EXAM

Practice Midterm 1

Math 132

February 22, 2004

ANSWERS

Problem 1. Let $h(t) = \dfrac{\sin^3(\pi t) + e^{\sqrt{t}}}{\arctan(1-t^2) - t}$ and let $G(x) = \displaystyle\int_0^x h(t)\,dt$.

(a) Find $\displaystyle\int_0^1 h'(t)\,dt$.

Answer:

By the fundamental theorem of calculus, we have

$$\int_0^1 h'(t)\,dt = h(1) - h(0) = \frac{\sin^3(\pi) + e^1}{\arctan(0) - 1} - \frac{\sin^3(0) + e^0}{\arctan(1) - 0} = \frac{e}{-1} - \frac{1}{\frac{\pi}{4}} = -\frac{4}{\pi} - e.$$

(b) Find $G'(0)$.

Answer:

By the fundamental theorm of calculus (the other part) we have $G'(0) = h(0) = -\frac{4}{\pi}$.

Problem 2. Determine whether the following integrals converge or diverge. Explain your answer completely. Find the exact answer if possible.

(a) $\displaystyle\int_2^\infty \frac{dx}{x\ln(x)}$

Answer:

Diverges. We integrate explicitly (using u substitution):

$$\int_2^\infty \frac{dx}{x\ln(x)} = \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln(x)} = \lim_{b\to\infty} \ln(\ln(x))\Big]_2^b = \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2)) = \infty.$$

(b) $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$

Answer:

Converges. Note that $\displaystyle\int_0^\infty \frac{dx}{1+x^3} = \int_0^1 \frac{dx}{1+x^3} + \int_1^\infty \frac{dx}{1+x^3}$ and $\displaystyle\int_0^1 \frac{dx}{1+x^3}$ is finite. Hence, $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$ converges if $\displaystyle\int_1^\infty \frac{dx}{1+x^3}$ converges. We now show that $\displaystyle\int_1^\infty \frac{dx}{1+x^3}$ converges by the comparison theorem. We have

$$0 < \frac{1}{1+x^3} < \frac{1}{1+x^2} < \frac{1}{x^2} \text{ for all } x > 1.$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2} = \frac{1}{3}$ which converges, the comparison theorem says that $\displaystyle\int_1^\infty \frac{dx}{1+x^3}$ converges.

(c) $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$

Answer:

Converges. We integrate explicitly (using integration by parts):

$$\int_0^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty}\int_0^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty} \frac{-1-x}{e^x}\Big]_0^b = \lim_{b\to\infty} \frac{-1-b}{e^b} - \frac{-1}{1} = 1.$$

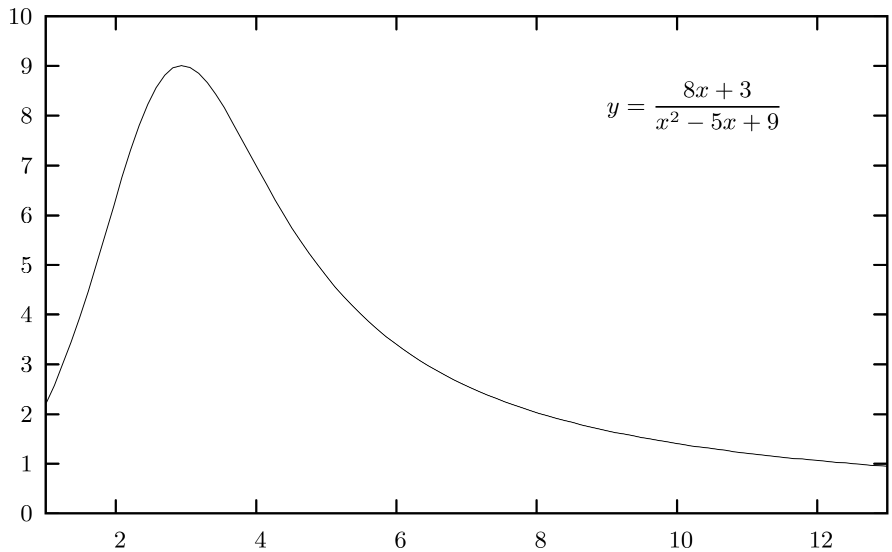
(d) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2-x}}$

Answer:

Converges. We integrate explicitly.

$$\int_0^2 \frac{dx}{\sqrt{2-x}} = \lim_{b\to 2}\int_0^2 \frac{dx}{\sqrt{2-x}} = \lim_{b\to 2} -2\sqrt{2-x}\Big]_0^b = 2\sqrt{2}.$$

Problem 3.



(a) Use the left hand rule with $n = 5$ to approximate $\int_2^{12} \frac{8x+3}{x^2-5x+9} dx$.

Answer:

Let's denote (for this part and every part) $\frac{8x+3}{x^2-5x+9}$ by $f(x)$. Then, the left hand rule with $n = 5$ gives

$$\int_2^{12} \frac{8x+3}{x^2-5x+9} dx \approx \frac{12-2}{5} \left(f(2) + f(4) + f(6) + f(8) + f(10)\right)$$
$$= 2\left(\frac{19}{3} + 7 + \frac{17}{5} + \frac{67}{33} + \frac{83}{59}\right) = 40.3408.$$

(b) Use the right hand rule with $n = 5$ to approximate $\int_2^{12} \frac{8x+3}{x^2-5x+9} dx$.

Answer:

$$\int_2^{12} \frac{8x+3}{x^2-5x+9} dx \approx \frac{12-2}{5} \left(f(4) + f(6) + f(8) + f(10) + f(12)\right)$$
$$= 2\left(7 + \frac{17}{5} + \frac{67}{33} + \frac{83}{59} + \frac{33}{31}\right) = 29.8032.$$

Problem 5. (Continued)

(c) Use the trapezoid hand rule with $n = 5$ to approximate $\int_2^{12} \frac{8x+3}{x^2-5x+9}dx$.

