Math 132 Solutions to Midterm 1 (instrumental)

$\boxed{\text{20 pts}}$ 1. $\displaystyle\int \frac{\ln(4t)}{t}\,dt$

Solution: Make the substitution $u = \ln(4t)$, so $du = \frac{1}{t}dt$. Then we have

$$\int \frac{\ln(4t)}{t}\,dt = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln 4t)^2 + C$$

$\boxed{\text{20 pts}}$ 2. $\displaystyle\int_1^2 x^2 \ln x\,dx$

Solution: We integrate by parts, letting $u = \ln x$ and $dv = x^2\,dx$.
Then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^3}{3}$, and we have

$$\begin{aligned}\int_1^2 x^2 \ln x\,dx = \frac{x^3}{3}\ln(x)\Big|_1^2 - \int_1^2 \frac{x^3}{3}\frac{dx}{x} &= \frac{2^3}{3}\ln 2 - \int_1^2 \frac{x^2}{3}\,dx \\ &= \frac{8\ln 2}{3} - \frac{x^3}{9}\Big|_1^2 = \frac{8\ln 2}{3} - \frac{7}{9}\end{aligned}$$

$\boxed{\text{20 pts}}$ 3. $\displaystyle\int \frac{x-1}{x^3+x}\,dx$

Solution: We can factor the denominator to rewrite the integral as $\displaystyle\int \frac{x-1}{x(x^2+1)}\,dx$ and then use partial fractions. We want to find A, B, C so that

$$\frac{x-1}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}.$$

So, we have $x - 1 = (Ax+B)x + C(x^2+1)$. Equating coefficients gives us the equations

$$A + C = 0 \qquad B = 1 \qquad C = -1$$

which tells us that $A = 1$. So we have

$$\int \frac{x-1}{x(x^2+1)}\,dx = \int \frac{x+1}{x^2+1} - \frac{1}{x}\,dx = \int \frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x}\,dx$$

Now we integrate (using $u = x^2+1$ and $du = 2xdx$ in the first integral) to get

$$\frac{1}{2}\ln(x^2+1) + \arctan x - \ln|x| + C.$$

20 pts 4. $\displaystyle\int_1^7 \frac{dr}{(4-r)^2}$

Solution: Observe that the integral is undefined when $r = 4$, so this is an improper integral since 4 between the limits of integration. So, we split the integral as

$$\int_1^4 \frac{dr}{(4-r)^2} + \int_4^7 \frac{dr}{(4-r)^2} = \lim_{M\to 4^-} \int_1^M \frac{dr}{(4-r)^2} + \lim_{L\to 4^+} \int_L^7 \frac{dr}{(4-r)^2}$$

Looking at the first integral, we have

$$\lim_{M\to 4^-} \int_1^M \frac{dr}{(4-r)^2} = \lim_{M\to 4^-} \frac{1}{4-r}\bigg|_1^M = \lim_{M\to 4^-} \frac{1}{4-M} - \frac{1}{3}$$

which diverges, as does the other integral. Hence, the integral is **Divergent**.

20 pts 5. $\displaystyle\int \frac{y+1}{\sqrt{9-y^2}}\,dy$

Solution: While this may look like an integral requiring a trig substition (and you can do it that way), it is actually easier to do by just separating the numerator. That is

$$\int \frac{y+1}{\sqrt{9-y^2}}\,dy = \int \frac{y}{\sqrt{9-y^2}}\,dy + \int \frac{1}{\sqrt{9-y^2}}\,dy$$

To do the first integral, let $u = 9 - y^2$, so $du = -2ydy$. For the second integral, factor out the 9 from the radical to recognize it as $\arcsin(y/3)$. Specifically, we have

$$-\int \frac{du}{2\sqrt{u}} + \int \frac{1}{3\sqrt{\left(1 - \left(\frac{y}{3}\right)^2\right)}}\,dy = -\sqrt{u} + \arcsin\left(\frac{y}{3}\right) + C = -\sqrt{9-y^2} + \arcsin\left(\frac{y}{3}\right) + C$$

If instead you did the trig subsitution, here's how it goes. Let $y = 3\sin\theta$, so $dy = 3\cos\theta\,d\theta$ and the integral becomes

$$\int \frac{3\sin\theta + 1}{\sqrt{9 - 9\sin^2\theta}}\,3\cos\theta\,d\theta = \int (3\sin\theta + 1)\,d\theta = -3\cos\theta + \theta + C.$$

Since $\theta = \arcsin(y/3)$, we need to simplify $\cos\theta$. Observe that if $\sin\theta = y/3$, then $\cos\theta = \frac{1}{3}\sqrt{9-y^2}$. So we have the solution

$$-\sqrt{9-y^2} + \arcsin\left(\frac{y}{3}\right) + C$$

$\boxed{\text{20 pts}}$ 6. $\displaystyle\int e^{2z}\cos(z)\,dz$

Solution: In this one, we integrate by parts twice, and solve. We can choose either the cosine or the exponential as u; I'll choose the exponential. So that I don't have to keep writing it, I'm going to call the desired integral $\mathcal{I}$.

