LECTURES ON LINEAR SERIES[1]

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with the assistance of

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Contents

Introduction

Linear series have long played a central role in algebraic geometry, and the basic results and techniques from their study form an essential part of the field's culture. However the past decade has witnessed two important new developments. First, vector bundles have emerged as powerful tools for analyzing linear series on curves and surfaces. These vector bundle techniques — the most important being Reider's method involving Bogomolov instability on surfaces — have led to considerable simplifications and extensions of classical results. A second major influence has come from the flowering of higher dimensional geometry. One now has a conjectural picture of how the most familiar facts about linear series on curves should extend to arbitrary smooth projective varieties, and some encouraging partial results have been proved. On the technical side, the cohomological machinery developed to study higher dimensional varieties — notably vanishing theorems for $\mathbf{Q}$-divisors — is proving to have applications even to concrete questions on surfaces. These new tools promise to play an important role in the future, but they have to a certain extent remained embedded in the research literature, and this has limited somewhat their accessibility to the novice or non-expert.

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The purpose of these lectures is to provide a down-to-earth and gentle introduction to some of these new ideas and techniques. While I hope that these notes may have something to say to seasoned geometers who wish to learn about recent work on linear series, I have particularly tried to gear the discussion to a novice audience. My intention was that with a little faith and effort, the material here should be accessible to anyone having finished the standard texts, e.g. [H2] or [GH1].* The underlying theme is the search for higher dimensional generalizations of the most basic theorems about linear series on algebraic curves, but to keep things elementary we work more or less entirely in dimension two. The philosophy is to illustrate in the setting of surfaces the various methods that have been used to attack these questions in general, and we end up repeatedly proving variants of one central result, namely Reider's theorem. Sticking to the case of surfaces allows one to eliminate many technical complexities, and some of the underlying ideas become particularly transparent. I hope that parts of the present notes might therefore provide a useful first introduction to the powerful and important cohomological tools of higher dimensional geometry, as well as to an active area of current research. To this end, I have included many exercises which sketch further developments and applications of the material discussed in the text. An overview of the questions we consider, and an outline of the contents of the lectures, will be found in §1.

It was not my intention to produce a balanced survey of work on linear series, and I have ignored a number of topics that might have fit very nicely. I am particularly cogniscent of the fact that the complex analytic side of the story is woefully under-represented here. Starting with Demailly's ground-breaking paper [De1], there has been an intriguing and fruitful interplay between algebraic and analytic approaches to many of these questions. I can only hope that the one-sided slant of the present notes will motivate someone more able than I to give an introductory account of the analytic viewpoint.

I'm grateful to O. Küchle, V. Maşek and G. Xu for helpful suggestions, and to F. Schreyer for bailing me out of a problem in a preliminary version of these lectures. G. Fernández del Busto — who served as my course assistant at the Park City Institute — produced a preliminary version of these notes and also contributed useful advice. I've written an exposition on linear series once before [L2], but the viewpoint here is rather different. This evolution is partly the result of contact with a number of mathematicians to whom I owe thanks. To begin with, H. Esnault and E. Viehweg initially explained to me a few years ago the philosophy of how vanishing for **Q**-divisors could be used to produce sections of adjoint bundles. I've also greatly profitted from suggestions and encouragement from J. Kollár, M. Reid and Y.-T. Siu. But above all I wish to acknowledge my debt to Lawrence Ein, with whom I have spent the last few years working on the questions discussed in these notes. He deserves a large share of the credit for any originality or utility the present lectures may possess.

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*The only pre-requisites not covered in [H2] that we use systematically are Chern classes of vector bundles, and vanishing theorems. However the latter might be taken on faith. Griffiths and Harris [GH1] provide more than enough background for everything that appears here, except that the reader might want to supplement their discussion of the classical Kodaira vanishing theorem with an account (e.g. [CKM], Lecture 8, or [Kol3], Chapters 9, 10, or [EV2]) of the generalization due to Kawamata and Viehweg.

§0. Notations and Conventions.

(0.1). We work throughout over the complex numbers $\mathbf{C}$. Varieties are assumed to be smooth and projective unless otherwise stated.

(0.2). We write K_X for the canonical divisor (class) of a smooth variety X. If $Z \subset X$ is a subvariety, $\mathcal{I}_Z \subset \mathcal{O}_X$ denotes its ideal sheaf.

(0.3). Given a line bundle L on X, much of our focus will be on the adjoint bundle $K_X + L$, for which it is traditional to use additive notation. But then to indicate the sheaf of sections of this bundle vanishing at a point $x \in X$, it seems most natural to write $\mathcal{O}_X(K_X+L)\otimes\mathcal{I}_x$. It soon becomes difficult to maintain notational consistency. We've finally surrendered to a certain amount of chaos concerning notation for line bundles, divisors and the corresponding invertible sheaves. If L is a line bundle and D is a divisor on X, we write $L(D)$, $L + D$ and $O_X(L + D)$ more or less interchangably. We hope that this will not lead to undue confusion or annoyance. The notation $|\mathcal{O}_X(L + D)|$ is reserved for the complete linear system of divisors of sections of $\mathcal{O}_X(L + D)$. We say that a line bundle (or a divisor, or a linear series) L is *free* (or *globally generated* or *basepoint-free*) at a point $x \in X$ if there is a section $s \in H^0(X, L)$ with $s(x) \neq 0$. L is *free* if it is free at every point of X

(0.4). If L is a line bundle on a smooth projective variety X of dimension n, we denote by L^n the top self-intersection $\int c_1(L)^n$ of the first Chern class of L. If $C \subset X$ is a curve in X, $L\cdot C$ denotes the intersection number of $c_1(L)$ with C. Numerical equivalence of divisors or line bundles is denoted by $\equiv$. Recall that L is *nef*, or *numerically effective* if $L \cdot C \geq 0$ for every irreducible curve $C \subset X$. By Kleiman's criterion (cf. [H1], Chapter 1), L is nef if and only it is in the closure of the cone of ample line bundles. Note that the pull-back of a nef line bundle under any morphism remains nef. A nef line bundle is *big* if $L^n > 0$. This is equivalent (for nef bundles) to saying that $h^0(X, kL)$ grows like k^n. (See [CKM] or [Mori] for alternative characterizations.)

§1. Background and Overview

The theme of these lectures is the search for higher dimensional generalizations of the most familiar and elementary facts about linear series on curves. In this section we introduce the basic questions, and give a brief overview of their history and current status. The detailed contents of the notes are summarized at the end of the section.

To set the stage, we start by recalling the story in dimension one:

Theorem 1.1. *Let C be a compact Riemann surface of genus g.*

(A). If $g \geq 2$ then the canonical bundle K_C is globally generated, and the pluri-canonical series $|mK_C|$ are very ample when $m \geq 3$.

(B). Let N be a line bundle on C, with $\deg(N) = d$. *If $d \geq 2g$ then N is globally generated, and if $d \geq 2g + 1$ then N is very ample.*

Proof. We focus on statement (B). Suppose that $d \geq 2g$, and let $P \in C$ be a fixed point. We need to show that there exists a section $s \in \Gamma(C, N)$ with $s(P) \neq 0$. Consider to this end the exact sequence of sheaves

$$0 \longrightarrow N(-P) \longrightarrow N \longrightarrow N \otimes \mathcal{O}_P \longrightarrow 0.$$

It induces in cohomology the exact sequence

$$H^0(C, N) \xrightarrow{e_P} H^0(C, N \otimes \mathcal{O}_P) \longrightarrow H^1(C, N(-P)) \ ,$$

with e_P evaluation at P. Now $H^0(C, N \otimes \mathcal{O}_P) \cong \mathbf{C}$, and it suffices to show that e_P is surjective. But this follows from the fact that $\deg N(-P) \geq 2g - 1$, since in this case we have that $H^1(C, N(-P)) = 0$. The proof of very ampleness is similar, as is statement (A). (cf. [H2], IV.3.2). $\square$

The attempt to generalize this theorem to higher dimensions is a very fundamental and interesting problem. In setting (A), it is relatively clear what to look for. The higher dimensional analogue of a curve of genus ≥ 2 is a variety of general type. Taking into account the expectation that one might want to work on a particularly tractable birational model, we can state the question as

Problem A. *Study pluri-canonical series $|mK_X|$ on "nice" varieties X of general type.*

By contrast, the higher dimensional analogue of statement (B) has only recently come into focus. The key is to rephrase (1.1)(B) without explicitly bringing in the genus of the curve. To this end, note that if N is a line bundle of degree $2g$ on a curve C of genus g, then we can write N in the form

$$N = K_C + 2A$$

for some ample line bundle A on C. Similarly, if $\deg(N) = 2g + 1$, then $N = K_C + 3A$. Hence the natural question generalizing statement (B) is

Problem B. *Let X be a smooth projective variety of dimension n. Study the adjoint linear series*

$$|K_X + L|$$

where L is a suitably positive line bundle on X.

For example, given an ample line bundle A on X, one might take $L = (n+1)A$ or $L = (n+2)A$. As a start, one would like to understand when the adjoint bundles in question are globally generated or very ample.

Pluricanonical mappings of surfaces of general type were studied by Kodaira [Kod] and Bombieri [Bomb] in the late 1960's and early 1970's. In 1988, many of their results emerged as special cases of a theorem of Reider [Rdr] concerning adjoint linear series. Precise statements appear in §2. Suffice it to say here that Problems A and B are by now quite well understood on surfaces.

In higher dimensions, naturally enough, much less is known. But in recent years two conjectures of Fujita have attracted a great deal of interest:

Fujita's Conjectures. [Fuj1] *Let X be a smooth projective variety of dimension n.*

(A). Assume that X is minimal and of general type, i.e. suppose that the canonical bundle K_X is nef and big. Then $\mathcal{O}_X(mK_X)$ is globally generated for $m \geq n+2$.

(B). Let A be an ample line bundle on X (which is now not assumed to be minimal or of general type). Then $K_X + (n+1)A$ is globally generated, and $K_X + (n+2)A$ is very ample.

For curves this is essentially the content of Theorem 1.1, and on surfaces these statements follow from Reider's theorem. But in general Fujita's conjectures remain open as of this writing. Existing results are of two sorts. First, there are effective statements in all dimensions due to Demailly [De1], Kollár [Kol2] and most recently Siu [Siu2], which however are exponential in the dimension n. In another direction, one can stick to the simplest case of global generation on threefolds, and ask for statements closer to the bounds predicted by Fujita. Theorems of this sort are given by Ein and Lazarsfeld in [EL2], with Maşek in [ELM], and by Fujita in [Fuj2]. We refer to Ein's paper [E] for an overview of some of this work. Siu's paper on Matsusaka's theorem [Siu1] develops some related ideas (cf. Exercise 7.7).

It is probably well to stress from the outset why it is that Problems A and B, while completely elementary on curves, become more subtle in higher dimensions. The proof of Theorem 1.1 reduces the question to the vanishing of a cohomology group, and this goes through in the general setting without any problem:

Proposition 1.2. *Let L be an ample line bundle on a smooth projective variety X (of arbitrary dimension). Then $K_X + L$ is globally generated if and only if*

$$H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x) = 0$$

for all points $x \in X$, where $\mathcal{I}_x$ is the ideal sheaf of $\{x\}$. $\square$

(There is of course an analogous criterion for $K_X + L$ to be very ample.) If X is a curve, then $\mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x$ is locally free, and one deduces the required vanishing from general facts about the cohomology of line bundles. However when X is a variety of dimension ≥ 2, the sheaf $\mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x$ is no longer invertible and it becomes considerably harder to control the group $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x)$. So in higher dimensions the issue in effect is to prove a Kodaira-type vanishing theorem for certain non-invertible sheaves.

The "classical" approach to these questions on surfaces is to apply vanishing theorems involving numerical connectedness on a blow-up of X to control the required cohomology groups. This is discussed in Reid's lectures. During the 1980's some new techniques emerged for studying linear series on curves and surfaces, revolving around the geometry of vector bundles. Methods along these lines were used for example to study projective normality of curves in [GL], and the geometry of curves on K3 surfaces in [L1]. But certainly the most important development was Reider's application [Rdr] of Bogomolov's instability theorem to study adjoint series on surfaces. Reider's method – which has its antecedents in a proof by

Mumford of Kodaira vanishing [Reid] – has largely superceeded the classical approach using numerical connectedness à la Ramanujam. We remark that a number of geometers have attempted to apply vector bundles in a similar manner to study linear series on varieties of dimension three and higher, but so far these efforts have not met with success.

At about the same time that vector bundle techniques were developed to study linear series on surfaces, Kawamata, Reid and Shokurov (among others) introduced some powerful but subtle cohomological techniques to analyze the asymptotic behavior of pluricanonical and other linear series on varieties of all dimensions. These techniques – which we shall refer to as the KRS package – are based on the Kawamata-Viewheg vanishing theorem for $\mathbf{Q}$-divisors, and form one of the central tools of contemporary higher dimensional geometry. A typical result here is a theorem of Kawamata and Shokurov to the effect that if X is a smooth minimal variety of general type, then $|mK_X|$ is free for all $m \gg 0$. We refer to [CKM] or [KMM] for references and an overview of this circle of ideas. Ein and the author remarked that one could use the KRS machine to recover (some of) Reider's results. This in turn opened the door to the higher dimensional theorems appearing for instance in [Kol2] and [EL2].

The goal of these lectures is to explain some of the new methods that have been introduced to study Problems A and B on surfaces and higher dimensional varieties. To keep things elementary, we will work more or less exclusively on surfaces, and the plan is to discuss one central result (viz. Reider's theorem) from many points of view. To begin with, the vector bundle techniques apply here, and we go through these in some detail. However we also present in the surface setting some of the methods developed for higher dimensions. Thus we devote considerable attention to the circle of ideas surrounding vanishing theorems for $\mathbf{Q}$-divisors. We also explain – still in the case of surfaces – some algebro-geometric analogues of Demailly's analytic approach, as well as some local results from [EL3] and [EKL].

The detailed contents of these notes is as follows. In §2 we give the statement of Reider's theorem, and present some of its applications. The following two sections are devoted to the proof of this result via vector bundles: §3 discusses Serre's method (and related techniques) for constructing vector bundles, and in §4 Bogomolov's instability theorem is applied to prove Reider's statement. The focus then turns to vanishing theorems. We consider in §5 the questions involving Seshadri constants and local positivity that arise if one tries to apply vanishing theorems in the most naive possible way to study adjoint series. Section 6 revolves around vanishing theorems for $\mathbf{Q}$-divisors and the Kawamata-Reid-Shokurov machine: we use these techniques to reprove (parts of) Reider's theorem, as well as the rank two case of Bogomolov's theorem. Finally, §7 is devoted to the algebro-geometric analogues of Demailly's approach to these questions. Further applications and extensions of the material are outlined in numerous exercises scattered throughout the notes.

§2. Reider's Theorem – Statement and Applications

In this section, we state Reider's theorem [Rdr], and present some simple applications. Reider's result forms the core of the present notes: most of the subsequent sections are devoted to various different proofs of the theorem or its corollaries. Here we try to convey some feeling for its power and scope.

The theorem in question gives a very precise geometric explanation for the failure of an adjoint bundle on a surface to be globally generated or very ample.

Theorem 2.1 ([Rdr]). *Let X be a smooth projective surface, and let L be a nef line bundle on X.*

(i). Assume that $L^2 \geq 5$, and that the adjoint series $|K_X + L|$ has a base-point at $x \in X$. Then there exists an effective divisor $D \subset X$ passing through x such that either

$$\begin{aligned} &D \cdot L = 0 \quad \text{and} \quad D^2 = -1; \quad \text{or} \\ &D \cdot L = 1 \quad \text{and} \quad D^2 = 0. \end{aligned} \tag{2.2}$$

(ii). If $L^2 \geq 10$, and if $x,\ y \in X$ are two points (possibly infinitely near) which fail to be separated by $|K_X + L|$, then there exists an effective divisor $D \subset X$ through x and y such that either

$$\begin{aligned} &D \cdot L = 0 \quad \text{and} \quad D^2 = -1 \quad \text{or} \quad -2; \quad \text{or} \\ &D \cdot L = 1 \quad \text{and} \quad D^2 = 0 \quad \text{or} \quad -1; \quad \text{or} \\ &D \cdot L = 2 \quad \text{and} \quad D^2 = 0 \end{aligned} \tag{2.3}$$

Note that we do not assume that the base point x in (i) is isolated. Hence subject to the numerical inequalities on the nef line bundle L, the statement applies as soon as $\mathcal{O}_X(K_X+L)$ fails to be globally generated. Similarly, (ii) serves as a criterion for the bundle in question to be very ample.

Example 2.4. Typical examples of a divisor satisfying (2.2) may be obtained by taking D to be an exceptional curve of the first kind (in the first case), or as the fibre of a ruled surface (in the second).

Exercise 2.5. (i). Show that if D is one of the divisors satisfying (2.2) described in Example 2.4, then D is necessarily in the base locus of $|K_X + L|$.

(ii). Find examples of divisors D satisfying the various possibilities in (2.3), for which $\mathcal{O}_X(K_X + L)$ indeed fails to be very ample.

(iii). Show that the numerical hypothesis on L^2 in Theorem 2.1 cannot in general be weakened. □

Reider's theorem leads to a simple numerical criterion for an adjoint bundle to be globally generated or very ample:

Corollary 2.6. *Let L be an ample line bundle on a smooth projective surface X. Assume that $L^2 \geq 5$, and that $L \cdot C \geq 2$ for all irreducible curves $C \subset X$. Then $\mathcal{O}_X(K_X + L)$ is globally generated. If $L^2 \geq 10$ and $L \cdot C \geq 3$ for all $C \subset X$, then $\mathcal{O}_X(K_X + L)$ is very ample.* $\square$

This in turn implies the dimension $n = 2$ case of Fujita's conjecture:

Corollary 2.7. *If A is an ample line bundle on X, then $|K_X + 3A|$ is free, and $|K_X + 4A|$ is very ample.*

Proof. In fact, take $L = 3A$. Then $L^2 = 9A^2 \geq 9$, and $L \cdot C = 3A \cdot C \geq 3$ for any curve $C \subset X$. So Corollary 2.6 shows that $\mathcal{O}_X(K_X + L)$ is globally generated, and similarly $\mathcal{O}_X(K_X + 4A)$ is very ample. $\square$

As we indicated in the previous section, attempts to generalize Reider's theorem to higher dimensions have mainly focused on extending the statements of Corollaries 2.6 and 2.7. At the moment, one doesn't even have any clear conjectures as to what might be the analogue of the more precise information contained in Theorem 2.1. However in the case of surfaces, these restrictions on the self-intersection of D are very powerful. Indeed, we will now see that they allow one to deduce from Reider's theorem many of the classical facts concerning pluricanonical models of surfaces of general type.

Let X be a surface of general type. It is very natural and important to try to understand the geometry of the pluricanonical rational mappings

$$\Phi_m = \Phi_{|mK|} : X \dashrightarrow \mathbf{P} = \mathbf{P}H^0(mK_X)$$

defined by multiples of the canonical bundle K_X. Here Reider's theorem leads to an extremely quick proof of some of the fundamental results of Kodaira and Bombieri:

Corolllary 2.8. ([Kod], [Bomb]) *Assume that X is minimal i.e. not the blowing up of some other smooth surface at a point. Then:*

(i). The bundle $\mathcal{O}_X(mK_X)$ is globally generated (i.e. Φ_m is a morphism) if $m \geq 4$, or if $m \geq 3$ and $K_X^2 \geq 2$.

