# Invariance of Morse Homology

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This is a note on showing that the Morse homology is independent of the chosen Morse function and Smale pseudo-gradient field. Let $f_0, f_1 : V \to \mathbb{R}$ be two Morse functions on a smooth compact manifold $V$ and $X_0, X_1$ adapted pseudo-gradients to $f_0, f_1$, respectively. We'll show there is a morphism of complexes that induces an isomorphism on homology:

$$\Phi_* : (C_*(f_0), \partial_{X_0}) \to (C_*(f_1), \partial_{X_1})$$

Outline of the proof:

1. We choose a function $F : V \times [0,1] \to \mathbb{R}$ such that

$$\begin{cases} F(x,s) = f_0 & s \in [0, 1/3] \\ F(x,s) = f_1 & s \in [2/3, 1] \end{cases}$$

   We call such a function an **interpolation**. This gives us a morphism $\Phi^F$ on chain complexes as above.

2. Let $(f_0, X_0) = (f_1, X_1)$. We show that $I = F(x,s) = f_0$ for all $x \in V$ and every $s$. Also, $\Phi^I = \mathrm{id}$.

3. Let $(f_2, X_2)$ be another Morse-Smale pair. Let $G$ be an interpolation between $f_1$ and $f_2$ stationary on $s \in [0, 1/3] \cup [2/3, 1]$ and $H$ an interpolation between $f_0$ and $f_2$ with the same properties. We prove that the morphisms

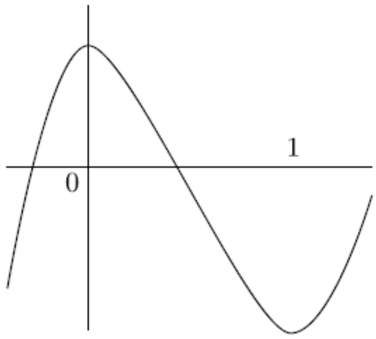
$$\Phi^G \circ \Phi^F, \Phi^H : (C_*(f_0), \partial_{X_0}) \to (C_*(f_2), \partial_{X_2})$$

   coincide on the homology level. Thus, if $(f_0, X_0) = (f_2, X_2)$, then $H = I$, $\Phi^H = \mathrm{id}$, and so $\Phi^F$ and $\Phi^G$ must be isomorphisms.

## 1 First Step

Let $A = [-1/3, 1/3], B = [2/3, 4/3], C = [-1/3, 4/3]$. We extend $F$ to $V \times C$ by letting $F(x,s) = f_0$ on $s \in A$ and $F(x,s) = f_1$ on $s \in B$. Let $g : \mathbb{R} \to \mathbb{R}$ be a Morse function whose critical points are 0 (max) and 1 (min) which is increasing on $(-\infty, 0)$ and $(1, +\infty)$. Let it also be sufficiently decreasing on $(0,1)$ such that

$$\forall x \in V, \forall s \in (0,1), \qquad \frac{\partial F}{\partial s}(x,s) + g'(s) < 0.$$




The Morse function $g$

These properties of $g$ makes it so that the function $\widetilde{F} = F + g : V \times C$ is a Morse function whose critical points are $\mathrm{Crit}(\widetilde{F}) = \mathrm{Crit}(f_0) \times \{0\} \cup \mathrm{Crit}(f_1) \times \{1\}$. This is so because $F = f_0$ on $A$ and $f_1$ on $B$ and $g'(0) = g'(1) = 0$. The sufficiently decreasing condition makes it so there are no critical points in the intermediary interval $(0, 1)$.

Moreover, if $a \in \mathrm{Crit}(f_0)$ and $b \in \mathrm{Crit}(f_1)$, then $\mathrm{Ind}_{\widetilde{F}}(a, 0) = \mathrm{Ind}_{f_0}(a) + 1$ while $\mathrm{Ind}_{\widetilde{F}}(b, 1) = \mathrm{Ind}_{f_1}(b)$. With a partition of unity, we can construct a pseudo-gradient field $X$ that is adapted to $\widetilde{F}$ and coincides with

$$\begin{cases} X_0 + \nabla g & \text{on } V \times A \\ X_1 + \nabla g & \text{on } V \times B. \end{cases}$$

$X$ is thus transverse to the boundary of $V \times [-1/3, 4/3]$. We may perturb $X$ by a $C^1$ small amount to get a Smale pseudo-gradient field. We call it $\widetilde{X}$. Moreover, we can do so in a way that $X$ is transverse to the slices $V \times \{s\}$, $s \in \{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\}$. The small perturbation also preserves the number of trajectories between critical points of consecutive index.

