Notes on Second Order Linear Differential Equations

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1. The general second order homogeneous linear differential equation with constant coefficients looks like

$$Ay'' + By' + Cy = 0,$$

where y is an unknown function of the variable x, and A, B, and C are constants. If $A = 0$ this becomes a first order linear equation, which we already know how to solve. So we will consider the case $A \neq 0$. We can divide through by A and obtain the equivalent equation

$$y'' + by' + cy = 0$$

where $b = B/A$ and $c = C/A$.

"Linear with constant coefficients" means that each term in the equation is a constant times y or a derivative of y. "Homogeneous" excludes equations like $y'' + by' + cy = f(x)$ which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

$$y = e^{\lambda x},$$

where λ is an unknown constant. Why? Because it works!

We substitute $y = e^{\lambda x}$ in our equation. This gives

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, we can divide through and get the equation

$$\lambda^2 + b\lambda + c = 0.$$

Whenever λ is a solution of this equation, $y = e^{\lambda x}$ will automatically be a solution of our original differential equation, and if λ is not a solution, then $y = e^{\lambda x}$ cannot solve the differential equation. So the substitution $y = e^{\lambda x}$ transforms the differential equation into an algebraic equation!

Example 1. Consider the differential equation

$$y'' - y = 0.$$

Plugging in $y = e^{\lambda x}$ give us the associated equation

$$\lambda^2 - 1 = 0,$$

which factors as

$$(\lambda+1)(\lambda-1)=0;$$

this equation has $\lambda = 1$ and $\lambda = -1$ as solutions. Both $y = e^x$ and $y = e^{-x}$ are solutions to the differential equation $y'' - y = 0$. (You should check this for yourself!)

Example 2. For the differential equation

$$y'' + y' - 2y = 0,$$

we look for the roots of the associated algebraic equation

$$\lambda^2 + \lambda - 2 = 0.$$

Since this factors as $(\lambda-1)(\lambda+2) = 0$, we get both $y = e^x$ and $y = e^{-2x}$ as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form

$$y'' + by' + cy = 0,$$

we need to find the roots of $\lambda^2 + b\lambda + c = 0$, which we can do using the quadratic formula to get

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If the *discriminant* $b^2 - 4c$ is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if y_1 and y_2 are solutions of the homogeneous differential equation $y'' + by' + cy = 0$, then so is the linear combination $py_1 + qy_2$ for any numbers p and q. This fact is easy to check (just plug $py_1 + qy_2$ into the equation and regroup terms; note that the coefficients b and c do not need to be constant for this to work. This means that for the differential equation in Example 1 ($y'' - y = 0$), any function of the form

$$pe^x + qe^{-x} \qquad \text{where } p \text{ and } q \text{ are any constants}$$

is a solution. Indeed, while we can't justify it here, *all* solutions are of this form. Similarly, in Example 2, the general solution of

$$y'' + y' - 2y = 0$$

is

$$y = pe^x + qe^{-2x}, \qquad \text{where } p \text{ and } q \text{ are constants.}$$

4. If the discriminant $b^2 - 4c$ is negative, then the equation $\lambda^2 + b\lambda + c = 0$ has no solutions, unless we enlarge the number field to include $i = \sqrt{-1}$, i.e. unless we work with complex numbers. If $b^2 - 4c < 0$, then since we can write any positive number as a square k^2, we let $k^2 = -(b^2 - 4c)$.

Then ik will be a square root of b^2-4c, since $(ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2-4c$. The solutions of the associated algebraic equation are then

$$\lambda_1 = \frac{-b+ik}{2}, \quad \lambda_2 = \frac{-b-ik}{2}.$$

Example 3. If we start with the differential equation $y''+y=0$ (so $b=0$ and $c=1$) the discriminant is $b^2-4c=-4$, so $2i$ is a square root of the discriminant and the solutions of the associated algebraic equation are $\lambda_1 = i$ and $\lambda_2 = -i$.

Example 4. If the differential equation is $y''+2y'+2y=0$ (so $b=2$ and $c=2$ and $b^2-4c = 4-8=-4$). In this case the solutions of the associated algebraic equation are $\lambda = (-2\pm 2i)/2$, i.e. $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$.

5. Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting $e^{\lambda x}$ as a function of x when λ is a complex number. Suppose λ has real part a and imaginary part ib, so that $\lambda = a+ib$ with a and b real numbers. Then

$$e^{\lambda x} = e^{(a+ib)x} = e^{ax}e^{ibx}$$

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor e^{ax} does not cause a problem, but what is e^{ibx}? Everything will work out if we take

$$e^{ibx} = \cos(bx) + i\sin(bx),$$

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

6. Let us try this formula with our examples.

Example 3. For $y''+y=0$ we found $\lambda_1 = i$ and $\lambda_2 = -i$, so the solutions are $y_1 = e^{ix}$ and $y_2 = e^{-ix}$. The formula gives us $y_1 = \cos x + i\sin x$ and $y_2 = \cos x - i\sin x$.

Our earlier observation that if y_1 and y_2 are solutions of the linear differential equation, then so is the combination py_1+qy_2 for any numbers p and q holds even if p and q are complex constants.

Using this fact with the solutions from our example, we notice that $\frac{1}{2}(y_1+y_2) = \cos x$ and $\frac{1}{2i}(y_1-y_2) = \sin x$ are both solutions. *When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions.* Using the fact about linear combinations again, we can say that $y = p\cos x + q\sin x$ is a solution for any p and q. This is the general solution. (It is also correct to call $y = pe^{ix}+qe^{-ix}$ the general solution; which one you use depends on the context.)

Example 4. $y''+2y'+2y=0$. We found $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$. Using the formula we have

$$y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i\sin x),$$

$$y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i\sin x).$$

Exactly as before we can take $\frac{1}{2}(y_1+y_2)$ and $\frac{1}{2i}(y_1-y_2)$ to get the real solutions $e^{-x}\cos x$ and $e^{-x}\sin x$. (Check that these functions both satisfy the differential equation!) The general solution will be $y = pe^{-x}\cos x + qe^{-x}\sin x$.

7. *Repeated roots.* Suppose the discriminant is zero: $b^2-4c=0$. Then the "characteristic equation" $\lambda^2+b\lambda+c=0$ has one root. In this case both $e^{\lambda x}$ **and** $xe^{\lambda x}$ are solutions of the differential equation.

Example 5. Consider the equation $y''+4y'+4y=0$. Here $b=c=4$. The discriminant is $b^2-4c = 4^2-4\times 4=0$. The only root is $\lambda=-2$. Check that **both** e^{-2x} and xe^{-2x} are solutions. The general solution is then $y=pe^{-2x}+qxe^{-2x}$.

8. *Initial Conditions.* For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition $y(0)=y_0$; in the same way the p and the q in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some "initial" value of x.

Example 5. Suppose that for the differential equation of Example 2, $y''+y'-2y=0$, we want a solution with $y(0)=1$ and $y'(0)=-1$. The general solution is $y=pe^{x}+qe^{-2x}$, since the two roots of the characteristic equation are 1 and -2. The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for p and q. In this case we have

$$1 = y(0) = pe^{0} + qe^{-2\times 0} = p+q$$

$$-1 = y'(0) = pe^{0} - 2qe^{-2\times 0} = p-2q.$$

This leads to the set of linear equations $p+q=1, p-2q=-1$ with solution $q=2/3, p=1/3$. You should check that the solution

$$y = \frac{1}{3}e^{x} + \frac{2}{3}e^{-2x}$$

satisfies the initial conditions.

Example 6. For the differential equation of Example 4, $y''+2y'+2y=0$, we found the general solution $y=pe^{-x}\cos x+qe^{-x}\sin x$. To find a solution satisfying the initial conditions $y(0)=-2$ and $y'(0)=1$ we proceed as in the last example:

$$-2 = y(0) = pe^{-0}\cos 0 + qe^{-0}\sin 0 = p$$

$$1 = y'(0) = -pe^{-0}\cos 0 - pe^{-0}\sin 0 - qe^{-0}\sin 0 + qe^{-0}\cos 0 = -p+q.$$

So $p=-2$ and $q=-1$. Again check that the solution

$$y = -2e^{-x}\cos x - e^{-x}\sin x$$

satisfies the initial conditions.