(ii). Φ_m is an embedding away from (-2)-curves if $m \geq 5$, or if $m \geq 4$ and $K_X^2 \geq 2$, or if $m \geq 3$ and $K_X^2 \geq 3$.

Remark. By an embedding away from (-2)-curves, we mean a morphism which is one-to-one and unramified away from a divisor $Z \subseteq X$ consisting of smooth rational curves with self-intersection -2. Catanese and Reider [C] have used Reider's method to prove the more precise theorem of Bombieri that K_V is very ample on the canonical model V of X obtained by blowing down all the $A - D - E$ cycles Z to rational double points.

Proof of Corollary 2.8. We will only consider the global generation of $|mK_X|$, leaving the proof of the second assertion as an exercise.

Recall that on a minimal surface of general type X, the canonical bundle K_X is nef [BPV], III.2.3. Hence Theorem 2.1 applies to $L = (m-1)K_X$. The numerical hypotheses of (2.1)(i) are satisfied thanks to the conditions on m and K_X^2. Thus if $|mK_X|$ has a base point, then there exists an effective divisor $D \subseteq X$ such that either $(m-1)K_X \cdot D = 1$ or else

$$(m-1)K_X \cdot D = 0 \text{ and } D^2 = -1.$$

The first possibilty is excluded by the assumption that $(m-1) \geq 2$. As for the second, if $K_X \cdot D = 0$ then $D^2 = D \cdot (D + K_X) \equiv 0 \pmod 2$ by adjunction, contradicting $D^2 = -1$. □

We refer to [Rdr] for further applications of Reider's method to pluricanonical mappings. The following exercises present some other applications of Theorem 2.1, and the conference proceedings [SBL] contain a sampling of some more recent developments.

Exercise 2.9. (**Embeddings of Abelian Surfaces.**) Prove the following result of Ramanan [Ram]. Let X be an abelian surface which contains no elliptic curves. If L is an ample line bundle on X such that $L^2 \geq 10$, then L is very ample. (It follows for example that on a sufficiently general abelian surface, a polarization of type $(1, d)$ is very ample if $d \geq 5$. In particular, taking d = 5, there exist smooth abelian surfaces in $\mathbf{P}^4$.) See [LB] Chapt. 10, §4, for more precise statements, and the following exercise for a generalization. The corresponding questions on higher dimensional abelian varieties are considered in [BLR] and [DHS]. □

Exercise 2.10. (**Linear Series on Minimal Surfaces of Kodaira Dimension Zero.**) In this exercise, X denotes a smooth projective surface whose canonical bundle K_X is numerically trivial. This hypothesis is satisfied e.g. by Abelian, K3 and Enriques surfaces; see [BPV], IV.1, for the complete list. Fix an ample line bundle L on X.

(i). Assume that $L^2 \geq 5$. Prove that $\mathcal{O}_X(K_X + L)$ fails to be globally generated if and only if there exists an irreducible curve $E \subset X$ with $p_a(E) = 1$ and $E \cdot L = 1$.

(ii). Assume that $L^2 \geq 10$, and that $|K_X + L|$ is free. Prove that $\mathcal{O}_X(K_X + L)$ fails to be very ample if and only if there exists a reduced curve $E \subset X$, with $p_a(E) = 1$ and $E \cdot L = 2$. [Note that the intersection form on X is even, i.e. D^2 is even for every divisor D on X.]

See [Rdr], Proposition 5, for a more precise statement due to Beauville. □

We conclude this section with some interesting open problems of an algebraic nature that arise in connection with Reider's theorem. Returning for an instant to the one-dimensional case, let N be a line bundle of degree d on a smooth projective curve C of genus g. A classical theorem of Castelnuovo, Mattuck and Mumford asserts that if $d \geq 2g+1$, then N is normally generated, i.e. the natural maps $Sym^m(H^0(N)) \longrightarrow H^0(mN)$ are surjective for all $m \geq 0$. Furthermore, if $d \geq 2g+2$, then in the embedding $C \subset \mathbf{P}\mathrm{H}^0(N)$ defined by N, the homogeneous ideal I_C of C is generated by quadrics. The famous theorems of

Noether and Petri (cf. [Mfd], [ACGH] Chapter III, [GL]) give analogous statements for the canonical bundle, and Green [Grn] has shown that at least conjecturally the whole picture extends to higher syzygies as well. (See also Exercise 3.5.)

The question then arises whether similar statements hold for adjoint and pluricanonical bundles on surfaces and higher dimensional varieties. In the two dimensional case, the natural thing to hope for here is the following:

Conjecture 2.11. (**Mukai**.) *Let A be an ample line bundle on a smooth projective surface X, and let P be any nef line bundle. Then $\mathcal{O}_X(K_X + 4A + P)$ is normally generated, and in the embedding $X \subset \mathbf{P}$ defined by $\mathcal{O}_X(K_X + 5A + P)$, the homogeneous ideal I_X of X is generated by quadrics.*

One would also like results in the spirit of (2.1) dealing with the adjoint bundles $K_X + L$ subject only to numerical conditions on L. For instance, one might hope that if L satisfies the hypotheses of (2.1)(ii), then $\mathcal{O}_X(K_X + L)$ is normally generated. There should also be analogous statements for higher syzygies.

When X is a ruled surface, Butler [But] proves that $K_X + 5A + P$ is projectively normal, and he also obtains results for generation by quadrics and higher syzygies. For "hyper-adjoint" bundles of the form $K + mB + P$ where B is *very ample*, results in arbitrary dimension are given by Ein and Lazarsfeld in [EL1]. In fact, if V is a smooth projective variety of dimension n, then $K_V + (n+1)B + P$ is normally generated, and $K_V + (n+2)B + P$ defines an embedding in which the homogeneous ideal of V is defined by quadrics. (Compare Exercise 5.15. [EL1] also contains analogous statements for higher syzygies.) However it seems that new ideas will be needed to tackle Conjecture 2.11. It would be already very interesting to get a result along the lines of (2.11) but with weaker numbers.

§3. Building Vector Bundles

This section is devoted to preparations for the first proof of Reider's theorem, via vector bundles. The strategy for using vector bundles to study linear series involves two steps:

- Encode the geometric data at hand in a vector bundle E; and
- Study the geometry of E (e.g. stability, or sub-bundles, or endomorphisms) to arrive at the desired conclusions.

Here we focus on the first point, and discuss techniques for constructing bundles. Specifically, after some warm-up with extentions of line bundles, we study a method introduced by Serre, which underlies Reider's argument. In the exercises, we sketch several other constructions and applications. A good reference for the general homological machinery appearing here is [GH1], Chapter 5, §3 and §4.

Extensions of Line Bundles.

To set the stage, we begin with the simplest technique for producing vector bundles, namely as extensions of invertible sheaves. Let X be an irreducible projective variety, and let L and M be line bundles on X. Recall that an *extension* of L by M is a short exact sequence of sheaves:

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0, \tag{3.1}$$

so that E is a rank two vector bundle on X. Two such extentions are *equivalent* if there is a map between them inducing the identity on the outer terms. The set of equivalence classes of such extensions can be given the structure of a complex vector space, denoted $\mathrm{Ext}^1(L, M)$. The zero element of $\mathrm{Ext}^1(L, M)$ corresponds to the split sequence. (Analogous considerations apply when L and M are arbitrary coherent sheaves, although of course in this case the middle term in (3.1) isn't in general locally free.)

Assuming as we are that L and M are line bundles, one has the basic isomorphism:

$$\mathrm{Ext}^1(L, M) = H^1(X, L^* \otimes M), \tag{3.2}$$

whose proof we outline in Exercise 3.3. For the present purposes, the importance of (3.2) is that the bundle E serves as a geometric realization of a class $e \in H^1(X, L^* \otimes M)$. This is illustrated in Exercise 3.5, where we indicate how ideas along these lines may be used to prove Noether's theorem and various generalizations concerning the projective normality of algebraic curves.

Exercise 3.3. (**Extension Classes** – Compare [H2] Ex. III.6.1, and [GH1], pp. 722 - 725.) We keep the notation just introduced. In particular, L and M are locally free sheaves of rank one on the projective variety X.

(i). We start by defining the map

$$\mathrm{Ext}^1(L, M) \longrightarrow H^1(X, L^* \otimes M) \tag{3.3.1}$$

required for (3.2). To this end, fix an extension of L by M. Prove that there exists a covering of X by open sets $\{U_i\}$ on which the restriction of (3.1) splits. Show that on the intersections $U_i \cap U_j$, the difference of two such local splittings determines homomorphisms $L(U_i \cap U_j) \longrightarrow M(U_i \cap U_j)$, which in turn yields a Čech cocycle in $Z^1(\{U_i\}, L^* \otimes M)$. Prove that the resulting cohomology class $e \in H^1(X, L^* \otimes M)$ — which is called the *extension class* of the extension (3.1) — is independent of the choices made, and vanishes if and only if (3.1) splits. This defines (3.3.1).

(ii). Show that the map (3.3.1) is an isomorphism. [It may be helpful to remark that one can take transition matrices for the bundle E appearing in (3.1) to be triangular, with the cocycle representing the extension class as the off-diagonal entry.]

(iii). Prove that map (3.3.1) can alternatively be defined as follows. Given (3.1), tensor through by L^* to get $0 \longrightarrow L^* \otimes M \longrightarrow L^* \otimes E \longrightarrow \mathcal{O}_X \longrightarrow 0$. Then the corresponding extension class is the image of the constant function $1_X \in H^0(X, \mathcal{O}_X)$ under the connecting homomorphism $\delta : H^0(X, \mathcal{O}_X) \longrightarrow H^1(X, L^* \otimes M)$.

(iv). Note that cup product determines a map $H^0(X, L) \otimes H^1(X, L^* \otimes M) \longrightarrow H^1(X, M)$, or equivalently a homomorphism

$$H^1(X, L^* \otimes M) \longrightarrow Hom(H^0(X, L), H^1(X, M)). \tag{3.3.2}$$

Verify that (3.3.2) may be interpreted as the map which takes an extension to the connecting homomorphism it determines. □

Exercise 3.4. Let E be a rank two vector bundle on $\mathbf{P}^n$ which admits a nowhere vanishing section. Prove that if $n \geq 2$, then E is a direct sum of line bundles. (The same statement is true when $n = 1$, but the proof is a little more delicate. In fact, a theorem of Grothendieck (cf. [OSS], I.2.1) states that any vector bundle on $\mathbf{P}^1$ splits as a direct sum of line bundles.) □

Exercise 3.5. (**Noether's Theorem and Generalizations**.) Let C be a smooth projective curve of genus $g \geq 2$. A classical theorem of Noether states that if C is non-hyperelliptic, then the natural maps

$$\rho_m : Sym^m(H^0(C, K_C)) \longrightarrow H^0(C, mK_C)$$

are surjective for $m \geq 2$. In other words, every pluri-canonical differential form can be expressed as a polynomial in one-forms, or equivalently, C is projectively normal in its canonical embeding $C \subset \mathbf{P}^{g-1}$. In this exercise we will outline a proof of this statement via vector bundles. There are quicker approaches to Noether's theorem, but with only a little extra effort the present argument yields also some substantial generalizations of the classical results (see (v) below). These appear in [GL], from which this exercise is adapted. For simplicity we will focus on the surjectivity of ρ_2; the case of ρ_m for $m > 2$ is similar but more elementary.

(i). Let C be any smooth curve of genus g, and assume that ρ_2 is not surjective. Then the map $H^0(C, \mathcal{O}(2K_C))^* \longrightarrow H^0(C, \mathcal{O}(K_C))^* \otimes H^0(C, \mathcal{O}(K_C))^*$ dual to multiplication has a non-trivial kernel. Using Serre duality and Exercise 3.3.(iv), show that there exists a non-split extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0 \tag{3.5.1}$$

which is exact on global sections. In particular, $h^0(C, E) = g + 1$. The plan is to show that the existence of such a bundle forces C to be hyperelliptic.

(ii). Now fix a point $P \in C$. Show that there exists a non-zero section $s \in H^0(C, E)$ vanishing at P. Denoting by D the effective divisor (containing P) on which s vanishes, one has an exact sequence:

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow E \longrightarrow \mathcal{O}_C(K_C - D) \longrightarrow 0. \tag{3.5.2}$$

Using the fact that (3.5.1) is non-split, show that $\mathcal{O}_C(D) \not\cong \mathcal{O}_C(K_C)$.

(iii). Compare the estimates on $h^0(C, E)$ obtained from (3.5.1) and (3.5.2) to prove that $deg(D) \leq 2r(D)$, where as usual $r(D) = h^0(C, \mathcal{O}_C(D)) - 1$. Finally, conclude from Clifford's theorem that C is hyperelliptic. This completes the proof of Noether's theorem.

(iv). Prove the theorem of Castelnouvo, Mattuck and Mumford that if L is a line bundle of degree $\geq 2g+1$ on a curve C of genus g, then L is normally generated, i.e. the maps $Sym^m(H^0(C, L)) \longrightarrow H^0(C, mL)$ are surjective for $m \geq 1$.

(v). A result going back to Segre (cf. [MS]) states that if F is a rank two vector bundle of degree d on C, then F contains a line sub-bundle $A \subset F$ with $deg(A) \geq [\frac{d-g+1}{2}]$. Using this, prove the theorem of [GL] that if L is a very ample line bundle with

$$deg(L) \geq 2g + 1 - 2h^1(L) - \text{Cliff}(C), \tag{3.5.3}$$

then L is normally generated. Here

$$\text{Cliff}(C) = \min\left\{deg(A) - 2r(A) \mid h^0(A) \geq 2, deg(A) \leq g-1\right\}$$

denotes the Clifford index of C, which measures how general C is from the point of view of moduli. (For instance, $\text{Cliff}(C) = 0$ if and only if C is hyperelliptic; $\text{Cliff}(C) = 1$ iff C is either trigonal or a smooth plane quintic; and if C is a general curve of genus g, then $\text{Cliff}(C) = [\frac{g-1}{2}]$.) Note that (3.5.3) contains the theorems of Noether and Castelnuovo-Mattuck-Mumford as special cases. $\square$

The Serre Construction.

We henceforth assume that X is a smooth projective surface. The bundles on X that arise as extensions of line bundles are rather special, as Exercise 3.4 suggests. Reider's theorem requires a more general construction. It was introduced by Serre, and applied notably in the analysis of codimension two subvarieties of projective space (cf. [OSS], I.5). The case of surfaces, which has a somewhat different flavor, was studied by Griffiths and Harris [GH2].

Consider to begin with a rank two vector bundle E on X, with $\det(E) = L$. Suppose that $s \in \Gamma(X, E)$ is a section of E that vanishes on a finite set. Denote by

$$Z = Z(s) \subset X$$

the zero-scheme of s: locally one may view s as given by a vector $s = (s_1, s_2)$ of regular functions, and Z is defined locally by the vanishing of s_1 and s_2. Remark that the Koszul complex associated to (E, s) determines a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot s} E \longrightarrow L \otimes \mathcal{I}_Z \longrightarrow 0 \tag{3.6}$$

resolving $L \otimes \mathcal{I}_Z$, where $\mathcal{I}_Z$ is the ideal sheaf of Z. Somewhat abusively, we will refer to (3.6) as the Koszul complex arising from s.

Serre addresses the possibility of making the inverse construction. Specifically, fix a line bundle L on X, plus a finite scheme $Z \subset X$. We pose

Question 3.7. *When does there exist a rank two vector bundle E on X, with $det(E) = L$, plus a section $s \in \Gamma(X, E)$ such that $Z(s) = Z$?*

Clearly there are local obstructions to finding E: for instance, Z must be locally defined by two equations. But there are also some very interesting global conditions, and these are the key to Reider's method.

Serre's idea is that one should try to construct E via (3.6), as an extension of $L \otimes \mathcal{I}_Z$ by $\mathcal{O}_X$. As above, we consider the group

$$\mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$$

parametrizing all such. Given $e \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$ we denote by $\mathcal{F}_e$ the sheaf arising from the corresponding extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}_e \longrightarrow L \otimes \mathcal{I}_Z \longrightarrow 0. \tag{3.8}$$

Thus $\mathcal{F}_e$ is a torsion-free $\mathcal{O}_X$-module of rank two. Suppose it happens that $\mathcal{F}_e$ is actually locally free for some $e \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$. Then the map $\mathcal{O}_X \longrightarrow \mathcal{F}_e$ determines a section $s \in \Gamma(X, \mathcal{F}_e)$ with (3.8) as the corresponding Koszul complex. In particular, $Z(s) = Z$. Therefore $(\mathcal{F}_e, s)$ gives the required solution to (3.7). Thus the essential point is to determine when there exists an element $e \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$ such that $\mathcal{F}_e$ is a vector bundle.

We begin with a criterion for the failure of $\mathcal{F}_e$ to be locally free.

Proposition 3.9. *Given an element $e \in \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$, the correponding sheaf $\mathcal{F}_e$ fails to be locally free if and only if there exists a proper subscheme $Z' \subsetneqq Z$ (possibly $Z' = \emptyset$) such that*

$$e \in Im \left\{ \mathrm{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X) \right\}. \tag{3.9.1}$$

Remark 3.10. Let us explicate the map in (3.9.1). If $Z' \subseteq Z$, then $L \otimes \mathcal{I}_Z \subseteq L \otimes \mathcal{I}_{Z'}$. Thus starting with an extension of $L \otimes \mathcal{I}_{Z'}$ by $\mathcal{O}_X$, one can pull it back in an evident way to get an extension of $L \otimes \mathcal{I}_Z$ by $\mathcal{O}_X$. This gives rise to the homomorphism appearing in (3.9.1).

We now give a proof of Proposition 3.9 drawing on some facts from local algebra. A more geometric approach to the main implication is sketched in Exercise 3.19.

Proof of Proposition 3.9. We start with some general remarks of a homological nature, referring to [OSS], II.1.1 (especially pp. 148-149), for details and proofs. Let $\mathcal{F}$ be a torsion-free sheaf on the smooth surface X. Then the set of points at which $\mathcal{F}$ fails to be locally free is finite or empty. Denote by $\mathcal{F}^{**}$ the double dual of $\mathcal{F}$. Then $\mathcal{F}^{**}$ is reflexive, hence a vector bundle. (In general, a reflexive sheaf on a smooth variety is locally free off a set of codimension ≥ 3.) Furthermore, the natural injection $\mu : \mathcal{F} \longrightarrow \mathcal{F}^{**}$ fails to be an isomorphism exactly over the set where $\mathcal{F}$ is not locally free. It is suggestive to think of $\mathcal{F}^{**}$ as a sort of "canonical desingularization" of $\mathcal{F}$.