Therefore, we can choose an $\widetilde{X}$ such that when $\widetilde{F}$ is restricted to $V \times A$, then

$$(C_*(\widetilde{F}|_{V\times A}), \partial_{\widetilde{X}}) = (C_*(f_0 + g)|_A, \partial_{X_0+\nabla g}) = (C_{*+1}(f_0), \partial_{X_0}).$$

Similarly, when restricting $\widetilde{F}$ to $V \times B$,

$$(C_*(\widetilde{F}|_{V\times B}), \partial_{\widetilde{X}}) = (C_*(f_1 + g)|_B, \partial_{X_1+\nabla g}) = (C_*(f_1), \partial_{X_1}).$$

Now, $\widetilde{X}$ has two types of trajectories that connect critical points of $\widetilde{F}$. (1) Those that remain in the interval $A$ or $B$, thereby, are trajectories of $X_0$ or $X_1$. (2) Those that go from a critical point of $f_0$ to a critical point of $f_1$ (they cross over). Therefore, we have $C_{k+1}(\widetilde{F}) = C_k(f_0) \oplus C_{k+1}(f_1)$. Thus, the differential

$$\partial_{\widetilde{X}} : C_k(f_0) \oplus C_{k+1}(f_1) \to C_{k-1}(f_0) \oplus C_k(f_1)$$

has a matrix of the form

$$\partial_{\widetilde{X}} = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1}. \end{pmatrix}$$

$\Phi^F$ is defined as you would expect: Let $n_{\widetilde{X}}(a, b)$ be the mod 2 count of the number of trajectories of $\widetilde{X}$ between $a \in \mathrm{Crit}_k(f_0)$ and $b \in \mathrm{Crit}_k(f_1)$. Then $\Phi^F(a) = \sum_b n_{\widetilde{X}}(a, b)b$. Technically, we're considering $(a, 0)$ and $(b, 1)$ as critical points of $\widetilde{F}$.

This $\partial_{\widetilde{X}}$ defines for us a Morse chain complex $(C_*(\widetilde{F}, \widetilde{X}))$ for the manifold $V \times [-1/3, 4/3]$. This $\partial^2_{\widetilde{X}} = 0$ which implies that

$$\Phi^F \circ \partial_{X_0} + \partial_{X_1} \circ \Phi^F = 0 \Longrightarrow \Phi^F \circ \partial_{X_0} = \partial_{X_1} \circ \Phi^F.$$

The last equality holds because we're considering $\mathbb{Z}_2$ coefficients. Thus, $\Phi^F$ is a chain complex morphism.

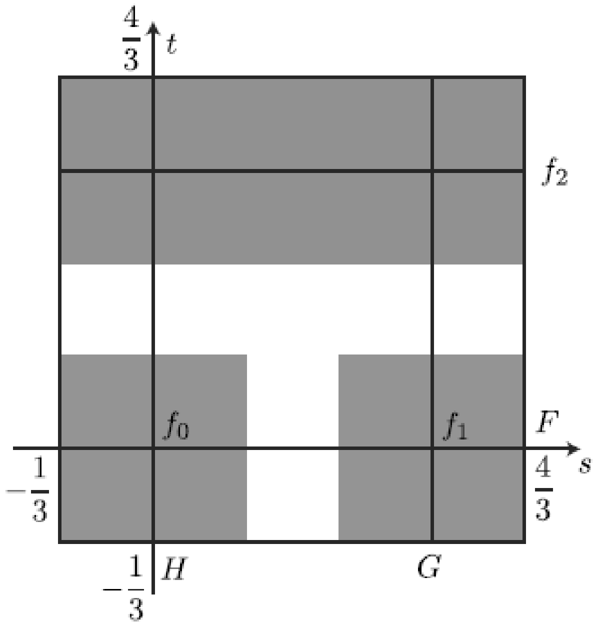
# 2 Second Step

Now we suppose that $(f_0, X_0) = (f_1, X_1)$ and $I(x, s) = f_0(x)$ for all $s$. Using the same $g$ as above, we get $X = X_0 + \nabla g$ is a Smale adapted pseudo-gradient field. Moreover, for every critical point $a$ of $f_0$, there is a unique trajectory from $(a, 0)$ to some $(c, 1)$ where $\mathrm{Ind}_{f_0}(a) = \mathrm{Ind}_{f_0}(c)$. This $(c, 1)$ is in fact $(a, 1)$ and the trajectory is the straight line $\ell_a : [-1/3, 4/3] \to V \times [-1/3, 4/3]$ $\ell_a(s) = (a, s)$. Thus, $\Phi^I(a) = a$ so $\Phi^I = \mathrm{id}$.

