore, an attracting point of period $m = 1$ is necessarily one of the two critical fixed points, zero or infinity.

Let us concentrate on the basin of zero. Construct the sets U_j as above. If there are no other critical points in this immediate basin, then it follows inductively that the U_j are all simply connected. If there is another critical point c in the immediate basin, then there will be a smallest j such that U_{j+1} contains c. Consider the Riemann-Hurwitz formula

$$\chi(U_{j+1}) \;=\; d\cdot\chi(U_j) - n\ ,$$

where $\chi(U_j) = 1$, d is the degree of the map $U_{j+1} \to U_j$, and $n = 2$ is the number of critical points in U_{j+1}. If $d = 3$, then U_{j+1} is simply connected, and again it follows inductively that the U_j are all simply connected. However, if $d = 2$ so that U_{j+1} is an annulus, then we would have a more complicated situation, as illustrated in Figure 7. Recall that any two disjoint closed annuli in the Riemann sphere must separate it into two simply connected regions plus one annulus. Hence one simply connected region E would have to share a boundary with U_{j+1}, and the antipodal region $\mathcal{A}(E)$ would share a boundary with $\mathcal{A}(U_{j+1})$. The remaining annulus A would then share a boundary with both U_{j+1} and $\mathcal{A}(U_{j+1})$, and hence would be self-antipodal. We must prove that this case cannot occur.

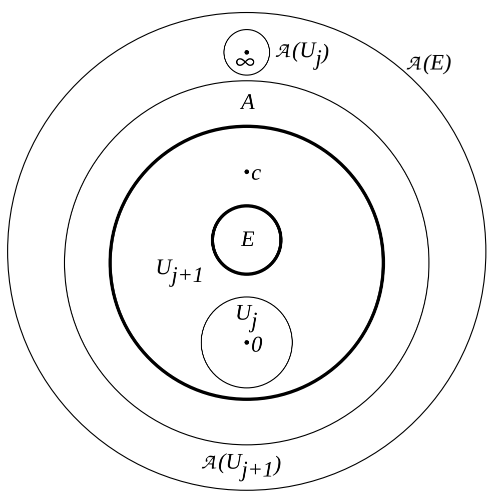

FIGURE 7. *Three concentric annuli. The innermost annulus U_{j+1} (with emphasized boundary) contains the disk U_j, while the outermost annulus $\mathcal{A}(U_{j+1})$ contains $\mathcal{A}(U_j)$. The annulus A lies between these two. Finally, the innermost region E and the outermost region $\mathcal{A}(E)$ are topological disks.*

Each of the two boundary curves of U_{j+1} (emphasized in Figure 7) must map homeomorphically onto ∂U_j. Thus one of the two boundary curves of A maps homeomorphically onto ∂U_j, and the other must map homeomorphically onto $\mathcal{A}(\partial U_j)$. Note that there are no critical points in A. The only possibility is that A maps homeomorphically onto the larger annulus $A' = \widehat{\mathbb{C}} \smallsetminus \big(\overline{U}_j \cup \mathcal{A}(\overline{U}_j)\big)$. This behavior is perfectly possible for a branched covering, but is impossible for a holomorphic map, since the modulus of A' must be strictly larger than the modulus of A. This contradiction completes the proof that each periodic Fatou component is either simply connected, or a Herman ring.

The Preperiodic Case. For any non-periodic Fatou component U, we know by Sullivan's nonwandering theorem that some forward image $f^{\circ k}(U)$ is periodic. Assuming inductively that $f(U)$ is an annulus or is simply connected, we must prove the same for U. If $f(U)$ is simply connected, then it is easy to check that there can be at most one critical point in U, hence as before it follows that U is simply connected. In the annulus case, the two free critical points necessarily belong to the Julia set, and hence cannot be in U. Therefore U is a finite unbranched covering space of $f(U)$, and hence is also an annulus. This completes the proof of Theorem 2.1. □

This result in the dynamic sphere has an immediate corollary in the parameter plane.

Corollary 2.2. *Every hyperbolic component, either in the q-plane or in the (q^2)-plane, is simply connected, with a unique critically finite point* (*which is called its* ***center***).

Proof. Since a hyperbolic map cannot have any Herman ring, it follows from Theorem 2.1 that its Julia set is connected. It then follows from the discussion in [**Mi4**, Theorems 9.3 and 7.13] that any hyperbolic component in the q-plane is simply connected, with a unique critically finite point. The corresponding statement for the (q^2)-plane then follows easily. □

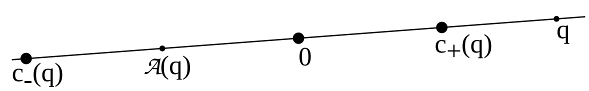

FIGURE 8.

We will need a more precise description of the four (not necessarily distinct) critical points of f_q. The critical fixed points zero and infinity are always present, and there are also two mutually antipodal ***free critical points***, given by the formula

$$c_\pm(q) \;=\; \left(\frac{a-3\pm\sqrt{9+10\,a+a^2}}{4\,a}\right) q\,, \qquad \text{where} \qquad a \;=\; |q|^2\,, \tag{2}$$

for $q \neq 0$, with $c_+(0) = 0$. The proof is a straightforward computation. It follows easily that the four critical points, as well as the zero q and the pole $\mathcal{A}(q)$, all lie on a straight line, with $c_+(q)$ between zero and q, as shown in Figure 8.

Setting $q = x + iy$, computation shows that the two ***free fixed points*** are given by

$$z_\pm(q) \;=\; i\Big(y \pm \sqrt{y^2+1}\Big) \;=\; i\Big(\mathrm{Im}(q) \;\pm\; \sqrt{\mathrm{Im}(q)^2\,+1}\Big)\,. \tag{3}$$

Thus $z_+(q)$ always lies on the positive imaginary axis, while its antipode $z_-(q)$ lies on the negative imaginary axis. As noted in the proof of Theorem 2.1, both of these free fixed points must be strictly repelling.

We can now give a preliminary description of the possible hyperbolic components.

Lemma 2.3. *Every hyperbolic component in the q^2-plane belongs to one of the following four types:*

- *The* ***central hyperbolic component*** $\mathcal{H}_0$ (the white region in Figure 1). *This is the unique component for which the Julia set is a Jordan curve separating the basins of zero and infinity. In this case, the critical point $c_+(q)$ necessarily lies in the immediate attractive basin of zero.* (Compare Figures 6, 18, and 24(left).)
- ***Mandelbrot Type****. Here there are two mutually antipodal attracting orbits of period two or more, with one free critical point in the immediate basin of each one.* (For an example in the parameter plane, see Figure 11, and for the corresponding Julia set see Figures 22 and 23.)
- ***Tricorn Type****. Here there is one self antipodal attracting orbit, necessarily of even period. The most conspicuous examples are the tongues which stretch out to infinity* (§5)*; but there are also small bounded tricorns* (see Figure 9).
- ***Capture Type****. Here the free critical points are not in the immediate basin of zero or infinity, but some forward image belongs in one basin or the other.* (These regions are either dark grey or white in Figures 1, 2, 3, 9, 11 according as the orbit of c_+ converges to ∞ or 0.)

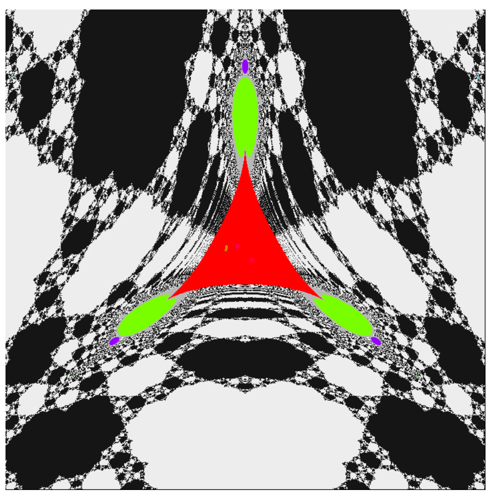

FIGURE 9. *Small period four tricorn in the q-plane, centered at $q \approx 2.006\, i$ on the imaginary axis. The white and black regions are capture components.*

Proof of Lemma 2.3. This is mostly a straightforward exercise—Since there are only two free critical points, whose orbits are necessarily antipodal to each other, there are not many possibilities. As noted in the proof of Theorem 2.1, the two fixed points in $\mathbb{C}\smallsetminus\{0\}$ must be strictly repelling. However, the uniqueness of $\mathcal{H}_0$ requires some proof. In particular, since $c_+(q)$ can be arbitrarily large, one might guess that there could be another component with $c_+(q)$ in the immediate basin of infinity. We must show that this is impossible.

According to Corollary 2.2, for the center of such a hyperbolic component, the point $c_+(q)$ would have to be precisely equal to infinity. But as $c_+(q)$ approaches infinity, according to Equation (2) the parameter q must also tend to infinity. Since $f(q) = 0$ and $f(\infty) = \infty$, there cannot be any such well behaved limit.

Examples illustrating all four cases, are easily provided. (Compare the figures cited above.) $\square$

3. The central hyperbolic component $\mathcal{H}_0$.

As noted in Lemma 2.3, the central hyperbolic component in the s-plane consists of all points $s = q^2$ such that the Julia set of f_q is a simple closed curve separating the basin of zero from the basin of infinity. The critical point $c_+(q)$ necessarily lies in the basin of zero, and $c_-(q)$ in the basin of infinity.

Lemma 3.1. *The open set* $\mathcal{H}_0$ *in the* s*-plane is canonically diffeomorphic to the open unit disk* $\mathbb{D}$ *by the map*

$$\widehat{\mathfrak{b}} : \mathcal{H}_0 \xrightarrow{\cong} \mathbb{D}$$

which satisfies $\widehat{\mathfrak{b}}(0) = 0$ *and carries each* $s = q^2 \neq 0$ *to the Böttcher coordinate*[5] *of the marked critical point* $c_+(q)$.

Thus there is a canonical way of putting a conformal structure on the open set $\mathcal{H}_0$, even though the entire s-plane does not have any natural conformal structure.

The proof of Lemma 3.1 will be based on the following remarks. Let $\mathcal{H}_0^{\rm poly}$ be the hyperbolic component centered at $z \mapsto z^3$ in the space of monic centered cubic polynomial maps. Then the mating of any two polynomials p and p' in $\mathcal{H}_0^{\rm poly}$ yields a well defined rational map $p \amalg p'$, which has just two Fatou components, conformally isomorphic to the bounded Fatou components of p and p' respectively, each with one attracting fixed point. Furthermore, there is a distinguished repelling fixed point in the Julia set, corresponding to the landing point of the zero rays for both p and p'. Let $\mathcal{H}_0^{\rm rat}$ be the hyperbolic component centered at the map $z \mapsto z^3$ in the moduli space for all cubic rational maps with one marked attracting fixed point and one marked repelling fixed point. Then the mating operation yields a well defined biholomorphic mapping

$$\mathcal{H}_0^{\rm poly} \times \mathcal{H}_0^{\rm poly} \xrightarrow{\cong} \mathcal{H}_0^{\rm rat} . \qquad (4)$$

(A similar statement holds for maps of any degree.)

This function (4) is clearly injective, since the rational map $p \amalg p'$ restricted to its "first" Fatou component is holomorphically conjugate to p restricted to its bounded Fatou component, and similarly for the "second" Fatou component and p'. The full statement can be proved for example using [**Mi4**, Theorems 6.1 and 9.4], which describes a canonical biholomorphic model for hyperbolic components, using either monic centered polynomials, or conjugacy classes of rational maps with

[5] A priori, the Böttcher coordinate is defined only in an open set containing $c_+(q)$ on its boundary. However, it has a well defined limiting value at this boundary point.

marked fixed points.[6] In the case of the hyperbolic component $\mathcal{H}_0^{\rm rat}$, this canonical biholomorphic model can be expressed as a Cartesian product, corresponding precisely to the function (4), as required.

Proof of Lemma 3.1. Since the map f_q restricted to the basin of zero, is holomorphically conjugate to a cubic polynomial map, restricted to the basin of an attracting fixed point, it suffices to prove the corresponding statement for cubic polynomials, which is proved for example in [**Mi3**, Lemma 3.6]. □

Remark 3.2. In order for a mating of two cubic polynomials to commute with the antipodal map, the second polynomial must be affinely conjugate to the "***negative-complex-conjugate***"

$$p^*(z) \;=\; -\overline{p(-\overline{z})}$$

of the first. For example, the negative-complex-conjugate of the polynomial $p(z) = z^3 + a\,z^2$, with a critical fixed point, is $p^*(z) = z^3 - \overline{a}\,z^2$. Since all maps f_q in $\mathcal{H}_0$ can be obtained as matings, it would be interesting to know just which maps in the boundary $\partial\mathcal{H}_0$ can also be described as matings.[7]

Remark 3.3. The diffeomorphism $\widehat{\mathfrak{b}}$ of Lemma 3.1 is not conformal. In fact, the embedding $q \mapsto f_q \cong p \amalg p^*$ of the q-plane into $\mathcal{H}_0^{\rm rat}$ is not holomorphic, so there is no reason for its projection into $\mathcal{H}_0^{\rm poly}$ to be holomorphic.

Using this canonical dynamically defined diffeomorphism $\widehat{\mathfrak{b}}$, each *internal angle* $\theta \in \mathbb{R}/\mathbb{Z}$ determines an ***internal parameter ray***

$$R_\theta \;=\; \widehat{\mathfrak{b}}^{-1}\Big(\{r\,e^{2\pi i\,\theta}\;;\;0 \le r < 1\}\Big) \;\subset\; \mathcal{H}_0\,.$$

Theorem 3.4. *If the angle $\theta \in \mathbb{Q}/\mathbb{Z}$ is periodic under doubling, then, in the limit as r tends to 1, the internal ray R_θ either:*

- **(a)** *lands at a parabolic point on the boundary $\partial\mathcal{H}_0$,*
- **(b)** *accumulates on a curve of parabolic boundary points, or*
- **(c)** *lands at a point on the circle at infinity, so that $|s| \to \infty$.*

In fact it seems that all three cases can occur. See Figures 11 and 25 for cases **(a)** and **(c)**. Numerical computations by Hiroyuki Inou suggest strongly that rays can accumulate on a curve of parabolic points without actually landing. (Compare Remark 6.12 and Figure 10. In the case of multicorns see [**IM**].)

Remark 3.5. In case **(b)**, we conjecture that the orbit is bounded away from the circle at infinity. If the angle θ is rational but not periodic, then we conjecture that R_θ lands on a critically finite parameter value.

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[6] The marking of just two fixed points, as specified above, can be extended easily to a well defined marking of all fixed points.

[7] Sharland ([**Sha**]) has shown that many such maps can be obtained as matings. For example, the landing points described in Theorem 3.4(a) are all matings. (Compare Figures 11, 22, 23.) In fact, we have not observed any Mandelbrot type set in our family which is not a mating.

For the proof of Theorem 3.4, we will need to make use of *dynamic internal rays.* For any $q \in \mathbb{C}$, let U_q be the basin of zero for the map f_q.

Definition 3.6. Assume that $q \neq 0$, so that the Böttcher map $\mathfrak{b}_q$ from a neighborhood of zero in U_q to a neighborhood of zero in the unit disk $\mathbb{D}$ is defined, and satisfies $\mathfrak{b}_q\big(f_q(z)\big) = \mathfrak{b}_q(z)^2$. Evidently $\mathfrak{b}_q$ extends diffeomorphically over a maximal connected open set $U_q^{\mathbf{vis}}$, namely the union of all smooth internal rays from the origin (not including their possible landing points). By definition, a point is ***visible*** from the origin if and only if it belongs to the closure $\overline{U}_q^{\mathbf{vis}}$ of this set.

If $s = q^2$ belongs to $\mathcal{H}_0$, then $U_q^{\mathbf{vis}}$ is a proper subset of U_q, and $\mathfrak{b}_q$ maps this subset $U_q^{\mathbf{vis}}$ onto a proper open subset of $\mathbb{D}$ which is obtained from the open disk by removing countably many radial slits near the boundary. (See Figure 12.) These correspond to the countably many internal rays from the origin in U_q which crash into critical or precritical points, and hence are bounded away from the boundary ∂U_q. However, most internal rays make it all the way to the boundary.

We will make use of the following constructions, which are adapted from Petersen and Ryd [**PR**]. Let $\Pi : \mathbb{H} \to \mathbb{D}\smallsetminus\{0\}$ be the universal covering map $\Pi(Z) = e^{2\pi i Z}$ from the upper half-plane to the punctured unit disk. Thus the doubling map $Z \mapsto 2Z$ in the upper half-plane corresponds to the squaring map $w \mapsto w^2$ in the disk, and hence to the cubic map $z \mapsto f_q(z)$ in the region $U_q^{\mathbf{vis}}$.

Every periodic angle θ can be written as a ratio a/b with b odd. Fix some arbitrary constant $h_0 > 0$. Let $\widehat{\theta} \in \mathbb{Q}$ represent $\theta \in \mathbb{Q}/\mathbb{Z}$, and let S^+ and S^- be the sectors in $\mathbb{H}$ defined by

$$S^{\pm} \;:=\; \Big\{x + iy \;;\; 0 \;<\; bh_0\,\big|x - \widehat{\theta}\,\big| \;<\; y\Big\}\;,$$

with $x > \widehat{\theta}$ for S^+ but $x < \widehat{\theta}$ for S^-. The inverse Böttcher map $\mathfrak{b}_q^{-1}$ extends over the open set $\mathfrak{b}_q(U_q^{\mathbf{vis}})$ (the dark grey region in Figure 12(right)). Correspondingly, the composition $\mathfrak{b}_q^{-1} \circ \Pi$ extends to all of the upper half-plane except for countably many vertical line segments. (Compare Figure 13.) More precisely, if $\widehat{\mathfrak{b}}(q^2) = \Pi(\widehat{\theta} + ih)$, so that $\Pi(\widehat{\theta} + ih)$ corresponds to the critical point, then these are the vertical line segments joining $Z = x + iy$ to its real part x, where

$$x \;=\; (\widehat{\theta} + k)/2^m\;, \qquad y \;=\; h/2^m\;.$$

Here k and m are integers with $m \geq 0$. Since $2^m\widehat{\theta} \equiv \widehat{\theta} \mod 1/b$, it follows that the horizontal distance

$$\left|\frac{\widehat{\theta} + k}{2^m} - \widehat{\theta}\right| \;=\; \frac{\big|\widehat{\theta} - 2^m\widehat{\theta} + k\big|}{2^m}$$

is either zero or $\geq 1/(2^m b)$. Hence the slope of the line joining $\widehat{\theta}$ to $x + iy$ is always less than or equal to $b\,h$. (Compare Figure 13.) In particular, we have the following statement.

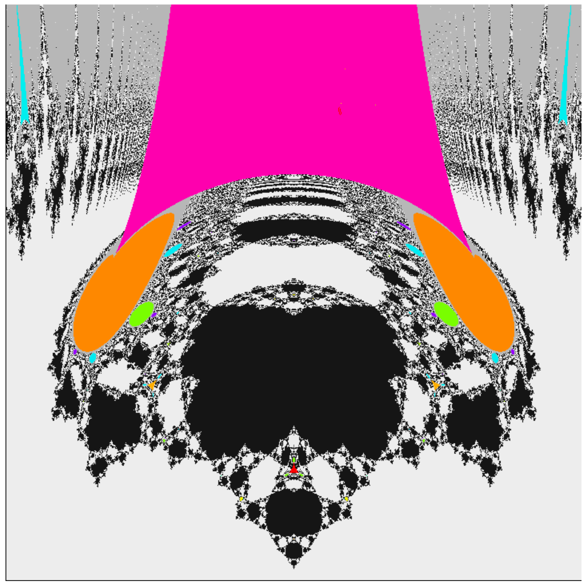

FIGURE 10. *Crown at the head of the upper $(1/2)$-tongue in the q-plane.*

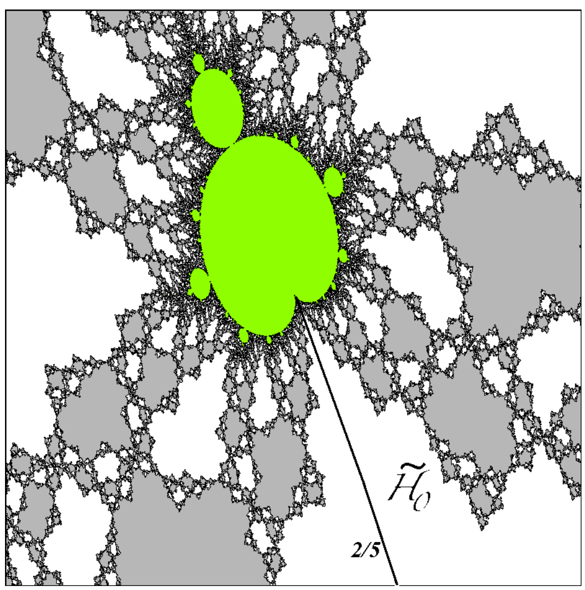

FIGURE 11. *Detail to the lower right of Figure* 10 *showing a small Mandelbrot set. The internal ray of angle* $2/5$ (drawn in by hand) *lands at the center of this figure, at the root point of this Mandelbrot set. Here* $\widetilde{\mathcal{H}}_0$ *is the white region to the lower right—The other white or grey regions are capture components.*[8]

Lemma 3.7. *If* $\widehat{\mathfrak{b}}(q^2) = \Pi(\widehat{\theta} + i\,h)$ *with* $h < h_0$*, then the map* $\mathfrak{b}_q^{-1} \circ \Pi$ *extends holomorphically to a map from the sector* S^+ *or* S^- *into the basin* $U_q^{\mathbf{vis}}$*.*

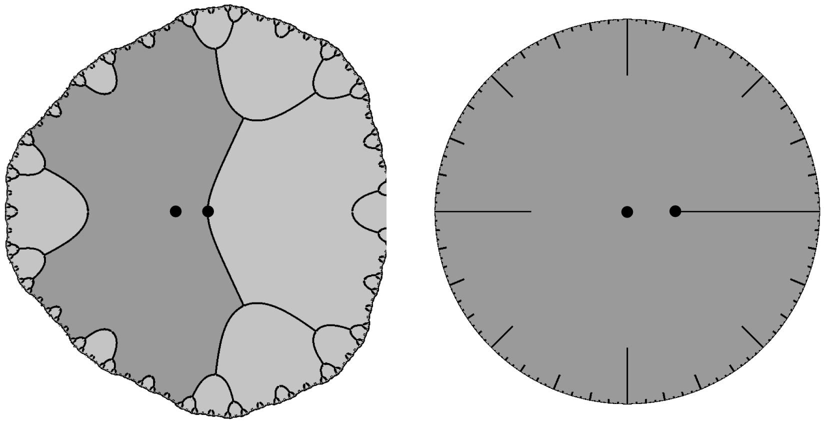

FIGURE 12. *On the left: the basin* U_q *. In this example,* $q > 0$ *so that the critical point* $c_+(q)$ *is on the positive real axis. Points which are visible from the origin are shown in dark gray, and the two critical points in* U_q *are shown as black dots. On the right: a corresponding picture in the disk of Böttcher coordinates. Note that the light grey regions on the left correspond only to slits on the right.*

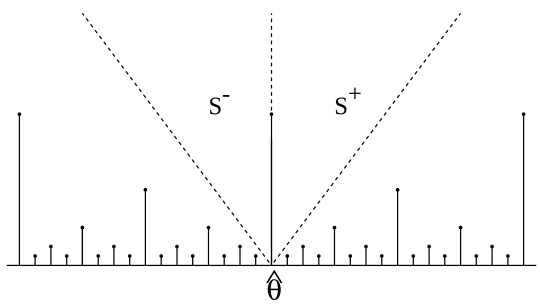

FIGURE 13. *The sectors* S^- *and* S^+ *in the upper half-plane.*

Proof. This follows from the discussion above. □

Now define the ***slanted rays*** $R_q^\pm \subset S^\pm \subset U_q$ by the formula

$$R_q^\pm := \left\{ \mathfrak{b}_q^{-1} \circ \Pi(x+iy) \quad ; \quad y = 2\, b\, h_0 \left| x - \widehat{\theta} \right| \right\} ,$$

with $x > \widehat{\theta}$ for R_q^+ but $x < \widehat{\theta}$ for R_q^-. Note that the definition makes sense as soon as $\mathfrak{b}_q^{-1} \circ \Pi$ extends to $S^\pm$. In particular, it makes sense when $q \notin \mathcal{H}_0$ or when $\widehat{\mathfrak{b}}(q^2) = \Pi(\widehat{\theta} + i\,h)$ with $h < h_0$. If the angle θ has period n under doubling, then evidently these slanted rays $R_q^\pm$ are periodic of period n for the map f_q.

Lemma 3.8. *The slanted rays* $R_q^\pm$ *land at periodic points* $z_q^\pm$ *in the boundary* ∂U_q.