Turning now to the situation of the Proposition, write $\mathcal{F} = \mathcal{F}_e$, and assume that $\mathcal{F}$ fails to be locally free. The extension defining $\mathcal{F}$, plus the map $\mu : \mathcal{F} \longrightarrow \mathcal{F}^{**}$, give rise to an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F} & \longrightarrow & L \otimes \mathcal{I}_Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mu} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F}^{**} & \longrightarrow & L \otimes \mathcal{I}_{Z'} & \longrightarrow & 0\ . \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \tau & = & \tau & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.11}$$

Here $\tau = \mathcal{F}^{**}/\mathcal{F}$ is a finite sheaf supported on the set where $\mathcal{F}$ fails to be locally free. The map $\mathcal{O}_X \longrightarrow \mathcal{F}^{**}$ determines a section $s' \in \Gamma(X, \mathcal{F}^{**})$ vanishing on a finite scheme $Z' =_{\text{def}} Z(s')$, and the second row of (3.11) is the corresponding Koszul complex. It follows from Remark 3.10 that if $e' \in \text{Ext}^1(L \otimes I_{Z'}, \mathcal{O}_X)$ denotes the extension class of the middle row, then e' maps to the given extension class $e \in \text{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$. So to prove the first implication in Proposition, it suffices to show that Z' is a proper subscheme of Z. But $\tau = \ \text{coker}(\mu) \neq 0$ since $\mathcal{F}$ is not locally free, and the right hand column of (3.11) then shows that $Z' \subsetneq Z$, as required.

Conversely, suppose that there exists a proper subscheme $Z' \subsetneq Z$ such that the given extension defining $\mathcal{F} = \mathcal{F}_e$ is induced from an extension $e' \in \text{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_X)$. Denote by $\mathcal{E}$ the double dual of the torsion-free sheaf $\mathcal{F}_{e'}$ determined by e', so that $\mathcal{E}$ is locally free of rank two. Arguing from a diagram much like (3.11), one finds that there exists an injective sheaf homomorphism $\nu : \mathcal{F} \longrightarrow \mathcal{E}$ with a non-trivial finite cokernel. But a generically injective map between two vector bundles of the same rank is either an isomorphism or drops rank along a divisor (locally defined as a determinant). Therefore $\mathcal{F}$ cannot be locally free. $\square$

We assume henceforth for simplicity that Z is a *reduced* finite scheme, i.e. that all the points of Z appear with "multiplicity one". Then we have:

Corollary 3.12. *There is an extension class $e \in \text{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$ with $\mathcal{F}_e$ locally free if and only if for every proper subset*

$$Z' \subsetneq Z \quad (\textit{including } Z' = \emptyset),$$

$\mathrm{Ext}^1(L \otimes \mathcal{I}_{Z'}, \mathcal{O}_X)$ *maps to a proper subspace of* $\mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X)$.

Proof. In fact, since Z is reduced there are only finitely many proper subschemes $Z' \subset Z$. $\square$

The next ingredient we'll need is:

Serre-Grothendieck Duality. *If* $\mathcal{G}$ *is any coherent sheaf on the smooth projective surface* X, *then there is a natural isomorphism*

$$\mathrm{Ext}^1(\mathcal{G}, \mathcal{O}_X(K_X)) \longrightarrow H^1(X, \mathcal{G})^*.$$

We refer to [H2], III.7, for the proof (cf. also [GH1], Chapter 5, §4). However let us at least define the map appearing in the statement. Represent a given element $e \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{O}_X(K_X))$ by an extension $0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{F}_e \longrightarrow \mathcal{G} \longrightarrow 0$. Then the connecting homomorphism defines a map

$$H^1(X, \mathcal{G}) \longrightarrow H^2(X, \mathcal{O}_X(K_X))$$

which, after fixing an identification $H^2(X, \mathcal{O}_X(K_X)) = \mathbf{C}$, may be viewed as an element of $H^1(X, \mathcal{G})^*$. (See also Exercise 3.19.(iv).)

Still assuming that Z is reduced, we may now state the answer to Question 3.7 as:

Theorem 3.13. [**GH2**] *There exists a rank two vector bundle* E *with det* $E = L$, *plus a section* $s \in \Gamma(E)$ *with* $Z(s) = Z$, *if and only if every section of* $\mathcal{O}_X(K_X + L)$ *vanishing at all but one of the points of* Z *also vanishes at the remaining point.*

Remark. When $Z = \{x\}$ consists of a single (reduced) point, the criterion is simply that x be a base-point of $\mathcal{O}_X(K_X + L)$.

Proof of Theorem 3.13. As we observed above, the question is equivalent to the existence of an extension (3.8) with $\mathcal{F}_e$ locally free. Note to begin with the formal fact that it is enough to test the condition in Corollary 3.12 for sets $Z' \subset Z$ obtained by deleting one point from Z. Hence there exists a locally free extension of $L \otimes \mathcal{I}_Z$ by $\mathcal{O}_X$ if and only if for every point $x \in Z$, the map

$$\mathrm{Ext}^1(L \otimes \mathcal{I}_{Z-\{x\}}, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X) \tag{*}$$

is non-surjective. Now

$$\mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X) = \mathrm{Ext}^1(\mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z, \mathcal{O}_X(K_X)),$$
$$\mathrm{Ext}^1(L \otimes \mathcal{I}_{Z-\{x\}}, \mathcal{O}_X) = \mathrm{Ext}^1(\mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z-\{x\}}, \mathcal{O}_X(K_X)).$$

So by Duality, the existence of a locally free extension is equivalent to the non-injectivity of

$$H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z) \longrightarrow H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z-\{x\}}) \tag{**}$$

for every $x \in Z$. But the map (**) sits in the long exact sequence on cohomology determined by the sheaf sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z \longrightarrow \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z-\{x\}} \longrightarrow \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_{\{x\}} \longrightarrow 0.$$

In particular, the non-injectivity of (**) is equivalent to evaluation at x giving the zero homomorphism $H^0(\mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z-\{x\}}) \longrightarrow H^0(\mathcal{O}_X(K_X + L) \otimes \mathcal{O}_{\{x\}}) = \mathbf{C}$. But this means exactly that every section of $\mathcal{O}_X(K_X + L)$ vanishing on $Z - \{x\}$ also vanishes at x, as claimed. $\square$

In the situation of Reider's Theorem 2.1(i), we have now achieved the goal of encoding geometric hypotheses on the linear series $|K_X + L|$ into the existence of a (hopefully!) interesting vector bundle on X:

Corollary 3.14. *Let L be any line bundle on the smoooth surface X, and suppose that $x \in X$ is a point at which every section of $\mathcal{O}_X(K_X + L)$ vanishes. Then there exists a rank two vector bundle E on X, with det $E = L$, which has a section $s \in \Gamma(X, E)$ vanshing precisely at x, i.e. $Z(s) = \{x\}$.* $\square$

It remains to use this bundle to produce the required curve D. We take this up in the next section. We conclude the present discussion with several exercises, some of which outline other applications of vector bundles to study linear series.

Exercise 3.15. (**Non-Reduced Zero-Schemes**.) Generalize Theorem 3.13 to allow for the possibility that Z is not reduced. In fact, assuming that $Z \subset X$ is a finite local complete intersection subscheme, show that the statement of Corollary 3.12 remains true if one deals with proper subschemes $Z' \subset Z$, with an analogous modification in the statement of Theorem 3.13. Deduce that if L is a line bundle on X such that $\mathcal{O}_X(K_X + L)$ fails to be very ample, then there is a rank two vector bundle E with det $E = L$ with a section vanishing on a finite scheme $Z \subset X$ of length two. [The crucial point is that a local Gorenstein ring – in particular, a local ring of Z – has a unique minimal non-zero ideal of dimension one, to wit the socle (cf. [C]). Hence a finite local complete intersection scheme does not contain continuous families of maximal proper subschemes.] $\square$

Exercise 3.16. (**Cayley-Bacharach Theorem**, [GH2].) Let X be a smooth surface, and let $C_1, C_2 \subset X$ be effective reduced divisors on X which meet transversely. Prove that if $D \in |K_X + C_1 + C_2|$ is a divisor passing through all but one of the points of $C_1 \cap C_2$, then D passes through the remaining one as well. Generalize to the case when the intersection of C_1 and C_2 is proper but possibly not transversal. $\square$

Exercise 3.17. (**Elementary Transformations.**) Let X be a smooth surface, let V be a vector bundle of rank e on X, and let $C \subset X$ be a smooth curve. We suppose given a line bundle A on C of degree d, plus a surjective map $\lambda : V|C \longrightarrow A$ from the restriction of V to C onto A. From these data, we construct a new rank e bundle F on X, as follows. We may view the invertible $\mathcal{O}_C$-module A as a torsion $\mathcal{O}_X$ - module; i.e. via "extension by zero", A becomes a coherent sheaf on X. We denote this sheaf by $\mathcal{A}$ to emphasize that it is not localy free on X. The composition $V \longrightarrow V|C \longrightarrow A$ determines a surjection of $\mathcal{O}_X$-modules $\bar{\lambda} : V \longrightarrow \mathcal{A}$, and we set $F = \ker\ \bar{\lambda}$. Thus one has the basic exact sequence:

$$0 \longrightarrow F \xrightarrow{\mu} V \longrightarrow \mathcal{A} \longrightarrow 0 \tag{3.17.1}$$

of sheaves on X.

(i). Prove that F is locally free on X, of rank e. By analogy with a classical construction on ruled surfaces (cf. [H2], V.5.7.1), F is called the *elementary transformation* of V determined by $\bar{\lambda} : V \longrightarrow \mathcal{A}$.

(ii). Show that the Chern classes of F are given by

$$\begin{aligned} c_1(F) &= c_1(V) - [C] \\ c_2(F) &= c_2(V) - c_1(V) \cdot [C] + d, \end{aligned} \tag{3.17.2}$$

where as above d is the degree of A, considered as a line bundle on C. [For c_1 one can argue for instance that the map μ in (3.17.1) is a homomorphism between two vector bundles of the same rank which drops rank on C. The formula for c_2 takes more work – for example, one could deduce it by working backwards from Riemann-Roch.]

(iii). Show that the transpose μ^* of μ sits in the exact sequence

$$0 \longrightarrow V^* \xrightarrow{\mu^*} F^* \longrightarrow \mathcal{B} \longrightarrow 0, \tag{3.17.3}$$

where $\mathcal{B}$ is the extension by zero of the line bundle $B = N_{C/X} \otimes A^* = \mathcal{O}_C(C) \otimes A^*$ on C. [This amounts to the assertion that $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X) = \mathcal{B}$. When $A = \mathcal{O}_C$ this is elementary, and in general one can argue that given a point $P \in C$, the statement is true for A if and only if it holds for $A(P)$.] $\square$

Exercise 3.18. (**Special Divisors on Curves on a K3 Surface.**) Let X be a K3 surface, and let $C \subset X$ be a smooth curve of genus g. In this exercise we outline a proof of the following

Theorem [L1]. *Assume that every curve in the linear series $|C|$ is reduced and irreducible. Then for every line bundle A on C, one has*

$$g(C) - h^0(A) \cdot h^1(A) \geq 0.$$

To give some background for this statement, suppose for a moment that C is any smooth curve of genus g. It is important to understand under what conditions on r, d, and g one can find a line bundle A on C of degree d with $h^0(A) \geq r + 1$. Classical parameter counts show that the family of all such line bundles has expected dimension $\rho = \rho(d, g, r) =_{def} g - (r+1)(g - d + r) \geq 0$. The existence of g^r_d's when $\rho \geq 0$ was established by Kempf and

Kleiman-Laksov in 1972. Classically, it was conjectured that if $\rho < 0$, then a *general* curve C of genus g carries no line bundles of degree d and $h^0 \geq r+1$. By Riemann-Roch, it is equivalent to predict:

(*) If A is any line bundle on a general curve C of genus g, then

$$g - h^0(A) \cdot h^1(A) \geq 0.$$

Griffiths and Harris proved this (and more) in 1980 using a rather elaborate degenerational argument. We refer to [Mfd] for a quick introduction to this circle of ideas, and to [ACGH] for a comprehensive overview and references. Returning to the Theorem above, the hypothesis on the linear series $|C|$ is certainly satisfied if $Pic(X) = \mathbf{Z} \cdot [C]$. On the other hand, one knows from Hodge theory that for any integer $g \geq 2$, there exists a K3 surface X whose Picard group is generated by the class of a curve $C \subset X$ of genus g. Thus the Theorem gives a quick proof of (*), without degenerations.

(i). Returning to the situation of the Theorem, let A be a line bundle of degree d on C, with $h^0(A) = r+1$, such that both A and $K_C \otimes A^*$ are globally generated. Then A is a quotient of the trivial vector bundle $\mathcal{O}_C^{r+1}$ on C, so we can make an elementary transformation of $V = \mathcal{O}_X^{r+1}$ to create a vector bundle $F = F(C, A)$ on X, of rank $r+1$, sitting in the exact sequence $0 \longrightarrow F \longrightarrow \mathcal{O}_X^{r+1} \longrightarrow \mathcal{A} \longrightarrow 0$. Set $E = F^*$. Prove that $H^1(E) = H^2(E) = 0$, and show that E is generated by its global sections.

(ii). Prove that the holomorphic Euler characteristic of $E \otimes E^*$ satisfies $\chi(E \otimes E^*) = 2h^0(E \otimes E^*) - h^1(E \otimes E^*) = 2 - 2\rho(A)$, where $\rho(A) = g(C) - h^0(A) \cdot h^1(A)$. [Use the Hirzebruch–Riemann–Roch formula $\chi(E \otimes E^*) = \int Td(X) \cdot ch(E \otimes E^*)$, together with the multiplicativity of the Chern character.]

(iii). Assume now that $\rho(A) < 0$. Then E has an endomorphism w which is not a multiple of the identity. Use this to construct a homomorphism $u : E \longrightarrow E$ which drops rank everywhere on X. [If λ is an eigenvalue of $w(x)$ for some $x \in X$, put $u = w - \lambda \cdot 1$.] Then consider the exact sequence $0 \longrightarrow \text{im } u \longrightarrow E \longrightarrow \text{coker } u \longrightarrow 0$. Using the fact that E is globally generated, show that in the Chow group $A_1(X) = Pic(X)$, $c_1(\text{im } u)$ and $c_1(\text{coker } u)$ are represented by non-zero effective curves. Then deduce that $|C|$ contains a reducible or multiple curve.

(iv). The previous step proves the Theorem when both A and $K_C \otimes A^*$ are globally generated. Show that the general case of the Theorem reduces to this one. $\square$

Exercise 3.19. (**Alternative Approach to Theorem 3.13**). We indicate here a proof of the existence statement in Theorem 3.13 which avoids explicit use of the sheaf-theoretic considerations appearing in the proof of Proposition 3.9. As in the statement of the Theorem, let $Z \subset X$ be a reduced finite scheme, say $Z = \{x_1, \ldots, x_r\}$. Let

$$f : Y = \text{Bl}_Z(X) \longrightarrow X$$

be the blowing-up of X along Z, and let $D \subset Y$ be the exceptional divisor. Thus $D = \sum_1^r D_i$, where $\mathbf{P}^1 \cong D_i \subset Y$ is the (-1)-curve lying over $x_i \in X$. The idea is that if the pair (E, s)

sought in (3.7) exists, then since $f^*(s) \in \Gamma(Y, f^*E)$ vanishes on D, the pull-back of (3.6) determines an extension $0 \longrightarrow \mathcal{O}_Y(D) \longrightarrow f^*E \longrightarrow f^*L(-D) \longrightarrow 0$ on Y. This suggests that we consider on Y extensions of $f^*L(-D)$ by $\mathcal{O}_Y(D)$. So fix $e \in \mathrm{Ext}^1(f^*L(-D), \mathcal{O}_Y(D))$ corresponding to the exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_Y(D) \longrightarrow V \longrightarrow f^*L(-D) \longrightarrow 0.$$

(i). Let $\mathbf{P}^1 \cong D_i \subset Y$ be one of the exceptional curves. Show that $V|D_i$ is an extension of $\mathcal{O}_{\mathbf{P}^1}(1)$ by $\mathcal{O}_{\mathbf{P}^1}(-1)$. Hence either $V|D_i \cong \mathcal{O}^2_{\mathbf{P}^1}$, or $V|D_i \cong \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$.

(ii). Prove that if $V|D_i \cong \mathcal{O}^2_{\mathbf{P}^1}$ for every i, then $V = f^*E$, where E is a rank two vector bundle on X with a section vanishing on Z. So in this case we are done. [See [OSS], I.2.2.6.]

(iii). Suppose that $V|D_i \cong \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ for some i. Then make an elementary transformation along the resulting map $\bar{\lambda} : V \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1)$ to define a new vector bundle V' on Y. Using the fact that the composition $\mathcal{O}_Y(D) \longrightarrow V \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1)$ is surjective, show that V' can be realized as an extension of $f^*L(-D)$ by $\mathcal{O}_Y(D-D_i)$ from which (*) is induced. Thus the given extension class e lies in the image of $\mathrm{Ext}^1(f^*L(-D), \mathcal{O}_Y(D-D_i)) \longrightarrow \mathrm{Ext}^1(f^*L(-D), \mathcal{O}_Y(D))$. (This is the analogue of Proposition 3.9.)

(iv). Recalling that $K_Y = f^*K_X + D$, reprove the existence statement in Theorem 3.13 using Serre duality for line bundles on Y. $\square$

§4. Reider's Theorem via Vector Bundles

In the previous section we constructed a vector bundle E encoding the failure of an adjoint linear series $|K_X + L|$ to be very ample or free. The next step is to study the geometry of E. Reider's basic tool to this end is Bogomolov's instability theorem. We start with a quick review of this fundamental result. Then we present the proof of Reider's theorem.

Bogomolov's Instability Theorem.

Let X be a smooth complex projective surface, and let E be a rank two vector bundle on X. It is always easy to construct very negative rank one subsheaves of X: in fact, if H is an ample divisor, then for all $n \gg 0$ there exist sheaf monomorphisms $\mathcal{O}_X(-nH) \hookrightarrow E$. The notion of instability refers to the exceptional situation in which E has an unusually positive subsheaf. Bogomolov's theorem is a numerical criterion for instability in terms of the Chern numbers of E.

We start with some formal definitions. Let $N(X)$ be the Néron-Severi vector space of X, i.e. the subspace of $H^2(X, \mathbf{R})$ generated by the classes of algebraic curves. The Hodge index theorem implies that the intersection form on $N(X)$ has type $(+, -, \ldots, -)$, so

$$C = \{\alpha \mid \alpha^2 > 0\} \subset N(X)$$

is a cone with two connected components. The *positive cone* $N(X)^+$ of X is by definition the component of C containing the classes of ample divisors.

Definition 4.1. Let E be a rank two bundle on X. One says that E is *Bogomolov unstable* if there exist a finite subscheme $Z \subset X$ (possibly empty), plus line bundles A and B on X sitting in an exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \otimes \mathcal{I}_Z \longrightarrow 0, \tag{4.1.1}$$

where

$$c_1(A) - c_1(B) \in N(X)^+. \tag{4.1.2}$$

Very concretely, (4.1.2) is equivalent to the conditions that $(A - B)^2 > 0$ and that $(A - B) \cdot H > 0$ for all ample divisors H. Note that the finite scheme Z in (4.1.1) is the scheme defined by the vanishing of the vector bundle map $A \longrightarrow E$. It is suggestive to think of (4.1.2) as meaning roughly speaking that "A is more positive than B".

Bogomolov's statement is the following:

Theorem 4.2. ([**Bog**] cf. also [**Reid**].) *Let E be a rank two vector bundle on a smooth projective surface X. If*

$$c_1(E)^2 - 4c_2(E) > 0, \tag{4.2.1}$$

then E is Bogomolov unstable.

We will outline a proof of Bogomolov's theorem from [FdB1] in §6. We refer to [Bog], [Gies] or [Mka] for other arguments, as well as the corresponding statement for bundles of higher rank. Shepherd-Barron [S-B] has studied the geometry of this result in positive characteristic, and Moriwaki [Mwk] has given some arithmetic analogues.