# 3 Third Step

Suppose we have the three interpolating functions $F, G, H$ from $f_0$ to $f_1$, $f_1$ to $f_2$, and $f_0$ to $f_2$, resp. We now construct an interpolation of these interpolations $K : V \times [-1/3, 4/3]^2 \to \mathbb{R}$ satisfying

$$K(x, s, t) = \begin{cases} H(x, t), & s \in [-1/3, 1/3] \\ G(x, t), & s \in [2/3, 4/3] \\ F(x, s), & t \in [-1/3, 1/3] \\ f_2(x), & t \in [2/3, 4/3] \end{cases}$$




Interpolating the Interpolations

We continue to use a Morse function $g : \mathbb{R} \to \mathbb{R}$ as above and require that

$$\frac{\partial K}{\partial s}(x, s, t) + g'(s) < 0 \text{ for all } (x, s, t) \in V \times (0, 1) \times [1/3, 4/3]$$

and

$$\frac{\partial K}{\partial s}(x, s, t) + g'(t) < 0 \text{ for all } (x, s, t) \in V \times [1/3, 4/3] \times (0, 1)$$

Lastly, let $\widetilde{K}(x, s, t) = K(x, s, t) + g(s) + g(t)$. The critical points of $\widetilde{K}$ are in the shaded regions of the figure above, where in those regions, $\widetilde{K}$ has the form $f_i(x) + g(s) + g(t)$, $i = 0, 1, 2$. Moreover, the critical points of $\widetilde{K}$ are exactly the union of $\mathrm{Crit}(f_0) \times \{0\} \times \{0\}$, $\mathrm{Crit}(f_1) \times \{1\} \times \{0\}$, $\mathrm{Crit}(f_2) \times \{0\} \times \{1\}$, and $\mathrm{Crit}(f_2) \times \{1\} \times \{1\}$. The indices are as follows:

- If $a \in \mathrm{Crit}(f_0)$, then $\mathrm{Ind}_{\widetilde{K}}((a,0,0)) = \mathrm{Ind}_{f_0}(a) + 2$.
- If $b \in \mathrm{Crit}(f_1)$, then $\mathrm{Ind}_{\widetilde{K}}((b,1,0)) = \mathrm{Ind}_{f_1}(b) + 1$.
- If $c \in \mathrm{Crit}(f_2)$, then $\mathrm{Ind}_{\widetilde{K}}((c,0,1)) = \mathrm{Ind}_{f_2}(c) + 1$ and $\mathrm{Ind}_{\widetilde{K}}((c,1,1)) = \mathrm{Ind}_{f_2}(c)$.

Let $X$ be the pseudo-gradient adapted to $F$ and $Y$ the one for $G$. Let $Z$ be a pseudo-gradient for $H(x,t) + g(t) : V \times [-1/3, 4/3] \to \mathbb{R}$. Using a partition of unity, construct a vector field $W$ adapted to $\widetilde{K}$ such that:

- For $s \in [-1/3, 1/3]$, $W(x,s,t) = Z(x,t) + \nabla g(s)$.
- For $s \in [2/3, 4/3]$, $W(x,s,t) = Y(x,t) + \nabla g(s)$.
- For $t \in [-1/3, 1/3]$, $W(x,s,t) = X(x,s) + \nabla g(t)$.
- For $t \in [2/3, 4/3]$, $W(x,s,t) = X_2 + \nabla g(s) + \nabla g(t)$.

We then perturb $W$ to some Smale $\widetilde{W}$, taking care to ensure that outside of $V \times [1/3, 2/3]^2$, the trajectories of $W$ connecting critical points of consecutive indices are in 1-1 correspondence with those of $\widetilde{W}$. We have

$$C_{k+1}(\widetilde{K}) = C_{k-1}(f_0) \oplus C_k(f_1) \oplus C_k(f_2) \oplus C_{k+1}(f_2).$$

Then $(C_*(\widetilde{K}), \partial_{\widetilde{W}})$ is a Morse chain complex for on $V \times [-1/3, 4/3]^2$. We may represent the differential in the following way (letting $S : C_{k-1}(f_0) \to C_k(f_2)$ be some map):

$$\partial_{\widetilde{W}} = \begin{pmatrix} \partial_{X_0} & 0 & 0 & 0 \\ \Phi^F & \partial_{X_1} & 0 & 0 \\ \Phi^H & 0 & \partial_{X_2} & 0 \\ S & \Phi^G & \mathrm{id} & \partial_{X_2} \end{pmatrix}$$

The fact that $\partial_{\widetilde{W}}^2 = 0$ means that $S \circ \partial_{X_0} + \Phi^G \circ \Phi^F + \Phi^H + \partial_{X_2} \circ S = 0$ or, because of $\mathbb{Z}_2$ coefficients, $\Phi^G \circ \Phi^F - \Phi^H = S \circ \partial_{X_0} + \partial_{X_2} \circ S$. This means that $S$ is a chain-homotopy and thus $\Phi^G \circ \Phi^F$ and $\Phi^H$ induce the same morphism on homology. Then, when $(f_0, X_0) = (f_2, X_2)$, this means $H = I$ and $\Phi^G \circ \Phi^F = \mathrm{id}$. Hence, we have isomorphims for Morse homology.