---

[8]The regions shown are roughly $[-.8, .8] \times [1.7, 3.3]$ in Figure 10, and $[.347, .377] \times [2.07, 2.10]$ in Figure 11.

Proof. For $\mathfrak{b}^{-1}\big(\Pi(Z)\big) \in R^{\pm}$ we can write

$$Z = \widehat{\theta} \pm \frac{y}{2bh_0} + iy\ .$$

The hyperbolic distance in $S^{\pm}$ between Z and $\widehat{\theta} + 2^n\big(\pm\frac{y}{2bh_0} + iy\big)$ does not depend on $y > 0$. Since the map $\mathfrak{b}_q^{-1} \circ \Pi : S^{\pm} \to U_q$ is a contraction for the corresponding hyperbolic metrics, it follows that the hyperbolic distance in U_q between $z \in R_q^{\pm}$ and $f^{\circ n}(z)$ is uniformly bounded.

Near the boundary of U_q, the ratio between the Euclidean metric and the hyperbolic metric is comparable to the distance to the boundary. So, as $z \in R_q^{\pm}$ tends to the boundary of U_q, we have

$$\Big|z - f_q^{\circ(2n+1)}(z)\Big| \ \leq\ \mathrm{O}\Big(\mathrm{d}(\mathrm{z},\, \partial \mathrm{U_q})\Big) \underset{\mathrm{z} \to \partial \mathrm{U_q}}{\longrightarrow} 0\,.$$

The set of accumulation points of the ray $R_q^{\pm}$ is therefore contained in the finite set consisting of periodic points of f_q of period dividing n, which forces the ray to land at such a point. □

Proof of Theorem 3.4. We now assume that $q_k = \mathfrak{b}^{-1}\big(\Pi(\theta + ih_k)\big)$ where the angle θ is periodic under doubling, and where $h_k \searrow 0$. If the sequence $\{|q_k|\}$ does not diverge to infinity then, extracting a subsequence if necessary, we may assume that

- $q_k \to q \in \partial\mathcal{H}_0$,
- $h_k < h_0$ for all k,
- the slanted rays $R_{q_k}^{\pm}$ have Hausdorff limits $\mathcal{R}^{\pm}$ and
- the associated periodic limit points $z_{q_k}^{\pm}$ have limits $z^{\pm} \in \mathcal{R}^{\pm}$.

The intersection of $\mathcal{R}^{\pm}$ with the basin U_q coincides with the slanted rays $R_q^{\pm}$. Since $q \notin U_q$, the Böttcher map extends to an isomorphism $\mathfrak{b}_q^{-1} : \mathbb{D} \to U_q$. It follows that both slanted rays $R_q^{\pm}$ land at a common periodic point $z_q := z_q^+ = z_q^-$, which is also the landing point of the associated unslanted ray. In fact the hyperbolic distance between corresponding points in $\mathbb{H}$ is bounded, hence the estimates above show that the Euclidean distance between corresponding points of the left and right slanted rays tend to zero. This common landing point cannot be repelling since otherwise, for nearby parameters q_k, the slanted rays $R_{q_k}^{\pm}$ would land on a common repelling periodic point z_{q_k} close to z_q. So, according to the Snail Lemma, z_q must be a parabolic point of f_q. (See for example [**Mi2**, Corollary 16.3].)

Thus we have proved that every finite accumulation point of a periodic parameter ray is parabolic. Let n be the period of θ under doubling. Since the slanted rays $R_q^{\pm}$ landing on z_q have period n, each connected component of the immediate basin of z_q has exact period n. (Compare [**GM**, Lemma 2.4].) In other words, if the period of this orbit is p and its multiplier is an k-th root of unity, then $n = p\,k$. The set of all parabolic points with ray period n is a real algebraic variety, hence it has finitely many connected components, each of which has dimension either zero or one. Since the set of accumulation points of the θ-parameter ray is connected, it must either be a single parabolic point, or a subset of a one-dimensional variety of parabolic points. This proves Theorem 3.4. □

The following closely related lemma will be useful in §5.

Lemma 3.9. *If $s = q^2 \in \partial\mathcal{H}_0$ is a finite accumulation point for the periodic parameter ray R_θ of period n, then the dynamic internal ray of angle θ in U_q lands at a parabolic point of period dividing n.*

Proof. This follows from the discussion above. □

4. Semiconjugacies and Rotation Numbers

In order to understand the dynamics of maps in $\mathcal{H}_0$ we will need to compare three different $\mathbb{R}/\mathbb{Z}$-valued invariants for suitable points $z \in \widehat{\mathbb{C}}$ in the dynamical space, and also two $\mathbb{R}/\mathbb{Z}$-valued invariants for non-zero points $s = q^2 \in$ © in the moduli space.

• For fixed $q \neq 0$, the ***internal angle*** of any point $z \neq 0$ on the dynamic internal ray $R^q_\theta \subset U_0^{\mathbf{vis}}$ is defined to be $\theta = \theta(z)$. Note that $\theta\big(f_q(z)\big) = 2\,\theta(z)$.

• The ***external angle*** of a point on the antipodal ray $\mathcal{A}(R^q_\theta) \subset \mathcal{A}(U_0^{\mathbf{vis}})$ will be defined as this same θ. Again $\theta\big(f_q(z)\big) = 2\,\theta(z)$.

• There is a topological conjugacy $z \mapsto \xi(z)$ from the Julia set $J = J(f_q)$ onto $\mathbb{R}/\mathbb{Z}$ satisfying $\xi\big(f_q(z)\big) = 3\,\xi(z)$. This is unique up to orientation, and up to the involution $\xi \leftrightarrow \xi + 1/2$ which interchanges the labeling of the two Julia fixed points $\xi(0)$ and $\xi(1/2)$. We will choose the orientation so that ξ increases by one as z travels once around J in the positive direction, as seen from the origin. For the relation between θ and ξ, see Lemma 4.17 which uses the notation $\xi(z) = x$.

• As discussed in §3, every point $s = q^2$ on the parameter ray $\mathbf{R}_{\boldsymbol{\theta}}$ can be assigned the ***internal parameter angle*** (or ***critical angle***) $\boldsymbol{\theta}(s) = \theta\big(c_+(q)\big) = \theta_0$.

• In this section we will work in the dynamic space for f_q to define the ***formal rotation number*** $\mathbf{t} \in \mathbb{R}/\mathbb{Z}$ associated with any non-zero point $s = q^2 \in \mathcal{H}_0$. Then we will study the relationship between the internal angle $\boldsymbol{\theta} = \boldsymbol{\theta}(s)$ and this formal rotation number $\mathbf{t}$. (See Theorems 4.27, 4.25.)

But first there will be a digression to describe some needed terminology.

Rotation Numbers and Rotation Sets.

Definition 4.1. Any continuous map $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ will be called a **circle map**. For any circle map, there exists a continuous ***lift***, $\widehat{g} : \mathbb{R} \to \mathbb{R}$, unique up to addition of an integer constant, such that the square

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widehat{g}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{g} & \mathbb{R}/\mathbb{Z} \end{array}$$

is commutative. Here the vertical arrows stand for the natural map from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$. This lift satisfies the identity

$$\widehat{g}(y+1) \;=\; \widehat{g}(y) \,+\, d\;,$$

where d is an integer constant called the ***degree*** of g. The map g will be called ***monotone*** if

$$y \,<\, y' \quad\Longrightarrow\quad \widehat{g}(y) \,\leq\, \widehat{g}(y')\;.$$

Lemma 4.2. *Any monotone circle map* g *of degree* $d=1$ *has a well defined* ***rotation number*** $\mathrm{rot}(g)\in\mathbb{R}/\mathbb{Z}$*, as defined below.*

Proof. Choose some lift $\widehat{g}$. We will first prove that the ***translation number***

$$\lim_{n\to\infty}\frac{\widehat{g}^{\circ n}(x)-x}{n}\ \in\ \mathbb{R}$$

is well defined and independent of x. For any circle map of degree one, setting $\underline{a}_n = \min_x\big(\widehat{g}^{\circ n}(x)-x\big)$, the inequality $\underline{a}_{m+n}\ge\underline{a}_m+\underline{a}_n$ is easily verified. It follows that $\underline{a}_{km+n}\ge k\,\underline{a}_m+\underline{a}_n$. Dividing by $h=km+n$ and passing to the limit as $k\to\infty$ for fixed $m>n$, it follows that $\liminf\ \underline{a}_h/h\ \ge\ \underline{a}_m/m$. Therefore the *lower translation number*

$$\underline{\mathrm{transl}}(\widehat{g})\ =\ \lim_{n\to\infty}\underline{a}_n/n\ =\ \sup_n\ \underline{a}_n/n$$

is well defined. Similarly, the *upper translation number*

$$\overline{\mathrm{transl}}(\widehat{g})\ =\ \lim_{n\to\infty}\overline{a}_n/n\ =\ \inf_n\ \overline{a}_n/n$$

is well defined, where $\overline{a}_n=\max_x\big(\widehat{g}^{\circ n}(x)-x\big)$. Finally, in the monotone case it is not hard to check that $|\overline{a}_n-\underline{a}_n|\le 1$, and hence that these upper and lower translation numbers are equal. By definition, the ***rotation number*** $\mathrm{rot}(g)$ is equal to the residue class of the translation number modulo $\mathbb{Z}$. □

Definition 4.3 (**Rotation Sets).** (Compare Goldberg [**G**].) Fixing some integer $D\ge 2$, let $\mathbf{m}_D:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ be the map $\mathbf{m}_D(x)=D\,x$. A compact $\mathbf{m}_D$-invariant set $X=\mathbf{m}_D(X)\subset\mathbb{R}/\mathbb{Z}$ is called a **rotation set** for $\mathbf{m}_D$ if there exists a monotone degree one circle map g_X such that $g_X(x)=m_D(x)$ for every $x\in X$. It is not hard to see that such a map g_X is unique up to homotopy. Hence the rotation number $\mathrm{rot}(g_X)$ is uniquely defined. The rotation number of X is then defined as $\mathrm{rot(X)=rot(g_X)}\in\mathbb{R}/\mathbb{Z}$.

Any connected component of the complement $(\mathbb{R}/\mathbb{Z})\smallsetminus X$ will be called a ***gap***. A gap Γ of length ℓ will be called a ***minor gap*** if $\ell<1/D$, and will be called a ***major gap*** of ***multiplicity*** $n\ge 1$, if its length satisfies $n\le D\,\ell<n+1$.

Lemma 4.4. *Every rotation set for* $\mathbf{m}_D$ *with non-zero rotation number has exactly* $D-1$ *major gaps, counted with multiplicity.*

Proof. If Γ is a minor gap of length ℓ then it is not hard to see that both $\mathbf{m}_D$ and the associated monotone map g send Γ to a longer gap of length $D\,\ell$. Hence, after iterating we must eventually reach some major gap.

Now let Γ be a major gap of length ℓ. Then there are two possibilities: If $D\,\ell$ is an integer n, then the image of Γ under $\mathbf{m}_D$ will wrap n times around the circle; but Γ must map to a single point under g. But if $D\,\ell=n+\ell'$ with $0<\ell'<1$, then the image $g(\Gamma)$ must be a gap Γ' of length ℓ'. One can then check that the image of Γ under $\mathbf{m}_D$ must cover Γ' $n+1$ times, and cover the rest of the circle n times.

Now it is not hard to see that a generic point of the circle has n preimages under $\mathbf{m}_D$ in each major gap of multiplicity n, and also one preimage in a minor gap. It follows that the sum of the multiplicities of the major gaps is equal to $D-1$, as asserted. □

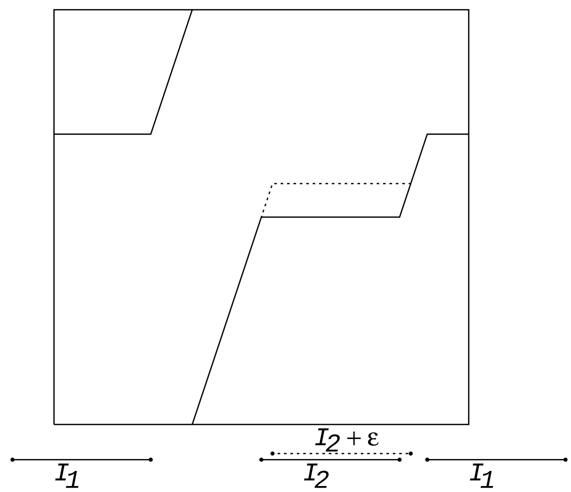

FIGURE 14. *A typical graph for* $g_{\mathcal{I}}$ *in the case* $D = 3$.

For any rotation set X, it follows from Lemma 4.4 that the complementary open set $(\mathbb{R}/\mathbb{Z}) \smallsetminus X$ contains $D-1$ disjoint open intervals of length $1/D$. Conversely, given $D-1$ disjoint open intervals of length $1/D$, we can construct an associated rotation set.

Lemma 4.5. *Let* $\mathcal{I} \subset \mathbb{R}/\mathbb{Z}$ *be an open set which is the union of disjoint open subintervals* $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_{D-1}$, *each of length precisely* $1/D$; *and let* $X_{\mathcal{I}} \subset \mathbb{R}/\mathbb{Z}$ *be the set of all points* $x \in \mathbb{R}/\mathbb{Z}$ *such that the forward orbit of* x *under* $\mathbf{m}_D$ *never enters* $\mathcal{I}$. *Then* $X_{\mathcal{I}}$ *is a rotation set for* $\mathbf{m}_D$.

Proof. The required monotone degree one map $g = g_{\mathcal{I}}$ will coincide with $\mathbf{m}_D$ outside of $\mathcal{I}$, and will map each subinterval $\mathcal{I}_i$ to a single point: namely the image under $\mathbf{m}_D$ of its two endpoints. The resulting map $g_{\mathcal{I}}$ is clearly monotone and continuous, and has degree one since the complement $(\mathbb{R}/\mathbb{Z}) \smallsetminus \mathcal{I}$ has length exactly $1/D$. □

Thus the rotation number of $X_{\mathcal{I}}$ is well defined. We will use the abbreviated notation $\mathrm{rot}(\mathcal{I})$ for this rotation number.

Remark 4.6. If we move one or more of the intervals $\mathcal{I}_i$ to the right by a small distance ϵ, as illustrated in Figure 14, then it is not hard to check that for each $x \in \mathbb{R}/\mathbb{Z}$ the image $g(x)$ increases by at most ϵD. Hence the rotation number $\mathrm{rot}(\mathcal{I})$ also increases by a number in the interval $[0, \epsilon D]$.

Remark 4.7 (**Rotation Sets under Doubling**). This construction is particularly transparent in the case $D = 2$. In fact, if $\mathcal{I}_c$ is the interval $(c, c+1/2) \subset \mathbb{R}/\mathbb{Z}$, then the correspondence

$$c \mapsto \mathrm{rot}(\mathcal{I}_c) \in \mathbb{R}/\mathbb{Z}$$

can itself be considered as a monotone circle map of degree one. Hence for each $t \in \mathbb{R}/\mathbb{Z}$ the collection of c with $\mathrm{rot}(\mathcal{I}_c) = t$ is non-vacuous: either a point or a closed interval. Let X be a rotation set with rotation number t. If t is rational with denominator p, then each gap in X must have period p under the associated monotone map g. Since there is only one major gap, there can only be one cycle of gaps, hence X must consist of a single orbit of period p. If ℓ_0 is the shortest gap length, then the sum of the gap lengths is

$$1 = \ell_0(1+2+2^2+\cdots+2^{p-1}) = (2^p-1)\ell_0 .$$

It follows that the shortest gap has length $\ell_0 = 1/(2^p-1)$; hence the longest gap has length $2^{p-1}/(2^p-1)$.

For each rational $t \in \mathbb{R}/\mathbb{Z}$ there is one and only one periodic orbit $X = X_t$ which has rotation number t under doubling. In fact we can compute X_t explicitly as follows. Evidently each point $x \in X_t$ is equal to the sum of the gap lengths between x and $2x$. But we have computed all of the gap lengths, and their cyclic order around the circle is determined by the rotation number t.

Note that an interval $\mathcal{I}_c$ of length $1/2$ has rotation number t if and only if it is contained in the longest gap for X_t which has length $2^{p-1}/(2^p-1) > 1/2$. Thus the interval of allowed c-values has length equal to

$$\frac{2^{p-1}}{2^p-1} - \frac{1}{2} = \frac{1}{2(2^p-1)} . \tag{5}$$

One can check[9] that the sum of this quantity (5) over all fractions $0 \le t = k/p < 1$ in lowest terms is precisely equal to $+1$. It follows that the set of c for which $\mathrm{rot}(\mathcal{I}_c)$ is irrational has measure zero.

Since the function $c \mapsto \mathrm{rot}(\mathcal{I}_c)$ is monotone, it follows that for each irrational t there is exactly one corresponding c-value. In particular, there is just one corresponding rotation set X_t. In the irrational case, it is not difficult to check that X_t has exactly one gap of length $1/2^i$ for each $i \ge 1$.

Remark 4.8 (Rotations Sets under Tripling). The situation for the tripling map $\mathbf{m}_3$ is more complicated. For example, there are three different minimal rotation sets with rotation number $1/2$, namely

$$\{1/8,\, 3/8\}, \quad \{1/4,\, 3/4\}, \quad \text{and} \quad \{5/8,\, 7/8\},$$

(There are also two maximal rotation sets with rotation number $1/2$.) However, if we restrict attention to rotation sets which are "self-antipodal", in the sense that $X = X + 1/2$, then the situation is completely analogous to the situation for the doubling map. In particular, a similar argument shows that there is one and only one such rotation set for each rotation number. This special case will play an important role in our discussion.

Remark 4.9 (Semiconjugacy). By a ***semiconjugacy*** from a circle map f to a circle map g we will always mean a monotone degree one map σ satisfying $g\big(\sigma(x)\big) = \sigma\big(f(x)\big)$ for all x. Lemmas 4.4 and 4.5 have precise analogs in the study of semiconjugacy. Thus if f is the map $\mathbf{m}_D$ and if g is monotone of degree $D-\delta$, then any such semiconjugacy has δ gaps of length $\ge 1/D$, counted with multiplicity. Conversely, given a union $\mathcal{I}$ of δ disjoint open intervals of length $1/D$, there is a uniquely defined monotone circle map $g_{\mathcal{I}}$ of degree δ which sends

[9] Using the identity $1/(2^p-1) = 1/2^p + 1/2^{2p} + 1/2^{3p} + \cdots$, one reduces the required sum to the form $\sum_{0\le k<p} 1/2^{p+1}$, or in other words to $\sum_1^\infty p/2^{p+1}$. Differentiating the identity $\sum_0^\infty x^p = 1/(1-x)$, we obtain $\sum p\, x^{p-1} = 1/(1-x)^2$. Now multiplying by x^2 and setting x equal to $1/2$ the conclusion follows.

each of these intervals to a point, but coincides with $\mathbf{m}_D$ elsewhere.[10] Evidently this construction describes a semiconjugacy from $\mathbf{m}_D$ to $g_{\mathcal{I}}$.

We now return to the study of our family of rational maps f_q.

Example 4.10. Consider the situation described in the proof of Theorem 3.4. Let $s = q^2$ be a non-zero point in the central hyperbolic component $\mathcal{H}_0$, and let U_q be the basin of zero for f_q. (Compare Figure 12.) Then U_q is a topological disk, and the map f_q on the Julia set ∂U_q is topologically conjugate to the tripling map $\mathbf{m}_3$ on $\mathbb{R}/\mathbb{Z}$. More explicitly, there is a topological conjugacy

$$\eta : \mathbb{R}/\mathbb{Z} \xrightarrow{\cong} \partial U_q \,, \qquad \text{satisfying} \qquad \eta(3\,x) = f_q\big(\eta(x)\big) \,,$$

which is unique up to the automorphism $x \leftrightarrow x + 1/2$ of $\mathbb{R}/\mathbb{Z}$. (Here we always assume that η is orientation preserving, where ∂U_q is oriented as the boundary of U_q.)

Any visible point of this Julia set is either the landing point of a unique internal ray, with angle say θ, or is the left or right limit of such landing points. (Compare Definition 3.6.) Recall that the map f_q carries the ray of angle θ to the ray of angle 2θ. Thus, if V is the set of parameter values $x \in \mathbb{R}/\mathbb{Z}$ corresponding to points of ∂U_q which are visible from the origin, then there is a well defined semiconjugacy $\phi_V : V \xrightarrow{\cong} \mathbb{R}/\mathbb{Z}$ which satisfies $\phi_V(3\,x) = 2\,\phi_V(x)$ for all $x \in V$. Here $\phi_V^{-1}(\theta) \in V$ is well defined except at the countably infinite set of internal angles which correspond to critical or precritical points in $U_q \smallsetminus \{0\}$, but ϕ_V^{-1} has a jump discontinuity at each θ in this countable set. The left and right limits

$$a(\theta) \;=\; \lim_{\epsilon \searrow 0} \phi_V^{-1}(\theta - \epsilon) \,, \qquad b(\theta) \;=\; \lim_{\epsilon \searrow 0} \phi_V^{-1}(\theta + \epsilon)$$

are well defined since ϕ_V^{-1} preserves the cyclic order.[11] (Compare Figure 15.)

Note that V is compact and invariant under multiplication by three, so that $V = \mathbf{m}_3(V)$. It follows that V is totally disconnected. (If V contained an interval I of length $\ell > 0$, then it would also contain successive images $f_q^{\circ n}(I)$ of length $3^n \ell$, and hence would have to be the entire circle.)

Each connected component of the complement $\mathbb{R}/\mathbb{Z} \smallsetminus V$ is an open interval $\big(a(\theta),\, b(\theta)\big)$ consisting of all x such that the three angles $a(\theta),\, x,\, b(\theta)$ are in positive cyclic order. Here θ varies over the countable set consisting of all critical or precritical internal angles. Each such open interval $\big(a(\theta),\, b(\theta)\big)$ is called a ***gap*** in the set of visible points.

Lemma 4.11. *If $\Gamma_\theta = (a, b)$ is any gap of length $\ell < 1/3$, then the tripling map $\mathbf{m}_3$ sends Γ_θ homeomorphically onto the gap $\Gamma_{2\theta} = (3a, 3b)$. However, if Γ_θ has length $\ell > 1/3$ (but necessarily less than $2/3$), then its image under $\mathbf{m}_3$ covers the gap $\Gamma_{2\theta} = (3a, 3b)$ twice, and covers the rest of the circle once. Finally, if Γ_θ has length equal to $1/3$, so that $3a = 3b$, then $\mathbf{m}_3$ wraps Γ_θ precisely once around the circle. In all cases, there is one and only one gap Γ_{θ_0} which has length $\ell(\Gamma_{\theta_0}) \geq 1/3$, and all other gaps are iterated preimages of Γ_{θ_0}.*

[10] It can be shown that any monotone circle map of degree $\delta > 1$ is semiconjgate to the standard map $\mathbf{m}_\delta$. This semiconjugacy is unique up to a choice of which fixed point maps to zero.

[11] Three points $x_1, x_2, x_3 \in \mathbb{R}/\mathbb{Z}$ are in ***positive cyclic order*** if we can choose lifts $\widehat{x}_j \in \mathbb{R}$ so that $\widehat{x}_1 < \widehat{x}_2 < \widehat{x}_3 < \widehat{x}_1 + 1$.

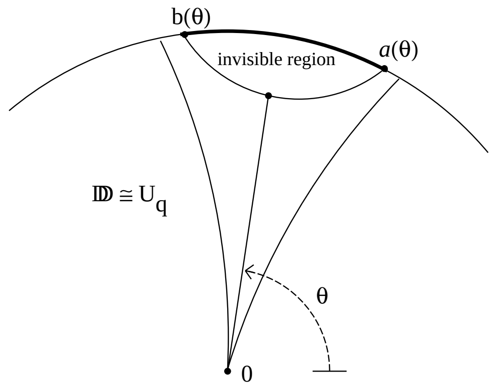

FIGURE 15. *Schematic picture after mapping the basin U_q conformally onto the unit disk, showing three internal rays. Here the middle internal ray of angle θ bifurcates from a critical or precritical point. The gap $\big(a(\theta),\, b(\theta)\big)$ has been emphasized.*

Here θ_0 can be characterized as the angle of the ray which bifurcates from the free critical point.

Proof. First suppose that the gap $\Gamma_\theta = (a,\, b)$ has length $\ell < 1/3$. Choose two neighboring internal rays of angle $\theta \pm \epsilon$ which land at visible points of ∂U_q, as shown in Figure 15. If ϵ is small enough, then the interval

$$I^+ \;=\; [\phi_V^{-1}(\theta - \epsilon)\,,\; \phi_V^{-1}(\theta + \epsilon)]$$

between the landing points will also have length $\ell^+ < 1/3$. Let Δ be the triangular region bounded by these two rays, together with this boundary interval I^+. Then f_q maps Δ homeomorphically onto the triangular region Δ' which is bounded by the internal rays of angle $2(\theta \pm \epsilon)$ together with the boundary segment of length $3\ell^+ < 1$ which joins their landing points. In fact f_q clearly maps the boundary $\partial\Delta$ homeomorphically onto $\partial\Delta'$. Since f_q is an open mapping, it must map the interior into the interior, necessarily with degree one. It follows that there can be no critical point in the interior of Δ, so the θ-ray must bifurcate off a precritical point.