Exercise 4.3. If E is a rank two bundle sitting in the exact sequence (4.1.1), show that the Chern classes of E are given by:

$$c_1(E) = c_1(A) + c_1(B); \qquad c_2(E) = c_1(A) \cdot c_1(B) + \text{ length}(Z).$$

[For c_2, note that one can think of (4.1.1) as determining a map $\mathcal{O}_X \longrightarrow E \otimes A^*$ with zero-scheme Z. In particular, $c_2(E \otimes A^*) = \text{ length}(Z)$.] □

Exercise 4.4. Remark that if E is a rank two vector bundle on X, and N is a line bundle, then E is Bogomolov unstable if and only if $E \otimes N$ is. Prove that up to scalar multiples, $c_1^2 - 4c_2$ is the only weight two polynomial in the Chern classes of E which is invariant under twisting by line bundles. Thus the essential content of Theorem 4.2 is that there is *some* numerical criterion for instability. □

Proof of Reider's Theorem.

We now turn to the proof of Reider's theorem. We follow the approach of [BFS], which simplifies somewhat Reider's original presentation.* We limit ourselves to the first statement of Reider's result, leaving the second as an exercise for the reader.

Proof of Theorem 2.1.(i). Fix a point $x \in X$, and suppose that L is a nef line bundle on X, with $L \cdot L \geq 5$, such that $\mathcal{O}_X(K_X + L)$ has a base-point at x. By Corollary 3.14, there exists a rank two vector bundle E sitting in the exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \otimes \mathcal{I}_x \longrightarrow 0.$$

Note from (4.3) that $\det E = L$ and $c_2(E) = 1$, and hence

$$c_1(E)^2 - 4c_2(E) = L \cdot L - 4 > 0.$$

Therefore Bogomolov's theorem applies, and one has an exact sequence

$$(**) \qquad 0 \longrightarrow A \longrightarrow E \longrightarrow B \otimes \mathcal{I}_Z \longrightarrow 0,$$

where Z is some finite subscheme of X, and $c_1(A) - c_1(B) \in N(X)^+$.

The plan now is to play off (*) against (**). Taking determinants, we find in the first place that $A + B = L$, whence $A - B = 2A - L$. Hence by definition of the positive cone:

$$(4.5) \qquad \begin{gathered} (2A - L)^2 > 0 \\ (2A - L) \cdot H > 0 \quad \forall \;\text{ ample divisors } H. \end{gathered}$$

Denote by

$$\alpha : A \longrightarrow L \otimes \mathcal{I}_x$$

the composition $A \hookrightarrow E \longrightarrow L \otimes \mathcal{I}_x$ determined by (**) and (*). We claim to begin with that $\alpha \neq 0$. In fact, in view of (*) it is enough to show that $\mathrm{Hom}(A, \mathcal{O}_X) = 0$, and this follows from the nefness of L and the second equation in (4.5). Then α is given by multiplication by (the equation of) an effective divisor $D \subset X$, with

$$x \in D \qquad \text{and} \qquad A = L - D.$$

It remains to show that D satisfies the numerical conclusions of Theorem 2.1.

To this end, we collect various inequalities. First:

$$(4.6) \qquad (L - 2D) \cdot L > 0.$$

In fact, note that $L - 2D = 2A - L$. Since L is nef, and hence a limit of ample divisors, (4.5) implies that in any event $(L - 2D) \cdot L \geq 0$. But if $(L - 2D) \cdot L = 0$, then $(L - 2D)^2 < 0$ by Hodge Index, and this contradicts (4.5).

*We remark that the argument given in [L2] contains an error.

Next:

$$(L^2)(D^2) \leq (L \cdot D)^2 \tag{4.7}$$

$$(L - D) \cdot D \leq 1. \tag{4.8}$$

Indeed (4.7) is a consequence of Hodge Index. As for (4.8), computing from (**) via (4.3), one finds $c_2(E) = (L - D) \cdot D +$ length Z. But $c_2(E) = 1$ and length $Z \geq 0$, and this gives (4.8). Finally, we claim:

$$2D^2 < L \cdot D. \tag{4.9}$$

Here we argue in cases. If $D^2 > 0$, then $L \cdot D \neq 0$ by Hodge index. Moreover

$$2(L \cdot D)(D^2) \underset{(4.6)}{<} (L^2)(D^2) \underset{(4.7)}{\leq} (L \cdot D)^2,$$

and (4.9) follows. Next, say $D^2 = 0$. Then Hodge index rules out the possibility that $L \cdot D = 0$. Therefore $L \cdot D > 0$, so again (4.9) is verified. Finally, if $D^2 < 0$, then (4.9) is trivial since $L \cdot D \geq 0$.

Combining (4.8) and (4.9), one finds:

$$L \cdot D - 1 \leq D^2 < \frac{L \cdot D}{2}.$$

But this is only possible if

$$L \cdot D = 0 \ , \ D^2 = -1; \quad \text{or}$$
$$L \cdot D = 1 \ , \ D^2 = 0.$$

This completes the proof. □

Exercise 4.10. (**Mumford's Proof of Vanishing.**) Let X be a smooth projective surface, and let L be a nef line bundle on X such that $L^2 > 0$. Prove that then $H^1(X, \mathcal{O}_X(K_X + L)) = 0$. [If not, there exists a non-split extension of L by $\mathcal{O}_X$, to which one can apply Bogomolov's theorem. See [Reid] for details.] □

Exercise 4.11. (**Higher Order Embeddings.**) This exercise is concerned with the following result of Beltrametti and Sommese:

Theorem. ([BS]) *Let X be a smooth surface, let L a nef line bundle on X, and let d be a positive integer such that $L^2 > 4d$. Then either the restriction*

$$e_Z : \Gamma(X, \mathcal{O}_X(K_X + L)) \longrightarrow \Gamma(Z, \mathcal{O}_Z(K_X + L))$$

is surjective for every subscheme $Z \subset X$ of length d, or else there exists an effective divisor $D \subset X$ such that

$$(*) \qquad L \cdot D - d \leq D^2 < \frac{1}{2}(L \cdot D).$$

(i). Assume that there exists a finite subscheme $Z \subset X$ of length d such that e_Z fails to be surjective. By induction, one can assume that $e_{Z'}$ is surjective for every proper subscheme $Z' \subset Z$. Show that then there exists a rank two vector bundle E on X, with $\det E = L$, having a section $s \in \Gamma(X, E)$ such that $Z(s) = Z$.

(ii). Arguing as in the proof of Reider's theorem, construct a divisor D satisfying (*).

(iii). Let L be an ample line bundle on X such that $L^2 > 4d$ and $L \cdot C \geq 2d$ for every irreducible curve $C \subset X$. Show that then the restriction e_Z is surjective for every finite subscheme $Z \subset X$ of length $\leq d$. $\square$

Exercise 4.12. (**Gonality of Complete Intersection Curves.**) Reider-type methods can sometimes be used to study linear series on subvarieties of codimension ≥ 2 in an ambient manifold. To illustrate the approach, we consider here a very concrete question in classical curve theory: what is the least degree required to express a complete intersection curve $C \subset \mathbf{P}^r$ as a branched covering $C \longrightarrow \mathbf{P}^1$ of the Riemann sphere? The answer is given in the following

Theorem. *Let $C \subset \mathbf{P}^r$ be a smooth complete intersection of hypersurfaces of degrees $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{r-1}$. Let A be a base-point free line bundle on C, of degree d, with $h^0(C, A) \geq 2$. Then $d \geq (a_1 - 1) \cdot a_2 \cdot \ldots \cdot a_{r-1}$.*

The idea of the argument is this: let $S \supset C$ be a general complete intersection surface of type $(a_2, \ldots, a_{r-1})$. As in Exercise 3.18, one can associate to A a rank two vector bundle F on S. One finds that if $d < (a_1 - 1)a_2 \cdots a_{r-1}$, then F is Bogomolov unstable. It is easy to get a contradiction provided one knows that the destabilizing subsheaf is of the form $\mathcal{O}_S(k)$, but this doesn't seem to be guaranteed. To remedy this, instead of working on a surface we work on a complete intersection threefold $X \supset C$, whose Picard group is controlled by the Lefschetz theorems. Related results, proved by more classical methods, appear in [CL] and [Bas], and the Theorem also connects with some of the conjectures in [EGH]. Paoletti [Paol] has extended the techniques of this exercise to deal with certain non-complete intersection curves. He proves the striking result that under suitable numerical hypotheses, the gonality of a space curve $C \subset \mathbf{P}^3$ is governed by its *Seshadri constant*, which roughly speaking measures how positive the hyperplane bundle $\mathcal{O}_{P^3}(1)$ is in a neighborhood of C (see §5).

(i). Put $\gamma = a_3 a_4 \cdots a_{r-1}$, and let $X \supset C$ be a general complete intersection threefold of type $(a_3, \ldots, a_{r-1})$. [If $r = 3$ take $X = \mathbf{P}^3$ and $\gamma = 1$.] Let $f : Y \longrightarrow X$ be the blowing-up of C, and let $E \subset Y$ be the exceptional divisor, with $\pi : E \longrightarrow C$ the natural map. Consider on E the globally generated line bundle $\mathcal{A} = \pi^* A$. Choosing a base-point free pencil in $\Gamma(E, \mathcal{A})$, we define in the usual way a rank two vector bundle $\mathcal{F}$ on Y via the sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Y^2 \longrightarrow \mathcal{A} \longrightarrow 0$. Compute the Chern classes of $\mathcal{F}$.

(ii). Denote by H the pull-back to Y of the hyperplane divisor on X, and for $0 \leq \epsilon \in \mathbf{Q}$ consider the $\mathbf{Q}$-divisor $D_\epsilon = (a_2 + \epsilon)H - E$. Show that $D =_{\text{def}} D_0 = a_2 H - E$ is globally generated and that D_ϵ is ample if $\epsilon > 0$. Now assume that $d < (a_1 - 1)a_2\gamma$. Prove that then for $0 < \epsilon \ll 1$:

$$\text{(*)} \qquad \big(c_1(\mathcal{F})^2 - 4c_2(\mathcal{F})\big) \cdot D_\epsilon = (a_1 - \epsilon)a_1 a_2 \gamma - 4d > 0.$$

(iii). Fixing ϵ for which (*) holds, an extension by Miyaoka [Mka] of Bogomolov's instability theorem implies that there exists a rank one subsheaf $\mathcal{L} \subset \mathcal{F}$ such that $(2c_1(\mathcal{L}) - c_1(\mathcal{F})) \cdot D_\epsilon \cdot D > 0$. Show that one can assume that $\mathcal{L}$ is locally free, and that the vector bundle map $\mathcal{L} \longrightarrow \mathcal{F}$ drops rank (if at all) on a codimension two subset $Z \subset Y$. Prove that $\mathcal{L} = \mathcal{O}_Y(-tH - \mu E)$ for some integers $t, \mu \in \mathbf{Z}$. [Recall that $Pic(X) = \mathbf{Z}$ thanks to the Lefschetz Hyperplane Theorem.]

(iv). Now let $S \in |a_2 H - E| = |D|$ be a general divisor, so that S is isomorphic to a complete intersection surface of type $(a_2, \ldots, a_{r-1})$ through C. Setting $F = \mathcal{F}|S$, show that $c_2(F) = d$, and that the restriction to S of the subsheaf $\mathcal{L} \hookrightarrow \mathcal{F}$ gives rise to an exact sequence $0 \longrightarrow \mathcal{O}_S(-sH) \longrightarrow F \longrightarrow \mathcal{O}_S((s - a_1)H) \otimes I_W \longrightarrow 0$, where $s = t + \mu a_1$, and $W \subset S$ is some finite subscheme. Use instability to prove that $2s < a_1$. Then estimate $c_2(F)$ to deduce that $a_1 < s + 1$. But $s > 0$ since $h^0(S, F) = 0$, and this gives a contradiction.

(v). Prove that the inequality in the Theorem is the best possible, in the sense that for any integers $2 \leq a_1 \leq \cdots \leq a_{r-1}$, there exists a complete intersection curve C that carries a base-point free pencil of degree $(a_1 - 1) \cdot a_2 \cdot \ldots \cdot a_{r-1}$. $\square$

§5. Vanishing Theorems and Local Positivity

The vector bundle methods described above seem essentially limited to surfaces. In higher dimensions, vanishing theorems are the only tools that have so far achieved significant success, and they will be the focus of the rest of these lectures. In this section we discuss the questions that arise if one tries to use vanishing in the most naive way to produce pluricanonical or adjoint divisors. In a word, one is led to study the "local positivity" of ample line bundles. We will see that one cannot hope to recover completely the known results (e.g. Corollary 2.7) in this fashion. However it turns out somewhat surprisingly that there are bounds on local positivity which apply at a *generic* point of a smooth surface, and these give results which in some respects go beyond Reider's theorem. While we try to explain in detail how vanishing theorems come into the picture, we content ourselves with just a sketch of the statements on local positivity.

Let X be a smooth projective surface, and let L be an ample (or nef and big) line bundle on X. We remarked in Proposition 1.2 that $\mathcal{O}_X(K_X + L)$ is free at a point $x \in X$ if and only if $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x) = 0$. Standard vanishing theorems can't directly apply here because the sheaf in question isn't locally free. The traditional first step around this problem is to blow up at x, which at least reduces the question to one involving only invertible sheaves.

So fix a point $x \in X$, let
$$f : Y = Bl_x(X) \longrightarrow X$$
be the blowing up of X at x, and denote by $E \subset Y$ the exceptional divisor. The first point to note is:

Lemma 5.1. *Let L be a line bundle on the smooth surface X, and let $r > 0$ be any positive integer. Then for all $i \geq 0$ there are isomorphisms*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x^r) = H^i(Y, \mathcal{O}_Y(K_Y + f^*L - (r+1)E)).$$

Proof. Observe first that $f_*(\mathcal{O}_Y(-rE)) = \mathcal{I}_x^r$, whereas $R^j f_*(\mathcal{O}_Y(-rE)) = 0$ for $j > 0$. In fact, via the inclusion $f_*(\mathcal{O}_Y(-rE)) \subseteq f_*(\mathcal{O}_Y) = \mathcal{O}_X$, we may identify the stalk of $f_*(\mathcal{O}_Y(-rE))$ at x as consisting of germs of functions on X whose pull-backs to Y vanish to order $\geq r$ along the exceptional divisor, which yields the first assertion. The second may be proven inductively by taking direct images of the sequence $0 \longrightarrow \mathcal{O}_Y(-rE) \longrightarrow \mathcal{O}_Y(-(r-1)E) \longrightarrow O_E(-(r-1)E) \longrightarrow 0$. (Compare [H2], V.3.4.) Now recall that $K_Y = f^*K_X + E$. Using the Leray spectral sequence and the projection formula, one finds:

$$\begin{aligned} H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x^r) &= H^i(Y, f^*(O_X(K_X + L)) \otimes \mathcal{O}_Y(-rE)) \\ &= H^i(Y, \mathcal{O}_Y(K_Y + f^*L - (r+1)E)), \end{aligned}$$

as claimed. $\square$

Recall next the statement of the basic:

Vanishing Theorem 5.2. *Let V be a smooth projective variety. If L is any ample line bundle on V, then*
$$H^i(V, \mathcal{O}_V(K_V + L)) = 0 \quad \text{for all } i > 0.$$
More generally, the same statement holds assuming only that L is nef and big. $\square$

The first assertion is of course the classical Kodaira vanishing theorem. The fact that it is enough that L be nef and big was proven by Kawamata [K1] and Viehweg [V]. As we will see, it is very convenient in applications only to have to check this weaker condition. We refer to [SS], Chapter VII, or [Kol1] for nice introductions to this extension of Kodaira vanishing.

The plan is to try to apply vanishing to the line bundle $f^*L - (r+1)E$ on Y. This requires knowing something about its positivity. Demailly [De2] has introduced a useful accounting mechanism for keeping track of what one needs:

Definition 5.3. With notation as above, the *Seshadri constant* of a nef line bundle L at x is the real number

$$\epsilon(L,x) = \sup\{\epsilon \geq 0 \mid f^*L - \epsilon \cdot E \text{ is nef}\}.$$

Here $f^*L - \epsilon E$ is considered as an $\mathbf{R}$-divisor on Y, and to say that it is nef means simply that $f^*L \cdot C' \geq \epsilon(E \cdot C')$ for all irreducible curves $C' \subset Y$. Needless to say, one can make the analogous definition on a smooth projective variety of any dimension.

The connection with Seshadri's criterion for ampleness (cf. [H1], Chapt. 1) occurs via:

Lemma 5.4. *One has*

$$\epsilon(L,x) = \inf_{C \ni x} \left\{ \frac{L \cdot C}{\mathrm{mult}_x(C)} \right\},$$

where the infimum is taken over all reduced and irreducible curves $C \subset X$ passing through x.

Proof. In fact, let $C \subset X$ be a reduced and irreducible curve with multiplicity m at x, and let $C' \subset Y$ denote the proper transform of C. Then $C' \cdot E = m$, and consequently for any ϵ:

$$(f^*L - \epsilon E) \cdot C' = L \cdot C - \epsilon m.$$

Hence if $f^*L - \epsilon E$ is nef, then $\epsilon \leq \frac{L \cdot C}{m}$. The reverse inequality is similar. $\square$

It is suggestive to think of the Seshadri constant $\epsilon(L,x)$ as measuring how positive L is locally near x. We present two exercises that convey this point, and refer to [De2], §6, for other interpretations.

Exercise 5.5. (i). Show that if L is a *very* ample line bundle on the surface X, then $\epsilon(L,x) \geq 1$ for all points $x \in X$.

(ii). Prove that the inequality in (i) holds assuming only that L is ample and globally generated.

(iii). Show that for any ample line bundle L on X, there exists a positive constant $\epsilon = \epsilon(L) > 0$ such that $\epsilon(L,x) \geq \epsilon$ for all $x \in X$. (This is the elementary half of Seshadri's criterion for ampleness.) $\square$

For the next exercise, and subsequent discussion, we need a definition. Given a line bundle B on X, and an integer $s \geq 0$, we say that the linear series $|B|$ *generates s-jets* at x if the natural map

$$H^0(X,B) \longrightarrow H^0(X, B \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is surjective. In other words, we ask that we be able to find a global section of B with arbitrarily prescribed s-jet at x. For instance $|B|$ generates 0-jets at every point of X if and only if it is free, and $|B|$ generates all 1-jets if and only if the differential $d\phi_{|B|}$ of the corresponding map $\phi_{|B|} : X \longrightarrow \mathbf{P}$ is everywhere injective.

