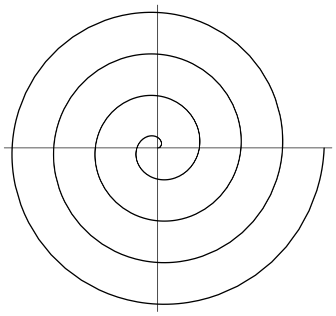
MATH 132 Solutions to Practice 2

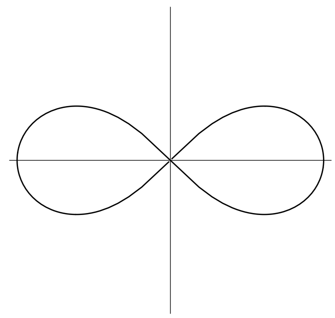
1234 pts 1. There are a bunch of problems about work and volume and stuff on the Spring 2010 exam. So those aren't here. Go do those, OK? If you already did them, here are the solutions.

47 pts 2. Match the following polar equations to their graphs. Please write the letter of the graph in the space preceeding the equation. Note that, although each graph is accurate, two different graphs may be drawn at different scales.

C $r = \cos 2\theta$ **A** $r = \theta \ \ (\theta > 0)$ **D** $\theta = \dfrac{\pi}{4}$ **E** $r = 1 + 2\sin\theta$



A



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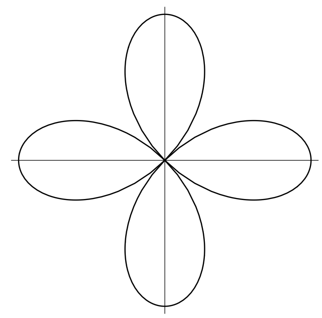

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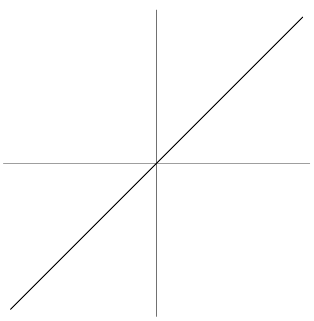

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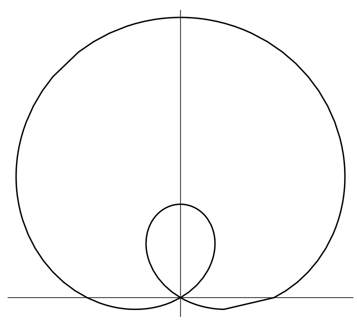

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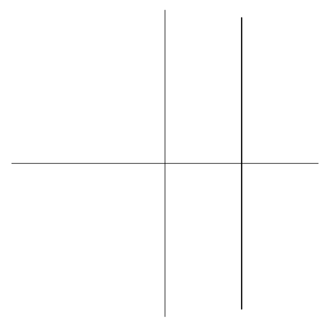

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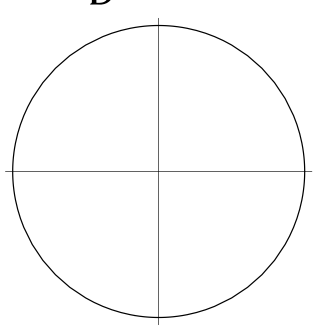

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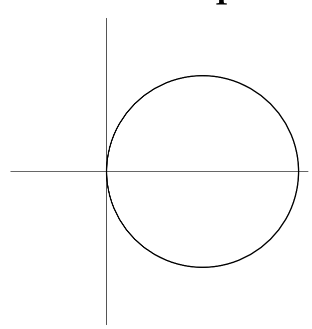
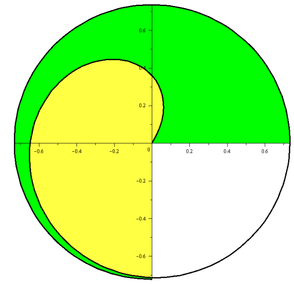


H



47 pts

3. At right is shown the graph of the polar curve
$$r = \frac{\ln\theta}{\sqrt{\theta}} \qquad 1 \le \theta \le \frac{7\pi}{2}$$
Calculate the area of the green region.