It follows that the gap Γ_θ of length ℓ maps homeomorphically onto the gap $\Gamma_{2\theta}$. Continuing taking forward images, we must eventually reach a gap Γ_{θ_0} of length $\ell_0 \ge 1/3$. There can be only one such gap. For if there were two of them, then every point of $\mathbb{R}/\mathbb{Z}$ would have one preimage in each under the tripling map. This is impossible, since every visible point must have two distinct visible preimages. (A similar argument shows that Γ_{θ_0} must have length $\ell_0 < 2/3$.) It follows that θ_0 must be the angle of the internal ray which bifurcates on the free critical point. The rest of the argument is straightforward. □

Remark 4.12. It follows easily that the map $\phi_V : V \to \mathbb{R}/\mathbb{Z}$ extends uniquely to a continuous circle map $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ of degree one which takes a constant value,

$$\varphi\big(a(\theta)\,,\, b(\theta)\big) \;=\; \theta\,,$$

on every connected component $\Gamma_\theta = \big(a(\theta), b(\theta)\big)$ of $\mathbb{R}/\mathbb{Z}\smallsetminus V$. (Compare Figure 16.) In fact, the preimage $\varphi^{-1}(\theta)$ is precisely equal to the closure $\overline{\Gamma}_\theta = \big[a(\theta)\,,\,b(\theta)\big]$.

Remark 4.13. For each gap Γ_θ of length ℓ, the preimage $\mathbf{m}_3^{-1}(\Gamma_\theta)$ has three connected components of length $\ell/3$. If $\ell < 1/2$, then two of these components are themselves gaps Γ_{θ_1} and Γ_{θ_2}, where $\mathbf{m}_2(\theta_1) = \mathbf{m}_2(\theta_2) = \theta$. (Hence $\theta_2 \equiv \theta_1 + 1/2 \pmod{\mathbb{Z}}$.) The third preimage is necessarily a subset of the largest gap Γ_{θ_0}. Similarly, for each visible x, the preimage $\mathbf{m}_3^{-1}(x)$ consists of two visible points, together with one point in the critical gap. Note also that the largest gap always contains one of the two fixed points, 0 and $1/2$.

Lemma 4.14. *If θ_0 is not periodic for $\mathbf{m}_2$, then the length of Γ_{θ_0} is $1/3$. If θ_0 is periodic of period p for $\mathbf{m}_2$, then the length of Γ_{θ_0} is* $\dfrac{3^{p-1}}{3^p - 1}$.

Proof. Let ℓ be the length of the largest gap $\Gamma_{\theta_0} = (a, b)$. As noted above, $1/3 \le \ell < 2/3$. In the case $\ell > 1/3$, the gap $\Gamma_{2\theta_0} = \big(\mathbf{m}_3(a),\ \mathbf{m}_3(b)\big)$ has length $3\,\ell - 1$. Since every gap is equal to Γ_{θ_0} or to some iterated preimage of Γ_{θ_0}, we must be in the periodic case, $\theta_0 = 2^p\theta_0 \in \mathbb{R}/\mathbb{Z}$ for some $p \ge 1$. Then $\mathbf{m}_3^{\circ p-1}(\Gamma_{2\theta_0}) = \Gamma_{\theta_0}$, which implies that

$$3^{p-1}(3\ell - 1) \;=\; \ell\,, \qquad \text{hence} \qquad \ell \;=\; \frac{3^{p-1}}{3^p - 1}\,.$$

In the remaining case, $\ell = 1/3$, there can be no gap $\Gamma_{2\theta}$. In other words, the internal ray of angle $2\theta_0$ passes through the critical value and lands at the point $\varphi^{-1}(2\theta_0)$. Thus, in this case, θ_0 is not periodic under doubling. □

Remark 4.15. The length of an arbitrary gap Γ_θ is equal to the length of Γ_{θ_0} divided by 3^m, where m is the smallest integer with $2^m\theta = \theta_0 \in \mathbb{R}/\mathbb{Z}$. Whether or not θ_0 is periodic, we can also write

$$\ell(\Gamma_\theta) \;=\; \sum_{2^m\theta = \theta_0} 1/3^{m+1}\,,$$

to be summed over *all* such m. In all cases, the sum of the lengths of all of the gaps is equal to $+1$. In other words, $V = \overline{V}$ is always a set of measure zero.

Remark 4.16 (Computation of φ^{-1}). Although it would be natural to work with the continuous function φ, it is actually easier to study the discontinuous function φ^{-1}, which can be explicitly computed as follows.

To fix our ideas, let us assume that $\widehat{\boldsymbol{\theta}} \ne 0$, so that we are not in the period one case. Then exactly one of the two fixed points of $\mathbf{m}_3$ must be visible.[12] They cannot both be visible, since the map $\mathbf{m}_2$ has only one fixed point. If neither were visible, then each one would belong to a gap, which would map onto a larger gap containing the same fixed point under $\mathbf{m}_3$. Since each gap eventually maps to the critical gap, it would follow that the critical gap contains both fixed points. But this is impossible: Since we have excluded the period one case, it follows from Lemma 4.14, that all gaps have length $\le\ 3^{2-1}/(3^2 - 1) < 1/2$.

[12] In the period one case the critical gap is either $(0,\, 1/2)$ or $(1/2,\, 1)$, and both fixed points 0 and $1/2$ are visible boundary points. (This case is illustrated in Figure 12. However note that the Julia set fixed points are near the top and bottom of this figure—not at the right and left.) In this case, the visible set V is a classical middle third Cantor set. The computation of φ^{-1} is not difficult, and will be left to the reader.

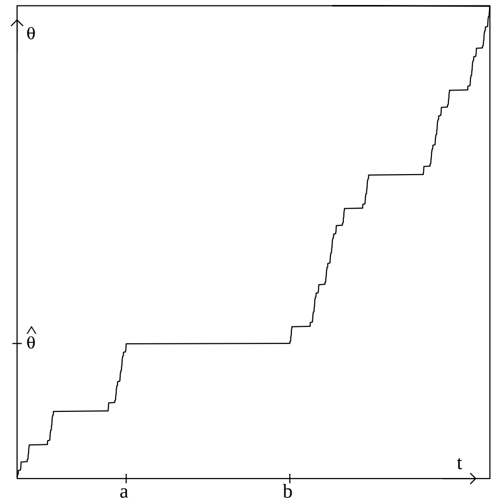

FIGURE 16. *Graph of the function* $t \mapsto \theta = \varphi(t)$ *for the case* $\widehat{\boldsymbol{\theta}} = 2/7$. *Note that* φ *maps the major gap* $\Gamma_{\widehat{\boldsymbol{\theta}}} = (a,\, b) = (6/26,\, 15/26)$ *to the point* $\widehat{\boldsymbol{\theta}}$. *In this example, since* $\widehat{\boldsymbol{\theta}}$ *has period* $p = 3$ *under doubling, it follows that* a *and* b *have period three under tripling.*

After adding $1/2$ to the coordinate x if necessary, we may assume that the fixed point zero is visible, and hence satisfies $\varphi(0) = 0$. The other fixed point $1/2$ must belong to some gap, and hence must belong to the critical gap $\Gamma_{\widehat{\boldsymbol{\theta}}}$. It will be convenient to describe the coordinate x by its base three expansion

$$x \;=\; .\,x_1x_2x_3\cdots \;=\; x_1/3 \,+\, x_2/9 \,+\, x_3/27 \,+\, \cdots\,,$$

where $x_m \in \{0,\,1,\,2\}$. Note that the map $x \mapsto \mathbf{m}_3(x) \in \mathbb{R}/\mathbb{Z}$ then corresponds to a left shift. Let us introduce the temporary notation

$$\mathbf{min} \;=\; \min(\widehat{\boldsymbol{\theta}},\ 1/2)\,, \qquad \mathbf{max} \;=\; \max(\widehat{\boldsymbol{\theta}},\ 1/2)\,,$$

so that $0 < \mathbf{min} \le \mathbf{max} < 1$.

Lemma 4.17. *If the internal ray of angle* θ *lands on* ∂U_q *then, assuming as above that* $\varphi(0) = 0$, *we have the formula*[13]

$$\varphi^{-1}(\theta) \;=\; x_1/3 \,+\, x_2/9 \,+\, x_3/27 \,+\, \cdots\,,$$

where x_m *takes the value zero, one, or two according as the angle* $2^m\theta$ *belongs to the interval* $[0,\,\mathbf{min})$, $[\mathbf{min}\,,\ \mathbf{max})$, *or* $[\mathbf{max}, 1)$ *modulo* $\mathbb{Z}$. *If* $2^m\theta = \widehat{\boldsymbol{\theta}}$ *in* $\mathbb{R}/\mathbb{Z}$, *then the function* φ^{-1} *has a jump discontinuity at* θ *of size equal to the length of the gap* Γ_θ*; but at all other points the function* φ^{-1} *is continuous.*

Proof. Since the three points 0, $1/3$, $2/3 \in \mathbb{R}/\mathbb{Z}$ all map to the visible point zero, it follows that two of the three are visible, and the third belongs to the

[13] Compare [**Mi2**, Lemma 2.9], where the same family of semiconjugacies is studied in a somewhat different context; namely the study of cubic polynomial maps with one critical point fixed but the other critical point outside its immediate basin.

critical gap $\Gamma_{\widehat{\boldsymbol{\theta}}}$. To fix our ideas, suppose that $2/3$ is visible. It then follows easily that

$$\varphi(1/3) \;=\; \widehat{\boldsymbol{\theta}} \;<\; \varphi(2/3) \;=\; 1/2\,.$$

Suppose that $x \;=\; x_1/3 + x_2/9 + \cdots$, where we may assume that the sequence $\{x_m\}$ does not terminate with a string of infinitely many twos. Then a brief computation shows that

$$\frac{x_1}{3} \;\le\; x \;<\; \frac{x_1+1}{3}\;,$$

Hence $\varphi(x)$ belongs to the interval

$$[0\,,\; \widehat{\boldsymbol{\theta}}) \quad \text{or} \quad [\widehat{\boldsymbol{\theta}},\; 1/2) \quad \text{or} \quad [1/2\,,\; 1) \qquad \text{mod } \mathbb{Z}$$

according as x_1 takes the value $0\,,1\,,$ or 2. Similarly, $\varphi(3^m x) = 2^m\varphi(x)$ belongs to one of these three intervals according to the value of x_{m+1}. The corresponding property for φ^{-1} follows immediately.

It might seem that this expression for φ^{-1} should have a discontinuity whenever $2^m\theta = 1/2$, and hence $2^n\theta = 0$ for $n > m$, but a brief computation shows that the discontinuity in $x_m/3^m$ is precisely canceled out by the discontinuity in $x_{m+1}/3^{m+1} + \cdots$. Further details will be left to the reader. □

Corollary 4.18. *The left hand endpoint* $a(\widehat{\boldsymbol{\theta}})$ *of the critical gap is continuous from the left as a function of the critical angle* $\widehat{\boldsymbol{\theta}}$. *Similarly, the right hand endpoint* $b(\widehat{\boldsymbol{\theta}})$ *is continuous from the right, Both* $a(\widehat{\boldsymbol{\theta}})$ *and* $b(\widehat{\boldsymbol{\theta}})$ *are monotone as functions of* $\widehat{\boldsymbol{\theta}}$.

Proof. This follows easily from the Lemma. □

Remark 4.19. In the aperiodic case, where the critical gap $\Gamma_{\widehat{\boldsymbol{\theta}}}$ has length $1/3$, the set V of Example 4.10 can clearly be identified with the set $X_{\mathcal{I}}$ of Lemma 4.5. Simply take the degrees to be $D = 3$ and $d = 2$, and take $\mathcal{I}$ to be the critical gap $\Gamma_{\widehat{\boldsymbol{\theta}}}$. More generally, if $\Gamma_{\widehat{\boldsymbol{\theta}}}$ has length $\ell > 1/3$, then we can take $\mathcal{I}$ to be *any* open interval of length $1/3$ which is contained in the critical gap.

Lemma 4.20. *If* $\mathcal{I}$ *is an open interval of length* $1/3$ *which is contained in* $\Gamma_{\widehat{\boldsymbol{\theta}}}$. *Then the set* V *of visible points is equal to the set* $X_{\mathcal{I}}$ *of Lemma* 4.5.

Proof. It is enough to observe that the orbit of any point that enters $\Gamma_{\widehat{\boldsymbol{\theta}}}$ eventually enters $\mathcal{I}$. Let H be either one of the two components of $\Gamma_{\widehat{\boldsymbol{\theta}}}\smallsetminus\mathcal{I}$. Then,

$$\text{Length}(H) \;\le\; \text{Length}(\Gamma_{\widehat{\boldsymbol{\theta}}}) - \text{Length}(\mathcal{I}) \;=\; \frac{3^{p-1}}{3^p-1} - \frac{1}{3} \;=\; \frac{1}{3(3^p-1)}$$

and therefore

$$(3^p-1)\text{Length}(H) \;<\; \frac{1}{3} \;=\; \text{Length}(\mathcal{I}).$$

It follows that

$$\text{Length}\big(\mathbf{m}_3^{\circ p}(H)\big) \;=\; 3^p\,\text{Length}(H) \;<\; \text{Length}(H) + \text{Length}(\mathcal{I}).$$

Since the boundary points of $\Gamma_{\widehat{\boldsymbol{\theta}}}$ are periodic of period p, this implies that

$$\mathbf{m}_3^{\circ p}(H) \;\subset\; H \cup \mathcal{I}\;.$$

Since $\mathbf{m}_3^{\circ p}$ multiplies distance by 3^p, the orbit of any point in H under $\mathbf{m}_3^{\circ p}$ eventually enters $\mathcal{I}$, as required. □

We now concentrate on the case $D = 3$. Thus we assume that $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where the $\mathcal{I}_i$ are disjoint open intervals of length $1/3$. Let $t = t_{\mathcal{I}} = \mathrm{rot}(g_{\mathcal{I}})$ be the associated rotation number. Set $X_1 = X_{\mathcal{I}_1}$, $X_2 = X_{\mathcal{I}_2}$, so that $X = X_{\mathcal{I}} = X_1 \cap X_2$. Note that there are semiconjugacies

$$\varphi_1 : X_1 \to \mathbb{R}/\mathbb{Z}\ , \qquad \varphi_2 : X_2 \to \mathbb{R}/\mathbb{Z}\ ,$$

both satisfying $\varphi_i(3\,x) = 2\varphi_i(x)$. In most cases, X is a minimal set under multiplication by 3, but there are exceptions:

Proposition 4.21. *If the rotation number $t = t_{\mathcal{I}}$ is irrational, then $X = X_{\mathcal{I}}$ is a minimal set under the action of $\mathbf{m}_3$. If t is rational and either $\varphi_1(\mathcal{I}_1)$ or $\varphi_2(\mathcal{I}_2)$ is not periodic with rotation number t under $\mathbf{m}_2$, then X consists of a single cycle for $\mathbf{m}_3$. However, if t is rational and both $\varphi_1(\mathcal{I}_1)$ and $\varphi_2(\mathcal{I}_2)$ are periodic with rotation number t under $\mathbf{m}_2$, then X may consist of either one or two cycles for $\mathbf{m}_3$.*

Proof. The semiconjugacies $\varphi_1 : X_1 \to \mathbb{R}/\mathbb{Z}$ and $\varphi_2 : X_2 \to \mathbb{R}/\mathbb{Z}$ send $X_1 \cap X_2$ to compact sets $Y_1 \subset \mathbb{R}/\mathbb{Z}$ and Y_2 which are invariant by $\mathbf{m}_2$ and have the same rotation number t. It follows from Remark 4.7, that there is a unique such compact set Y. (Compare [**G**].) Therefore $Y_1 = Y_2 = Y$.

$$\begin{tikzcd}
\mathbb{R}/Z \arrow[d, "\varphi_1"'] & \supset & X_1 & \supset & X \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \subset & X_2 & \subset & \mathbb{R}/\mathbb{Z} \arrow[d, "\varphi_2"] \\
\mathbb{R}/\mathbb{Z} & & \supset & Y_1 & = & Y_2 & \subset & & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

If t is irrational, Y is a Cantor set and the orbit of any $\theta \in Y$ under $\mathbf{m}_2$ is dense in Y. It follows that the orbit of any $x \in X$ under $\mathbf{m}_3$ is dense in X.

Assume now that t is rational. Then Y is a finite set. If $\theta_1 := \varphi_1(\mathcal{I}_1)$ does not belong to Y, then $\varphi_1 : X \to Y$ is a homeomorphism. In that case, X consists of a single cycle for $\mathbf{m}_3$. The result is similar if $\theta_2 := \varphi_2(\mathcal{I}_2)$ does not belong to Y.

Let us finally assume that t is rational and both θ_1 and θ_2 belong to the cycle Y of $\mathbf{m}_2$. Then $\varphi_1^{-1}(Y) \subset X_1$ is the union of 2 cycles: the two cycles of the periodic points in the boundary of Γ_{θ_1}. Similarly, $\varphi_2^{-1}(Y) \subset X_2$ is the union of 2 cycles: the two cycles of the periodic points in the boundary of Γ_{θ_2}. Either $\varphi_1^{-1}(Y) = \varphi_2^{-1}(Y)$ and X contains two periodic cycles; or $\varphi_1^{-1}(Y) \neq \varphi_2^{-1}(Y)$ and X contains a single periodic cycle. $\square$

Self-Antipodal Rotation Sets. Consider a point $s \neq 0$ in the central hyperbolic component $\mathcal{H}_0$, and choose one of the two holomorphically conjugate maps f_q with $q^2 = s$. Let $V_1 \subset \mathbb{R}/\mathbb{Z}$ be the compact set of parameters corresponding to points in the Julia set of f_q which are visible from the origin. (Compare Example 4.10.) Then $V_2 = V_1 + 1/2$ will parametrize the points which are visible from infinity, and the intersection $X = V_1 \cap V_2$ will parametrize points visible from both zero and infinity. (See Figures 17 and 18 for an example.)

Let Γ_1 be the major gap in $\mathbb{R}/\mathbb{Z} \smallsetminus V_1$ and let $\Gamma_2 = \Gamma_1 + 1/2$. As in Lemma 4.20, we can choose an open interval $\mathcal{I}_1$ of length $1/3$ within Γ_1. Let $\mathcal{I}_2 = \mathcal{I}_1 + 1/2 \subset \Gamma_2$. It follows from Lemma 4.20 that $V_1 = X_{\mathcal{I}_1}$ and $V_2 = X_{\mathcal{I}_2}$,

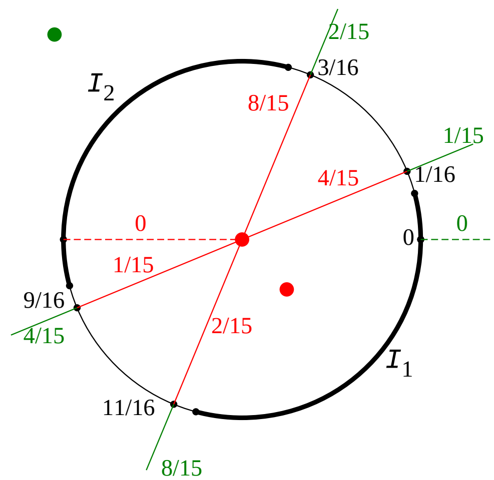

FIGURE 17. *Here* $\mathcal{I}_2$ *can be any open interval* $(x,\, x+1/3)$ *with* $3/16 \le x < x+1/3 \le 9/16$, *and* $\mathcal{I}_1 = \mathcal{I}_2 + 1/2$. *The associated rotation number is* $1/4$. *The interior angles* (*in red*) *and the exterior angles* (*in green*) *double under the map, while the angles around the Julia set* (*in black*) *multiply by* 3. *Heavy dots indicate critical points.*

and it then follows from Lemma 4.5 that the set $X = V_1 \cap V_2$ has a well defined rotation number under angle tripling.

Definition 4.22 (**Formal Rotation Number: the dynamic definition**)**.** For any point $s \in \mathcal{H}_0 \smallsetminus \{0\}$, the rotation number $\mathbf{t}$ associated with the set

$$X \;=\; V_1 \cap V_2 \;\subset\; \mathbb{R}/\mathbb{Z}\,,$$

representing points in the Julia set which are visible from both zero and infinity, will be called the ***formal rotation number***[14] of s.

It is not hard to show that the image $\varphi_1(X) = \varphi_2(X) \subset \mathbb{R}/\mathbb{Z}$ is a rotation set under angle doubling with the same rotation number $\mathbf{t}$. Intuitively, this means the entire configuration consisting of the rays from zero and infinity with angles in this $\mathbf{m}_2$-rotation set, together with their common landing points in the Julia set, has a well defined rotation number $\mathbf{t}$ under the map f_q. For example, in Figure 18 the light grey region is the union of interior rays, while the dark grey region is the union of rays from infinity. These two regions have four common boundary points, labeled by the points of X. The entire configuration of four rays from zero and four rays from infinity maps onto itself under f_q with combinatorial rotation number $\mathbf{t}$ equal to $1/4$.

Note that the entire major gap Γ_1 maps to a single point under φ_1. The image $\varphi_1(\Gamma_1) = \varphi_2(\Gamma_2)$ can be identified with the critical angle $\boldsymbol{\theta} \in \mathbb{R}/\mathbb{Z}$.

Lemma 4.23. *The formal rotation number* $\mathbf{t}$ *is continuous and monotone* (*but not strictly monotone*) *as a function* $\psi(\boldsymbol{\theta})$ *of the critical angle* $\boldsymbol{\theta}$. *Furthermore,*

[14]This definition will be extended to points outside of $\mathcal{H}_0$ in Definition 5.4. (See also Remark 6.13.)

FIGURE 18. *A corresponding Julia set, showing the four points visible from both zero and infinity for a map with formal rotation number 1/4. The white points are not visible from either zero or infinity, The angles of the rays from infinity, counterclockwise starting from the top, are 1/15, 2/15, 4/15 and 8/15, while the corresponding rays from zero have angles 4/15, 8/15, 1/15 and 2/15. The heavy dots represent the free critical points.*

as $\boldsymbol{\theta}$ *increases from zero to one, the formal rotation number also increases from zero to one.*

Proof. First note that the function $\boldsymbol{\theta} \mapsto \mathbf{t} = \psi(\boldsymbol{\theta})$ is well defined and single valued. In fact, given $\boldsymbol{\theta}$ the associated major gap Γ is well defined. (Compare Figure 15.) The associated $\mathbf{t}$ can then be described as the rotation number under $\mathbf{m}_3$ for the set of orbits which never enter the set $\Gamma \cup (\Gamma + 1/2)$. Thus $\mathbf{t}$ is uniquely determined. Setting $\Gamma = \big(a(\boldsymbol{\theta}),\, b(\boldsymbol{\theta})\big)$, and setting $\mathcal{I} = \big(a(\boldsymbol{\theta}),\, a(\boldsymbol{\theta}) + 1/3\big) \subset \Gamma$, we have $V_1 = X_{\mathcal{I}}$ by Lemma 4.20. Hence continuity from the left follows from Corollary 4.18 together with Remark 4.6. Continuity from the right and monotonicity follow by a similar argument. $\square$

We can describe this function $\theta \mapsto \mathbf{t} = \psi(\theta)$ explicitly as follows. (See Theorem 4.25 for a more conceptual description.) As in Remark 4.16, it is easier to describe the inverse function, which is graphed in Figure 19.

Definition 4.24 (**Balance**)**.** For each $t \in \mathbb{R}/\mathbb{Z}$, let X_t be the unique rotation set with rotation number t under the doubling map $\mathbf{m}_2$. If t is rational with denominator $p > 1$, then the points of X_t can be listed in numerical order within the open interval $(0, 1)$ as $x_1 < x_2 < \cdots < x_p$. If p is odd, then there is a unique middle element $x_{(p+1)/2}$ in this list. By definition, this middle element will be called the ***balanced*** angle in this rotation set. (For the special case $p = 1$, the unique element $0 \in X_0$ will also be called balanced.)

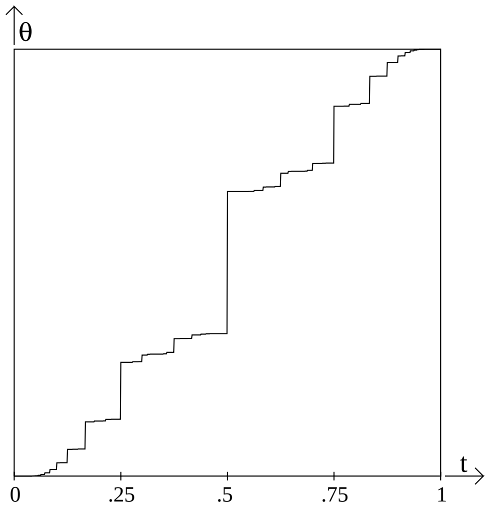

FIGURE 19. *Graph of the critical angle* $\boldsymbol{\theta}$ *as a function of the formal rotation number* $\mathbf{t}$.