Exercise 5.6. ([De2], Theorem 6.4). Given a line bundle B on X, let $s(B,x)$ be the largest integer such that the global sections in $H^0(X,B)$ generate s-jets at $x \in X$. Returning to an ample line bundle L on X, put

$$\sigma(L,x) = \limsup_{k\to\infty} \frac{1}{k} s(kL,x).$$

Show that $\epsilon(L,x) = \sigma(L,x)$ for every $x \in X$. [Compare Proposition 5.10 below.] $\square$

We now have:

Proposition 5.7. *Let L be an ample line bundle on the smooth surface X, and let $x \in X$ be a fixed point. If $\epsilon(L,x) > 2$, then $\mathcal{O}_X(K_X + L)$ is free at x. More generally, if $\epsilon(L,x) > s+2$ for some integer $s \geq 0$, then the linear series $|K_X + L|$ generates s-jets at x. The same statement holds if $\epsilon(L,x) = s+2$ provided that $L^2 > (s+2)^2$.*

Proof. To prove that $|K_X + L|$ generates s-jets at x, it is sufficient to establish the vanishing

$$(*) \qquad H^1(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{I}_x^{s+1}) = 0.$$

As before, let $f : Y \longrightarrow X$ be the blowing up of X at x, and let $E \subset Y$ be the exceptional divisor. Setting $\epsilon = \epsilon(L,x)$, one has on Y the numerical equivalence of $\mathbf{R}$-divisor classes:

$$f^*L - (s+2)E \equiv \frac{s+2}{\epsilon}(f^*L - \epsilon E) + (1 - \frac{s+2}{\epsilon})f^*L.$$

The first term on the right is nef, and the second is nef and big since $\epsilon > s+2$. Hence the vanishing theorem (5.2) applies to $f^*L - (s+2)E$, and (*) then follows from Lemma 5.1. If $\epsilon = s+2$, then $f^*L - (s+2)E$ is nef, and the inequality in the statement of the Proposition implies that it is big. $\square$

Corollary 5.8. *Let A be an ample line bundle on the surface X such that $\epsilon(A,x) \geq 1$ at some point $x \in X$. Then for all $s \geq 0$ the linear series $|K_X + (s+3)A|$ generates s-jets at x.* $\square$

Exercise 5.9. Show that if $\epsilon(L,x) > 4$ for all $x \in X$, then $\mathcal{O}_X(K_X + L)$ is very ample. State and prove the analogues of (5.1), (5.7) and (5.8) on a smooth projective variety V of arbitrary dimension. $\square$

As one might expect, it can happen that $\mathcal{O}_X(K_X + L)$ is free at x without this being accounted for by the bound $\epsilon(L,x) \geq 2$. (See Proposition 5.12 below.) However by a small variant of [De2], Theorem 6.4, statements such as (5.8) for jets of arbitrarily high order are *equivalent* to inequalities on the Seshadri constant:

Proposition 5.10. *Let A be an ample line bundle on X, $x \in X$ a fixed point, and $\epsilon > 0$ a real number. Suppose that for all $s \gg 0$, the linear series $|K_X + rA|$ generates s-jets as soon as $r > (s+2)/\epsilon$. Then $\epsilon(A, x) \geq \epsilon$.*

Proof. Let $C \ni x$ be a reduced and irreducible curve, with $\mathrm{mult}_x(C) = m$. Fix $s \gg 0$, and let $r = r(s)$ be the least integer $> (s+2)/\epsilon$. Then $|K_X + rA|$ generates s-jets, and consequently we can find a curve $D = D_s \in |K_X + rA|$, with $\mathrm{mult}_x(D) = s$, having a prescribed tangent cone at x. In particular, we can choose D so that the tangent cones to C and D at x meet properly in T_xX, and since C is irreducible it follows that C and D themselves meet properly. Therefore

$$C \cdot (K_X + rA) \geq \ \mathrm{mult}_x(C) \cdot \ \mathrm{mult}_x(D) = ms,$$

whence

$$\frac{C \cdot A}{m} \geq \frac{s}{r} - \frac{C \cdot K_X}{rm}.$$

But we can assume that $r \leq (\frac{s+2}{\epsilon}) + 1$, and the result follows from Lemma 5.4 upon letting $s \to \infty$. $\square$

So far this discussion has been quite formal. It remains to say something about the actual behavior of the Seshadri constants $\epsilon(L, x)$. As measures of local positivity, these are in any event very interesting invariants, quite apart from the potential application to adjoint series. We may summarize the story on surfaces in the following two statements:

Theorem 5.11. [EL3]. *Let L be an ample line bundle on a smooth projective surface X. Then $\epsilon(L, x) \geq 1$ for all except perhaps countably many points $x \in X$, and moreover if $L^2 > 1$ then the set of exceptional points is finite. If $L^2 \geq 5$ and $L \cdot C \geq 2$ for all curves $C \subset X$, then $\epsilon(L, x) \geq 2$ for all but finitely many $x \in X$.*

Proposition 5.12. (**Miranda**). *Given $\epsilon > 0$, there exists a surface X, a point $x \in X$, and an ample line bundle L on X such that $\epsilon(L, x) \leq \epsilon$.*

It follows for instance from (5.7) and the Theorem that if $L^2 \geq 5$ and $L \cdot C \geq 2$ for all $C \subset X$ then $\mathcal{O}_X(K_X + L)$ is free off a finite set. But of course we know from Reider's theorem (Corollary 2.6) that in fact $\mathcal{O}_X(K_X + L)$ is *everywhere* globally generated. So from this point of view, one may think of Seshadri constants as giving local Reider-type results, which however are valid only at a general point of the surface X. On the other hand, the statements coming from (5.7) and (5.11) for higher order jets are necessarily stronger than the uniform results deduced from Reider's method. (See Exercise 5.14 for a summary.)

We start by outlining Miranda's construction of examples of small Seshadri constants.

Proof of Proposition 5.12. Let $D \subset \mathbf{P}^2$ be an irreducible plane curve of degree $d \gg 0$ with a point $x \in D$ of multiplicity m. Let D' be a second irreducible curve of degree d, meeting D transversely. Choosing D' generally, we may suppose that all the curves in the pencil spanned by D and D' are irreducible. Blow up the base-points of this pencil to obtain a surface X, admitting a map $\pi : X \longrightarrow \mathbf{P}^1$ with irreducible fibres, among them $D \subset X$. Observe that π has a section $S \subset X$ (viz. an exceptional curve of the blowing up $X \longrightarrow \mathbf{P}^2$) meeting D transversely at one point. Fix an integer $a \geq 2$. It follows from the Nakai criterion that the divisor $L = aD + S$ on X is ample. But $L \cdot D = 1$ whereas $\text{mult}_x(D) = m$, so $\epsilon(L, x) \leq \frac{1}{m}$. Note that by taking suitable a we can even make L^2 arbitrary large, and by taking L to be a multiple of $aD + S$ we can arrange that the intersection numbers $L \cdot C$ of L with irreducible curves $C \subset X$ be bounded below by any preassigned integer. $\square$

Finally, we sketch without full details the main idea of the proof of Theorem 5.11. The argument is very elementary, but is essentially limited to surfaces.

Idea of proof of Theorem 5.11. We focus on the first statement, and we use the characterization (5.4) of Seshadri constants. The main point, which was inspired by [Xu], is to view the question variationally. In fact, the set

$$\{(C, x) \mid C \subseteq X \text{ is a reduced, irreducible curve with } \text{mult}_x(C) > L \cdot C\}$$

consists of at most countably many algebraic families. The first statement of the theorem will follow if we show that each of these families is discrete, i.e. that pairs (C, x) forcing $\epsilon(L, x) < 1$ are rigid.

Suppose to the contrary that (C_t, x_t) is a non-trivial one-parameter family of reduced and irreducible curves $C_t \subseteq X$ and points $x_t \in C_t$ with $\text{mult}_{x_t}(C_t) > L \cdot C_t$. Let $C = C_{t^*}$ and $x = x_{t^*}$ for general t^*, and set $m = \text{mult}_{x_{t^*}}(C_{t^*})$. A local computation involving deformation theory of singular curves shows that

$$C^2 \geq m(m-1).$$

(In brief, the given deformation of C determines a section $\rho(\frac{d}{dt}) \in H^0(\mathcal{O}_C(C))$, and one shows that since the deformation preserves the m-fold point of C, $\rho(\frac{d}{dt})$ vanishes to order $\geq m-1$ at x.) But $L^2 \cdot C^2 \leq (L \cdot C)^2$ by the Hodge index theorem, and since $L \cdot C \leq m - 1$ by assumption, we find that

$$m(m-1) \leq L^2 \cdot C^2 \leq (C)^2 \leq (m-1)^2.$$

But this is a contradiction when $m > 1$, and the first statement follows. $\square$

Remark 5.13. Theorem 5.11 has recently been extended to varieties of arbitrary dimension, although the numerical bound obtained is somewhat weaker. In fact, it is shown in [EKL] that if L is an ample line bundle on a smooth projective variety of dimension n, then $\epsilon(L, x) \geq \frac{1}{n}$ for all $x \in X$ outside a countable union of proper subvarieties. The idea is to choose a divisor $D = D_x \in |kL|$ $(k \gg 0)$ with large multiplicity at x, and to study in effect the higher order deformation theory of the pair (D_x, C_x), where $C_x \subset X$ is a Seshadri-exceptional curve at x. The argument is inspired on the one hand by some techniques that come up in the theory of diophantine approximation, and on the other hand by the methods used to prove boundedness of Fano varieties of given dimension.

Exercise 5.14. (**Higher Jets and Adjoint Series.**) It is interesting to summarize the statements on separation of jets that come out of these discussions. Let A be an ample line bundle on the smooth projectve surface X. Prove the following:

(i). The adjoint series $|K_X + (s+3)A|$ generates s-jets at a sufficiently general point $x \in X$.

(ii). There cannot exist a linear function $f(s)$ such that for all X, A and $s \gg 0$, $|K_X + f(s)A|$ generates s-jets at *every* point $x \in X$.

(iii). There exists a *quadratic* function $f(s)$ (independent of X and A) such that for $s \gg 0$, $|K_X + f(s)A|$ generates s-jets at every point $x \in X$. In fact, for $s \geq 1$ one can take $f(s) = (s+1)(s+2)$. [Use Exercise 4.11.(iii).] $\square$

Exercise 5.15. (**Normal Generation of "Hyper-adjoint" series,** [BEL],§3, [ABS].) Let V be a smooth projective variety of dimension n, and let B be a *very ample* line bundle on V.

(i). Prove that $K_V + (n+1)B$ is globally generated, and that $K_V + (n+2)B$ is very ample. [For the first statement, consider a general divisor $W \in |B|$ and argue by induction on n.]

(ii). Show that $\mathcal{O}_V(K_V + (n+1)B)$ is normally generated, i.e. that the homomorphisms

$$Sym^m(H^0(\mathcal{O}_V(K_V + (n+1)B))) \longrightarrow H^0(\mathcal{O}_V(m(K_V + (n+1)B)))$$

are surjective for $m \geq 0$. [The case $m = 2$ is the essential one. Put $N = \mathcal{O}_V(K_V + (n+1)B)$, and consider on $V \times V$ the line bundle $F = pr_1^*(N) \otimes pr_2^*(N)$. It suffices to prove that $H^1(V \times V, F \otimes \mathcal{I}_\Delta) = 0$, where $\Delta \subset V \times V$ is the diagonal. Using the hypothesis that B is very ample, show that $pr_1^*(B) \otimes pr_2^*(B) \otimes \mathcal{I}_\Delta$ is globally generated. Now apply vanishing on the blow-up of $V \times V$ along Δ.] See [EL1] for some generalizations involving defining equations and higher syzygies, and [BEL] for some similar elementary applications of vanishing theorems to study the equations defining projective varieties. $\square$

Exercise 5.16. (**Seshadri Constants Along Finite Sets**.) This exercise gives an outline of some unpublished work of **Geng Xu**, and we thank him for his permission to include it. Let X be a smooth surface, and let $Z \subset X$ be a finite set consisting of r points, say $Z = \{x_1, \ldots, x_r\}$. Given an effective divisor $D \subset X$, define $\mathrm{mult}_Z(D) = \sum \mathrm{mult}_{x_i}(D)$. If L is a nef line bundle on X, we define the Seshadri constant of L along Z to be the real number

$$\epsilon(L, Z) = \inf_{C \subset X} \left\{ \frac{L \cdot C}{\mathrm{mult}_Z(C)} \right\},$$

the infimum being taken over all reduced and irreducible curves $C \subset X$ (not necessarily passing through Z).

(i). Assume that L is nef and $L^2 > r$. Show that if Z consists of r sufficiently general points, then $\epsilon(L, Z) \geq 1$. [If not, let $C \subset X$ be a Seshadri exceptional curve which deforms with all local deformations of the x_i. Put $m_i = \mathrm{mult}_{x_i}(C) \geq 0$. We may assume that C passes through x_1, so $m_1 \geq 1$, and by hypothesis $L \cdot C \leq (m_1 - 1) + m_2 + \cdots + m_r$. As C moves in an algebraic family generically covering X, $L \cdot C \geq 1$. Consider the deformation obtained by letting x_1 move and fixing $x_2, \ldots, x_r$. Since C is reduced and irreducible, one finds as in [EL3], §1, the inequality:

$$C^2 \geq (m_1 - 1)m_1 + m_2^2 + \cdots + m_r^2 \geq (m_1 - 1)^2 + m_2^2 + \cdots + m_r^2.$$

Hodge index and the hypothesis $L^2 \geq r + 1$ then gives

$$(r+1)\left((m_1 - 1)^2 + m_2^2 + \cdots + m_r^2\right) \leq (m_1 - 1 + m_2 + \cdots + m_r)^2,$$

and by minimizing the the difference of the two sides one sees that this is impossible.]

(ii). In the situation of (i), suppose that $D \subset X$ is any effective divisor (possibly non-reduced or reducible). Show that then $\mathrm{mult}_Z(D) \leq L \cdot D$.

See Exercise 7.8 for an interesting application of this result. □

§6. Adjoint Series and Bogomolov Instability via Vanishing.

We saw in the previous section that one can't hope to prove known and expected results on linear series using only the most naive application of vanishing theorems for line bundles. Our purpose here is to show how more subtle vanishing theorems, for **Q**-divisors, do lead to (parts of) Reider's theorem. This argument appears in [EL2], §1, and follows the approach pioneered by Kawamata, Reid, Shokurov *et al.* in connection with the minimal model program. Many of the complexities of the general KRS machine disappear on surfaces, and the ideas become particularly transparent. We hope that the present discussion can serve as a low-key introduction to this important and powerful tool. Completing this circle of ideas, Fernández del Busto [FdB1] has shown that one can use the approach of [EL2] to give a new proof of Bogomolov's instability theorem. We sketch the argument at the end of the section, and in Exercises 6.23 and 6.24.

By way of motivation, let L be an ample line bundle on a smooth projective surface X, and consider the problem of constructing a section of $\mathcal{O}_X(K_X + L)$ which is non-vanishing at some point $x \in X$. The "classical" approach might be to prove something along the following lines:

Exercise 6.1. Assume that $L^2 \geq 5$, and suppose there exists a reduced irreducible divisor

$$D \in |L| \;\text{ such that } q = \; \mathrm{mult}_x(D) \geq 2.$$

Then $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x) = 0$, and consequently $\mathcal{O}_X(K_X + L)$ is free at x. [Let $f : Y \longrightarrow X$ be the blowing up of X at x, with $E \subset X$ the exceptional divisor. Let $D' \subset Y$ be the proper transform of D, so that $f^*D - 2E \equiv D' + (q-2)E$ is effective and has positive self-intersection. Prove that $f^*D - 2E$ is numerically 1-connected, and deduce from Ramanujam vanishing ([BPV], IV.8.2) that $H^1(Y, \mathcal{O}_Y(K_Y + f^*L - 2E)) = 0$. Then use Lemma 5.1 to conclude.] $\square$

The essential drawback to this statement is that in the situations of interest there is no reason even to suppose that $|L|$ is non-empty. The advantage of the KRS method is that it lets one make an *asymptotic* construction. Specifically, we will take $k \gg 0$ and work with a divisor

$$D \in |kL| \ \text{ such that } \mathrm{mult}_x(D) > 2k.$$

There is no problem in producing D if $L^2 \geq 5$ and k is sufficiently large. Then one will want to "divide D by k", and this is where **Q**-divisors come into the picture.

Vanishing Theorems for Q-Divisors.

We start with some notation and definitions. Let V be a smooth projective variety of dimension n. A **Q**-*divisor* on V is simply a **Q**-linear combination of prime divisors:

$$M = \sum a_i D_i \quad (a_i \in \mathbf{Q}).$$

The multiplicity $\mathrm{mult}_x M$ of M at a point $x \in X$ is taken to be $\sum a_i \mathrm{mult}_x D_i$. Assuming that the D_i are distinct, we define the *round-up* of M to be the integral divisor $\ulcorner M \urcorner = \sum \ulcorner a_i \urcorner D_i$, where $\ulcorner a_i \urcorner$ denotes the least integer $\geq a_i$. The *integer part*, or *round-down* $[M]$ is defined similarly, and the *fractional part* $\{M\}$ of M is $\{M\} = M - [M]$. There is a **Q**-valued intersection theory involving **Q**-divisors, defined in the evident way by first clearing denominators, and one has the usual functorial operations such as pull-backs under morphisms. This gives rise to the notion of numerical equivalence of **Q**-divisors, which we continue to denote by $\equiv$. We say that M is *nef* if $M \cdot C \geq 0$ for all irreducible curves $C \subset V$, and M is *ample* if the statement of Nakai's criterion holds. Equivalently, M is nef or ample if mM is so, for some positive integer $m > 0$ such that mM is an integral divisor. If M is nef, it is in addition *big* if $(M^n) > 0$.

The basic result is:

Kawamata-Viehweg Vanishing Theorem 6.2. *Let M be a nef and big* **Q**-*divisor on the smooth projective variety V. Assume that the fractional part $\{M\}$ of M is supported on a divisor with global normal crossings. Then*

$$H^i(V, \mathcal{O}_V(K_V + \ulcorner M \urcorner)) = 0 \quad \textit{for} \quad i > 0.$$

In other words, the Theorem gives a vanishing for (integer) divisors of the form:

$$K_V + (\text{ nef and big } \mathbf{Q}\text{-divisor }) + \Delta,$$

where Δ is an effective fractional divisor (i.e. $[\Delta] = 0$) with normal crossing support. The statement may appear non-intuitive at first blush, but we will see momentarily that this is precisely the tool needed to generalize the argument sketched in Exercise 6.1. We won't prove (6.2) in these notes. The original arguments of [K1] and [V] used covering constructions to deduce the result from vanishing for integer divisors. The normal crossing hypothesis is used to contol the singularities introduced upon passing to a covering. A number of direct proofs have since been given, one based on connections with logarithmic singularities [EV1], another on Hodge theory for twisted coefficient systems [Kol1], and a third involving singular metrics on line bundles [De2]. We refer to [Kol1], [CKM], Chapter 8, [Kol3], Chapters 9 - 10, and [EV2] for good discussions. It is worth emphasizing that Theorem 6.2 is by now not much harder to prove than the classical Kodaira vanishing theorem.

Remark 6.3. Since rounding does not in general respect linear equivalence, it is essential that the fractional part of the $\mathbf{Q}$-divisor M appearing in (6.2) be defined as an actual divisor, and not merely as an element in $Pic(X) \otimes \mathbf{Q}$. However we will often identify two $\mathbf{Q}$-divisors if their fractional parts coincide and their integer parts are linearly equivalent. By the same token, we will deal with "hybrid" objects of the form $L + D$ where L is a line bundle (defined up to isomorphism) and D is a $\mathbf{Q}$-divisor.

In the hope of conveying right away some feeling for how Theorem 6.2 is used, we prove a criterion extending Exercise 6.1. It asserts roughly that the existence of a divisor in $|kL|$ with an "almost isolated" singularity of high multiplicity gives rise to a non-vanishing section of $\mathcal{O}_X(K_X + L)$. This result will play an important role in our proof of Reider's theorem. In the argument below, we will make use of the fact — to be established in Exercise 6.6 — that on a surface, one can ignore the normal crossing hypothesis in (6.2).