Solution: Note that for $1 \le \theta \le \frac{3\pi}{2}$, the curve $r = \frac{\ln\theta}{\sqrt{\theta}}$ traces out the inner curve, so $\frac{1}{2}\int_1^{3\pi/2}\left(\frac{\ln\theta}{\sqrt{\theta}}\right)^2 d\theta$ represents the yellow area.

Similarly, the integral $\frac{1}{2}\int_{2\pi}^{7\pi/2}\left(\frac{\ln\theta}{\sqrt{\theta}}\right)^2 d\theta$ represents the green and yellow areas.

This means that the green area will be given by their difference, i.e.
$$\frac{1}{2}\int_{2\pi}^{7\pi/2}\frac{\ln^2\theta}{\theta}\,d\theta - \frac{1}{2}\int_1^{3\pi/2}\frac{\ln^2\theta}{\theta}\,d\theta.$$
Let $u = \ln\theta$, so $du = 1/\theta\,d\theta$. We also adjust the limits of integration and get
$$\begin{aligned}\frac{1}{2}\int_{\ln(2\pi)}^{\ln(7\pi/2)} u^2\,du - \frac{1}{2}\int_0^{\ln(3\pi/2)} u^2\,du &= \frac{1}{2}\left(\left.\frac{u^3}{3}\right|_{\ln(2\pi)}^{\ln(7\pi/2)} - \left.\frac{u^3}{3}\right|_0^{\ln(3\pi/2)}\right)\\ &= \frac{1}{6}\left((\ln(7\pi/2))^3 - (\ln(2\pi))^3 - (\ln(3\pi/2))^3\right)\end{aligned}$$

4. Find the limit of each of the following infinite sequences, if it converges. If the sequence does not converge, say so. Justify your answer.

4 pts

(a) $\left\{\frac{2^n}{3^{n+1}}\right\}_{n=0}^{\infty} = \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$

Solution:
$$\lim_{n\to\infty}\frac{2^n}{3^{n+1}} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n = 0.$$
The sequence converges to 0.

4 pts

(b) $\left\{\frac{(n+5)(n+6)(n+12)}{2n^2+24n+36}\right\}_{n=0}^{\infty} = 10, \frac{273}{31}, \frac{196}{23}, \frac{60}{7}, \ldots$

Solution: Since $\frac{(n+5)(n+6)(n+12)}{2n^2+24n+36} > \frac{n^3}{2n^2+24n+36}$ and $\lim_{n\to\infty}\frac{n^3}{2n^2+24n+36} = +\infty$, so the sequence diverges.

4 pts

(c) $\left\{1 + \left(\frac{-1}{3}\right)^n\right\}_{n=0}^{\infty} = 1, \frac{2}{3}, \frac{10}{9}, \frac{26}{27}, \ldots$

Solution:
$$\lim_{n\to\infty}\left(1 + \left(\frac{-1}{3}\right)^n\right) = 1 + 0 = 1,$$
so the sequence converges to 1.

5. Find the sum of the following infinite series

8 pts (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2 \cdot 3^n} = \frac{1}{6} - \frac{1}{18} + \frac{1}{54} - \ldots$

Solution: Factor $\frac{1}{6}$ out to obtain a geometric series with ratio $-1/3$:

$$\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n = \frac{1}{6} \cdot \frac{1}{1-\left(\frac{-1}{3}\right)} = \frac{1}{6} \cdot \frac{1}{4/3} = \frac{1}{8}.$$

8 pts (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{2}{3} + \frac{1}{4} + \frac{2}{15} + \frac{1}{12} + \ldots$

Solution: We use partial fractions to rewrite $\frac{2}{n(n+2)}$ as $\frac{1}{n} - \frac{1}{n+2}$. So we have

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots$$

This is a telescoping series; the second term of a_n cancels the first term of a_{n+2}. Only 1 and $\frac{1}{2}$, remain so the sum is $\dfrac{3}{2}$.

For each of the inifinite sums below, state whether it converges absolutely, converges (but not absolutely), or diverges. To receive any credit, you must justify your answer fully.

47 pts 6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \frac{1}{2\ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \ldots$

Solution: We use the integral test.

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{u \to \infty} \ln u - \ln(\ln 2) = +\infty.$$

Since the integral diverges, the series also diverges.

47 pts 7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{2} - \ldots$

Solution: This is an alternating series. Since the terms of the series are decreasing in absolute value, and $\lim_{n\to\infty} 1/\sqrt{n} = 0$, the series converges.