On the other hand, if p is even, then there is no middle element. However, there is a unique pair $\{x_{p/2}\,,\,x_{1+p/2}\}$ in the middle. By definition, this will be called the ***balanced-pair*** for this rotation set.

Finally, if t is irrational, then X_t is topologically a Cantor set. However, every orbit in X_t is uniformly distributed with respect to a uniquely defined invariant probability measure. By definition, the ***balanced point*** in this case is the unique x such that both $X_t\cap(0,x)$ and $X_t\cap(x,1)$ have measure $1/2$. (The proof that x is unique depends on the easily verified statement that x cannot fall in a gap.)

Here is a conceptual description of this function $\mathbf{t}\mapsto\boldsymbol{\theta}$.

Theorem 4.25. *If* $\mathbf{t}$ *is either irrational, or rational with odd denominator, then* $\boldsymbol{\theta}$ *is equal to the unique balanced angle in the rotation set* $X_{\mathbf{t}}$ *. However, if* $\mathbf{t}$ *is rational with even denominator, then there is an entire closed interval of corresponding* $\boldsymbol{\theta}$*-values. This interval is bounded by the unique balanced-pair in* X_t *.*

We first prove one special case of this theorem.

Lemma 4.26. *Suppose that the formal rotation number is* $\mathbf{t}=m/(2n{+}1)$ *, rational with odd denominator. Then the corresponding critical angle* $\boldsymbol{\theta}_{\mathbf{t}}$ *is characterized by the following two properties.*

- $\boldsymbol{\theta}_{\mathbf{t}}$ *is periodic under doubling with rotation number* $\mathbf{t}$ *.*
- $\boldsymbol{\theta}_{\mathbf{t}}$ *is balanced. If* $n>0$ *this means that exactly* n *of the points of the orbit* $\{2^k\boldsymbol{\theta}_{\mathbf{t}}\}$ *belong to the open interval* $(0,\,\boldsymbol{\theta}_{\mathbf{t}})$ $\bmod 1$ *, and* n *belong to the interval* $(\boldsymbol{\theta}_{\mathbf{t}},\,1)$ $\bmod 1$ *.*

(Compare Definition 4.24.) Figure 20 provides an example to illustrate this Lemma. Here $\widehat{\boldsymbol{\theta}}=2/7$, and the rotation number is $\mathbf{t}=1/3$. Exactly one point of the orbit of $\widehat{\boldsymbol{\theta}}$ under doubling belongs to the open interval $(0,\,\widehat{\boldsymbol{\theta}})$, and exactly one

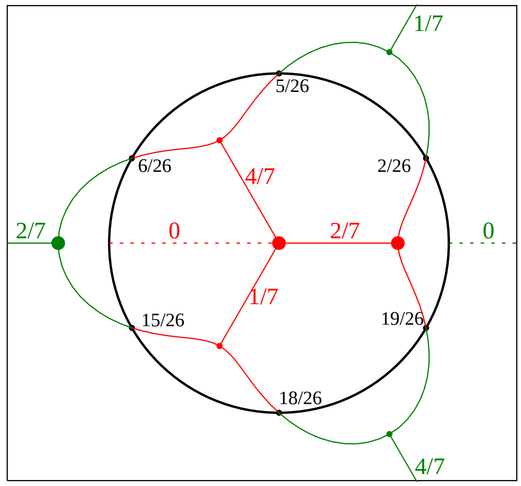

FIGURE 20. *The circle in this cartoon represents the Julia set (which is actually a quasicircle) for a map belonging to the* $2/7$ *ray in* $\mathcal{H}_0$. *The heavy dots represent the three finite critical points. As in Figure* 17, *the interior angles* (*red*) *and exterior angles* (*green*) *are periodic under doubling, while the angles around the Julia set are periodic under tripling. Angles are not to scale.*

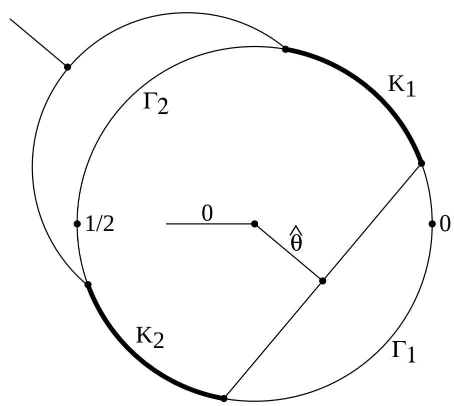

FIGURE 21. *Schematic diagram for any example with rotation number* $\mathbf{t} \neq 0$. (*Here, unlike Figure 20, angles are portrayed correctly.*)

point belongs to $(\widehat{\boldsymbol{\theta}}, 1)$. The major gap Γ_1 for points visible from the origin is the arc $[6/26, 15/26]$ of length $9/26$. (Compare Figure 16.) Similarly the major gap $\Gamma_2 = \Gamma_1 + 1/2$ for points visible from infinity is the arc from $19/26 \equiv -7/26$ to $2/26$.

Proof of Lemma 4.26. According to [**G**], there is only one $\mathbf{m}_2$-periodic orbit with rotation number $\mathbf{t}$, and clearly such a periodic orbit is balanced with respect to exactly one of its points. (Compare Remark 4.7 and Definition 4.24.)

To show that $\widehat{\boldsymbol{\theta}}$ has these two properties, consider the associated $\mathbf{m}_3$-rotation set $X = V_1 \cap V_2$. Since X is self-antipodal with odd period, it must consist of two mutually antipodal periodic orbits. As illustrated in Figure 20, we can map X to the unit circle in such a way that $\mathbf{m}_3$ corresponds to the rotation $z \mapsto e^{2\pi i \mathbf{t}} z$. If the rotation number is non-zero, so that $n \geq 1$, then each of the two periodic orbits must map to the vertices of a regular $(2n+1)$-gon. As in Figure 20, one of the two fixed points $0,\, 1/2$ in the Julia set must lie in the interval Γ_1, and the other must lie in $\Gamma_1 + 1/2$. It follows that $2n+1$ of the points of X lie in the interval $(0,\, 1/2)$, and the other $2n+1$ must lie in $(1/2,\, 1)$. It follows easily that n of the points of the orbit of $\widehat{\boldsymbol{\theta}}$ must lie in the open interval $(0,\, \widehat{\boldsymbol{\theta}})$ and n must lie in $(\widehat{\boldsymbol{\theta}},\, 1)$, as required.

The case of rotation number zero, as illustrated in Figure 12, is somewhat different. In this case, the rotation set X for $\mathbf{m}_3$ consists of the two fixed points 0 and $1/2$, The corresponding rotation set Y for $\mathbf{m}_2$ consists of the single fixed point zero, which is balanced by definition. $\square$

Proof of Theorem 4.25. The general idea of the argument can be described as follows. Since the case $\mathbf{t} = 0$ is easily dealt with, we will assume that $\mathbf{t} \neq 0$. Let Γ_1 be the critical gap for rays from zero, and let $\Gamma_2 = \Gamma_1 + 1/2$ be the corresponding gap for rays from infinity. Then one component K_1 of $\mathbb{R}/\mathbb{Z} \smallsetminus (\Gamma_1 \cup \Gamma_2)$ is contained in the open interval $(0,\, 1/2)$ and the other, $K_2 = K_1 + 1/2$, is contained in $(1/2,\, 1)$. (Compare Figure 21.) Thus any self-antipodal set which is disjoint from $\Gamma_1 \cup \Gamma_2$ must have half of its points in K_1 and the other half in K_2. However, all points of K_1 correspond to internal angles in the interval $(0,\, \widehat{\boldsymbol{\theta}})$, while all points of K_2 correspond to internal angles in the interval $(\widehat{\boldsymbol{\theta}},\, 1)$. (Here we are assuming the convention that $0 \in \Gamma_1$ and hence $1/2 \in \Gamma_2$.) In each case, this observation will lead to the appropriate concept of balance. First assume that $\mathbf{t}$ is rational.

Case 1 (The Generic Case). If $\boldsymbol{\theta}$ does not belong to Y, then X consists of a single cycle of $\mathbf{m}_3$ by Proposition 4.21. (This is the case illustrated in Figure 18.) This cycle is invariant by the antipodal map, therefore has even period, hence $\mathbf{t}$ is rational with even denominator. In addition, by symmetry, the number of points of X in K_1 is equal to the number of points of X in K_2. Therefore the number of points in Y in $(0,\, \boldsymbol{\theta})$ is equal to the number of points of Y in $(\boldsymbol{\theta},\, 1)$. (In other words, the set Y is "*balanced*" with respect to $\boldsymbol{\theta}$ in the sense that half of its elements belong to the interval $(0\,,\, \boldsymbol{\theta})$ and half belong to $(\boldsymbol{\theta},\, 1)$.) If $\{\theta^-,\, \theta^+\} \subset Y$ is the unique ***balanced pair*** for this rotation number, in the sense of Definition 4.24, this means that $\theta^- < \boldsymbol{\theta} < \theta^+$. As an example, for rotation number $1/4$ with $Y = \{1/15,\, 2/15,\, 4/15,\, 8/15\}$, the balanced pair consists of $2/15$ and $4/15$. Thus in this case the orbit is "balanced" with respect to $\boldsymbol{\theta}$ if and only if $2/15 < \boldsymbol{\theta} < 4/15$.

Case 2. Now suppose that $\boldsymbol{\theta}$ belongs to Y, and that the rotation set X consists of a single periodic orbit under $\mathbf{m}_3$. Since this orbit is invariant under the antipodal map $t \leftrightarrow t + 1/2$, it must have even period p. Furthermore $p/2$ of these points must lie in $(0,\, 1/2)$ and $p/2$ in $(1/2,\, 1)$. The image of this orbit in Y is not quite balanced with respect to $\boldsymbol{\theta}$ since one of the two open intervals $(0,\, \boldsymbol{\theta})$ and $(\boldsymbol{\theta},\, 1)$ must contain only $p/2 - 1$ points. However, it does follow that $\boldsymbol{\theta}$ is one of the balanced pair for Y. (Examples of such balanced pairs are $\{1/3,\, 2/3\}$ and $\{2/15,\, 4/15\}$.)

More explicitly, since $\boldsymbol{\theta}$ is periodic of period p, it follows that both endpoints of the critical gap Γ_1 are also periodic of period p. However, only one of these two endpoints is also visible from infinity, and hence belongs to X.

Case 3. Suppose that $\boldsymbol{\theta} \in Y$ and that X contains more than one periodic orbit. According to [**G**, Theorem 7], any $\mathbf{m}_3$-rotation set with more than one periodic orbit must consist of exactly two orbits of odd period. Thus we are in the case covered by Lemma 4.26. (This case is unique in that both of the end points of Γ_1 or of Γ_2 are visible from both zero and infinity.)

Case 4. Finally suppose that $\mathbf{t}$ is irrational. The set X carries a unique probability measure μ invariant by $\mathbf{m}_3$ and the push-forward $\varphi_*\mu$ under either φ_1 or φ_2 is the unique probability measure carried on Y and invariant by $\mathbf{m}_2$. By symmetry, $\mu(K_1) = \mu(K_2) = 1/2$. Therefore,

$$\varphi_*\mu[0,\ \boldsymbol{\theta}] \;=\; \varphi_*\mu[\boldsymbol{\theta},\ 1] \;=\; 1/2.$$

By definition, this means that the angle $\boldsymbol{\theta}$ is balanced. □

For actual computation, just as in Lemma 4.17, it is more convenient to work with the inverse function.

Theorem 4.27. *The inverse function* $\mathbf{t} \mapsto \boldsymbol{\theta}_{\mathbf{t}} = \psi^{-1}(\mathbf{t})$ *is strictly monotone, and is discontinuous at* $\mathbf{t}$ *if and only if* $\mathbf{t}$ *is rational with even denominator. The base two expansion of its right hand limit* $\psi^{-1}(\mathbf{t}^+) = \lim_{\epsilon\searrow 0} \psi^{-1}(\mathbf{t}+\epsilon)$ *can be written as*

$$\theta_{\mathbf{t}}^+ \;=\; \psi^{-1}(\mathbf{t}^+) \;=\; \sum_{k=0}^{\infty} \frac{b_k}{2^{k+1}} \tag{6}$$

where

$$b_k \;:=\; \begin{cases} 0 & \text{if} \quad \operatorname{frac}(1/2 + k\,\mathbf{t}) \;\in\; [0,\, 1-\mathbf{t})\,, \\ 1 & \text{if} \quad \operatorname{frac}\big(1/2 + k\,\mathbf{t}\big) \;\in\; [1-\mathbf{t},\, 1)\,. \end{cases} \tag{7}$$

Here $\operatorname{frac}(x)$ *denotes the fractional part of* x, *with* $0 \le \operatorname{frac}(x) < 1$ *and with* $\operatorname{frac}(x) \equiv x \pmod{\mathbb{Z}}$.

Remark 4.28. It follows that the "discontinuity" of this function at $\mathbf{t}$, that is the difference $\Delta\boldsymbol{\theta}$ between the right and left limits, is equal to the sum of all powers 2^{-k-2} such that $1/2 + k\mathbf{t} \equiv 1 - \mathbf{t} \pmod{\mathbb{Z}}$. If $\mathbf{t}$ is a fraction with even denominator $2n$, then this discontinuity takes the value

$$\Delta\boldsymbol{\theta} \;=\; 2^{n-1}/(2^{2n}-1)\,,$$

while in all other cases it is zero.

The proof of Theorem 4.27 will depend on Lemma 4.26.

Proof of Theorem 4.27. It is not hard to check that the function defined by (6) and (7) is strictly monotone, with discontinuities only at rationals with even denominator, and that it increases from zero to one as $\mathbf{t}$ increases from zero to one. Since rational numbers with odd denominator are everywhere dense, it suffices to check that this expression coincides with $\varphi^{-1}(\mathbf{t})$ in the special case where $\mathbf{t} = m/(2n+1)$ is rational with odd denominator. In that case, each $1/2 + k\mathbf{t}$ is rational with even denominator, and hence cannot coincide with either zero or $1 - \mathbf{t}$. The cyclic order of the $2n+1$ points $t_k := 1/2 + k\mathbf{t} \in \mathbb{R}/\mathbb{Z}$ must be the same as the cyclic order of the $2n+1$ points $\theta_k := 2^k\boldsymbol{\theta} \in \mathbb{R}/\mathbb{Z}$. This proves that $\boldsymbol{\theta}$ is periodic under $\mathbf{m}_2$, with the required rotation number. Since the arc $(0,\, 1/2)$ and $(1/2,\, 1)$ each contain n of the points t_k, it follows that the corresponding arcs $(0,\, \boldsymbol{\theta})$ and $(\boldsymbol{\theta},\, 1)$ each contain n of the points θ_k. □

5. Fjords

Let $\mathbf{t} = m/(2n+1) \in \mathbb{R}/\mathbb{Z}$ be rational with odd denominator. It will be convenient to use the notation $\theta = \theta(\mathbf{t}) \in \mathbb{R}/\mathbb{Z}$ for the unique angle which satisfies the equation $\psi(\theta) = \mathbf{t}$. In other words, $\theta(\mathbf{t})$ is the unique angle which is balanced, with rotation number $\mathbf{t}$ under doubling. (Compare Theorem 4.27 and Lemma 4.26.) This section will sharpen Theorem 3.4 by proving the following result.

Theorem 5.1. *With $\mathbf{t}$ and $\theta(\mathbf{t})$ as above, the internal ray in $\mathcal{H}_0$ of angle $\theta(\mathbf{t})$ lands at the point $\infty_{\mathbf{t}}$ on the circle of points at infinity.*

(We will see in Corollary 5.5 that no other internal ray has this property.) Intuitively, we should think of the $\theta(\mathbf{t})$ ray as passing to infinity ***through the*** $\mathbf{t}$***-fjord***. (Compare Figure 25.)

Following is an example to illustrate the proof.

Example 5.2. As in Figure 20, let f_q be a map such that $s = q^2 \in \mathcal{H}_0$ belongs to the $\theta = 2/7$ ray, which has a rotation number equal to $1/3$ under doubling. This angle is balanced, that is, its orbit $2/7 \mapsto 4/7 \mapsto 1/7$ contains one number in the interval $(0,\, 2/7)$ and one number in $(2/7,\, 1)$, hence the formal rotation number $\mathbf{t} = \mathbf{t}(s)$ is also $1/3$. The dynamic ray of angle $2/7$ bounces off the critical point c_+, but it has well defined left and right limits (equal to the landing points of corresponding slanted rays). These have Julia set coordinates $x = 6/26$ and $15/26$, as can be computed by taking left and right limits in Lemma 4.17. Evidently these x-coordinates are periodic under tripling. Similarly, the internal ray of angle $2^n \cdot 2/7$ bifurcates off a precritical point, and the associated slanting rays land at points on the Julia set with coordinates $3^n \cdot 6/26$ and $3^n \cdot 15/26$ in $\mathbb{R}/\mathbb{Z}$.

Now suppose that we let s tend as far as possible along this parameter ray. If it accumulates at some finite point of $\partial\mathcal{H}_0$, then we would expect the two landing points with coordinates $x = 6/26$ and $15/26$ to come together towards a parabolic limit point, thus pinching off a parabolic basin containing c_+. Similarly, we would expect the points with $x = 18/26$ and $19/26$ to merge, and also the points with $x = 2/26$ and $5/26$, thus yielding a parabolic orbit of period 3.

Now look at the same picture from the viewpoint of the point at infinity (but keeping the same parametrization of ∂U_q). (Recall that the action of the antipodal map on the Julia set is given by $x \leftrightarrow x + 1/2 \mod 1$.) Then a similar argument shows for example, that the points with coordinate $x = 19/26$ and $2/26$ should pinch together. *That is, all six of the indicated landing points on the Julia set should pinch together, yielding just one fixed point, which must be invariant under the antipodal map.* Since this is clearly impossible, it follows that the parameter ray of angle $2/7$ cannot accumulate on any finite point of $\partial\mathcal{H}_0$. Hence it must diverge to the circle at infinity.

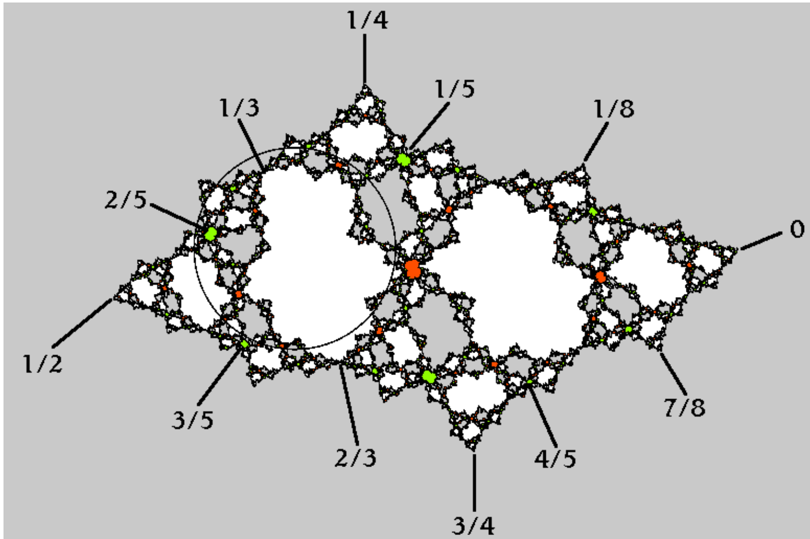

FIGURE 22. *Julia set for the landing point of the parameter ray of angle* $2/5$*, with some external angles labeled. The unit circle has been cray in to fix the scale.*

Note that this difficulty, with too many landing points which must pinch together, occurs only in the balanced case. Figures 22 and 23 show the Julia set for the landing point of the $2/5$ parameter ray. Here we have a period four cycle of parabolic basins shown in red, which are accessible from the basin of zero with internal angles $2/5 \to 4/5 \to 3/5 \to 1/5$; and an antipodal cycle of basins, shown in green, which are accessible from the basin of infinity with external angles $2/5 \to 4/5 \to 3/5 \to 1/5$. There is no conflict between the required identifications. The corresponding picture in the q-parameter plane in shown in Figure 11.

Proof of Theorem 5.1. The argument in the general case is completely analogous to the argument in Example 5.2. Let f_q be a map representing a point of the $\boldsymbol{\theta}$-ray, where $\boldsymbol{\theta}$ is periodic under doubling and balanced, with rotation number $\mathbf{t} = m/(2n+1)$. Since the case $\mathbf{t} = n = 0$ is straightforward, we may assume that $2n+1 \geq 3$. The associated left hand slanted rays land at a period $(2n+1)$ orbit $\mathcal{O}_0^-$ in the Julia set, while the right hand slanted rays land on an orbit $\mathcal{O}_0^+$. According to Goldberg [**G**], each such orbit $\mathcal{O}$ is uniquely determined by its rotation number, together with the number $N(\mathcal{O})$ of orbit points for which the coordinate x lies in the semicircle $0 < x < 1/2$. (Thus the number in the complementary semicircle $1/2 < x < 1$ is equal to $2n+1-N(\mathcal{O})$. Here 0 and $1/2$ represent the coordinates of the two fixed points.) Since $\boldsymbol{\theta}$ is balanced with $2n+1 \geq 3$, it is not hard to check that $N(\mathcal{O})$ must be either n or $n+1$. Now consider the antipodal picture, as viewed from infinity. Since the antipodal map corresponds to $x \leftrightarrow x + 1/2$, we see that

$$N(\mathcal{O}_\infty^\pm) \;=\; 2n+1 \;-\; N(\mathcal{O}_0^\pm) \;=\; N(\mathcal{O}_0^\mp)\,,$$

Therefore, it follows from Goldberg's result that the orbit $\mathcal{O}_\infty^\pm$ is identical with $\mathcal{O}_0^\mp$.

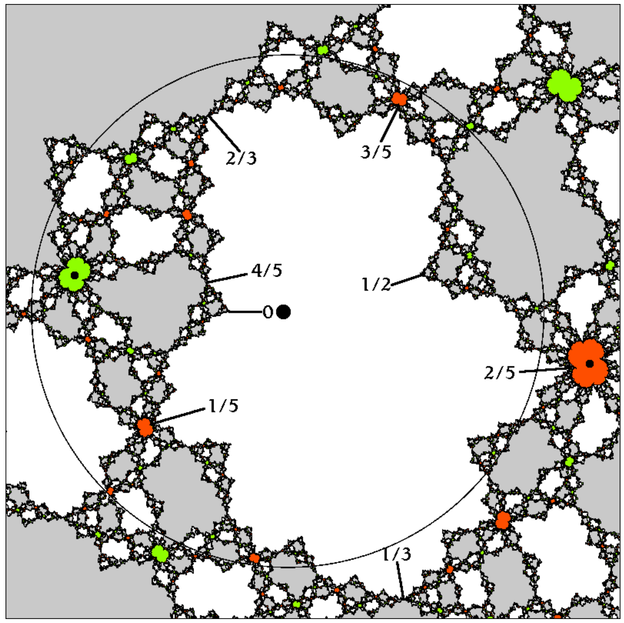

FIGURE 23. *Detail of Figure 22, with some internal angles labeled.*

Now consider a sequence of points q_k such that each $s_k = q_k^2$ belongs to the $\boldsymbol{\theta}$ parameter ray. If this sequence accumulates on a finite point of $\partial\mathcal{H}_0$, then, passing to a subsequence, we can assume that q_k tends to a finite limit q. As we approach this limit point, the right hand limit orbit $\mathcal{O}_0^+$ must converge to the left hand limit orbit $\mathcal{O}_0^-$ which is equal to $\mathcal{O}_\infty^+$. In other words, the limit orbit must be self-antipodal. But this is impossible, since the period of this limit orbit must divide $2n+1$, and hence be odd. Thus the $\boldsymbol{\theta}$ parameter ray has no finite accumulation points, and hence must diverge towards the circle at infinity.