Proposition 6.4. *Let L be a nef and big line bundle on a smooth projective surface X. Fix a point $x \in X$, and an integer $s \geq 0$. Suppose that for some $k > 0$ there exists a divisor*

$$D \in |kL| \quad \textit{with} \quad q =_{\text{def}} \textit{mult}_x(D) > (s+2)k,$$

plus an open neighborhood $U \ni x$ of x in X such that

$$\textit{mult}_y(D) < q/(s+2) \quad \textit{for all} \quad y \in U - \{x\}.$$

Then

$$H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x^{s+1}) = 0,$$

i.e. $|K_X + L|$ generates s-jets at x.

Proof. Write $D = \sum d_i D_i$, where the D_i are distinct prime divisors. The upper bound on $\text{mult}_y(D)$ implies that if D_i passes through x, then $d_i < q/(s+2)$. For simplicity we assume for the time being that every component of D passes through x; the changes necessary in general will be sketched at the end of the proof. Consider as before the blowing-up $f : Y \longrightarrow X$ of X at x. Let $E \subset Y$ be the exceptional divisor, and denote by $D_i' \subset Y$ the proper transform of D_i. Thus $f^*D = qE + \sum d_i D_i'$. The idea of the proof is to study the $\mathbf{Q}$-divisor

$$(*) \qquad M = f^*L - \left(\frac{s+2}{q}\right) f^*D = f^*L - (s+2)E - \sum \left(\frac{s+2}{q}\right) d_i D_i'.$$

The first point to note is that by hypothesis

$$\left(\frac{s+2}{q}\right) d_i < 1 \ \text{ for all } i,$$

and therefore

$$K_Y + \ulcorner M \urcorner = K_Y + f^*L - (s+2)E.$$

On the other hand, one has the numerical equivalence:

$$M \equiv f^*L - \left(\frac{s+2}{q}\right) f^*D \equiv \left(1 - \left(\frac{s+2}{q}\right) k\right) f^*L,$$

and as $q > k(s+2)$, it follows that M is nef and big. Since we are on a surface we don't need to worry about normal crossings, and therefore Theorem 6.2 gives $H^1(Y, \mathcal{O}_Y(K_Y + f^*L - (s+2)E)) = 0$. We conclude with Lemma 5.1.

It remains to treat the possibility that not all of the components of D pass through x. In this case, we only know that $d_i < q/(s+2)$ for those i such that $D_i \ni x$. Defining M as in (*), it follows that $K_Y + \ulcorner M \urcorner = f^*L - (s+2)E - N'$, where $N' \subset Y$ is an effective (or zero) integral divisor whose support is disjoint from E. Let $N = f_*N'$ be the corresponding divisor on X, so that $x \notin \operatorname{supp} N$. Then as above vanishing gives $H^1(X, \mathcal{O}_X(K_X + L - N) \otimes \mathcal{I}_x^{s+1}) = 0$, i.e. we can find a section of $\mathcal{O}_X(K_X + L - N)$ with arbitrarily prescribed s-jet at x. But one has an inclusion $H^0(\mathcal{O}_X(K_X + L - N)) \subset H^0(\mathcal{O}_X(K_X + L))$, and since N doesn't pass through x, it follows that $|\mathcal{O}_X(K_X + L)|$ also generates s-jets at x, as required. $\square$

Remark 6.5. There is an analogous statement in higher dimensions:

(6.5.1). *Let V be a smooth projective variety of dimension n, and let L be an ample line bundle on V. Given a point $x \in V$, assume that there exists a divisor $D \in |kL|$ with*

$$\mathit{mult}_x D > (n+s)k$$

but $\mathit{mult}_y D \leq k$ for y in a punctured neighborhood of x. Then $|K_X + L|$ generates s-jets at x.

We learned of (6.5.1) from Siu. Esnault and Viehweg give a proof in [EV2], (7.5), (7.7). The argument is more involved than in the surface case, because in general one has to pass to an embedded resolution of D in order to apply Kawamata-Viehweg vanishing. Note also that in order to verify the upper bound on $\operatorname{mult}_y(D)$ in dimension three or higher, it is not enough to know simply that the components of D appear with low multiplicity.

As we have already remarked, it is a happy fact that one can ignore the normal crossing hypothesis in Theorem 6.2 when working on surfaces:

Exercise 6.6. (**Sakai's Lemma**, [Sak1], cf. [EL2], §1.) Let X be a smooth surface, and let M be any big and nef $\mathbf{Q}$-divisor on X. Then

$$H^i(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner)) = 0 \ \text{ for } i > 0.$$

[A simple if somewhat inefficient argument proceeds as follows. By a sucession of blowings up at points, one constructs a map $\phi : X_1 \longrightarrow X$ such that the fractional part $\{\phi^* M\}$ of $\phi^* M$ is supported on a normal crossing divisor, and hence $H^i(X_1, \mathcal{O}_{X_1}(K_{X_1} + \ulcorner \phi^* M \urcorner)) = 0$ for $i > 0$. Working step by step down from X_1, it is then enough to prove the following:

Let $f : Y \longrightarrow X$ be the blowing up of a smooth surface at a point $x \in X$, and let M be a $\mathbf{Q}$-divisor on X. If $H^i(Y, \mathcal{O}_Y(K_Y + \ulcorner f^* M \urcorner)) = 0$ for some $i > 0$, then $H^i(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner)) = 0$.

In fact, show that

$$K_Y + \ulcorner f^* M \urcorner = f^*(K_X + \ulcorner M \urcorner) - pE$$

for some $p \geq -1$. Then

$$H^i(Y, \mathcal{O}_Y(K_Y + \ulcorner f^* M \urcorner)) = H^i(X, \mathcal{O}_X(K_X + \ulcorner M \urcorner) \otimes \mathcal{I}_x^p)$$

for all i, where we make the convention that $\mathcal{I}_x^p = \mathcal{O}_X$ if $p = -1$. The assertion follows.] $\square$

Exercise 6.7. (**Algebro-Geometric Multiplier Ideals**, [EV2], (7.4), (7.5).) Let M be any nef and big $\mathbf{Q}$-divisor on a smooth projective variety V of dimension n. Prove that there exists an ideal sheaf $\mathcal{J}_M \subset \mathcal{O}_V$, with $\mathcal{O}_V / \mathcal{J}_M$ supported in codimension ≥ 2, such that

$$H^i(V, \mathcal{O}_V(K_V + \ulcorner M \urcorner) \otimes \mathcal{J}_M) = 0 \quad \text{for } i > 0. \tag{6.7.1}$$

[Let $f : W \longrightarrow V$ be an embedded resolution of $\operatorname{supp}\{M\}$. One can write

$$K_W + \ulcorner f^* M \urcorner = f^*(K_V + \ulcorner M \urcorner) + P - N,$$

where P and N are relatively prime effective f-exceptional divisors on W. Then set $\mathcal{J}_M = f_* \mathcal{O}_W(-N)$. For (6.7.1), show first that $f_* \mathcal{O}_P(P) = 0$. Then argue e.g. as in the proof of Grauert-Riemenschneider vanishing in [Kol1], Corollary 11, to prove that $R^i f_*(K_W + \ulcorner f^* M \urcorner) = 0$ for $i > 0$.] Esnault and Viehweg show that $\mathcal{J}_M$ is actually independent of the resolution chosen. This is the algebro-geometric analogue — and a special case of — the multiplier ideal sheaf associated to a line bundle with a singular metric (cf. [De2] and [Siu2]), and (6.7.1) is a special case of Nadel's vanishing theorem [Nad]. One thinks of $\mathcal{J}_M$ as measuring the singularities of the fractional part $\{M\}$ of M. Its co-support is contained in the locus of points at which $\operatorname{supp}\{M\}$ fails to be a normal crossing divisor. (Compare [Kol1], Theorem 19$'$.) $\square$

Exercise 6.8. (**Singularities of Plane Curves.**) In this exercise, $C \subset \mathbf{P}^2$ is a reduced plane curve of degree d.

(i). Let $\Sigma = \operatorname{Sing} C$ be the singular locus of C, considered as a reduced subscheme of $\mathbf{P}^2$. Use Kawamata-Viehweg vanishing to prove the classical theorem that Σ imposes independent conditions on curves of degrees $k \geq d - 2$, i.e.

$$H^1(\mathbf{P}^2, \mathcal{I}_\Sigma(k)) = 0 \quad \text{for } k \geq d - 2. \tag{*}$$

[Write $\Sigma = \{x_1, \ldots, x_m\}$, and consider the blowing-up $f : Y \longrightarrow \mathbf{P}^2$ of $\mathbf{P}^2$ along Σ, with $E = \sum E_i$ the exceptional divisor, and H the pull-back to Y of the hyperplane divisor on $\mathbf{P}^2$. Let $C' \subset Y$ be the proper transform of C, and put $q_i = \operatorname{mult}_{x_i}(C)$, so that $f^*C = C' + \sum q_i E_i$. If $q_i \geq 3$ for all i, then one can argue much as in the proof of Proposition 6.4. In case some $q_i = 2$ one has to make small perturbations of the divisors in question, as follows. Fix first a small rational number $0 < \epsilon \ll 1$ such that $H - \epsilon E$ is ample. Then choose $a < 1$ such that $a > 2/(q_i + \epsilon)$ for all i, and consider the $\mathbf{Q}$-divisor $M = (d+1)H - aC' - \sum a(q_i + \epsilon)E_i$. The numerical equivalence

$$M \equiv (d+1)(1-a)H + a(H - \sum \epsilon E_i) + a(dH - C' - \sum q_i E_i)$$

shows that M is big and nef (in fact ample). Now apply (6.2).] Observe that this argument shows that one can replace the ideal $\mathcal{I}_\Sigma$ in (*) by $\mathcal{J} = \mathcal{I}_{x_1}^{q_1 - 1} \cdots \mathcal{I}_{x_m}^{q_m - 1}$.

(ii). Suppose now that C has a certain number of simple cusps (i.e. $z^2 = w^3$ in local analytic coordinates): let $\Delta = \{x_1, \ldots, x_\kappa\}$ be the set of such. Prove the theorem of Zariski [Z], §6, that

$$H^1(\mathbf{P}^2, \mathcal{I}_\Delta(k)) = 0 \quad \text{for } k > \frac{5}{6}d - 3.$$

It follows for example that a septic curve can have no more than ten simple cusps. [Work on a blowing-up $f : Y \longrightarrow \mathbf{P}^2$ which is an embedded resolution of C over the cusps.] It is interesting to note that Zariski proves this result by studying the irregularity of cyclic coverings of $\mathbf{P}^2$ branched along C. One can see the approach to Kawamata-Viehweg vanishing via covering constructions as a vast generalization of this idea. Building in part on work of Esnault [Es], Sakai [Sak3] has applied these and other techniques to study the singularities of plane curves. $\square$

Exercise 6.9. (**Variant of Proposition 6.4.**) Let L be an ample line bundle on a smooth projective surface X, and suppose that $Z \subset X$ is a finite set. Assume given a divisor $D \in |kL|$ for some $k > 0$, say $D = \sum d_i D_i$, plus integers $s \geq 0$ and $q > (s+2)k$, such that $\operatorname{mult}_x(D) \geq q$ for all $x \in Z$, and with $(s+2)d_i \leq q$ whenever D_i meets Z. Prove that then $H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z^{s+1}) = 0$ for $i > 0$. [If $(s+2)d_i < q$, proceed as in the proof of Proposition 6.4. In general, as in Exercise 6.8.(i), one can introduce small perturbations, as follows. Let $f : Y \longrightarrow X$ be the blowing-up of Z, with exceptional divisor E, and fix $\epsilon > 0$ such that $f^*L - \epsilon E$ is ample. For rational numbers $0 < b \ll 1$ and $0 < 1 - a \ll 1$, consider

$$\begin{aligned} M &= f^*\left(L - a\left(\frac{s+2}{q}\right)D\right) - b\epsilon E \\ &\equiv \left((1-b) - \frac{a(s+2)}{q}k\right) f^*L + b\left(f^*L - \epsilon E\right). \end{aligned}$$

If $0 < b < 1 - k(s+2)/q$ and $1 - a < b\epsilon/(s+2)$, then M is ample, and every component of E has coefficient $\leq -(s+2)$ in M.] $\square$

Reider's Theorem Revisited.

Our next goal is to use Kawamata-Viehweg vanishing to give another proof of part of Reider's theorem. Specifically, we focus on the following statement:

Proposition 6.10. *Let X be a smooth projective surface, $x \in X$ a fixed point, and L a nef line bundle on X. Assume that $L^2 \geq 5$, and that*

$$L \cdot C \geq 2 \quad \text{for all curves } C \subset X \quad \text{such that } C \ni x.$$

Then $\mathcal{O}_X(K_X + L)$ has a section which doesn't vanish at x.

The plan is this: the hypothesis $L^2 \geq 5$ allows us to construct a divisor $D \in |kL|$ ($k \gg 0$) with high multiplicity at x. If D has small multiplicity at nearby points, then Proposition 6.4 applies. In the contrary case, D contains a component D_0 appearing with high multiplicity. The philosophy of the KRS method is that vanishing still gives a useful surjectivity statement. The problem is then reduced to producing a section on D_0, and this is attacked using the lower bound on $L \cdot D_0$. We remark that Sakai [Sak2], using some ideas of Serrano, has given a cohomological proof of Reider's theorem via Miyaoka's vanishing theorem for Zariski decompositions of linear series on surfaces. While the present approach has the advantage of using mainly general techniques from higher dimensional geometry, it would be interesting to understand more clearly than one does at the moment the precise connections between the two proofs. (One can see the construction below as the first step in producing the Zariski decomposition of the relevant linear series.)

Proof of Proposition 6.10. The first step is to show that for $k \gg 0$ there exists a divisor

$$D \in |kL| \quad \text{with} \quad q =_{\text{def}} \ \text{mult}_x D > 2k.$$

This is an elementary parameter count. In fact, consider the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{2k+1}) \longrightarrow H^0(\mathcal{O}_X(kL)) \longrightarrow H^0(\mathcal{O}_X(kL) \otimes \mathcal{O}_X/\mathcal{I}_x^{2k+1}).$$

We are looking for the divisor of a non-zero section in the group on the left. Since L is nef and $L^2 \geq 5$, a standard argument using Riemann-Roch (cf. [H2], V.1.8, or [SS], p. 146) shows that:

$$h^0(\mathcal{O}_X(kL)) = \frac{k^2 L^2}{2} + o(k^2) \geq \frac{5}{2} k^2 + o(k^2).$$

On the other hand,

$$h^0(\mathcal{O}_X(kL) \otimes \mathcal{O}_X/\mathcal{I}_x^{2k+1}) = \binom{2k+2}{2} = \frac{4k^2}{2} + o(k^2).$$

Hence $H^0(\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{2k+1}) \neq 0$ for $k \gg 0$, as required.

Fix such a divisor D, and write

$$D = \sum d_i D_i + F,$$

where $\{D_i\}$ are the components of D passing through x, and F is the effective (or zero) divisor consisting of those components of D disjoint from x. If $q > 2d_i$ for all i, then Proposition 6.4 applies, and we are done.

Assume next that $q < 2d_i$ for some i. Note that

$$q = \sum d_i \ \mathrm{mult}_x(D_i),$$

so in the first place $q \geq d_i$ for all i. It follows also that there is a *unique* component D_i — call it D_0 — of maximal mutiplicity $d_0 > q/2$. Furthermore, D_0 is necessarily smooth at x. Consider now the $\mathbf{Q}$-divisor $M = L - \frac{1}{d_0}D$ on X. Then

$$K_X + \ulcorner M \urcorner = K_X + L - D_0 - N,$$

where $N = [\frac{1}{d_0}F]$ is an effective divisor supported away from x. We have the numerical equivalence

$$M \equiv L - \left(\frac{1}{d_0}\right) kL \equiv \left(1 - \frac{k}{d_0}\right) L,$$

and since $d_0 > q/2 > k$, it follows that M is nef and big. Keeping in mind that we don't need to worry about normal crossings, Theorem 6.2 gives the vanishing $H^1(X, \mathcal{O}_X(K_X + L - D_0 - N)) = 0$. Therefore the restriction

$$H^0(X, \mathcal{O}_X(K_X + L - N)) \longrightarrow H^0(D_0, \mathcal{O}_X(K_X + L - N) \mid D_0) \tag{6.11}$$

is surjective.

Observe next that it is enough to show that $\mathcal{O}_X(K_X + L - N) \mid D_0$ has a section $\bar{t}$ which does not vanish at x. For then thanks to the surjectivity of (6.11), $\bar{t}$ lifts to a section $t \in H^0(X, \mathcal{O}_X(K_X + L - N))$ with $t(x) \neq 0$. Since $x \notin \mathrm{supp}\, N$, t gives rise to a section $s \in H^0(X, \mathcal{O}_X(K_X + L))$ which is non-zero at x, as desired.

The existence of the required section $\bar{t}$ will in turn follow if we verify:

$$(L - D_0 - N) \cdot D_0 > 1. \tag{6.12}$$

For then, since $(L - D_0 - N) \cdot D_0$ is in any event an integer, it will follow that the restriction $\mathcal{O}_X(K_X + L - N) \mid D_0$ is of the form $\mathcal{O}_{D_0}(K_{D_0} + B)$ for some line bundle B of degree ≥ 2, and hence is free at x thanks to Theorem 1.1. (Note that the possible singularities of D_0 don't cause any problems here since D_0 is Gorenstein and x is a smooth point.) As for (6.12), note that $\ulcorner M \urcorner - M = \sum_{i \geq 1} \frac{d_i}{d_0} D_i + \Delta$, where $\Delta = \ulcorner \frac{1}{d_0} F \urcorner - \frac{1}{d_0} F$ is an effective divisor which meets D_0 properly. Thus

$$L - D_0 - N = \ulcorner M \urcorner \equiv M + \sum_{i \geq 1} \frac{d_i}{d_0} D_i + \Delta.$$

But $L \cdot D_0 \geq 2$ by assumption, $D_i \cdot D_0 \geq mult_x(D_i)$ for $i \geq 1$, and $M \equiv (1 - \frac{k}{d_0})L$. Thus:

$$\begin{aligned}(L - D_0 - N) \cdot D_0 &= \left(1 - \frac{k}{d_0}\right) L \cdot D_0 + \sum_{i \geq 1} \left(\frac{d_i}{d_0}\right) D_i \cdot D_0 + \Delta \cdot D_0 \\ &\geq 2\left(1 - \frac{k}{d_0}\right) + \sum_{i \geq 1} \frac{d_i}{d_0} \operatorname{mult}_x D_i \\ &= 2\left(1 - \frac{k}{d_0}\right) + \left(\frac{q - d_0}{d_0}\right) = 1 + \left(\frac{q - 2k}{d_0}\right).\end{aligned}$$

Recalling that $q > 2k$, (6.12) follows.

It remains to treat the possibility that $q = 2d_i$ for some i. When L is ample one can invoke Exercise 6.9 (i.e. introduce small perturbations of the divisors in question and argue as in Proposition 6.4). In general, apply Exercise 6.13 below. We leave details to the reader. $\square$

Remark. This is the model of the argument used in [EL2], [ELM] and [Fuj2] to study global generation of adjoint linear series on threefolds. Given an ample line bundle L on a smooth projective threefold V, plus a point $x \in V$, one starts by taking a divisor $D \in |kL|$ ($k \gg 0$) with high multiplicity at x. If D has an (almost) isolated singularity at x, then one concludes at once as in Remark 6.5. Otherwise — very roughly speaking — the KRS approach reduces one to producing a section on the "most singular locus" Z of D, which in the case at hand is either a curve or a component appearing with particularly high multiplicity in D. Then one applies Reider-type statements for $\mathbf{Q}$-divisors on Z. However there are considerable technical difficulties stemming in part from the fact that one has to start by passing to an embedded resolution of D, since already in dimension three one can't ignore the normal crossing hypothesis in (6.2). Unfortunately, these problems have so far blocked the possibility of extending the argument to dimensions four or more.