We let $u = e^{2z} \quad dv = \cos z dz \quad$ so $\quad du = 2e^{2z}dz \quad v = \sin z$. Then we have

$$\mathcal{I} = uv - v\,du = e^{2z}\sin z - 2\int e^{2z}\sin z dz$$

Integrate by parts again ($u = e^{2z} \quad dv = \sin z dz$ and $du = 2e^{2z}dz \quad v = -\cos z$) gives

$$\mathcal{I} = e^{2z}\sin z - 2\left(-e^{2z}\cos z + 2\int e^{2z}\cos z dz\right) = e^{2z}\sin z + 2e^{2z}\cos z - 4\mathcal{I}$$

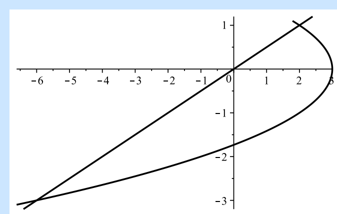
Solving for $\mathcal{I}$ gives us

$$5\mathcal{I} = e^{2z}(\sin z + 2\cos z) \qquad \text{or} \qquad \mathcal{I} = \frac{e^{2z}}{5}(\sin z + 2\cos z)$$

$\boxed{\text{20 pts}}$ 7. Find the area between the curves $y = \dfrac{x}{2}$ and $y^2 = 3 - x$.

Solution: Unfortunately, due to a glitch the figure was printed incorrectly on the exam. The correct figure (with axes labeled) at right, and the web version now is correct.

First, we need to determine where the two curves cross. It is easier if we rewrite the curves as $2y = x$ and $x = 3 - y^2$. Then we must have $2y = 3 - y^2$, or $y^2 + 2y - 3 = 0$, and the curves cross when $y = -3$ or $y = 1$.
Solving for x gives $x = -6$ and $x = 2$.



The integral is simpler if we integrate with respect to y. In this case, the parabola is always on the right, so the area is

$$\int_{-3}^{1} 3 - y^2 - 2y\,dy = 3y - \frac{1}{3}y^3 - y^2\Big|_{-3}^{1} = (3 - 1/3 - 1) - (-9 + 27/3 - 9) = \frac{32}{3}$$

If you insist on integrating with respect to x, not that you have to do two integrals: one for $-6 \le x \le 2$, and another for $2 \le x \le 3$. Specifically, the area is given by

$$\int_{-6}^{2} \frac{x}{2} - (-\sqrt{3-x})\,dx + \int_{2}^{3} \sqrt{3-x} - (-\sqrt{3-x})\,dx.$$

You should still get $\frac{32}{3}$ as the answer, though.

20 pts 8. Does the improper integral $\displaystyle\int_2^\infty \frac{x^{1/2}-1}{5x^2+6}\,dx$ converge? Fully justify your answer (note that if the integral converges, you need not give its value).

Solution: Yes, $\boxed{\textbf{it converges.}}$ Note that

$$\frac{x^{1/2}-1}{5x^2+6} < \frac{x^{1/2}-1}{5x^2} < \frac{x^{1/2}}{5x^2} < \frac{1}{x^{3/2}}.$$

$\displaystyle\int_2^\infty \frac{dx}{x^{3/2}}$ converges, since

$$\int_2^\infty \frac{dx}{x^{3/2}} = \lim_{M\to\infty} \left.\frac{-2}{x^{1/2}}\right|_2^M = \lim_{M\to\infty} \frac{-2}{\sqrt{M}} + \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}.$$

By the comparison test, the original integral converges.

9. Since $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln 2$, we can approximate $\ln 2$ using only addition, multiplication, and division by approximating the integral numerically.

10 pts (a) Use Simpson's rule with 1 interval ($n = 2$) to estimate $\ln 2 = \displaystyle\int_1^2 \frac{1}{x}\,dx$. You don't have to add up the fractions.

Solution: Since there is one interval and total length 1, we use the points $x_0 = 1$, $x_1 = \frac{3}{2}$, and $x_2 = 2$. So we have

$$S_2 = \frac{1}{3}\cdot\frac{1}{2}\Big(f(1) + 4f(3/2) + f(2)\Big) = \frac{1}{6}\left(1 + 4\left(\frac{2}{3}\right) + \frac{1}{2}\right) = \boxed{\frac{25}{36}} \approx 0.69444$$

Since $\ln 2 \approx 0.693147$, this isn't too bad.

10 pts (b) What n do we need to estimate $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln 2$ within $\frac{1}{1000}$ using Simpson's rule?[1]

Solution: We use the information in the footnote. We need to determine n so that where K is the maximum of the absolute value of the fourth derivative of $f(x) = 1/x$ for x between 0 and 1. Since $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, $f^{(3)}(x) = -6/x^4$, and $f^{(4)}(x) = 24/x^4$, we should take $K = 24$.

Now we need to solve $\dfrac{24}{180n^4} < \frac{1}{1000}$. Rewriting, we have $n^4 > 24000/180 = 400/3$, or $3n^4 > 400$. We can just try a few values of n, remembering that n must be even. if $n = 2$, we have $3\cdot 16 = 48$ (too small), and $n = 4$ gives $3\cdot 256$, which is bigger than 400. So, $\boxed{\text{n=4}}$ does the job.

[1] Use the following estimate for E_S: If $|f^{(4)}(x)| \le K$ then $E_S \le K\,\dfrac{(b-a)^5}{180n^4}$.
It might be useful to know that $2^4 = 16$, $3^4 = 81$, $4^4 = 256$, $5^4 = 625$, $6^4 = 1296$, $7^4 = 2401$, $8^4 = 4096$, $9^4 = 6561$, and $10^4 = 10000$. Or not.