In order to determine the precise limit on the circle at infinity, we make a change of coordinate, bringing this circle at infinity into the finite plane. To this end, let $u = z/q$, so that our map $z \mapsto f_q(z) = z^2(q-z)/(1+\overline{q}z)$ takes the form

$$u \;\mapsto\; \frac{f_q(qu)}{q} \;=\; \frac{q^2u^2(1-u)}{1+q\overline{q}u}\;.$$

Dividing numerator and denominator by $q\,\overline{q}\,u$, and introducing the new parameters

$$\lambda \;=\; \frac{q}{\overline{q}} \;=\; \frac{s}{|s|} \qquad \text{and} \qquad b \;=\; \frac{1}{|q|^2} \;=\; \frac{1}{|s|}\;,$$

this takes the form

$$u \;\mapsto\; \frac{\lambda u(1-u)}{1+b/u}\;. \tag{8}$$

Thus, as b tends to zero, for u in the punctured sphere $\widehat{\mathbb{C}}\smallsetminus\{0\}$, this cubic rational map (8) converges locally uniformly to the quadratic polynomial map

$$u \;\mapsto\; \lambda\, u(1-u)\;. \tag{9}$$

Here λ, the multiplier of the fixed point at zero, lies on the unit circle. (Compare Figure 24.)

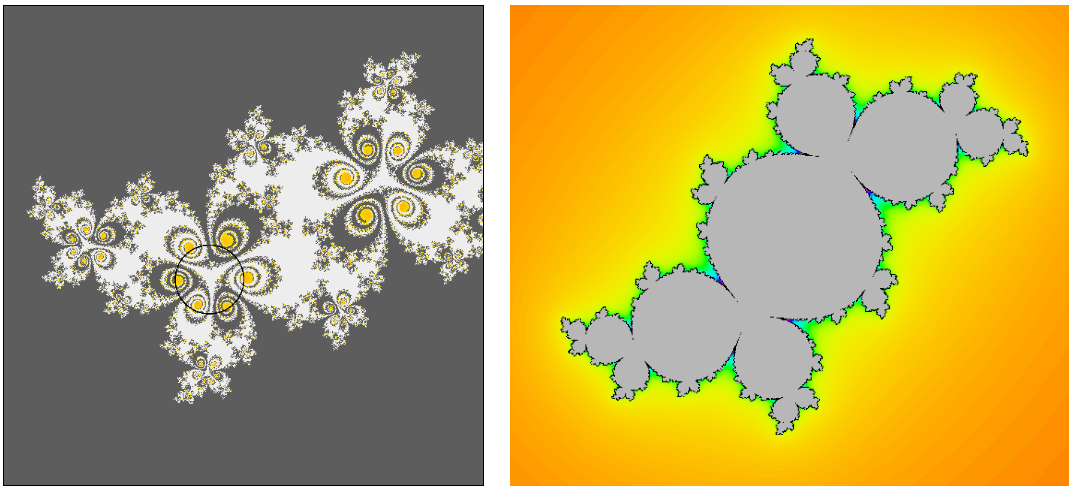

FIGURE 24. *On the left: Julia set for a map* f_q *in the* $2/3$ *fjord. On the right: the limit of this Julia set suitably renormalized, as we tend to infinity through the* $2/3$ *fjord, is the Julia set for the quadratic map* $w \mapsto e^{2\pi i\, 2/3} w(1-w)$. *In the limit the unit circle (as drawn in on the left) shrinks to a parabolic fixed point.*

Now consider a sequence of points s_k on the $\boldsymbol{\theta}$ parameter ray, with $|s_k|$ diverging to infinity. Passing to a subsequence, we may assume that the points $\lambda_k = s_k/|s_k|$ on the unit circle converge to some λ, while $b_k = 1/|s_k|$ converges to zero. As in the discussion above, we can consider associated left and right slanted limit rays in the dynamic plane for the map (9). There are two possible cases. If one of these limit rays has an accumulation point in $\widehat{\mathbb{C}} \smallsetminus \{0\}$, then arguing as above, both slanted rays must land at this point. Hence it would have to be parabolic, which is impossible since a quadratic polynomial can have at most one non-repelling periodic orbit.

It follows that both limit rays must land at $u = 0$. Since these rays are permuted with rotation number $\mathbf{t} = m/(2n+1)$, this is possible only if λ is precisely equal to $e^{2\pi i\, t}$. This shows that the $\boldsymbol{\theta}$ ray must land at the point $\infty_{\mathbf{t}}$ on the circle at infinity, and completes the proof of Theorem 5.1. $\square$

Formal Rotation Number: The Parameter Space Definition. As a corollary of this theorem, we can generalize the concept of "formal rotation number", as defined in §4, to all points of $\mathbb{C} \smallsetminus \{0\}$. (For a possible dynamic interpretations of this generalized definition, see Remark 6.13, as well as Corollary 7.12.)

Let $I \subset \mathbb{R}/\mathbb{Z}$ be a closed interval such that the endpoints $\mathbf{t}_1$ and $\mathbf{t}_2$ are rational numbers with odd denominator, and let $\triangleleft(I) \subset \mathbb{C}$ be the compact triangular region bounded by the unique rays which join $\infty_{\mathbf{t}_1}$ and $\infty_{\mathbf{t}_2}$ to the origin, together with the arc at infinity consisting of all points $\boldsymbol{\infty}_{\mathbf{t}}$ with $\mathbf{t} \in I$. (Compare Figure 25.)

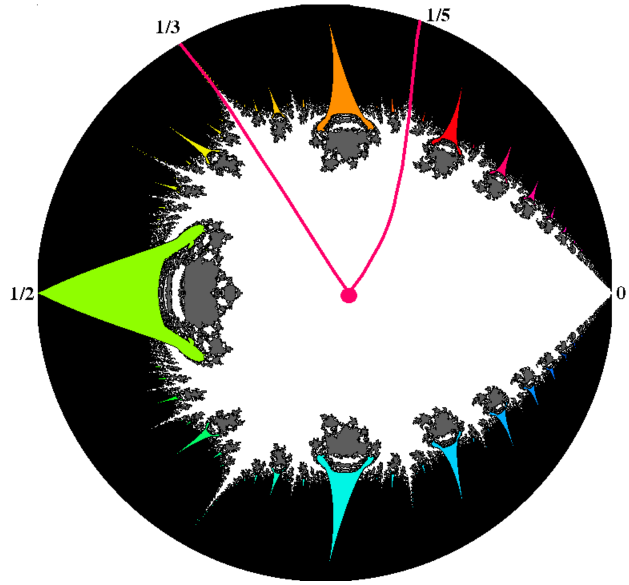

FIGURE 25. *Unit disk model for the circled s-plane, with two rays from the origin to the circle at infinity sketched in.*

For any angle $\mathbf{t} \in \mathbb{R}/Z$, let $\triangleleft(\mathbf{t})$ be the intersection of the compact sets $\triangleleft(I)$ as I varies over all closed intervals in $\mathbb{R}/\mathbb{Z}$ which contain $\mathbf{t}$ and have odd-denominator rational endpoints.

Lemma 5.3. *Each $\triangleleft(\mathbf{t})$ is a compact connected set which intersects the circle at infinity only at the point $\infty_{\mathbf{t}}$. These sets $\triangleleft(\mathbf{t})$ intersect at the origin, but otherwise are pairwise disjoint. In particular, the intersections $\triangleleft(\mathbf{t}) \cap (©\smallsetminus\mathcal{H}_0)$ are pairwise disjoint compact connected sets.* (*Here it is essential that we include points at infinity.*)

Proof. The argument is straightforward, making use of the fact that $\mathcal{H}_0$ is the union of an increasing sequence of open sets bounded by equipotentials. $\square$

Definition 5.4. Points in $\triangleleft(\mathbf{t})\smallsetminus\{0\}$ have ***formal rotation number*** equal to $\mathbf{t}$. It is not difficult to check that this is compatible with Definition 4.22 in the special case of points which belong to $\mathcal{H}_0$. Furthermore, it is not difficult to check that this formal rotation number is continuous as a map from $©\smallsetminus\{0\}$ to $\mathbb{R}/\mathbb{Z}$ (or equivalently as a continuous retraction of $©\smallsetminus\{0\}$ onto the circle at infinity).

One immediate consequence of Lemma 5.3 is the following.

Corollary 5.5. *For t rational with odd denominator, the ray $R_{\boldsymbol{\theta}(\mathbf{t})}$ of Theorem 5.1 is the only internal ray in $\mathcal{H}_0$ which accumulates on the point $\infty_{\mathbf{t}}$.*

In fact, for any $\mathbf{t}' \neq \mathbf{t}$ the ray $R_{\boldsymbol{\theta}(\mathbf{t}')}$ is contained in a compact set $\triangleleft(\mathbf{t}')$ which does not contain $\infty_{\mathbf{t}}$. $\square$

Remark 5.6. (Accessibility and Fjords.) A topological boundary point

$$s_0 \in \partial\mathcal{H}_0 \subset ©$$

is said to be ***accessible*** from $\mathcal{H}_0$ if there is a continuous path $p : [0,1] \to \copyright$ such that p maps the half-open interval $(0,1]$ into $\mathcal{H}_0$, and such that $p(0) = s_0$. Define a ***channel*** to s_0 within $\mathcal{H}_0$ to be a function $\mathfrak{c}$ which assigns to each open neighborhood U of s_0 in $\copyright$ a connected component $\mathfrak{c}(U)$ of the intersection $U \cap \mathcal{H}_0$ satisfying the condition that

$$U \supset U' \quad \Longrightarrow \quad \mathfrak{c}(U) \supset \mathfrak{c}(U') \ .$$

(If $U_1 \supset U_2 \supset \cdots$ is a basic set of neighborhoods shrinking down to s_0, then clearly the channel $\mathfrak{c}$ is uniquely determined by the sequence $\mathfrak{c}(U_1) \supset \mathfrak{c}(U_2) \supset \cdots$. Here, we can choose the U_j to be simply connected, so that the $\mathfrak{c}(U_j)$ will also be simply connected.)

By definition, the access path p lands on s_0 ***through*** the channel $\mathfrak{c}$ if for every neighborhood U there is an $\epsilon > 0$ so that $p(\tau) \in \mathfrak{c}(U)$ for $\tau < \epsilon$. It is not hard to see that every access path determines a unique channel, and that for every channel $\mathfrak{c}$ there exits one and only one homotopy class of access paths which land at s_0 through $\mathfrak{c}$.

The word ***fjord*** will be reserved for a channel to a point on the circle at infinity which is determined by an internal ray which lands at that point.

Remark 5.7. (Are there Irrational Fjords ?) It seems likely, that there are uncountably many irrational rotation numbers $\mathbf{t} \in \mathbb{R}/\mathbb{Z}$ such that the ray $R_{\boldsymbol{\theta}(\mathbf{t})} \subset \mathcal{H}_0$ lands at the point $\infty_{\mathbf{t}}$ on the circle at infinity. In this case, we would say that the ray lands through an ***irrational fjord***. What we can actually prove is the following statement.

Lemma 5.8. *There exist countably many dense open sets $W_n \subset \mathbb{R}/\mathbb{Z}$ such that, for any $\mathbf{t} \in \bigcap W_n$, the ray $R_{\boldsymbol{\theta}(\mathbf{t})}$ accumulates at the point $\infty_{\mathbf{t}}$ on the circle at infinity.*

However, we do not know how to prove that this ray actually lands at $\infty_{\mathbf{t}}$. The set of all accumulation points for the ray is necessarily a compact connected subset of $\lhd(\mathbf{t}) \cap \partial\mathcal{H}_0$, but it could contain more than one point. In that case, all neighboring fjords and tongues would have to undergo very wild oscillations as they approach the circle at infinity.

Proof of Lemma 5.8. Let V_n be the open set consisting of all internal angles $\boldsymbol{\theta}$ such that the internal ray $R_{\boldsymbol{\theta}}$ contains points s with $|s| > n$. A corresponding open set W_n of formal rotation numbers can be constructed as follows. For each rational $r \in \mathbb{R}/\mathbb{Z}$ with odd denominator, choose an open neighborhood which is small enough so that every associated internal angle is contained in V_n. The union of these open neighborhoods will be the required dense open set W_n. For any $\mathbf{t}$ in $\bigcap W_n$, the associated internal ray $R_{\boldsymbol{\theta}(\mathbf{t})}$ will come arbitrarily close to the circle at infinity. Arguing as in Corollary 5.5, we see that $\infty_{\mathbf{t}}$ is the only point of $\partial\copyright$ where this ray can accumulate. $\square$

If the following is true, then we can prove a sharper result. It will be convenient to use the notation $\mathfrak{X}$ for the compact set $\copyright \smallsetminus \mathcal{H}_0$, and the notation $\mathfrak{X}_t$ for the intersection $\mathfrak{X} \cap \lhd(t)$. Thus $\mathfrak{X}$ is the disjoint union of the compact connected sets $\mathfrak{X}_t$.

Conjecture 5.9. *If t is rational with odd denominator, then the set $\mathfrak{X}_t$ consists of the single point ∞_t.*

Assuming this statement, we can prove that a generic ray actually lands, as follows. Let $\overline{\mathbb{D}}_n$ be the disk consisting of all $s \in \mathbb{C}$ with $|s| \le n$, and let $W_n \subset \mathbb{R}/\mathbb{Z}$ be the open set consisting of all t for which $\mathfrak{X}_t \cap \overline{\mathbb{D}}_n = \emptyset$. Then W_n is dense since it contains all rational numbers with odd denominator. For any $t \in \bigcap_n W_n$, the set $\mathfrak{X}_t$ evidently consists of the single point ∞_t, and it follows easily that the associated ray $R_{\boldsymbol{\theta}(t)}$ lands at this point. $\square$

6. Tongues

By definition, a ***tongue*** in our family is an unbounded hyperbolic component consisting of maps with a self-antipodal periodic orbit, necessarily of even period. Corresponding to each rational rotation number m/n with even denominator, there are two symmetric tongues in the q parameter plane. (These are the colored regions in Figure 1.) As usual, we will find it more convenient to work in the moduli space with coordinate $s = q^2$. Thus the two symmetric hyperbolic components in the q-plane correspond to a single hyperbolic component, to be denoted by $\mathcal{H}(m/n) \subset \copyright$, in the circled s-plane. (Compare Figure 25.) Here is a precise statement.

Theorem 6.1. *For each $m/n \in \mathbb{Q}/\mathbb{Z}$ with even denominator, there is a hyperbolic component $\mathcal{H}(m/n)$ in the s-plane which has the point at infinity $\infty_{m/n}$ as a boundary point. Any representative map f_q for this component has a self-antipodal cycle of attracting Fatou components of period n. These can be numbered as $U_1, U_2, \ldots U_n$, with subscripts in $\mathbb{Z}/n$, so that $f_q(U_j) = U_{j+m}$, and so that each boundary ∂U_j intersects $\partial U_{j'}$ if and only if $j' \equiv j \pm 1$,*

(Compare Figures 26 and 5.) Briefly we will say that the period n Fatou components are arranged in a ***ring*** with ***combinatorial rotation number*** m/n. It follows easily from this theorem that m/n coincides with the formal rotation number as defined in Definition 5.4. (Compare Remark 6.13.)

Remark 6.2. Conjecturally the tongues $\mathcal{H}(m/n)$ and the central hyperbolic component $\mathcal{H}_0$ are the only unbounded hyperbolic components.

For the proof of this theorem, it will be convenient to introduce the rotated dynamic coordinate

$$w \;=\; z/k \qquad \text{where} \qquad k \;=\; q/|q| \;.$$

Then a brief computation shows that the map $z \mapsto f_q(z) = z^2(q-z)/(1+\overline{q}z)$ corresponds to the map

$$w \;\mapsto\; F_{t,a}(w) \;=\; \frac{f_q(k\,w)}{k} \;=\; e^{2\pi i t}\, w^2 \,\frac{1 - a\,w}{w + a} \;, \tag{10}$$

where

$$a \;=\; 1/|q| \;>\; 0\;, \qquad \text{and where} \qquad e^{2\pi i t} \;=\; q/\overline{q} \quad \text{with} \quad t \in \mathbb{R}/\mathbb{Z}\;.$$

Note that any point $\infty_{\mathbf{t}}$ on the circle at infinity in the s-plane, corresponds to the point $(\mathbf{t},\, 0)$ on the boundary $a = 0$ of the $(\mathbf{t},\, a)$-half-plane (consisting of real pairs with $\mathbf{t}$ well defined modulo $\mathbb{Z}$ and with $a \ge 0$). As $(\mathbf{t},\, a)$ converges

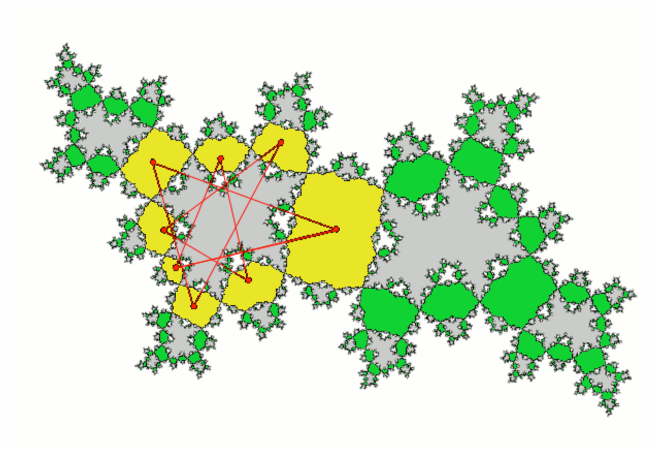

Figure 26. *The Julia set of a map in the hyperbolic component $\mathcal{H}(3/8)$. There is an attracting cycle of period 8 preserved by the antipodal map. The basin of 0 is light grey. The basin of infinity is white. The basin of the attracting cycle of period 8 is colored. The immediate basin (yellow) consists in 8 simply connected domains organized in a chain separating 0 and ∞. Each component of the immediate basin is mapped to the one which is 3 steps further counterclockwise. The red lines join successive points in the attracting cycle.*

to $(\mathbf{t}_0, 0)$, note that the map (10) converges, uniformly on compact subsets of $\mathbb{C}\smallsetminus\{0\}$, to the rotation $w \mapsto e^{2\pi i\mathbf{t}_0}w$. (Compare Lemma 1.2.)

The proof of Theorem 6.1 can now be outlined as follows. The first step is a more precise study of the asymptotic behavior of the map $F_{t,\,a}$ as a tends to zero. This, together with the implicit function theorem, enables the construction of two self-antipodal period n orbits, provided that a is close to zero. The first of these orbits is attracting, so that its immediate basin consists of n Fatou components, arranged in a ring around the origin. The second is repelling, and consists of the common boundary points for consecutive components of this immediate basin.

Rescaling Limits. Let V_1 be the open set consisting of all triples $(t,\, a,\, w) \in \mathbb{C}^3$ such that

$$|a| \;<\; |w| \;<\; 1/|a|\;,$$

so that w is bounded away from zero and infinity for each choice of a with $0 < |a| < 1$. Then the map

$$(t,\, a,\, w) \;\mapsto\; F_{t,a}(w) \;=\; e^{2\pi i t} w^2 \frac{1 - a\,w}{w\,+\,a}$$

is defined and holomorphic for $(t,\, a,\, w) \in V_1$, with values in $\mathbb{C}\smallsetminus\{0\}$. Now define a sequence of subsets $V_1 \supset V_2 \supset V_3 \supset \cdots$ by setting

$$V_{k+1} \;=\; \{(t,\, a,\, w) \in V_k\;;\; \big(t,\, a,\, F_{t,a}(w)\big) \in V_k\}\;. \tag{11}$$

Then the map $(t,\, a,\, w) \mapsto F_{t,a}^{\circ k}(w)$ is holomorphic as a function from V_k to $\mathbb{C}\smallsetminus\{0\}$. Note that each V_k is an open set which contains the product $\mathbb{R}\times\{0\}\times(\mathbb{C}\smallsetminus\{0\})$.

We will also make temporary use of two other coordinates changes. If we rotate and change the scale by setting $u = z/q = a\,w$, then our map takes the form

$$u \mapsto g_{t,\,b}(u) = e^{2\pi i t}\, u\, \frac{1 - u}{1 + b/u}\ , \tag{12}$$

where $b = a^2$. For $u \in \mathbb{C}\smallsetminus\{0\}$, this converges locally uniformly to the quadratic polynomial

$$u \mapsto g_t(u) = e^{2\pi i t}\, u(1 - u)$$

as $b \to 0$, as in equations (8) and (9) . On the other hand, if we introduce the coordinate $\eta = a/w$, which tends to zero as w tends to infinity, then the map is conjugate to

$$\eta \mapsto h_{t,b}(\eta) = \frac{a}{F_{t,a}(a/\eta)} = e^{-2\pi i t}\, \frac{\eta^2(1 + \eta)}{\eta - b}\ .$$

In this case, the limit as $b \to 0$ is the quadratic polynomial

$$h_t(\eta) = e^{-2\pi i t}\, \eta(1 + \eta)\ ,$$

with an indifferent fixed point at $\eta = 0$ (corresponding to $w = \infty$).

Now suppose that we fix $t = m/n$, where m and n are always assumed to be coprime. Then the polynomial $g_t^{\circ n}$ has a parabolic fixed point of multiplicity $n+1$ at the origin, so that

$$g_t^{\circ n}(u) = u\cdot\big(1 + C\,u^n + \mathrm{O}(u^{n+1})\big) \tag{13}$$

where C is a non-zero complex number. Similarly, the map $h_t^{\circ n}$ has a parabolic fixed point of multiplicity $n+1$ at $\eta = 0$, with

$$h_t^{\circ n}(\eta) = \eta\cdot\big(1 + C'\,\eta^n + \mathrm{O}(\eta^{n+1})\big)\ . \tag{14}$$

In fact, using the identity $h_t(\eta) = -g_{-t}(-\eta)$, it is not hard to check that C' is equal to the complex conjugate of C up to sign:

$$C' = (-1)^n\, \overline{C}\,.$$

We will sometimes use the notations $C = C_\lambda$ and $C' = C'_\lambda$ where $\lambda = e^{2\pi i m/n}$.

The Banerjee Polynomials. The following constructions are based on Banerjee [**Ba**]. It will be convenient to work with the function

$$\Phi(t, a, w) := \frac{F_{t,a}^{\circ n}(w) - w}{w}\ ,$$

which is defined and holomorphic for $(t, a, w) \in V_n$. For this section, n can be either even or odd.

Proposition 6.3. *Let m and $n > 0$ be coprime. For each k between 0 and $n-1$ there is a unique complex polynomial $B_k(a)$ of degree at most k, satisfying $B_k(0) = m/n$, such that, for each fixed $w \neq 0$,*

$$\Phi\big(B_k(a),\ a,\ w\big) = \mathrm{O}(a^{k+1}) \qquad \textit{as} \quad a \to 0\,. \tag{15}$$

A completely equivalent statement would be that the function $\Phi\big(B_k(a),\ a,\ w\big)$ can be expressed as a product $a^{k+1}\Psi(a,\ w)$, where Ψ is defined and holomorphic throughout V_n.

The proof of Proposition 6.3 starts as follows. For the case $k = 0$ we take $B_0(a) := m/n$. The required estimate $\Phi(m/n, a, w) = \mathrm{O}(a)$ follows easily since Φ is a holomorphic function with $\Phi(m/n,\ 0,\ w) = 0$.

Now suppose inductively that we are given a polynomial B_{k-1} satisfying (15) for some given $k-1 < n$. Then we can write $\Phi(B_{k-1}(a),\ a,\ w) = a^k\Psi_k(a,\ w)$ where Ψ_k is defined and holomorphic on the open set consisting of all (a, w) such that $\big(B_{k-1}(a),\ a,\ w\big) \in V_n$. Setting $\varphi_k(w) := \Psi_k(0, w)$, it follows easily that $\Psi_k(a, w) = \varphi_k(w) + \mathrm{O}(a)$, and hence that

$$\Phi\big(B_{k-1}(a),\ a,\ w\big) \;=\; a^k\,\varphi_k(w) \;+\; \mathrm{O}(a^{k+1})\ , \tag{16}$$

where the function φ_k is holomorphic throughout $\mathbb{C}\smallsetminus\{0\}$.

In fact, if $k < n$ we will show that this function φ_k must be constant. As a first step we will prove the following.

Lemma 6.4. *For $k \le n$, the function φ_k is meromorphic, with poles of order at most k at zero and infinity.*

Proof. If $|w| \gg 1 \gg |u|$, then the triple $(B_{k-1}(u/w),\ u/w,\ w)$ belongs to V_n, hence the expression

$$\Phi\big(B_k(u/w),\ u/w,\ w\big)$$

depends holomorphically on u and w. Expanding as a power series in u, we can write

$$\Phi\big(B_k(u/w),\ u/w,\ w\big) \;=\; \sum_{j=0}^{\infty}\beta_j(w)u^j\ ,$$

where each coefficient $\beta_j(w)$ is a holomorphic function on some region $|w| >$ constant > 1. On the other hand, according to the estimate (16), we can write

$$\begin{aligned}\Phi\big(B_{k-1}(u/w),\ u/w,\ w\big) \;&=\; (u/w)^k\varphi_k(w) \;+\; \mathrm{O}(u^{k+1}/w^{k+1})\\ &=\; u^k\,\frac{\varphi_k(w)}{w^k} \;+\; \mathrm{O}(u^{k+1})\ .\end{aligned} \tag{17}$$

as $u \to 0$ for each w. Hence the first non-zero coefficient is $\beta_k(w) = \varphi_k(w)/w^k$.