Exercise 6.13. Let R be a big and nef $\mathbf{Q}$-divisor on a smooth projective surface X, and let $E_1, \ldots, E_k$ be distinct irreducible curves on X which do not appear in the fractional part of R. Assume that $R \cdot E_i > 0$ for all $1 \leq i \leq k$. Prove that then

$$H^i(X, \mathcal{O}_X(K_X + \ulcorner R \urcorner + E_1 + \cdots + E_k)) = 0$$

for $i > 0$. [Use induction on k.] $\square$

Exercise 6.14. (**Reider-type Theorems for Q-divisors**, [EL2], §2.) The proof of Proposition 6.9 can be extended to give analogous statements for (round-ups of) $\mathbf{Q}$-divisors. Let X be a smooth projective surface.

(i). Suppose that M is an ample $\mathbf{Q}$-divisor on X such that

$$M^2 > 4 \text{ and } M \cdot C \geq 2 \text{ for all irreducible curves } C \subset X.$$

Show that then $\mathcal{O}_X(K_X + \ulcorner M \urcorner)$ is globally generated. [Fix $x \in X$ and then a divisor $D \in |kM|$ for sufficiently divisible $k \gg 0$ such that $q = \mathrm{mult}_x(D) > 2k$. Write $D = \sum d_i D_i + F$ and $M = \ulcorner M \urcorner - \sum a_i D_i - G$ $(0 \leq a_i < 1)$ where the D_i all pass through x, and F, G are effective divisors disjoint from x. Let $p = \mathrm{mult}_x(\ulcorner M \urcorner - M) = \sum a_i \ \mathrm{mult}_x D_i$. If $(2-p)/q < (1-a_i)/d_i$ for all i, blow up x and argue as in the proof of Proposition 6.4. If the reverse inequality holds for some i, adapt the proof given for Proposition 6.10.]

(ii). For applications, it is useful to have a relative statement. Thus consider a surjective map $h : X \longrightarrow X_0$ where X_0 is a complete irreducible surface. Let M be a big and nef $\mathbf{Q}$-divisor on X, and fix a point $x_0 \in X_0$. Suppose that $\beta_1, \beta_2 > 0$ are positive rational numbers such that

$$\begin{aligned} M^2 &> (\beta_2)^2 \\ M \cdot C &\geq \ \beta_1 \ \ \forall \text{ curves } C \subset X \text{ s.t. } h(C) \text{ is a curve through } x_0. \end{aligned}$$

Assume that $\mathcal{O}_X(K_X + \ulcorner M \urcorner) | \Gamma \cong \mathcal{O}_\Gamma$ for every effective divisor $\Gamma \subset X$ such that $h(\Gamma) = x_0$. Suppose also that

$$\beta_2 \geq 2, \quad \beta_1 \left(1 - \frac{2}{\beta_2}\right) \geq 1.$$

Show that then $\mathcal{O}_X(K_X + \ulcorner M \urcorner)$ has a section which is non-vanishing at some point $x \in h^{-1}(x_0)$. [See [EL2], Theorem 2.3.] $\square$

Exercise 6.15. (**Reider-type Theorem for Normal Surfaces**, [ELM], §1, compare [Sak2].) Let S be a complete normal surface. Recall that Mumford has defined a $\mathbf{Q}$-valued intersection product for Weil divisors on S. [In brief, let $f : T \longrightarrow S$ be a resolution of S. Given a Weil divisor D on S, let D' denote its proper transform on T. Then there is a unique f-exceptional $\mathbf{Q}$-divisor Δ on T such that

$$(D' + \Delta) \cdot E = 0$$

for all f-exceptional divisors E on T. Mumford first defines $f^*D = D' + \Delta$. Given two Weil-divisors D_1, D_2 on S, one then sets $D_1 \cdot D_2 = f^*D_1 \cdot f^*D_2 \in \mathbf{Q}$.] In particular the usual definition for nefness makes sense for Weil divisors on S. Similar considerations of course hold for $\mathbf{Q}$-divisors on S. Recall also that the canonical divisor K_S exists as a Weil divisor (class) on S.

Now suppose that M is a nef $\mathbf{Q}$-divisor on S, and let $\beta_1, \beta_2 > 0$ be rational numbers such that

$$M^2 > (\beta_2)^2 \quad , \quad M \cdot C \geq \beta_1 \ \ \forall \text{ curves } C \subset S.$$

Assume that $K_S + \ulcorner M \urcorner$ is Cartier, and that

$$(*) \qquad \beta_2 \geq 2 \quad , \quad \beta_1 \left(1 - \frac{2}{\beta_2}\right) \geq 1.$$

Prove that then $\mathcal{O}_S(K_S + \ulcorner M \urcorner)$ is globally generated. The inequalities (*) are satisfied for example if $M^2 > 16$ and $M \cdot C \geq 2$ for all curves C. We don't know whether the conclusion holds assuming only that $M^2 > 4$ and $M \cdot C \geq 2$. [Let $f : T \longrightarrow S$ be the minimal resolution of S, and define a divisor W on T by the equation $K_T + \ulcorner f^* M \urcorner = f^*(K_S + \ulcorner M \urcorner) - W$. Prove that W is integral and exceptional. Show that every component of $f^* \ulcorner M \urcorner - \ulcorner f^* M \urcorner$ has coefficient > -1, and deduce that W is effective (or zero). Now apply Exercise 6.14.(ii).] $\square$

Exercise 6.16. (**Very ampleness for Q-divisors**, [Mas].) Let X be a smooth projective surface, and let M be a $\mathbf{Q}$-divisor on X. Assume that

$$M^2 > 18 \quad \text{and} \quad M \cdot C \geq 3 \text{ for all effective curves } C \subset X.$$

Prove that then $\mathcal{O}_X(K_X + \ulcorner M \urcorner)$ is very ample. [The linear series in question is globally generated thanks to Exercise 6.14.(i), and the first step is to argue that given distinct points $x, y \in X$, one can find a divisor $\Lambda \in |K_X + \ulcorner M \urcorner|$ passing through one but not the other of the points. To this end, choose $D \in |kM|$, for k very large and divisible, such that $q_x = \mathrm{mult}_x(D) > 3k$ and $q_y = \mathrm{mult}_y(D) > 3k$. Write $\Delta =_{\mathrm{def}} \ulcorner M \urcorner - M = \sum a_j D_j$ and $D = \sum r_j D_j$, set $\mu_x = \mathrm{mult}_x(\Delta)$, $\mu_y = \mathrm{mult}_y(\Delta)$, and put

$$c_x = \min \left\{ \frac{2 - \mu_x}{q_x}, \frac{1 - a_j}{r_j} \;\middle|\; D_j \ni x \right\},$$

with c_y defined similarly. Define $c = \max\{c_x, c_y\}$, and consider the $\mathbf{Q}$-divisor $M - cD = \ulcorner M \urcorner - \Delta - cD$. Assuming for concreteness that $c = c_x$, one argues in cases according to whether the value of c_x is determined by the multiplicities at x, or whether it is accounted for by a component D_j – say D_0 – of D passing through x. In the latter instance, one further distinguishes between whether or not D_0 goes also through y. The argument for separating tangent directions at x is similar, but requires a little more thought. See [Mas] for details, as well as for a statement involving more general numerical hypotheses in the spirit of Exercise 6.14.(ii).] $\square$

Exercise 6.17. (**Shokurov's Non-Vanishing Theorem**, [Sho], cf. [CKM], Lecture 13.) In this exercise, we outline the proof of a "toy" version of Shokurov's Non-Vanishing Theorem. In the case of surfaces treated here, the statement is a consequence of Riemann-Roch. However the argument we indicate, suitably modified, also works in higher dimensions. (As the reader will note, it was the inspiration for the proof of Proposition 6.10 as well as [EL2].) The result we aim for is the following:

Theorem. *Let X be a smooth projective surface, and let L be a nef line bundle on X. Assume that*

$$A =_{\mathrm{def}} L - K_X \quad \textit{is ample.}$$

Then $H^0(X, \mathcal{O}_X(mL)) \neq 0$ for all $m \gg 0$.

(i). If L is numerically trivial, show that the statement follows directly from a computation of $\chi(X, \mathcal{O}_X(mL))$. Hence we may assume that L is not numerically trivial. Deduce that then the intersection number $(pL - K_X)^2$ is an increasing function of p.

(ii). Fix a general point $x \in X$. Show that one can take p sufficiently large so that for $k \gg 0$ there exists a divisor $D \in |k(pL - K_X)|$ such that $q =_{\text{def}} \text{mult}_x(D) > 2k$. Write $D = \sum d_i D_i$, and set $d = \max\{d_i\}$. Given a rational number $c > 0$, consider the $\mathbf{Q}$-divisor $N(m,c) = mL - K_X - cD$. The numerical equivalence

$$mL - K_X - cD \equiv (1 - ck)\,((L - K_X) + (p-1)L) + (m-p)L$$

shows that $N(m,c)$ is ample if $m \geq p$ and $ck < 1$.

(iii). If $q > 2d$, let $f : Y \longrightarrow X$ be the blowing-up of X at x. Argue as in Proposition 6.4 with the $\mathbf{Q}$-divisor $f^*N(m, \frac{2}{q})$ on Y to deduce the desired non-vanishing.

(iv). Assume $q \leq 2d$. Let D_0 be a component of D of maximal multiplicity $d_0 = d$, and consider the $\mathbf{Q}$-divisor $N(m, \frac{1}{d_0})$. Deduce from Vanishing and Exercise 6.13 that if $m \geq p$, then the restriction map

$$(*) \qquad H^0(X, \mathcal{O}_X(mL)) \longrightarrow H^0(D_0, \mathcal{O}_{D_0}(mL))$$

is surjective.

(v). In the situation of (iv), show that if $m \geq p$, then

$$(**) \qquad \mathcal{O}_{D_0}(mL) = \mathcal{O}_{D_0}(K_{D_0} + A_0)$$

for some ample line bundle A_0 on D_0. Conclude that $H^0(D_0, \mathcal{O}_{D_0}(mL)) \neq 0$ for $m \geq p$, and show that the theorem follows. [If $p_a(D_0) \geq 1$ the required non-vanishing follows from Riemann-Roch. When D_0 is rational, note that the bundle on the right in (**) can't have degree -1 since its degree is divisible by $m \gg 0$.] $\square$

Exercise 6.18. (**Kawamata's Basepoint-free Theorem**, [K2], [K3], cf. [CKM] Lecture 10.) This exercise is devoted to a stripped-down version on surfaces of a fundamental theorem of Kawamata:

Theorem. *Let X be a smooth surface, and let L be a nef line bundle on X such that $A =_{\text{def}} L - K_X$ is ample. Then the linear series $|mL|$ is free for all $m \gg 0$.*

(i). Observe to begin with it follows from the Non-Vanishing Theorem (Exercise 6.17) that $H^0(X, \mathcal{O}_X(mL)) \neq 0$ for all $m \gg 0$. Let $B(m)$ denote the reduced base locus of $|mL|$. Noting that $B(c^a) \subseteq B(c^b)$ whenever $a > b$ and $c \gg 0$, show that the sequence of subsets $B(c^n)$ stabilizes for $n \gg 0$. Denoting the limit by $B(c)$, show that it is enough to prove that $B(c) = \emptyset$ for every $c \gg 0$.

(ii). Suppose first that for some $m \geq 1$ the linear series $|mL|$ has only isolated basepoints. Show that then $|(3m + a)L|$ is free for all $a \geq 1$. [If the base-locus of $|mL|$ consists of a finite set $Z \subset X$, then there exists a reduced divisor $D \in |3mL|$ with $\text{mult}_x(D) \geq 3$ for every $x \in Z$. Let $f : Y \longrightarrow X$ be the blowing-up of Z, and consider the pull-back to Y of the $\mathbf{Q}$-divisor

$$(3m + a)\,L - \frac{2}{3}D \equiv K_X + A + (m + a - 1)L.$$

Use vanishing to deduce that $H^1(X, \mathcal{O}_X((3m+a)L \otimes \mathcal{I}_Z)) = 0$.]

Next, fix some large integer p and write

$$|pL| = |M| + \sum r_i F_i,$$

where $F = \sum r_i F_i$ is the fixed divisor of $|pL|$, and $|M|$ is the moving part of $|pL|$, so that $|M|$ has at most isolated fixed points. Assuming $F \neq 0$, we will argue that if F_0 is a component appearing with maximal multiplicity in F, then F_0 is no longer a fixed component of $|mL|$ for $m \gg 0$. In view of (i) and (ii), the theorem will follow. Turning to details, let $D \in |pL|$ be a general divisor. Then $D = D_1 + \sum r_i F_i$ where D_1 is reduced. After re-indexing if necessary we can suppose that $r_0 = \max r_i$. Consider the $\mathbf{Q}$-divisor $N(m) = mL - K_X - \frac{1}{r_0}D$. Since L is nef, the numerical equivalence

$$N(m) \equiv \left(m - 1 - \frac{p}{r_0}\right) L + A$$

shows that $N(m)$ is ample if $m > p + 1$.

(iii). Keeping the notation just introduced, use Vanishing and Exercise 6.13 to show that the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \longrightarrow H^0(F_0, \mathcal{O}_{F_0}(mL)) \tag{*}$$

is surjective when $m > p + 1$. Arguing as in Exercise 6.17.(v), show that the group on the right in (*) is non-vanishing. Conclude that F_0 is not in the base locus of $|mL|$ for $m > p + 1$. □

Exercise 6.19. (**Pluricanonical Series**.) With some extra work, one can show that the conclusion of Kawamata's Basepoint-free theorem (Exercise 6.18) remains valid assuming only that $L - K_X$ is nef and big (cf. [CKM], Lecture 10). We grant this fact here.

(i). Show that if X is a minimal surface of general type, then $\mathcal{O}_X(mK_X)$ is globally generated for all $m \gg 0$. Deduce that there exists a normal surface X_0, a surjective birational morphism $h : X \longrightarrow X_0$, plus an ample line bundle L_0 on X_0 such that $\mathcal{O}_X(K_X) = h^*L_0$. ($X_0$ is called the *canonical model* of X. Compare [EV2], §7.1, and [W].)

(ii). With X as in (i), use Exercise 6.14.(ii) to prove that $\mathcal{O}_X(mK_X)$ is globally generated for $m \geq 5$.

This is the model of the arguments used to study pluricanonical series on threefolds in [EL2] and [ELM]. Note however that the bound in (ii) is slightly less than optimal. □

Bogomolov's Theorem.

Fernández del Busto [FdB1] has shown that the sort of argument used to prove Propositions 6.4 and 6.10 leads to a new approach to Bogomolov's Instability Theorem 4.2. This completes in a very nice way the circle of ideas linking linear series on surfaces, vector bundles and vanishing theorems, and we present here an outline of his proof. We shall content ourselves with explaining the principal steps. Therefore we'll make some simplifying assumptions, and relegate some of the numerical calculations to exercises.

To begin with, recall the statement:

Bogomolov's Instability Theorem. *Let E be a rank two vector bundle on a smooth projective surface X. Assume that*

$$c_1(E)^2 - 4c_2(E) > 0.$$

Then there is a saturated invertible subsheaf $A \hookrightarrow E$ such that if $L = \det E$, then

$$(2A - L)^2 > 0 \quad \text{and} \quad (2A - L) \cdot H > 0 \text{ for some ample divisor } H.$$

By a saturated subsheaf we mean here that the vector bundle map $A \longrightarrow E$ vanishes only on a finite set. (See Definition 4.1.)

We start with some preliminary reductions and remarks. Note first that since Bogomolov's theorem is invariant under twisting, we are free to tensor E by a high multiple of an ample line bundle. Therefore we may assume that E is globally generated, that its determinant is ample, and that $c_2(E) > 0$. Let $s \in \Gamma(X, E)$ be a general section. Then the zero-scheme $Z = Z(s)$ is reduced, and the Koszul complex (3.6) determined by s realizes E as an extension

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot s} E \longrightarrow L \otimes \mathcal{I}_Z \longrightarrow 0. \tag{6.20}$$

Here $L = \det(E)$ is an ample line bundle on X, and $c_2(E) = \#Z > 0$.

The strategy of the proof is now very simple. The numerical hypothesis $c_1^2 > 4c_2$ guarantees the existence of a divisor $D \in |kL|$ ($k \gg 0$) with high multiplicity at the points of Z. If D has small multiplicity away from Z, then the argument of Proposition 6.4 would yield the vanishing of $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z)$. But since E is locally free, Theorem 3.13 shows that the group in question is non-zero. Therefore, as in the proof of (6.10), D must contain some distinguished components appearing with high multiplicity. One uses these components to construct a divisor $\Gamma \subset X$ passing through Z. Vanishing will imply that the inclusion $\mathcal{O}_X(L - \Gamma) \hookrightarrow L \otimes \mathcal{I}_Z$ lifts to an embedding $\mathcal{O}_X(L - \Gamma) \subset E$, and then one argues that the saturation of this subsheaf destabilizes E. One new feature of this argument is that while in Proposition 6.10 we worked with an arbitrary divisor D, here we must be careful to choose s and D rather generally.

Turning to the details, the first point is:

Lemma 6.21. *For any section $s \in \Gamma(X, E)$ as above, with reduced zero-scheme $Z = Z(s)$, there exists a divisor $D \in |kL|$ ($k \gg 0$) such that $\mathrm{mult}_x(D) > 2k$ for every $x \in Z(s)$.*

Moreover, by choosing s and D sufficiently generally, we can assume that D satisfies the following

Uniform Multiplicity Property (UMP). *For any rational number $\delta > 0$, the multiplicity of the integer part $[\delta D]$ is the same at every point of $Z(s)$.*

Proof. For the first assertion, one counts dimensions much as in the proof of Proposition 6.10. Specifically, by Riemann-Roch:

$$h^0(X, \mathcal{O}_X(kL)) = \frac{k^2}{2}L^2 + o(k^2) = \frac{k^2}{2}c_1(E)^2 + o(k^2).$$

On the other hand, the number of conditions required to impose multiplicity $\geq 2k+1$ at each point of Z is:

$$\binom{2k+2}{2}(\#Z) = \binom{2k+2}{2}c_2(E) = \frac{4k^2}{2}c_2(E) + o(k^2).$$

Since $c_1(E)^2 > 4c_2(E)$ by hypothesis, for $k \gg 0$ the required divisor will exist.

Turning to the second assertion, let $S \subset \mathbf{P}H^0(E)$ denote the open subset parametrizing sections of E with finite zero-schemes, and consider the incidence correspondence

$$X \times S \supset \mathcal{Z} = \left\{ (x, [s]) \mid x \in Z(s) \right\}.$$

The assumption that E is globally generated implies that $\mathcal{Z}$ is an open subset of a projective bundle over X, and hence is irreducible. Note also that it is finite over S, and in particular $\dim(\mathcal{Z}) = \dim(S)$. By an evident globalization of the parameter count just made, one can construct a divisor $X \times S \supset \mathcal{D} \supset \mathcal{Z}$ such that the fibre $D_s \subset X$ of $\mathcal{D}$ over $s \in S$ is a divisor in the linear series $|kL|$ having multiplicity $> 2k$ at every point of $Z_s = Z(s)$.