However, it does not converge absolutely, since if we take the absolute value of each term, we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = 1/2$. Since $p \leq 1$, the series of absolute values diverges.

$\boxed{\text{47 pts}}$ 8. $\displaystyle\sum_{n=0}^{\infty}(-1)^n 2^{-1/n^2} = 1 - \frac{1}{2} + \frac{1}{\sqrt[4]{2}} - \dots$

Solution: Note that $\lim_{n\to\infty} 2^{-1/n^2} = 1$, so the limit of a_n is not 0. Hence the infinite sum diverges.

$\boxed{\text{47 pts}}$ 9. $\displaystyle\sum_{n=0}^{\infty}\frac{e^n}{n!} = 1 + e + \frac{e^2}{2} + \frac{e^3}{6} + \frac{e^4}{24} + \dots$

Solution: We can use the ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{e^{n+1}/(n+1)!}{e^n/n!} = \lim_{n\to\infty}\frac{e^{n+1}n!}{e^n(n+1)!} = \lim_{n\to\infty}\frac{e}{n+1} = 0.$$

Since the ratio is less than 1, the series converges absolutely.

$\boxed{\text{47 pts}}$ 10. $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{n^2-n+3} = \frac{1}{3} + \frac{2}{3} + \frac{3}{5} + \frac{4}{9} + \dots$

Solution: We can use the limit comparison test to compare this series to the harmonic series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty}\frac{1/n}{(n+1)/(n^2-n+3)} = \lim_{n\to\infty}\frac{n^2-n+3}{n^2+n} = 1$$

Since the limit of the ratio exists and is nonzero, the two series do the same thing. Since the harmonic series diverges (p-series with $p=1$), the original sum also diverges.

$\boxed{\text{47 pts}}$ 11. Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{n+1} = 1 + \frac{x-2}{2} + \frac{(x-2)^2}{3} + \frac{(x-2)^3}{4} + \dots$$

Solution: First, we use the ratio test to determine the radius of convergence:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}/(n+2)}{(x-2)^n/(n+1)}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}(n+1)}{(x-2)^n(n+2)}\right| \\ &= \lim_{n\to\infty}\left|(x-2)\frac{n+1}{n+2}\right| = |x-2|\end{aligned}$$

The ratio will be less than 1 (and the series will converge absolutely) when $|x-2| < 1$, or $1 < x < 3$.

Now we need to determine what happens when $x = 1$ and $x = 3$.

When $x = 3$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1}$, which diverges (via limit comparison with $\sum \frac{1}{n}$, or the integral test).

When $x = 1$, the resulting series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, which is a convergent alternating series by the alternating series test (since $\lim 1/(n+1) = 0$).

This means the interval of convergence is $[1, 3)$.

$\boxed{\text{47 pts}}$ 12. Find the interval of convergence for the power series corresponding to the Bessel function

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2306} + \dots$$

Solution: Again, we use the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}/2^{2n+2}((n+1)!)^2}{(-1)^n x^{2n}/2^{2n}(n!)^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^{2n+2} 2^{2n} (n!)^2}{x^{2n} 2^{2n+2}((n+1)!)^2} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2^2(n+1)^2} \right| = 0. \end{aligned}$$

This is less than 1 for any x, so the series converges absolutely for all x. That is, the interval of convergence is $(-\infty, +\infty)$.

$\boxed{\text{47 pts}}$ 13. (a) Show that the infinite sum $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots$ converges.

Solution: This is an alternating series with terms decreasing in absolute value. So we can apply the alternating series test. Since $\lim_{n\to\infty} \frac{4}{2n+1} = 0$, the series converges.

(b) In fact, the series converges to π. What value of N do we need to use to ensure that $\sum_{n=0}^{N} \frac{(-1)^n \cdot 4}{2n+1}$ is within $\frac{1}{100}$ of π?

Solution: If we stop at N terms, the sum will be off by no more than $\frac{4}{2N+3}$. So, we want N so that

$$\frac{4}{2N+3} < \frac{1}{100} \qquad \text{or} \qquad 400 < 2N + 3$$

This means we need $397/2 < N$, so we must add at least 199 terms to ensure we are within $1/100$ of π.