Now recall that the map $F_{t,a}$ is conjugate to g_{t,a^2}, where $F_{t,a}(w) = g_{t,\,a^2}(aw)/a$. (Compare equation (12).) Substituting u/w for a and $B_k(u/w)$ for t, we see that $\Phi\big(B_k(u/w),\ u/w,\ w\big)$ is equal to

$$\frac{F^{\circ n}_{B_k(u/w),\,u/w}(w) - w}{w} \;=\; \frac{g^{\circ n}_{B_k(u/w),\,u^2/w^2}(u) - u}{u}\ . \tag{18}$$

As w tends to infinity, this tends to $\big(g^{\circ n}_{m/n,\,0}(u) - u\big)/u$ which, according to (13) has the form $Cu^n + \mathrm{O}(u^{n+1}) = \mathrm{O}(u^n)$. Therefore, all of the coefficients $\beta_j(w)$ with $j < n$ must converge to zero as $|w|$ tends to infinity. In particular, if $k < n$, this proves that $\varphi_k(w)/w^k$ tends to zero as $|w| \to \infty$, which proves half of Lemma 6.4. (For $k = n$, the argument shows only that $\varphi_k(w)/w^k$ is bounded as $|w| \to \infty$.)

To study behavior as $w \to 0$ we introduce the function

$$\Phi(B_k(\eta\, w),\, \eta\, w,\, w)\ ,$$

which is defined and holomorphic for $|\eta| \ll 1$ and $0 < |w| \ll 1$. According to (16), we have the estimate

$$\Phi(B_k(\eta\, w),\, \eta\, w,\, w) \;=\; \eta^k w^k \varphi_k(w) \,+\, \mathrm{O}(\eta^{k+1})\ ,$$

Expanding this function as a power series $\sum_{j=0}^{\infty} \gamma_j(w)\eta^j$, we see that the coefficient $\gamma_k(w)$ is equal to $w^k\varphi_k(w)$. On the other hand, using the conjugation identity

$$F^{\circ n}_{t,a}(w) \;=\; \frac{a}{h^{\circ n}_{t,a^2}(a/w)}\ ,$$

we see that

$$\Phi(B_k(\eta w),\, \eta w,\, w) \;=\; \frac{\eta}{h^{\circ n}_{t,\eta^2 w^2}(\eta)} \,-\, 1\,.$$

where $t = B_k(\eta w)$ hence, using (14),

$$\lim_{w \to 0} \Phi(B_k(\eta w),\, \eta w,\, w) \;=\; \frac{1}{1 + C'\eta^n + \mathrm{O}(\eta^{n+1})} - 1 \;=\; -C'\eta^n \,+\, \mathrm{O}(\eta^{n+1})\ .$$

If $k < n$ then it follows that $\gamma_k(w)$ converges to zero as $w \to 0$. This implies that $w^k\varphi_k(w)$ also converges to zero as $w \to 0$, which completes the proof of Lemma 6.4. □

On the other hand, we will prove the identity

$$\varphi_k(\lambda w) = \varphi_k(w) \qquad \text{for all} \quad w \neq 0\ , \tag{19}$$

where $\lambda := e^{2\pi i m/n}$. Consider the one-parameter family of maps

$$G_a(w) \;:=\; F_{B_k(a),\,a}(w)\ .$$

Note that $G_a(w) = \lambda\, w + O(a)$ and hence $\varphi_k\big(G_a(w)\big) \;=\; \varphi_k(\lambda\, w) + \mathrm{O}(a)$. Since

$$G^{\circ n}_a(w) \;=\; F^{\circ n}_{B_k(a),\,a}(w) \;=\; w \,+\, w\,\Phi\big(B_k(a), a, w\big) \;=\; w \,+\, w\, a^k \varphi_k(w) + \mathrm{O}(a^{k+1})\ .$$

It follows that

$$\begin{aligned} G^{\circ n}_a\big(G_a(w)\big) \quad &= \quad G_a(w) \,+\, G_a(w)\, a^k\, \varphi_k(G_a(w)) + \mathrm{O}(a^{k+1}) \\ &= \quad G_a(w) \,+\, \lambda\, w\, a^k\, \varphi_k(\lambda\, w) + \mathrm{O}(a^{k+1})\ . \end{aligned}$$

Similarly, $G'_a(w) = \lambda + O(a)$, and it follows that

$$\begin{aligned} G_a\big(G^{\circ n}_a(w)\big) \quad &= \quad G_a\big(w \,+\, w\, a^k \varphi_k(w) + \mathrm{O}(a^{k+1})\big) \\ &= \quad G_a(w) \,+\, G'_a(w) w\, a^k \varphi_k(w) + \mathrm{O}(a^{k+1}) \\ &= \quad G_a(w) + \lambda\, w\, a^k\, \varphi_k(w) \,+\, \mathrm{O}(a^{k+1})\ . \end{aligned}$$

Setting these two expressions equal, we obtain the required identity (19).

Thus φ_k commutes with multiplication by an n-th root of unity. It follows that the order of any pole must be a multiple of n. Combining this statement with Lemma 6.4, we see that φ_k has no poles, and hence is constant, for $k < n$.

Proof of Proposition 6.3. It is not difficult to check that

$$\frac{\partial \Phi(t, a, w)}{\partial t} \;=\; 2\pi i n\, e^{2\pi i n t} \,+\, \mathrm{O}(a)\ .$$

It follows easily from this, together with the estimate (16), that

$$\Phi(B_{k-1}(a)+\tau\, a^k,\ a,\ w) \ = \ a^k(2\pi i n\tau+\varphi_k) \ + \ \mathrm{O}(a^{k+1})$$

for any $\tau\in\mathbb{C}$. Thus, if we define c_k by the equation

$$2\pi i n\, c_k+\varphi_k \ = \ 0\,, \tag{20}$$

and set $B_k(a):=B_{k-1}(a)+c_k\, a^k$, then

$$\Phi(B_k(a),\ a,\ w) \ = \ \mathrm{O}(a^{k+1})\ .$$

This completes the induction and proves Proposition 6.3. □

Thus for each $k<n$ we have constructed a polynomial $B_k(a)=\sum_{j=0}^{k} c_j a^j$ with the property that $\Phi(B_k(a),\, a,\, w) \ = \ \mathrm{O}(a^{k+1})$ for each w. The statement for $k=n$ is slightly more complicated:

Theorem 6.5. *There is a polynomial $B(a)=B_n(a)=\sum c_k a^k$ of degree at most n, with*

$$\Phi\big(B(a),\, a,\, w\big) \ = \ a^n\Big(Cw^n \ - \ C'w^{-n}\Big) \ + \ \mathrm{O}(a^{n+1})\ , \tag{21}$$

where $C'=(-1)^n\overline{C}$. Furthermore, the coefficients c_k are real, and zero for k odd, so that $B(a)$ is a real even polynomial.

Proof. Setting

$$\Phi(B_{n-1}(a),\, a,\, w) \ = \ a^n\varphi_n(w) \ + \ \mathrm{O}(a^{n+1})\ ,$$

and arguing as above, we see easily that $\varphi_n(w)$ can be written as a rational function of w^n of the form $\varphi_n(w) \ = \ -2\pi i c_n + Cw^n - C'w^{-n}$ for some constant c_n. Setting $B(a)=B_n(a)+c_n w^n$, the required identity (21) follows.

We will prove by induction that the coefficients c_k are real, and zero for k odd. The polynomial $B_0(a)=m/n$ certainly has real coefficients. Suppose inductively that $B_k(a)$ has real coefficients. Then for real values of a the rational function $w\mapsto G_a(w) \ = \ F_{B_k(a),\,a}(w)$ commutes with the antipodal map $\mathcal{A}$. Hence

$$G_a^{\circ n}\big(\mathcal{A}(w)\big) \ = \ \mathcal{A}(w)\big(1+a^k\varphi_k\big(\mathcal{A}(w)\big)+\mathrm{O}(a^{k+1})\big)$$

is equal to

$$\mathcal{A}\big(G_a^{\circ n}(w)\big) \ = \ \frac{\mathcal{A}(w)}{1+a^k\overline{\varphi_k(w)}+\mathrm{O}(a^{k+1})} \ = \mathcal{A}(w)\big(1-a^k\overline{\varphi_k(w)}+\mathrm{O}(a^{k+1}\big)\ ,$$

hence

$$\varphi_k(-1/\overline{w}) \ = \ -\overline{\varphi_k(w)}\ . \tag{22}$$

For $k<n$ it follows that the constant φ_k is pure imaginary; hence by the identity (20) it follows that c_k is real. Similarly, for n even, since the function $w\mapsto Cw^n-\overline{C}w^{-n}$ automatically satisfies the identity (22), it follows easily that c_n is real.

On the other hand, since $\Phi(t,a,w) \ = \ \Phi(t,-a,-w)$, it follows that $(-a)^k\varphi_k(w) \ = \ a^k\varphi_k(-w)$, and it then follows easily that $c_k \ = \ 0$ for all odd $k\le n$. This completes the proof of Theorem 6.5. □

Periodic Cycles. From now on, we assume that n is even. Setting as before,

$$G_a := F_{B(a),a},$$

so that $G_0(w) = \lambda w$ and

$$G_a^{\circ n}(w) = w + a^n w\left(C_\lambda w^n - \frac{\overline{C_\lambda}}{w^n}\right) + \mathrm{O}(a^n).$$

We shall use the Implicit Function Theorem to identify particular cycles of period n for the map G_a. Let us consider the analytic map

$$\widehat{\Phi}(a,w) := \Phi(B(a),a,w)/a^n = \frac{G_a^{\circ n}(w) - w}{a^n w}$$

which is defined and holomorphic on the open set consisting of all (a,w) with $(B(a),\, a,\, w) \in V_n$. (Compare equation (11).) Note that for $a \neq 0$, if $\widehat{\Phi}(a,w) = 0$, then w is a periodic point of G_a of period dividing n.

The map

$$\widehat{\Phi}(0,w) = C_\lambda w^n - \overline{C_\lambda} w^{-n}$$

vanishes precisely at the points w which satisfy

$$w^{2n} = \frac{\overline{C_\lambda}}{C_\lambda} = \frac{|C_\lambda|^2}{C_\lambda^2}.$$

Let us denote by $w_j^\pm$, $j \in \mathbb{Z}/n\mathbb{Z}$, the n-th roots of $\pm|C_\lambda|/C_\lambda$, ordered cyclically on the unit circle, so that w_j^+ lies in between w_j^- and w_{j+1}^-. Note that

$$\frac{\partial\widehat{\Phi}(0,w)}{\partial w}\Big|_{0,w_j^\pm} = \frac{n}{w_j^\pm}\cdot\left(C_\lambda\cdot(w_j^\pm)^n + \frac{\overline{C_\lambda}}{(w_j^\pm)^n}\right) = \pm\frac{2n|C_\lambda|}{w_j^\pm} \neq 0.$$

According to the Implicit Function Theorem, there are $2n$ associated functions $\zeta_j^\pm(a)$, defined and holomorphic near $a = 0$, solutions of

$$\widehat{\Phi}\big(a,\zeta_j^\pm(a)\big) = 0 \quad\text{and}\quad \zeta_j^\pm(0) = w_j^\pm.$$

Lemma 6.6. *The point $\zeta_j^+(a)$, $j \in \mathbb{Z}/n\mathbb{Z}$, form a cycle $\mathcal{C}^+(a)$ of period n for G_a and the points $\zeta_j^-(a)$, $j \in \mathbb{Z}/n\mathbb{Z}$, form a cycle $\mathcal{C}^-(a)$ of period n for G_a. Each cycle is preserved by the antipodal map.*

Proof. For $a \neq 0$, the points $\zeta_j^\pm(a)$ are fixed points of $G_a^{\circ n}$. The map G_a permutes the fixed points of $G_a^{\circ n}$. As a consequence, $G_a\big(\zeta_j^\pm(a)\big)$ is a solution of $\widehat{\Phi}(a,w) = 0$. If $\lambda = e^{2\pi i m/n}$, then for $a = 0$, we have $G_0(w_j^\pm) = w_{j+m}^\pm$. It follows from the uniqueness of solutions provided by the Implicit Function Theorem that

$$G_a\big(\zeta_j^\pm(a)\big) = \zeta_{j+m}^\pm(a).$$

Similarly, the antipodal map permutes the cycles of G_a. Since it preserves the cycles $\mathcal{C}^+(0)$ and $\mathcal{C}^-(0)$ of G_0, it follows again from the uniqueness of solutions provided by the Implicit Function Theorem that the antipodal map preserves the cycles $\mathcal{C}^+(a)$ and $\mathcal{C}^-(a)$. □

Lemma 6.7. *If a is sufficiently close to 0, the cycle $\mathcal{C}^+(a)$ is repelling and the cycle $\mathcal{C}^-(a)$ is attracting.*

Proof. Observe that

$$(G_a^{\circ n})'(w) = 1 + (n+1)a^n C_\lambda w^n + (n-1)a^n \,\overline{C}_\lambda / w^n \ .$$

Using the estimate

$$(\zeta_j^{\pm})^n \ = \ \pm|C_\lambda|/C_\lambda + \mathrm{O}(a) \ ,$$

it follows that

$$(G_a^{\circ n})'\big(\zeta_j^{\pm}(a)\big) = 1 \pm 2n|C_\lambda|a^n + \mathrm{O}(a^{n+1}). \quad \square$$

Remark 6.8. For odd values of n there is a similar pair of period n orbits near the unit circle. However the multipliers, instead of being approximately $1 \pm \epsilon$, are rather approximately $1 \pm i\epsilon$. Compare Figure 24 (left).

Proof of Theorem 6.1. We now give a more global description of the dynamics of G_a for small real values of a. For each $j \in \mathbb{Z}/n\mathbb{Z}$, denote by $\mathcal{B}_j(a)$ the immediate basin of the attracting fixed point $\zeta_j^-(a)$ under the map $G_a^{\circ n}$. It follows from Theorem 2.1 that this basin is simply connected.

Proposition 6.9. *For a sufficiently small and for $j \in \mathbb{Z}/n\mathbb{Z}$, we have that*

$$\zeta_j^+(a) \in \partial\mathcal{B}_j(a) \cap \partial\mathcal{B}_{j+1}(a).$$

More precisely, $\zeta_j^+(a)$ is accessible from both $\mathcal{B}_j(a)$ and $\mathcal{B}_{j+1}(a)$.

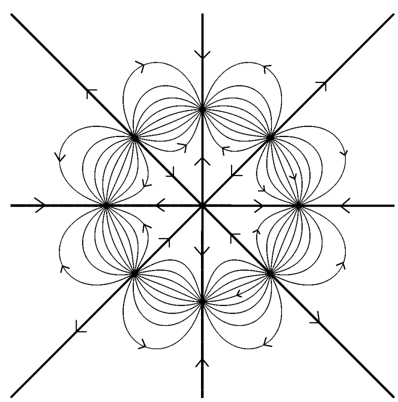

Figure 27. *Some trajectories of the vector field Ξ. The trajectories which go to the poles 0 and ∞ in finite time are emphasized. The picture is drawn for $n = 4$ and $C_\lambda/|C_\lambda| = -1$.*

The proof will depend on approximating the iterates of $G_a^{\circ n}$ by a smooth flow. Let Ξ be the holomorphic vector field defined on $\mathbb{C}\smallsetminus\{0\}$ by

$$\Xi := w\left(C_\lambda w^n - \frac{\overline{C_\lambda}}{w^n}\right)\frac{\mathrm{d}}{\mathrm{d}w},$$

and let Υ_s be the associated flow, satisfying

$$\frac{\partial \Upsilon_s(w)}{\partial s} \ = \ w\left(C_\lambda w^n - \frac{\overline{C_\lambda}}{w^n}\right), \qquad \text{with} \qquad \Upsilon_0(w) = w \ , \tag{23}$$

so that $\Upsilon_{s_1+s_2} = \Upsilon_{s_1} \circ \Upsilon_{s_2}$ wherever $\Upsilon_{s_1} \circ \Upsilon_{s_2}$ is defined. Compare Figure 27. Note that this flow carries the unit circle into itself. The fixed points, with $C_\lambda w^n = \overline{C}_\lambda / w^n$, divide the unit circle into $2n$ arcs, each leading from a repelling fixed point to an attracting fixed point.

For a close enough to 0, the map $G_a^{\circ n}$ may be thought as a kind of Euler's method with step a^n for the differential equation (23). In particular

$$G_a^{\circ n}(w) \;=\; \Upsilon_{a^n}(w) \,+\, \mathrm{O}(a^{n+1}).$$

Intuitively, the orbit of $G_a^{\circ n}$ from w converges to the flow through w as $a \to 0$. More precisely:

Lemma 6.10. *Let X be a compact subset of $\mathbb{C}\smallsetminus\{0\}$. Suppose that the flow $\Upsilon_s(w)$ is defined for all $w \in X$ and all $0 \le s \le s_0$. Then as a tends to zero and as ka^n tends to $s \in [0,\, s_0]$, the iterate $G_a^{\circ kn}(w)$ converges uniformly to $\Upsilon_s(w)$ for all $w \in X$.*

Proof. This follows from standard estimates in the theory of ordinary differential equations. See for example [**HW**, Theorem 4.4.1]. □

Proof of Proposition 6.9. We will show, for example, that $\zeta_j^+(a)$ is accessible from the immediate basin $\mathcal{B}_j(a)$ provided that a is small enough. Choose a small open disk D^+ centered at $\zeta_j^+(0)$ and a small open disk D^- centered at $\zeta_j^-(0)$. Then for a small, each fixed point $\zeta_j^\pm(a)$ will be contained in the corresponding disk $D^\pm$ which will serve as a Kœnigs linearizing neighborhood, with

$$G_a^{\circ n}(D^+) \supset D^+ \qquad \text{and} \qquad G_a^{\circ n}(D^-) \subset D^- \; .$$

Choose a point w_0 on the unit circle which belongs to D^+, and such that $\Upsilon_{s_0}(w_0)$ belongs to D^- for suitably chosen s_0. Then for a sufficiently small we can choose k so that $G_a^{\circ kn}$ also belongs to D^-. This proves that $\zeta_j^+(a)$ is accessible from the full attracting basin of $\zeta_j^-(a)$. To prove that it is accessible from the immediate basin, we must work just a little harder.

For a small, the image $G_a^{\circ n}(w_0)$ will also belong to D^+. Let $L \subset D^+$ be a line segment joining these points. If a is small enough, then

(1) the image $\Upsilon_s(L)$ will be contained in D^-, and furthermore
(2) the image $G_a^{\circ kn}(L)$ will also be contained in D^-.

Now the union of the k images $G_a^{\circ in}(L)$ with $0 \le i < k$ will form a path from w_0 to the point $G_a^{\circ kn}(w_0)$, all lying within the full attracting basin, and therefore all lying within the immediate basin $\mathcal{B}_j(a)$. This completes the proof of Proposition 6.9. Evidently Theorem 6.1, as stated at the beginning of this section, follows as an immediate consequence. □

Here is an important further property of the associated Julia sets.

Proposition 6.11. *For a sufficiently small, the immediate basin of ∞ and the immediate basin of 0 for G_a are quasidisks whose boundaries intersect precisely along the repelling cycle $\mathcal{C}^+(a)$. Hence the same statement is true for any map f_q representing an element of the hyperbolic component $\mathcal{H}(m/n)$.*

Sketch of proof. Figure 28 illustrates a compact region $W \subset \mathbb{C}$ with the property that the vector field Ξ points into W at every boundary point. (The detailed construction is given at the end of this proof.) If a is small enough, then it follows easily that $G_a^{\circ n}$ maps a neighborhood of ∂W diffeomorphically into the interior, which will be denoted by $\mathring{W}$. However, for the following construction, we

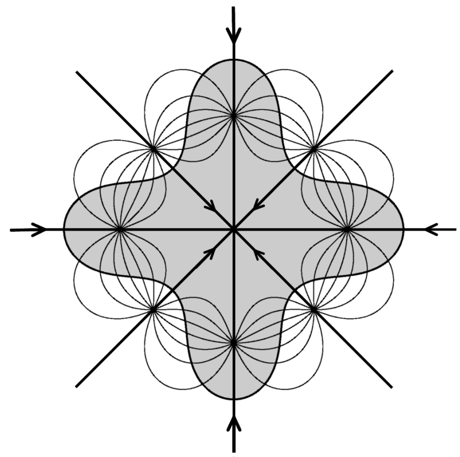

FIGURE 28. *The topological disk W. Some trajectories of the vector field Ξ are shown.*

will need the sharper statement that there is a subset $W' \subset W$ such that G_a itself maps $\partial W'$ into $\overset{\circ}{W}'$. The proof follows.

Let W_1 be the intersection of the compact sets $G_a^{\circ j}(W)$ for $0 \le j < n$. Then it is not hard to see that W_1 is simply connected, with

$$G_a(\partial W_1) \subset W_1 \qquad \text{and with} \qquad G_a^{\circ n}(\partial W_1) \subset \overset{\circ}{W}_1 \,.$$

Let A be the intersection $G_a(\partial W_1) \cap \partial W_1$. If A is vacuous, we are done. Otherwise, choose $W_2 \subset W_1$ so that $W_2 \supset G_a(W_1) \cap W_1$, and so that ∂W_2 consists of the compact set A, together with one or more arcs lying within the open set $\overset{\circ}{W}_1 \smallsetminus G_a(W_1)$. (Compare Figure 29.) Then we will show that

$$G_a^{\circ n-1}(\partial W_2) \subset \overset{\circ}{W}_2 \,.$$

For otherwise, there would be a partial orbit $G_a : w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n$ which is contained in ∂W_2. In fact these w_j would all have to be contained in the subset $A \subset \partial W_2$. For $j < n$ this follows since any point of $\partial W_2 \smallsetminus A$ belongs to $\overset{\circ}{W}_1$ and hence maps into $\overset{\circ}{W}_2$. For $j = n$ it follows since all points outside of A in $G_a(A)$ are contained in the interior $\overset{\circ}{W}_2$. On the other hand, if such a sequence existed, then we could add an initial element $w_0 \in W_1$ with $G_a(w_0) = w_1$. This would yield a partial orbit $w_0 \mapsto w_1 \mapsto \cdots \mapsto w_n$ of length $n+1$ which is contained in ∂W_1, which is impossible. Repeating this construction inductively, we find simply connected regions $W_1 \supset W_2 \supset \cdots \supset W_n$ such that $G_a^{\circ n+1-k}(\partial W_k) \subset \overset{\circ}{W}_k$. Thus W_n is the required region with $G_a(\partial W_n) \subset \overset{\circ}{W}_n$.

We may then perform a quasiconformal surgery to replace the dynamics of G_a within W_n by the dynamics of the linear contraction $w \mapsto w/2$. Since the region W_n contains two of the four critical points of G_a, the result will be of degree two, conformally conjugate to the quadratic polynomial map $w \mapsto w^2 + w/2$. This shows that G_a is quasiconformally conjugate to the quadratic polynomial $w \mapsto w/2 + w^2$

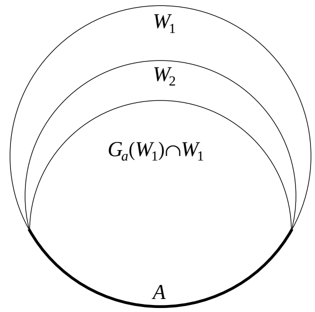

FIGURE 29. *Schematic picture illustrating the construction of* W_2.

on the set of points whose orbit never intersects W_n, in particular on the closure of the immediate basin of ∞ for G_a. The conclusion of Proposition 6.11 then follows easily for G_a, and hence for any map in the hyperbolic component $\mathcal{H}(m/n)$.

To complete the proof of this proposition, we must describe the construction of the initial compact region $W \subset \mathbb{C}$. We introduce two new functions: Let $\vartheta : \mathbb{C} \smallsetminus \{0\} \to \mathbb{C} \smallsetminus \{0\}$ and $\varrho : \mathbb{C} \smallsetminus \{0\} \to \widehat{\mathbb{C}} \smallsetminus \{-1, 1\}$ be defined by

$$\zeta = \vartheta(w) := \frac{C_\lambda}{|C_\lambda|} w^n \qquad \text{and} \qquad \tau = \varrho(\zeta) := \frac{\zeta + 1}{\zeta - 1}.$$

The introduction of the covering map ϑ has the purpose of eliminating the symmetry obtained by the fact that the vector field Ξ is invariant under the rotation, $G_0(w) = \lambda w$. Letting $\zeta = \vartheta(w)$, we obtain:

$$\frac{d\zeta}{\zeta} = n \frac{dw}{w} \qquad \text{and} \qquad \vartheta_* \Xi = n|C|(\zeta^2 - 1)\frac{d}{d\zeta}.$$

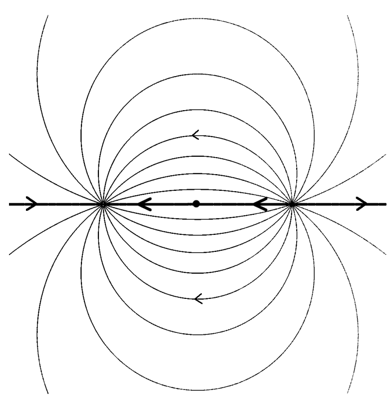

FIGURE 30. *Some trajectories of the vector field* $\vartheta_*\Xi$. *Note that Figure* 27 *is an* n*-fold branched covering of this figure with* $n = 4$, *branched at zero* (*indicated by the central dot*) *and at* ∞.