Answer:

$$\int_2^{12} \frac{8x+3}{x^2-5x+9}dx \approx \frac{12-2}{(2)(5)}\left(f(2)+2f(4)+2f(6)+2f(8)+2f(10)+f(12)\right)$$
$$= 1\left(\frac{19}{3}+2(7)+2\left(\frac{17}{5}\right)+2\left(\frac{67}{33}\right)+2\left(\frac{83}{59}\right)+\frac{33}{31}\right) = 35.072.$$

(d) Use the midpoint rule with $n = 5$ to approximate $\int_2^{12} \frac{8x+3}{x^2-5x+9}dx$.

$$\int_2^{12} \frac{8x+3}{x^2-5x+9}dx \approx \frac{12-2}{5}\left(f(3)+f(5)+f(7)+f(9)+f(11)\right)$$
$$= 2\left(9+\frac{43}{9}+\frac{59}{23}+\frac{5}{3}+\frac{91}{75}\right) = 38.446.$$

(e) Use Simpson's rule with $n = 4$ to approximate $\int_2^{10} \frac{8x+3}{x^2-5x+9}dx$.

$$\int_2^{10} \frac{8x+3}{x^2-5x+9}dx \approx \frac{12-2}{(3)(5)}\left(f(2)+4f(4)+2f(6)+4f(8)+f(10)\right)$$
$$= \frac{2}{3}\left(\frac{19}{3}+(4)(7)+2\left(\frac{17}{5}\right)+4\left(\frac{67}{33}\right)+\frac{83}{59}\right) = 33.7742$$

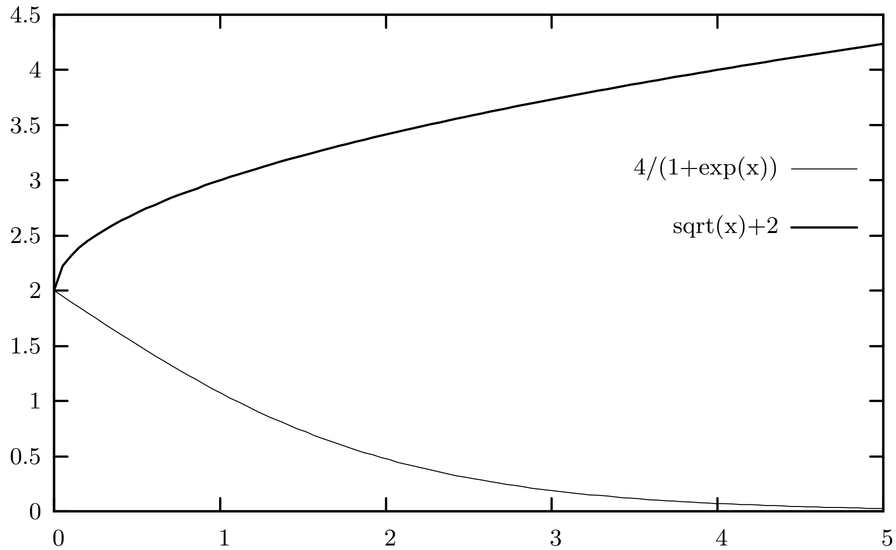
For your information, this integral can be computed exactly (though it's rather tedious). One obtains the precise antiderivative

$$\int \frac{8x+3}{x^2-5x+9} = \frac{46}{\sqrt{11}}\arctan\left(\frac{2x-5}{\sqrt{11}}\right) + 4\ln(x^2-5x+9)$$

and the integrals (exact to four decimal places)

$$\int_2^{12} \frac{8x+3}{x^2-5x+9}dx = 37.1868 \textit{ and } \int_2^{10} \frac{8x+3}{x^2-5x+9}dx = 34.754$$

Problem 6. **[20 points]** Consider the region trapped by the two curves $y = \dfrac{4}{1+e^x}$ and $y = \sqrt{x} + 2$ and between the lines $x = 0$ and $x = 5$. Here is a picture of the region:



(a) Use an integral to express the volume of the solid formed by rotating this region around the x-axis. Do not evaluate the integral.

Answer:

We use "washers."

$$Volume = \int_0^5 \pi \left(\sqrt{x} + 2\right)^2 - \pi \left(\frac{4}{1+e^x}\right)^2 dx$$

To use shells here is a little more complicated. First, we solve for x in terms of y:

$$y = \frac{4}{1+e^x} \Leftrightarrow x = \ln\left(\frac{4}{y} - 1\right) \text{ and } y = \sqrt{x} + 2 \Leftrightarrow x = (y-2)^2.$$

$$V = \int_{y=0}^{y=\sqrt{5}+2} 2\pi r h \, dy = \int_0^2 2\pi \left(5 - \ln\left(\frac{4}{y} - 1\right)\right) y \, dy + \int_2^{\sqrt{5}+2} 2\pi \left(5 - (y-2)^2\right) y \, dy.$$

(b) Use an integral to express the volume of the solid formed by rotating this region around the line $x = 5$. Do not evaluate the integral.

Answer:

Using shells:

$$Volume = \int_0^5 2\pi(5-x)\left(\sqrt{2} + x - \frac{4}{1+e^x}\right) dx.$$

We can use washers (well, discs) here:

$$V = \int_{y=0}^{y=\sqrt{5}+2} \pi r^2 \, dy = \int_0^2 \pi \left(5 - \ln\left(\frac{4}{y} - 1\right)\right)^2 dy + \int_2^{\sqrt{5}+2} \pi \left(5 - (y-2)^2\right)^2 dy.$$