The Möbius transformation ϱ takes the vector field $\vartheta_*\Xi$ of Figure 30 to a radial vector field, equal to $-\tau\, d/d\tau$ multiplied by a positive constant. More precisely, if $\tau = \varrho(\zeta)$, then

$$d\tau = \frac{-2}{(\zeta-1)^2}d\zeta \qquad \text{and} \qquad \varrho_*\vartheta_*\Xi \;=\; -2n|C|\tau\frac{d}{d\tau}\;.$$

In particular, the vector field $(\varrho\circ\vartheta)_*\Xi$ is radial, with a source at ∞ and a sink at 0. Now let $U = U(1/2,\,1)$, be the disk of center 1/2 and radius 1. Then the set $W := (\varrho\circ\vartheta)^{-1}(U)$ satisfies the required conditions. (See Figures 28 and 31.) This completes the proof of Proposition 6.11. □

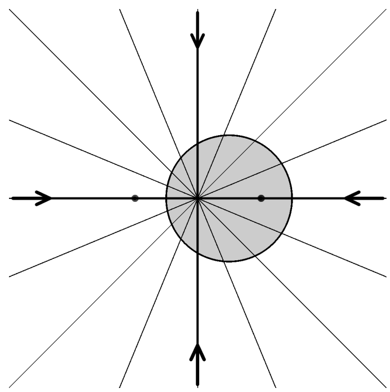

FIGURE 31. *The disk* $U = U(1/2,1)$ *of center* $1/2$ *and radius* 1*, and the inward radial vector field* $-\tau\, d/d\tau$*. The branch points at* ±1 *are indicated by dots. The preimage* $W = (\varrho\circ\vartheta)^{-1}(U)$ *is depicted in Figure* 28*.*

Remark 6.12. For any map f_q in the tongue $\mathcal{H}(m/n)$, let $U_1, U_2, \ldots, U_n$ be the periodic Fatou components, numbered in cyclic order as in Theorem 6.1, where the subscripts are integers modulo n. Then each U_j has a unique attracting periodic point $p_0 = p_0(U_j)$ in its interior, and three repelling periodic points p_1, p_2, p_3 on its boundary. These can be numbered so that $p_3(U_j) = p_1(U_{j+1})$. (Thus p_1 and p_3 are shared, but $p_2(U_j)$ is a boundary point only of U_j.)

The boundary $\partial\mathcal{H}(m/n)$ is made of three smooth curves of parabolic points, which can be numbered as C_1, C_2, or C_3 according as p_0 merges with p_1, p_2 or p_3 in the associated dynamic plane. The end points of these curves consist of two cusp points in the finite parameter plane: one at $\overline{C}_1\cap\overline{C}_2$, such that p_0 merges with both p_1 and p_2, and one at $\overline{C}_2\cap\overline{C}_3$, such that p_0 merges with both p_2 and p_3. There is also a third cusp point where in some sense p_0 merges with p_1 and p_3; but this can never happen in the finite parameter plane. In fact, p_1 and p_3 are distinct points of a common self-antipodal period n orbit. If they came together, this entire orbit would collapse to a self-antipodal fixed point, which is impossible. Thus the third cusp point $\overline{C}_1\cap\overline{C}_3$ must be identified with the point at infinity in $\partial\mathcal{H}(m/n)$.

Empirically, the finite boundary curve C_2 is always separated from the principal hyperbolic component $\mathcal{H}_0$ by a cluster of infinitely many other hyperbolic components. However, the two infinite boundary curves C_1 and C_3 seem to contain entire bare intervals which have $\mathcal{H}_0$ on one side and the hyperbolic component $\mathcal{H}(m/n)$ on the other. (Compare Figure 10.)

Now recall from Theorem 4.27 that the interval consisting of internal angles with formal rotation number $t = m/n$ can be explicitly computed as

$$\left[\psi^{-1}(t^-)\,,\ \psi^{-1}(t^+)\right].$$

(As an example, for the (1/2)-tongue as illustrated in Figure 10, this interval is $[1/3,\ 2/3]$.) Conjecturally the internal rays of angle $\psi^{-1}(t^{\pm})$ always accumulate on the bare stretches of C_1 and C_3. They may land at one single point of C_1 or C_3, but it seems more likely that they oscillate, with an entire interval of accumulation points.[15]

Remark 6.13. Suppose that $s = q^2$ belongs to the tongue $\mathcal{H}(m/n)$. Then it follows from Proposition 6.11 that the n points in the Julia set of f_q which are visible from both zero and infinity are the landing points of rays from zero or infinity having rotation number m/n, in precise analogy with Definition 4.22. It seems likely that the same statement is true more generally, for any f_q such that q^2 has formal rotation number m/n (in other words, for points in the associated crown). More generally one could consider the case of completely arbitrary rotation number. Is the same statement always true provided that there is no Herman ring which would serve as an obstruction? In particular, is it always possible to compute the formal rotation number for a given map by dynamic properties of the map itself, without comparing it with other maps?

Perhaps the following statement might be used as a step towards this goal. Fixing some f_q, let $\mathcal{B}_0$ and $\mathcal{B}_\infty$ be the immediate basins of zero and infinity.

Theorem 6.14. *For any map f_q in our family, either:*

(1) *The closures $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_\infty$ have a non-vacuous intersection; or else*

(2) *these two basins are separarted by a Herman ring A such that one of the two boundary curves of A is contained in $\overline{\mathcal{B}}_0$ and the other is contained in $\overline{\mathcal{B}}_\infty$.*

Proof. Assume $\overline{\mathcal{B}}_0 \cap \overline{\mathcal{B}}_\infty = \emptyset$. Since $\widehat{\mathbb{C}}$ is connected, there must be at least one component A of $\widehat{\mathbb{C}} \smallsetminus (\overline{\mathcal{B}}_0 \cup \overline{\mathcal{B}}_\infty)$ whose closure intersects both $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_\infty$. If this set A were simply connected, then the boundary ∂A would be a connected subset of $\overline{\mathcal{B}}_0 \cup \overline{\mathcal{B}}_\infty$ which intersects both of these sets, which is impossible. The complement of A cannot have more than two components, since each complementary component must contain either $\mathcal{B}_0$ or $\mathcal{B}_\infty$. Therefore A is an annulus with boundary components $C_0 \subset \overline{\mathcal{B}}_0$ and $C_\infty \subset \overline{\mathcal{B}}_\infty$. Evidently A is unique, and hence is self-antipodal.

Note that $f(A) \supset A$: This follows easily from the fact that f is an open map with $f(C_0) \subset \partial\mathcal{B}_0$ and $f(C_\infty) \subset \partial\mathcal{B}_\infty$.

We will prove that A is a fixed Herman ring for f. The map $f : f^{-1}(A) \to A$ is a proper map of degree 3. The argument will be divided into three cases according as $f^{-1}(A)$ has 1, 2 or 3 connected components.

Three Components. In this case, one of the three components, say A', must be preserved by the antipodal map (because A is preserved by the antipodal map).

[15] In fact, computer pictures by Hiroyuki Inou illustrate such oscillations for the $1/3$ rd ray. (Compare [**IM**] as well as [**HS**], [**MNS**], [**KN**] for analogous results in related cases.)

Then $f : A' \to A$ has degree 1 and is therefore an isomorphism. In particular, the modulus of A' is equal to the modulus of A. Since the annuli A and A' are both self-antipodal, they must intersect. (This follows from the fact that the antipodal map interchanges the two complementary components of A.) In fact A' must actually be contained in A. Otherwise A' would have to intersect the boundary ∂A, which is impossible since it would imply that $f(A') = A$ intersects $f(\partial A) \subset (\overline{\mathcal{B}}_0 \cup \overline{\mathcal{B}}_\infty)$. Therefore, using the modulus equality, it follows that $A' = A$. Thus the annulus A is invariant by f, which shows it is a fixed Herman ring for f.

Two Components. In this case, one of the two components would map to A with degree 1 and the other would map with degree 3. Each must be preserved by the antipodal map, which is impossible since any two self-antipodal annuli must intersect.

One Component. It remains to prove that $f^{-1}(A)$ cannot be connected. We will use a length-area argument that was suggested to us by Misha Lyubich. The argument is based on the following observation.

Lemma 6.15. *A generic curve joining the two boundary components of A has a preimage by f which intersects both boundary components of A.*

Here a "generic" curve means one that does not pass through a critical value, so that each of its three preimages is a continuous lifting of the curve.

Proof. We may assume that $\mathcal{B}_0$ contains only one critical point, since the case where it contains two is well understood (§3). Thus a generic point of $\mathcal{B}_0$ has two preimages in $\mathcal{B}_0$ and a third preimage in a disjoint copy $\mathcal{B}'_0$. The same is true for the closure $\overline{\mathcal{B}}_0$ (except that there may be an intersection point in $\overline{\mathcal{B}}_0 \cap \overline{\mathcal{B}}'_0$, necessarily a critical point).

A generic path γ through A from $z_0 \in C_0$ to $z_\infty \in C_\infty$ has three preimages. Two of the three start at points of $\overline{\mathcal{B}}_0$ and two of them end at points of $\overline{\mathcal{B}}_\infty$. Thus at least one must cross from $\overline{\mathcal{B}}_0$ to $\overline{\mathcal{B}}_\infty$, passing through A. □

Let μ be an extremal (flat) metric on A. Since there are points in A whose image is not in A (otherwise A would be a Fatou component, thus a Herman ring), we have[16]

$$\mathrm{Area}_{f_*\mu}(A) \le c \cdot \mathrm{Area}_\mu(A)$$

for some constant $c < 1$. Next, let γ be a curve joining the two boundary components of A and let γ' be a preimage of γ which intersects both boundaries of A. Then we can choose a segment $\gamma'_0 \subset \gamma' \cap A$ which joins the two boundaries. We can easily check that

$$\mathrm{Length}_{f_*\mu}(\gamma) \;\ge\; \mathrm{Length}_\mu(\gamma') \;\ge\; \mathrm{Length}_\mu(\gamma'_0).$$

[16]The notations need some explanation. The metric μ can be written as $g(z)|dz|$, where $g(z) > 0$ for all $z \in A$. It is convenient to set $g(z) = 0$ for $z \notin A$. The associated measure can be written as $g(z)^2\Lambda_z$, where Λ_z is the standard Lebesgue measure on the z-plane. The push-forward of this measure can be written as $\sum_{f(z)=w} \left|g(z)/f'(z)\right|^2 \Lambda_w$, to be summed over the three preimages of w. Thus, for compatibility, we must define $f_*\mu$ to be the metric

$$\left(\sum_{f(z)=w} \left|g(z)/f'(z)\right|^2\right)^{1/2} d|w| \;.$$

As a consequence,

$$\frac{\mathrm{Length}^2_{f_*\mu}(\gamma)}{\mathrm{Area}_{f_*\mu}(A)} \ge \frac{1}{c}\frac{\mathrm{Length}^2_{\mu}(\gamma_0')}{\mathrm{Area}_{\mu}(A)} \ge \frac{1}{c}\mathrm{modulus}(A).$$

Here $f_*\mu$ is a non-trivial metric on all of A since $f(A) \supset A$ Since this inequality holds for almost all curves γ joining the two boundary components of A, we have

$$\mathrm{modulus}(A) \ge \frac{1}{c}\mathrm{modulus}(A),$$

which is a contradiction. □

7. The Herman Ring Locus

This section will prove that Herman rings are fixed and will describe the locus of all such rings in parameter space.

Theorem 7.1. *Let f be any cubic map which commutes with the antipodal map. Then f has at most one Herman ring, which must have period one, and hence must consist of a single self-antipodal annulus which separates zero from infinity.*

(Compare Figures 4 and 32.) In fact Shishikura showed that a cubic map can have at most one cycle of Herman rings. (See [**Sh1**, Theorem 3].) The proof that the period of this cycle must be one in the case of a cubic map which commutes with the antipodal map is inspired by [**Sh2**]. It will make use of the following.

Remark 7.2 (Some Elementary Topology). If the antipodal map $\mathcal{A}$ of $\widehat{\mathbb{C}}$ sends a Jordan curve γ onto itself, then it interchanges the two components of $\widehat{\mathbb{C}}\smallsetminus\gamma$. (The only other possibility would be that $\mathcal{A}$ maps each complementary component E to itself. But this is impossible by the Brouwer Fixed Point Theorem, which asserts that every continuous map from the closed disk $\overline{E}$ to itself must have a fixed point.) It follows immediately that any two $\mathcal{A}$-invariant Jordan curves must intersect each other. Furthermore, any two $\mathcal{A}$-invariant connected open sets W_1 and W_2 must intersect each other. In fact, each W_j must contain an $\mathcal{A}$-invariant Jordan curve. To see this, choose a path P joining some point to its antipode within W_j. The union $P\cup\mathcal{A}(P)$ will then be an $\mathcal{A}$-invariant closed curve. This particular curve may have self-intersections; but it is not difficult to eliminate them. In particular, assuming that W_j has a hyperbolic metric, we can choose an $\mathcal{A}$-invariant curve of minimal length with respect to the Poincaré metric on W. This will be a closed geodesic, without self-intersections, and the conclusion follows.

Proof of Theorem 7.1. Start by considering a cycle of Herman rings H_j of period $p \ge 1$,

$$H_0 \overset{f}{\mapsto} H_1 \overset{f}{\mapsto} \cdots \overset{f}{\mapsto} H_p = H_0\,. \tag{24}$$

Since f commutes with the antipodal map $\mathcal{A}$, it follows that

$$\mathcal{A}(H_0) \overset{f}{\mapsto} \mathcal{A}(H_1) \overset{f}{\mapsto} \cdots \overset{f}{\mapsto} \mathcal{A}(H_p) = \mathcal{A}(H_0)$$

also forms a cycle of Herman rings. Since there is at most one cycle of Herman rings by [**Sh1**], the cycle of Herman rings of Equation (24) is globally preserved by the antipodal map. Let γ_j be the ***core curve*** of H_j, that is, the unique simple closed geodesic for H_j with its hyperbolic metric, with $j \in \mathbb{Z}/p\mathbb{Z}$; and let Γ be the union of these core curves.

Case 1. Suppose that $\mathcal{A}$ maps one of these curves γ_j to itself. Then it certainly also maps $f(\gamma_j)$ to itself. But by Remark 7.2 this is only possible if $\gamma_j = f(\gamma_j)$, so that the period is $p = 1$.

Case 2. No core curve is preserved by $\mathcal{A}$. In this case, there must be an even number of core curves γ, so p is even. Hence, the number of connected components of $\widehat{\mathbb{C}} \smallsetminus \Gamma$, which equals $p+1$, is odd. At least one of these must be preserved by $\mathcal{A}$. In fact there can be only one, by Remark 7.2.

Since the open set $V = f^{-1}(W)$ cannot intersect any core curve, each connected component V_j of V must be contained in one of the $p+1$ complementary components. Note that each V_j maps onto W by a (possibly branched) covering map of some degree $d_j \geq 1$, with $\sum_j d_j = 3$. The entire set V cannot be contained in W, since otherwise the sequence of iterates of f restricted to the complement $\widehat{\mathbb{C}} \smallsetminus \overline{W}$ would be normal. This is impossible, since $\widehat{\mathbb{C}} \smallsetminus \overline{W}$ contains boundary components of Herman rings, belonging to the Julia set.

Thus at least one component V_1 of V is disjoint from W, and its antipode $V_2 = \mathcal{A}(V_1)$ must belong to a different one of the $p+1$ components. Since $d_1 + d_2 \leq 3$, with $d_1 = d_2$, we must have $d_1 = d_2 = 1$. Hence there must be a third component V_3, also mapping with degree one, which must be contained in W.

Note that V_3 cannot be equal to W, for otherwise $f(W) = f(V_3)$ would be equal to W, hence W would be part of the Fatou set, which is impossible.

But the hypothesis that V_3 is a proper subset of W also leads to a contradiction, as follows. Let $\delta \subset W$ be an $\mathcal{A}$-invariant closed curve of minimal length L with respect to the hyperbolic metric on W. Pulling δ back under $f : V_3 \xrightarrow{\cong} W$, we obtain an $\mathcal{A}$-invariant curve which has length L with respect to the hyperbolic metric on V_3, and hence has length strictly less than L with respect to the W-metric. This contradiction completes the proof of Theorem 7.1. $\square$

Lemma 7.3. *There is a map f_q in our family with a Herman ring of rotation number t if and only if t is a Brjuno number.*

Proof. Suppose that f_q has a Herman ring H with rotation number t. Then $H = f_q(H)$ by Theorem 7.1. We can construct a quadratic rational map having a Siegel disk with the same rotation number by Shishikura surgery. (See [**Sh1**] or [**BF**].) Since this quadratic map will have a critical fixed point, it will be conformally conjugate to a polynomial. According to Yoccoz [**Yo**], this implies that t is a Brjuno number.

The converse proof is also by Shishikura surgery; but we need to be careful in carrying out this surgery in order to obtain a self-antipodal cubic map. It follows from Brjuno's work that for every Brjuno number β the quadratic map $g(z) = z^2 + e^{2\pi i\beta} z$ possesses a Siegel disk Δ, centered at the origin, with rotation number β. Choose a conformal isomorphism $\eta : \mathbb{D}_r \xrightarrow{\cong} \Delta$ where $\mathbb{D}_r$ is the round disk of some radius $r > 1$, so that the rotation $w \mapsto e^{2\pi i\beta} w$ of $\mathbb{D}_r$ corresponds to the map g on Δ. Let $M = \widehat{\mathbb{C}} \smallsetminus \eta(\overline{\mathbb{D}}_{1/r})$, and let M' be the same manifold but with opposite conformal structure. Form the disjoint union $M \sqcup M'$, and for each w in the annulus $1/r < |w| < r$ identify the image $\eta(w) \in M$ with the point corresponding to $\eta(-1/\overline{w})$ in M'. The result will be a Riemann surface $M \cup M'$ of genus zero, with a fixed point free antiholomorphic involution $\mathcal{A}'$ which interchanges M and M'. By the Uniformization Theorem, this surface is conformally isomorphic to the Riemann sphere, and it is not difficult to choose the conformal isomorphism so that $\mathcal{A}'$ corresponds to the antipodal map.

The inverse image $g^{-1}(\eta(\mathbb{D}_1) \subset \mathbb{C}$ consists of $\eta(\mathbb{D}_1)$ itself, together with a disjoint simply connected set U. Then g is well defined as a map from $\widehat{\mathbb{C}} \smallsetminus \eta(\mathbb{D}_1) \subset M$ into $M \subset M \cup M'$. We can easily extend to a C^1-smooth map F from $M \cup M'$ to itself which embeds U into M' and which commutes with $\mathcal{A}'$. Now, following Shishikura, we can put back the conformal structure on $M \cup M'$ to all iterated preimages of U and $\mathcal{A}'(U)$ to obtain a measurable conformal structure which is both F-invariant and $\mathcal{A}'$-invariant. Applying the measurable Riemann mapping theorem, F will become a cubic rational map with a ring of rotation number β, which commutes with the antipodal map and has a critical fixed point, and hence belongs to our family. □

Before going further, we need several preliminary results. Given $h \in (0, +\infty]$, let

$$A_h := \{z \in \mathbb{C}/\mathbb{Z} \; ; \; |\operatorname{Im}(z)| < h\} \;\subset\; \mathbb{C}/\mathbb{Z}$$

be the standard annulus of modulus $2h$, Let $\mathfrak{D}_h$ denote the set of univalent maps $\Phi : A_h \to \mathbb{C}/\mathbb{Z}$ isotopic to the inclusion map, and satisfying $\Phi(0) = 0$.

Lemma 7.4. *For all finite $h > 0$, the space $\mathfrak{D}_h$ is compact, using the topology of locally uniform convergence.*

Proof. We must show that every sequence of elements $\Phi_j \in \mathfrak{D}_h$ contains a convergent subsequence. For each Φ_j, since the image $\Phi_j(A_h)$ cannot contain the closure $\overline{A}_h$, we can choose a point $z_j \in \overline{A}_h \smallsetminus \Phi_j(A_h)$. After passing to an infinite subsequence, we may assume that the z_j converge to some limit $z_0 \in \overline{A}_j$. Next note that each of the perturbed maps $z \mapsto \Phi_j(z) + z_0 - z_j$ takes values in the hyperbolic surface $\mathbb{C}/\mathbb{Z} \smallsetminus \{z_0\}$. It follows that these form a normal family of maps on A_h; hence we can choose a convergent subsequence. Since the differences $z_0 - z_j$ tend to zero, it follows that the corresponding maps Φ_j converge to the same limit. □

Lemma 7.5. *For any $\epsilon > 0$ there exists a finite $h > 0$ such that for all $\Phi \in \mathfrak{D}_h$ and all $z \in A_1$ we have*

$$|\Phi(z) - z| < \varepsilon \quad \textit{and} \quad |\Phi'(z) - 1| < \varepsilon \; . \tag{25}$$

Proof. Suppose for example that for each $n > 0$ we could find a map $\Phi_n \in \mathfrak{D}_n$ and a point $z_n \in A_1$ such that $|\Phi_n(z_n) - z_n| \geq \varepsilon$. Choose infinite sets $I_1 \supset I_2 \supset \cdots$ of positive integers such that, as n tends to infinity within any given I_j, the maps Φ_n converge uniformly throughout the annulus A_j. Now let I' consist of the first element of I_1, the second element of I_2, and so on. Then as n tends to infinity through elements of I' the corresponding maps Φ_n converge locally uniformly throughout all of $\mathbb{C}/\mathbb{Z}$ to some limit map in $\mathfrak{D}_\infty$. But it is easy to see that the identity map of $A_\infty = \mathbb{C}/\mathbb{Z}$ is the unique element of $\mathfrak{D}_\infty$. Thus the Φ_n with $n \in I'$ must converge locally uniformly to the identity map. Since the corresponding points z_n all belong to the compact set $\overline{A}_1$, this contradicts the hypothesis that $|\Phi_n(z_n) - z_n| \geq \varepsilon$. The proof for the derivatives Φ' is similar, using the fact that uniform convergence implies uniform convergence of derivatives. □

Lemma 7.6. *For all $\varepsilon > 0$, there is an $h \in (0,+\infty)$ such that the following holds. Let $F : A \to \mathbb{C}/\mathbb{Z}$ be holomorphic on some open set $A \subset \mathbb{C}/\mathbb{Z}$ containing A_h. For $j \in \{1,2\}$, assume that $\tau_j \in \mathbb{C}/\mathbb{Z}$ is such that $F + \tau_j$ has a Herman ring H_j containing A_h with rotation number β_j. Then,*

$$\big|(\beta_1 - \beta_2) - (\tau_1 - \tau_2)\big| \le \varepsilon \cdot |\tau_1 - \tau_2|.$$

Proof. Given $\varepsilon \in (0, 1/3)$, let $h > 1$ be sufficiently large so that (25) holds. Let $j \in \{1, 2\}$. By assumption, $F + \tau_j$ has a Herman ring H_j of rotation number β_j containing A_h. Let $\Phi_j : H_j \to \mathbb{C}/\mathbb{Z}$ be the linearizing map conjugating $F + \tau_j$ to the rotation $z \mapsto z + \beta_j$, normalized by $\Phi_j(0) = 0$:

$$\begin{tikzcd} H_j \arrow[r, "F+\tau_j"] \arrow[d, "\Phi_j"'] & H_j \arrow[d, "\Phi_j"] \\ \mathbb{C}/\mathbb{Z} \arrow[r, "+\beta_j"] & \mathbb{C}/\mathbb{Z}\,. \end{tikzcd}$$

Note that Φ_j restricted to A_h is an element of $\mathfrak{D}_h$.

On the one hand, the curve $\Phi_1^{-1}(\mathbb{R}/\mathbb{Z})$ is contained in A_ε. Set

$$\Gamma := \Phi_2 \circ \Phi_1^{-1}(\mathbb{R}/\mathbb{Z}) \subset \mathbb{C}/\mathbb{Z}$$

and let $G : \Gamma \to \Gamma$ be defined by $G \circ \Phi_2 = \Phi_2 \circ (F + \tau_1)$:

$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[d, "z\mapsto z+\beta_1"'] & \Phi_1^{-1}(\mathbb{R}/\mathbb{Z}) \arrow[l, "\Phi_1"'] \arrow[r, "\Phi_2"] \arrow[d, "F+\tau_1"'] & \Gamma \arrow[d, "G"] \\ \mathbb{R}/\mathbb{Z} & \Phi_1^{-1}(\mathbb{R}/\mathbb{Z}) \arrow[l, "\Phi_1"] \arrow[r, "\Phi_2"'] & \Gamma. \end{tikzcd}$$

On the other hand, $\Phi_2(A_1)$ contains $A_{1-\varepsilon}$. If $z \in \Phi_1^{-1}(\mathbb{R}/\mathbb{Z}) \subset A_\varepsilon$, then $z \in H_2$, and $F(z) + \tau_2 \in H_2$,

$$\Phi_2(z) \in A_{2\varepsilon} \quad \text{and} \quad \Phi_2\big(F(z) + \tau_2\big) = \Phi_2(z) + \beta_2 \in A_{2\varepsilon}.$$

Since $\varepsilon < 1/3$, we have $A_{2\varepsilon} \subset A_{1-\varepsilon} \subset \Phi_2(A_1)$ and so, $F(z) + \tau_2 \in A_1$.

Now, if $w := \Phi_2(z) \in \Gamma$, we have

$$G(w) - w - \beta_2 = \Phi_2\big(F(z) + \tau_1\big) - \Phi_2(z) - \beta_2 = \Phi_2\big(F(z) + \tau_1\big) - \Phi_2\big(F(z) + \tau_2\big).$$

Since $F(z) + \tau_1$ and $F(z) + \tau_2$ both belong to A_1 and since

$$\sup_{z \in A_1} \big|\Phi_2'(z) - 1\big| < \varepsilon,$$

the Mean Value Theorem applied to $z \mapsto \Phi_2(z) - z$ yields

$$\big|G(w) - w - \beta_2 - (\tau_1 - \tau_2)\big| \le \varepsilon \cdot |\tau_1 - \tau_2|.$$

Finally, the rotation number β_1 of G is an average of $G(w) - w$ along the curve Γ, so that

$$\big|(\beta_1 - \beta_2) - (\tau_1 - \tau_2)\big| \le \varepsilon \cdot |\tau_1 - \tau_2|$$

as required. □

Corollary 7.7. *There is an* $h \in (0, +\infty)$ *such that for any map* $F : A \to \mathbb{C}/\mathbb{Z}$ *which is holomorphic on some open set* $A \subset \mathbb{C}/\mathbb{Z}$ *containing* A_h *and any irrational rotation number* $\beta \in \mathbb{R}/\mathbb{Z}$*, there is at most one* $\tau \in \mathbb{C}/\mathbb{Z}$ *such that* $F + \tau$ *has a Herman ring of rotation number* β *containing* A_h*.*

Proof. Apply the preceding lemma with $\beta_1 = \beta_2$ and $\varepsilon < 1$, which forces $\tau_1 = \tau_2$. □

Definition 7.8. By the ***Herman ring locus*** we mean the subset $\mathcal{R}$ of the q-parameter plane or of the (q^2)-parameter plane consisting of all q for which the map f_q has a Herman ring $H = f_q(H)$. Any connected component of $\mathcal{R}$ will be called a **hair** .

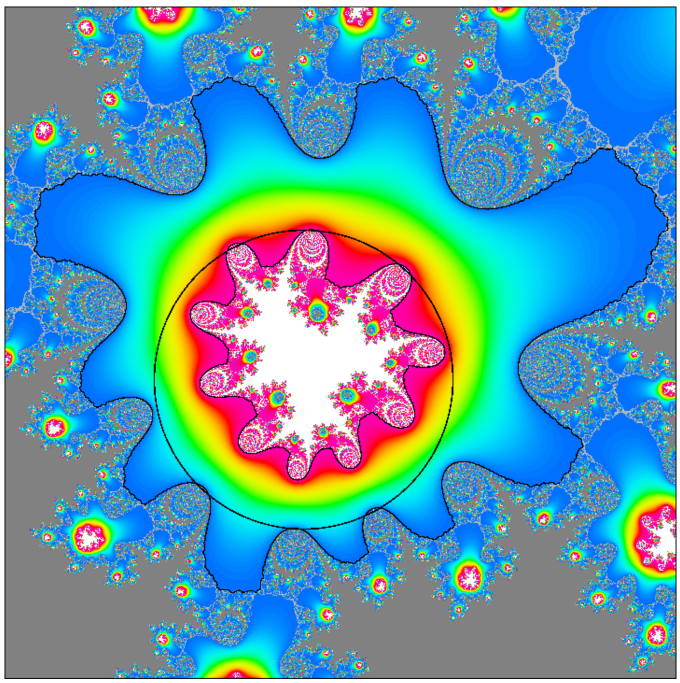

FIGURE 32. *Julia set for* f_q *with* $q \approx 4 + 3\,\boldsymbol{i}$*. The inner white region is the basin of zero; and the outer darker region is the basin of infinity; while the colored region consists of a Herman ring* H *together with its preimages. In general, the map carries each locus of constant color into itself. However, the critical orbits have been colored black to outline the ring, and the unit circle has been colored black to fix the scale. Note that any configuration inside the circle has an antipodal image outside the circle. The modulus* $\mathrm{m}(H)$ *is* 0.08428*, and the rotation number is* $\rho = 0.70013$ (*both truncated to five digits*)*. One interesting feature is the pair of repelling period ten orbits, one inside and one outside, which crowd the ring. This behavior depends on the fact that the rotation number is very close to the fraction 7/10 with denominator ten.*

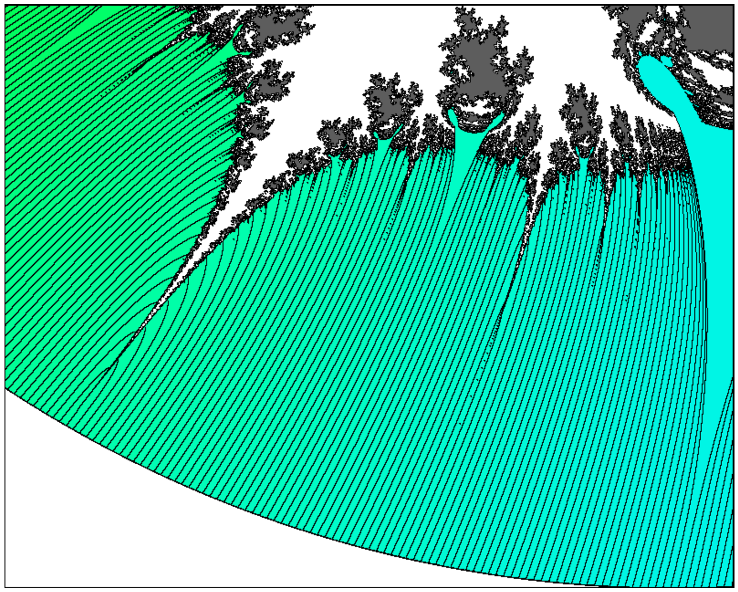

FIGURE 33. *Curves of constant rotation number* (*roughly between* 0.6 *and* 0.76), *using the disk model for parameter space as in Figures* 3, 25. *We can think of these curves as typical hairs* $\mathscr{H}_\beta$*'s*.

Hair Theorem 7.9. *For each Brjuno number β there exists a unique hair $\mathscr{H}_\beta$ in the (q^2)-plane consisting of maps with a Herman ring of rotation number β. Each hair $\mathscr{H}_\beta$ is analytically and regularly parametrized by the modulus $\mathrm{m}(H)$ of its Herman ring; and this modulus takes all values in the interval $0 < \mathrm{m}(H) < \infty$, where $|q|^2 \to \infty$ as $\mathrm{m}(H) \to \infty$.*

Remark 7.10. *The fact that the parametrization of each hair $\mathscr{H}_\beta$ is regular, that is the derivative does not vanish, is due to Epstein in a more general context. For completeness, we present the proof in our particular situation.*

As a first step, we have the following statement.

Lemma 7.11. *Each f_q having a Herman ring is contained in an analytic one parameter family of maps $f_{q(\mathrm{m})}$ having a Herman ring with the same rotation number, where $0 < \mathrm{m} < \infty$ is the modulus.*

Proof by Surgery. Let $\phi : H \to A_{\mathrm{m}_0/2}$ be an isomorphism, where m_0 is the modulus of the Herman ring H for the given f_q. Given any $\mathrm{m} > 0$ we can replace H by a Herman ring H_m of modulus m by a deformation argument, as follows. Consider the diffeomorphism $\psi_\mathrm{m} : A_{\mathrm{m}_0/2} \to A_{\mathrm{m}/2}$ defined by

$$\psi_\mathrm{m}(x + iy) = x + i\frac{\mathrm{m}}{\mathrm{m}_0}y.$$

Pulling the conformal structure of $A_{\mathrm{m}/2}$ back under the diffeomorphism

$$\psi_\mathrm{m} \circ \phi : H \to A_{\mathrm{m}/2},$$

and then back to all iterated preimages of H, we obtain a measurable f_q-invariant conformal structure of bounded dilatation. By the Measurable Riemann Mapping Theorem, there is a homeomorphism h_{m} which fixes 0 and ∞ and sends this exotic conformal structure to the standard conformal structure. The map

$$g_{\mathrm{m}} = h_{\mathrm{m}} \circ f_q \circ h_{\mathrm{m}}^{-1}$$

is a cubic rational map with a Herman ring of modulus m, and with two critical fixed points at 0 and ∞. Furthermore, g_{m} commutes with the fixed point free anti-holomorphic map $h_{\mathrm{m}} \circ \mathcal{A} \circ h_{\mathrm{m}}^{-1}$, and hence is conjugate to a map $f_{q(\mathrm{m})}$ in our family by a scaling map. Without loss of generality, we may assume that $g_{\mathrm{m}} = f_{q(\mathrm{m})}$, that is h_{m} conjugates f_q to $f_{q(\mathrm{m})}$.

We now prove that $q^2(\mathrm{m})$ depends analytically on m. The exotic complex structure depends analytically on m. It follows that for any pair of points z_1 and z_2 in $\mathbb{C} \setminus \{0\}$, the ratio $h_{\mathrm{m}}(z_1)/h_{\mathrm{m}}(z_2)$ depends analytic on m. The map $f_{q(\mathrm{m})}$ has two fixed points $z_\pm\big(q(\mathrm{m})\big) = h_{\mathrm{m}}\big(z_\pm(q)\big)$ in $\mathbb{C} \setminus \{0\}$. Those are roots of the polynomial $z^2 - 2\,\mathrm{Re}\big(q(\mathrm{m})\big)z - 1 = 0$. In particular, their product is -1. It follows that

$$q^2(\mathrm{m}) = \frac{q^2(\mathrm{m})}{z_+\big(q(\mathrm{m})\big) \cdot z_-\big(q(\mathrm{m})\big)} = \frac{h_{\mathrm{m}}(q)}{h_{\mathrm{m}}\big(z_+(q)\big)} \cdot \frac{h_{\mathrm{m}}(q)}{h_{\mathrm{m}}\big(z_-(q)\big)}.$$

As observed previously, both terms in the product depend analytically on m. Replacing $q(\mathrm{m})$ by $-q(\mathrm{m})$ when necessary, we may assume that $q(\mathrm{m})$ depends continuously (and so, analytically) on m. □

We now prove that the curve $\mathrm{m} \mapsto q(\mathrm{m})$ only depends on the choice of Brjuno number β, not on the choice of map f_q having a Herman ring with rotation number β. Our proof is based on Corollary 7.7.

First, assume $H \in \mathbb{C} \setminus \{0\}$ is a Herman ring of modulus $\mathrm{m} > 1/2$. According to [**BDH**], H contains a round annulus $\{z \in \mathbb{C}\ ;\ r < |z| < R\}$ of modulus $\mathrm{m} - 1/2$. The Herman ring H is invariant by the antipodal map. So, if H contains the circle $C_r := \big\{z \in \mathbb{C}\ ;\ |z| = z\big\}$, it also contains the circle $C_{1/r}$ and the whole round annulus bounded by those circles. So, if $\mathrm{m} > 2h + 1/2$, then H contains the round annulus

$$\mathbf{O}_h := \big\{z \in \mathbb{C}\ ;\ e^{-2\pi h} < |z| < e^{2\pi h}\big\}.$$

Second, let us consider the coordinate $w = z \cdot q/|q|$ in which f_q is conjugate to the normal form

$$w \mapsto F_{t,\,a}(w) = e^{2\pi i t} w^2 \frac{1 - aw}{w + a}$$

with $a = 1/|q|$ and $e^{2\pi i t} = q/\bar{q}$ as in equation (10). Identifying $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C} \setminus \{0\}$, Corollary 7.7 implies that there is an $h_0 \in (0, +\infty)$ such that for a given $a \in (0, +\infty)$ and a given $\beta \in \mathbb{R}/\mathbb{Z}$, there is at most one t for which $F_{t,a}$ has a Herman ring of rotation number β containing $\mathbf{O}_{h_0}$. For a given $a > 0$ and $t \in \mathbb{R}/\mathbb{Z}$, there are exactly exactly two parameters q on the circle centered at 0 with radius $1/a$ for which $q/\bar{q} = e^{2\pi i t}$. So, within each circle centered at 0 in the q^2-plane, there is at most one parameter for which f_q has a Herman ring of rotation number β and modulus greater than $2h_0 + 1/2$.

Let $\mathrm{m} \mapsto q(\mathrm{m})$ be a curve provided by Lemma 7.11. If $\mathrm{m} = 2h + 1/2$ with $h > h_0$, then $\big|q(\mathrm{m})\big| \geq e^{2\pi h}$, since $q(\mathrm{m})$ is outside the Herman ring H_{m} which contains $\mathbf{O}_h$. In particular, $\big|q(\mathrm{m})\big| \to +\infty$ as $\mathrm{m} \to +\infty$. It follows that all the curves $\mathrm{m} \mapsto q^2(\mathrm{m})$ provided by Lemma 7.11 coincide for m large enough. Since those curves are analytic, they coincide for all m.

We finally prove that the derivative of the map $\mathrm{m} \mapsto q(\mathrm{m})$ does not vanish. We shall proceed by contradiction, assuming that $q'(\mathrm{m}_0) = 0$. Set

$$\dot{f} = \frac{\partial f_{q(\mathrm{m})}}{\partial \mathrm{m}}\Big|_{\mathrm{m}=\mathrm{m}_0}, \quad \dot{h} = \frac{\partial h_{\mathrm{m}}}{\partial \mathrm{m}}\Big|_{\mathrm{m}=\mathrm{m}_0},$$

$$\dot{\phi} = \frac{\partial \phi_{\mathrm{m}}}{\partial \mathrm{m}}\Big|_{\mathrm{m}=\mathrm{m}_0} \quad \text{and} \quad \dot{\psi} = \frac{\partial \psi_{\mathrm{m}}}{\partial \mathrm{m}}\Big|_{\mathrm{m}=\mathrm{m}_0} = i\frac{y}{m_0}\frac{\mathrm{d}}{\mathrm{d}z}.$$

First, the assumption that $q'(\mathrm{m}_0) = 0$ means that $\dot{f} = 0$. Second, $\dot{h}$ is a quasiconformal vector field. On H_0, we have $\phi_{\mathrm{m}} \circ h_{\mathrm{m}} = \psi_{\mathrm{m}} \circ \phi$ with $(\phi_m : H_{\mathrm{m}} \to A_{\mathrm{m}/2})$ an analytic family of conformal isomorphism. It follows that

$$\dot{\phi} + D\phi \circ \dot{h} = \dot{\psi} \circ \phi$$

so that

$$\frac{\dot{\phi}}{\phi'} + \dot{h} = \phi^*(\dot{\psi}) \quad \text{and} \quad \bar{\partial}\dot{h} = \phi^*(\bar{\partial}\dot{\psi}) = \phi^*\left(-\frac{1}{2m_0}\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}\right).$$

As a consequence

$$\int_H \bar{\partial}\dot{h} \cdot \phi^*(\mathrm{d}z^2) = \int_{A_{\mathrm{m}_0/2}} -\frac{1}{2\mathrm{m}_0}\mathrm{d}z\mathrm{d}\bar{z} = i.$$

The assumption that $q'(\mathrm{m}_0) = 0$ means that $\dot{f} = 0$. Derivating $h_{\mathrm{m}} \circ f_q = f_{q(\mathrm{m})} \circ h_{\mathrm{m}}$ with respect to m and evaluating at $\mathrm{m} = \mathrm{m}_0$ yields

$$\dot{h} \circ f_q = \dot{f} + Df_q \circ \dot{h} = Df_q \circ \dot{h}.$$

In other words, the vector field $\dot{h}$ is invariant by f_q, that is $\dot{h} = f_q^*\dot{h}$. It follows that $\dot{h}$ vanishes on the set of repelling cycles. Since $\dot{h}$ is continuous and repelling periodic points are dense in the Julia set of f_q, we deduce that $\dot{h}$ vanishes on the Julia set, in particular on the boundary of H. Since $\phi^*(\mathrm{d}z^2)$ is holomorphic and integrable on H, a use of Ahlfors' mollifier yields (see the proof of (i) in [**GL**, Theorem 9])

$$\int_H \bar{\partial}\dot{h} \cdot \phi^*(\mathrm{d}z^2) = 0.$$

We have a contradiction since $i \neq 0$. This completes the proof of Theorem 7.9. □

Corollary 7.12. *If $f_{\pm q}$ has a Herman ring of rotation number β, then the formal rotation number of Definition* 5.4 *is also equal to β.*

Proof. This follows since the closure of the hair $\mathscr{H}_\beta$ is a connected set containing the point ∞_β on the circle at infinity. □

Corollary 7.13. *The complement $\mathfrak{X} = \text{©} \setminus \mathcal{H}_0$ is not locally connected.*

Proof. Let $\overline{\mathbb{D}}_r$ denote the disk $\{s \in \mathbb{C}\ ;\ |s| \leq r\}$. Note that $\overline{\mathbb{D}}_r$ intersects uncountably many connected components of $\mathfrak{X}$ whenever the radius r is sufficiently large. This follows easily from the fact that there are uncountably many Brjuno numbers β, and that each hair $\mathscr{H}_\beta$ intersects $\mathbb{D}_r$ whenever r is large enough. If $\{x_j\}$ is an infinite sequence of points of $\overline{\mathbb{D}}_r$ belonging to distinct components of $\mathfrak{X}$, then it follows that $\mathfrak{X}$ is not locally connected at any accumulation point of $\{x_j\}$. □

Density of the Herman Ring Locus. Given $h > 0$, again let A_h be the annulus consisting of all points $z \in \mathbb{C}/\mathbb{Z}$ such that $\left|\mathrm{Im}(z)\right| < h$; and let $\mathcal{D}_h$ denote the set of univalent maps $F : A_h \to \mathbb{C}/\mathbb{Z}$ isotopic to the inclusion map, and satisfying $F(0) = 0$.

Given $F \in \mathcal{D}_h$, denote by $\mathscr{R}_F$ the set of parameters $\tau \in \mathbb{C}/\mathbb{Z}$ such that $F + \tau$ has a Herman ring which is homotopic to $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$. Although this set may be empty in general, a theorem of Risler implies that this is never the case as long as h is large enough. Risler's theorem can be restated as follows. (See [**Ri**, Théorème 5].) Let Θ be the set of Brjuno numbers.

Theorem 7.14 (**Risler**). *There is a function $\mathscr{B} : \Theta \to \mathbb{R}$ such that the following holds. For all $\boldsymbol{\theta} \in \Theta$, all $h \geq \mathscr{B}(\boldsymbol{\theta})$ and all $F \in \mathcal{D}_h$, there is a $\tau \in \mathbb{C}/\mathbb{Z}$ such that $F + \tau$ has a Herman ring with rotation number $\boldsymbol{\theta}$ containing the annulus $A_{h-\mathscr{B}(\boldsymbol{\theta})}$.*

Here the product $2\pi\mathscr{B}$ is a function whose difference from the Brjuno function[17] is bounded. Let $\mathfrak{H}^1(E)$ be the Hausdorff 1-dimensional measure of a set $E \subset \mathbb{C}/\mathbb{Z}$:

$$\mathfrak{H}^1(E) := \lim_{\delta \to 0} \mathfrak{H}^1_\delta(E) \quad \text{with} \quad \mathfrak{H}^1_\delta(E) := \inf \sum_{i=1}^{+\infty} |U_i| \ ,$$

where the infimum is taken over all countable covers $\{U_i\}$ of E by sets of diameter $|U_i| \leq \delta$. If $E \subset \mathbb{R}/\mathbb{Z}$, then $\mathfrak{H}^1(E)$ coincides with the Lebesgue measure $\mathrm{Leb}(E)$ of E.

Our goal is to use Risler's result to prove the following statement, which generalizes a result of Herman [**He**].

Proposition 7.15. *Given $\rho < 1$, if h is sufficiently large, then $\mathfrak{H}^1(\mathscr{R}_F) \geq \rho$ for any map $F \in \mathcal{D}_h$.*

Applying this statement to the family of maps $\{f_q\}$, and arguing as in the proof of Theorem 7.9, it is not difficult to prove the following.

Corollary 7.16. *The Lebesgue measure of the Herman ring locus, intersected with the circle $|q| = r$, converges to $+1$ as $r \to \infty$.*

The proof of Proposition 7.15 will depend on the following remarks. Note that each $F \in \mathcal{D}_h$ lift to a univalent map

$$\widetilde{F} : \{z \in \mathbb{C}\, ;\, |\mathrm{Im}(z)| < h\} \ \to \ \mathbb{C}$$

satisfying

$$\widetilde{F}(0) = 0 \qquad \text{and} \qquad \widetilde{F}(z+1) = \widetilde{F}(z) + 1 \ . \tag{26}$$

Now given any compact $K \subset \mathbb{C}$, if h is very large, then it follows from the Koebe distortion theorem that $\widetilde{F}$ is approximately linear on K. In particular, the derivative $\widetilde{F}'$ is approximately constant on K. if we assume that $[0,\,1] \subset K$, then in view of (26), this derivative must be approximately $+1$ throughout K.

Proof of Proposition 7.15. Fix ρ and set $\kappa := 1/\sqrt{\rho}$. Without loss of generality, we may assume that $4/9 < \rho < 1$, and hence that $\kappa < 3/2$. For $h > 0$, set

$$\Theta_h := \{\boldsymbol{\theta} \in \Theta \ ;\ \mathscr{B}(\boldsymbol{\theta}) < h\}.$$

[17] See for example [**BC**].

Since $\mathrm{Leb}(\Theta) = 1$ and since Θ is the union of the sets Θ_h (increasing union as h increases), we can choose $h_0 > 0$ sufficiently large so that for all $h > h_0$,

$$\mathrm{Leb}(\Theta_h) \geq 1/\kappa. \tag{27}$$

Furthermore, by the remarks above, there exists $h_1 > 2$ such that for all $h > h_1$, all $F \in \mathcal{D}_h$, and all z in the compact set $\overline{A}_1$, we have

$$|F'(z) - 1| \ < \ \kappa - 1 \ < \ 1/2\,.$$

Let $h_1 \in (0, +\infty)$ be the constant provided by Lemma 7.6 with $\varepsilon := \kappa - 1$.

From now on, we assume that $h > h_1 + h_0$, and choose some fixed $F \in \mathcal{D}_h$. According to Risler, if $\boldsymbol{\theta} \in \Theta_{h_0}$ then there is a parameter τ such that $F + \tau$ has a Herman ring with rotation number $\boldsymbol{\theta}$ containing the annulus

$$A_{h - \mathscr{B}(\widehat{\boldsymbol{\theta}})} \supset A_{h_1}.$$

We denote by $\mathscr{T} = \mathscr{T}_F \subset \mathscr{R}_F$ the corresponding set of parameters τ as $\boldsymbol{\theta}$ ranges in Θ_{h_0} and by $\mathrm{Rot}(F + \tau)$ the rotation number $\boldsymbol{\theta} \in \Theta_{h_0}$ associated to τ. Note that the set $\mathscr{T}$ is a priori not closed.

According to Lemma 7.6 and our choice of h_1, the function $\mathrm{Rot} : \mathscr{T} \to \Theta_{h_0}$ is κ-Lipschitz:

$$\left|\mathrm{Rot}(F + \tau_1) - \mathrm{Rot}(F + \tau_2)\right| < \kappa \cdot |\tau_2 - \tau_1| \qquad \text{for all} \quad \tau_1, \tau_2 \in \mathscr{T}.$$

It follows immediately from this with (27) that

$$1/\kappa \leq \mathrm{Leb}(\Theta_{h_0}) = \mathfrak{H}^1(\Theta_{h_0}) \leq \kappa \cdot \mathfrak{H}^1(\mathscr{T}),$$

so that

$$\mathfrak{H}^1(\mathscr{R}_F) \geq \mathfrak{H}^1(\mathscr{T}) \geq 1/\kappa^2 = \rho.$$

This completes the proof of Proposition 7.15. □

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