To establish the UMP, it is enough to show that given an integer $p > 0$, if $[\delta D_s]$ has multiplicity $\geq p$ at *some* point $x \in Z(s)$ for general $s \in S$, then it has multiplicity $\geq p$ at *every* point $x \in Z(s)$. Consider to this end the Zariski-closed set

$$\mathcal{Z} \supset \mathcal{Z}_p = \left\{ (x, [s]) \in \mathcal{Z} \mid \mathrm{mult}_x([\delta\mathcal{D}]_s) \geq p \right\}.$$

By assumption $\mathcal{Z}_p$ dominates S, and hence $\dim(\mathcal{Z}_p) \geq \dim(S) = \dim(\mathcal{Z})$. $\mathcal{Z}$ being irreducible, it follows that $\mathcal{Z}_p = \mathcal{Z}$. Noting that $\mathrm{mult}_x([\delta\mathcal{D}]_s) = \mathrm{mult}_x([\delta D_s])$ for general s, this means exactly that $[\delta D_s]$ has multiplicity $\geq p$ for every $x \in Z(s)$. The Lemma follows. $\square$

Remark. Observe that the second assertion of the Lemma is essentially a monodromy argument in the spirit of [Har]. Specifically, fix a general "reference section" $s_0 \in \Gamma(X, E)$ with finite reduced zero-scheme Z_0. Letting s vary over the open subset $U \subset S$ parametrizing sections with reduced zero-loci, one obtains a monodromy action on the points of Z_0. The irreducibility of $\mathcal{Z}$ implies that this action is transitive. Since the divisors D_s also vary with

s, it follows that for general $s \in U$, D_s cannot be used to distinguish among the points of $Z(s)$. Therefore D_s must have the same multiplicity at every point of $Z(s)$, and similarly for $[\delta D_s]$.

Returning to the proof of Bogomolov's theorem, fix s and $D \in |kL|$ ($k \gg 0$), with $D \supset Z = Z(s)$, as in Lemma 6.21. In particular, we suppose that D satisfies the Uniform Multiplicity Property. Therefore D has the same multiplicity at every point of Z, say

$$q = \mathrm{mult}_x(D) \ \text{ for every } x \in Z.$$

Write $D = \sum d_i D_i$. We henceforth *make the simplifying assumption* that every component D_i of D meets Z. (See Exercise 6.24 for the general case.) Then set

$$d = \max\{d_i\}.$$

We assert next that $2d > q$. In fact, suppose to the contrary that $2d \leq q$. Then an evident generalization of Proposition 6.4 (i.e. Exercise 6.9) implies that

$$(*) \qquad H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z) = 0.$$

On the other hand, since E is locally free and $Z \neq \emptyset$, the extension class of the Koszul complex (6.20) must be non-zero. Therefore

$$\mathrm{Ext}^1(L \otimes \mathcal{I}_Z, \mathcal{O}_X) = H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_Z)^* \neq 0,$$

a contradiction. (Compare (3.12), (3.13).) Hence $2d > q$, as claimed.

Consider now the divisor

$$D_0 = \left[\sum \frac{d_i}{d} D_i \right].$$

Then D_0 is reduced, and D_0 meets Z thanks to our assumption that every component D_i passes through at least one point of Z. It follows from the (UMP) that in fact $Z \subset D_0$. Furthermore, D_0 is smooth at every point of Z. D_0 plays the role of the curve Γ appearing in the overview of the proof given above.

Since

$$\frac{1}{d} < \frac{2}{q} < \frac{1}{k},$$

the $\mathbf{Q}$-divisor $L - \frac{1}{d}D$ is ample. Kawamata-Viehweg vanishing (6.2), along with Sakai's Lemma (Ex. 6.6), therefore applies to show that

$$(**) \qquad H^1(X, \mathcal{O}_X(K_X + L - D_0)) = \mathrm{Ext}^1(\mathcal{O}_X(L - D_0), \mathcal{O}_X)^* = 0.$$

Now we can pull back the given extension (6.20) under the natural inclusion $\mathcal{O}_X(L - D_0) \subset \mathcal{O}_X(L) \otimes \mathcal{I}_Z$ to obtain an extension of $\mathcal{O}_X(L - D_0)$ by $\mathcal{O}_X$. But the vanishing (**) implies that the latter extension splits. Therefore the injection $\mathcal{O}_X(L-D_0) \subset \mathcal{O}_X(L) \otimes \mathcal{I}_Z$ lifts to a sheaf monomorphism $\mathcal{O}_X(L-D_0) \hookrightarrow E$:

$$\begin{tikzcd}
 & & \mathcal{O}_X(L-D_0) \arrow[r, equal] \arrow[d] & \mathcal{O}_X(L-D_0) \arrow[d, "\cdot D_0"] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\cdot s"] & E \arrow[r] & \mathcal{O}_X(L) \otimes \mathcal{I}_Z \arrow[r] & 0.
\end{tikzcd}$$

In other words, we have produced an "unexpected" invertible subsheaf of E. To complete the proof, Fernández del Busto shows that $\mathcal{O}_X(L-D_0)$ is a saturated subsheaf of E, and that it destabilizes E. The destabilization is equivalent to proving the inequalities:

$$\begin{aligned} (L-2D_0)^2 &> 0 \\ (L-2D_0)\cdot L &> 0, \end{aligned} \tag{6.22}$$

and we outline the required calculations in the following Exercise.

Exercise 6.23. Keeping the notations and assumptions made above, we sketch the verification of (6.22).

(i). Suppose given a decomposition $Z = Z_1 \coprod Z_2$ of Z into two disjoint subsets, with $Z_1 \neq \emptyset$. Prove that Z_1 cannot impose independent conditions on the sections of $\mathcal{O}_X(K_X+L)$ vanishing on Z_2, i.e. that the evaluation homomorphism

$$H^0(X, \mathcal{O}_X(K_X+L)\otimes \mathcal{I}_{Z_2}) \longrightarrow H^0(Z_1, \mathcal{O}_{Z_1}(K_X+L))$$

cannot be surjective. [Use Theorem 3.13.]

(ii). Prove that

$$(L-D_0)\cdot D_0 \leq \#Z.$$

[Each point of Z lies on exactly one component of D_0, and every component of D_0 contains at least one point of Z. So it suffices to show that if D' is a component of D_0, and $Z' = D' \cap Z$, then $(L-D_0)\cdot D' \leq \#Z'$. If on the contrary $(L-D_0)\cdot D' > \deg Z'$, then the points of Z' impose independent conditions on the linear series $|\mathcal{O}_{D'}(K_{D'}+L-D_0)|$. Use (**) and (i) to arrive at a contradiction.]

(iii). Show that

$$(L-D_0)\cdot D_0 \geq \left(1-\frac{k}{d}\right) L\cdot D_0 + (\#Z)\left(\frac{q}{d}-1\right).$$

[$L - D_0 \equiv (1-\frac{k}{d})L + D^*$, where $D^* = \sum \frac{d_i}{d} D_i - D_0$ is an effective $\mathbf{Q}$-divisor which does not contain any components in common with D_0. Estimate $D^*\cdot D_0$ as in the proof of (6.12).]

(iv). Show that

$$\#Z > \frac{1}{2} L\cdot D_0.$$

[Combine (ii) and (iii).]

(v). Prove that $\mathcal{O}_X(L-D_0)$ is a saturated subsheaf of E, and establish the inequalities (6.22). [For the saturation, use the fact that every component of D_0 contains a point of Z lying only on that component . The first inequality in (6.22) then follows formally from the hypothesis $c_1(E)^2 > 4c_2(E)$ by a suitable computation of $c_2(E)$, much as in (4.8). As for the second, one has $L^2 > 4(\#Z) > 2L \cdot D_0$.] $\square$

Exercise 6.24. Eliminate from the argument above the simplifying assumption that every component of D meets Z. [Take D satisfying Lemma 6.21, and write $D = \sum d_i D_i + F$, where every D_i meets Z and F is disjoint from Z. Defining q and d as above, one still has $2d > q$. Choose a minimal subdivisor $\Delta \subseteq [\frac{1}{d}F]$ for which one has the vanishing

$$H^1(X, \mathcal{O}_X(K_X + L - D_0 - \Delta)) = 0,$$

where $D_0 = \sum \frac{d_i}{d} D_i$. Set $\Gamma = D_0 + \Delta$. Then replace D_0 in the argument above with Γ. See [FdB1] for details.] $\square$

§7. Algebro-Geometric Analogue of Demailly's Approach

The method of Kawamata-Reid-Shokurov discussed in the previous section is essentially inductive in nature. To produce a section of $\mathcal{O}_X(K_X + L)$, for example, one starts by constructing a divisor $D \in |kL|$ for $k \gg 0$ with large multiplicity at a given point $x \in X$. If D has small multiplicity at neighboring points $y \in X$ — which is the "good" case — then one gets a vanishing which directly yields the required section (Proposition 6.4). If on the contrary D has a "bad" component D_0 appearing with high multiplicity, then the KRS machine reduces the problem to constructing a section on D_0. Unfortuately, already in the three dimensional situation of [EL2] and [ELM], this inductive step involves considerable technical difficulties.

Demailly's strategy in [De1] is quite different. In effect he puts his efforts into showing that under suitable hypotheses one can produce a divisor $D \in |kL|$ with an "almost isolated" singular point, thereby avoiding any inductions. (We're being somewhat metaphorical here: strictly speaking, Demailly does not work directly with divisors.) The argument of [De1] becomes essentially analytic at one point, but it has recently emerged in work with Ein and Nakamaye [ELN] that there are quite simple algebro-geometric analogues of Demailly's underlying geometric ideas. In the present section we discuss this approach in the case of surfaces, where (as usual) the picture is particularly clear.

Let X be a smooth projective surface, and consider a fixed point $x \in X$. We start with a definition:

Definition 7.1. Given a line bundle B on X, and a divisor $D \in |kB|$ for some $k > 0$, we say that D has an *almost isolated singularity of index* $> r$ at x if

$$\mathrm{mult}_x(D) > kr,$$

and if there exists a neighborhood $U \ni x$ of x in X such that

$$\mathrm{mult}_y D < k \text{ for } y \in U - \{x\}.$$

In other words, we require that the $\mathbf{Q}$-divisor $\frac{1}{k}D \equiv B$ have multiplicity $> r$ at x, and < 1 in a punctured neighborhood of x. Similarly, one can discuss an almost isolated singularity of index $\geq r$.

We may then rephrase Proposition 6.4 as:

Proposition 7.2. *Let X be a smooth surface, L a big and nef line bundle on X, and $s \geq 0$ an integer. If for some $k > 0$ there exists a divisor $D \in |kL|$ with an almost isolated singularity of index $> s+2$ at a point $x \in X$, then $|\mathcal{O}_X(K_X + L)|$ generates s-jets at x.* $\square$

So we need to find a way to produce divisors with almost isolated singularities.

The potential obstruction to constructing such divisors is easily described, especially on surfaces. Specifically, in the situation of the Proposition consider the linear series

$$|\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{k(s+2)+1}|$$

of all divisors $D \in |kL|$ having multiplicity $> k(s+2)$ at x. Let F_k denote the fixed divisor of this linear series, and let Σ_k denote its moving part, so that Σ_k has at most isolated base points. Thus

$$(*) \qquad |\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{k(s+2)+1}| = \Sigma_k + F_k,$$

i.e. every divisor $D \in |\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{k(s+2)+1}|$ is of the form $D = D' + F_k$ for some $D' \in \Sigma_k$. Now by Bertini's theorem, a general divisor $D' \in \Sigma_k$ is reduced. Therefore a general element $D \in |\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{k(s+2)+1}|$ will fail to have an almost isolated singular point at x if and only if the fixed divisor F_k has components of multiplicty $\geq k$ that pass through x. So the problem is prove an upper bound on the coefficients of the components of F_k for $k \gg 0$. The strategy roughly speaking will be to use a lower bound on the dimension of the linear series on the left in (*) to deduce an upper bound on the degree of F_k.

It is useful to generalize slightly. Let Y be a smooth projective surface, and let B be a big line bundle on Y. (In the application, Y will be a blow-up of X.) Denote by F_k the fixed divisor of the complete linear series $|kB|$ and put $M_k = kB - F_k$, so that

$$|kB| = |M_k| + F_k$$

for all k. The essential point is:

Proposition 7.3. *Assume that there exists a rational number $\rho > 0$ such that*

$$h^0(Y, \mathcal{O}_Y(kB)) \geq \rho \frac{k^2}{2} + o(k^2) \quad \textit{for } k \gg 0.$$

Then for $k \gg 0$:

$$M_k^2 \geq \rho k^2 + o(k^2).$$

To understand the intuition here, suppose it were to happen that $M_k = kM$ for some fixed nef and big divisor M. Then

$$\rho \frac{k^2}{2} + o(k^2) \leq h^0(\mathcal{O}_Y(kB)) = h^0(\mathcal{O}_Y(kM)) = \frac{k^2}{2} M^2 + o(k^2)$$

and the desired inequality follows. The content of the Proposition is that the same inequality holds in general.

The following argument is due to Fernández del Busto [FdB2]; the original proof of (7.3) was more cumbersome.

Proof of Proposition 7.3. Fix a very ample divisor H on Y with the property that $K_Y + H$ is very ample, and fix a general divisor $D \in |K_Y + H|$. Then multiplication by D defines for every k an inclusion of sheaves $\mathcal{O}_Y(M_k) \subset \mathcal{O}_Y(K_Y + H + M_k)$. In particular,

$$(*) \qquad h^0(Y, \mathcal{O}_Y(K_Y + H + M_k)) \geq h^0(Y, \mathcal{O}_Y(M_k)) = h^0(Y, \mathcal{O}_Y(kB))$$

for all k. Now M_k is nef, hence $H + M_k$ is ample, so by Vanishing and Riemann-Roch:

$$\begin{aligned} h^0(Y, O_Y(K_Y + H + M_k)) &= \chi(Y, \mathcal{O}_Y(K_Y + H + M_k)) \\ &= \frac{(K_Y + H + M_k) \cdot (H + M_k)}{2} + \chi(Y, \mathcal{O}_Y) \\ &= \frac{M_k \cdot M_k}{2} + \frac{M_k \cdot (K_Y + 2H)}{2} + c(Y, H), \end{aligned}$$

where $c(Y, H)$ is a constant not depending on k. On the other hand, since $K_Y + H$ and H are very ample, there exists a divisor $D' \in |K_Y + 2H|$ which meets F_k properly. Therefore

$$0 \leq M_k \cdot (K_Y + 2H) \leq kB \cdot (K_Y + 2H).$$

Hence

$$h^0(Y, O_Y(K_Y + H + M_k)) = \frac{M_k \cdot M_k}{2} + o(k^2),$$

and the Proposition follows from (*) and the hypothesis on $h^0(\mathcal{O}_Y(kB))$. $\square$

Remark. One may view Proposition 7.3 as a numerical counterpart to the existence of a Zariski decomposition for B. It would be very interesting to know to what extent an analogue of (7.3) remains true on higher dimensional varieties. The difficulty of course is that the decomposition $kB = F_k + M_k$ of $|kB|$ into fixed and moving parts in general only exists on a blowing-up Y_k of Y, and Y_k will depend on k. While it may be too much to hope for a result for an arbitrary big bundle B, Demailly's work suggests that (7.3) should extend to all dimensions at least for the sort of bundle occurring in the proof of Theorem 7.4. See [ELN] for a slightly different approach.

We will apply Proposition 7.3 in the spirit of [De1], §8, to prove:

Theorem 7.4. *Let X be a smooth projective surface, $x \in X$ a fixed point, and $s \geq 0$ a non-negative integer. Suppose that L is a nef line bundle on X such that*

$$L^2 \geq (s+2)^2 + 1 \quad \text{and} \quad L \cdot C \geq s^2 + 3s + 3$$

for all curves $C \subset X$ passing through x. Then for $k \gg 0$ there exists a divisor $D \in |kL|$ having an almost isolated singularity of index $> (s+2)$ at x.

Propositon 7.2 then implies:

Corollary 7.5. *If L is a nef line bundle such that $L^2 \geq 5$ and $L \cdot C \geq 3$ for all curves $C \ni x$, then $\mathcal{O}_X(K_X + L)$ has a section which doesn't vanish at x. More generally, if L satisfies the inequalities in Theorem 7.4, then $|K_X + L|$ generates s-jets at x.* $\square$

See Exercise 7.6 for an analogous criterion for $\mathcal{O}_X(K_X + L)$ to be very ample. Note that the numbers are a little weaker than the optimal bounds (Corollary 2.6) coming from Reider's theorem.

Proof of Theorem 7.4. To simplify the exposition, we first show how to produce a divisor $D \in |kL|$ with an almost isolated singularity of index $\geq (s+2)$ at x. At the end of the argument we will indicate the (routine) changes necessary to get index $> (s+2)$.

Let $f : Y \longrightarrow X$ be the blowing up of X at x, with $E \subset Y$ the exceptional divisor, and put $B = f^*L - (s+2)E$. Then

$$H^0(Y, \mathcal{O}_Y(kB)) = H^0(X, \mathcal{O}_X(kL) \otimes \mathcal{I}_x^{k(s+2)}),$$

so by the usual dimension count (cf. proof of Proposition 6.10):

$$\begin{aligned} h^0(Y, \mathcal{O}_Y(kB)) &\geq \frac{k^2}{2} L^2 - \binom{k(s+2)+1}{2} + o(k^2) \\ &= \left(L^2 - (s+2)^2\right) \frac{k^2}{2} + o(k^2). \end{aligned}$$

Consider as above the decomposition $|kB| = |M_k| + F_k$ of $|kB|$ into its moving and fixed parts. Proposition 7.3 yields the inequality

$$M_k^2 \geq \left(L^2 - (s+2)^2\right) k^2 + o(k^2).$$

Hence by Hodge Index:

$$M_k \cdot f^*L \geq \sqrt{(L^2)\,(M_k^2)}$$
$$\geq k\left(\sqrt{(L^2)\,(L^2-(s+2)^2)}\right) + o(k).$$

Recalling that $F_k = f^*(kL) - k(s+2)E - M_k$, it follows that

(*) $$F_k \cdot f^*L \leq k\left(L^2 - \sqrt{(L^2)\,(L^2-(s+2)^2)}\right) + o(k).$$

Now if $f_s(x)$ denotes the function

$$f_s(x) = x - \sqrt{x\,(x-(s+2)^2)},$$

one finds that $f_s(x) < s^2+3s+3$ when $x \geq (s+2)^2+1$. Hence it follows from (*) that for $k \gg 0$,

(**) $$F_k \cdot f^*L < (s^2+3s+3)k$$

provided that $L^2 \geq (s+2)^2+1$.

On the other hand, consider the image

$$\overline{F}_k = f_*F_k \subset X$$

of F_k in X. Then $\overline{F}_k$ is the fixed divisor of the linear series $|\mathcal{O}_X(kL)\otimes\mathcal{I}_x^{k(s+2)}|$. It follows from (**) that for $k \gg 0$:

$$\overline{F}_k \cdot L < (s^2+3s+3)k.$$

But by hypothesis, every effective curve $F \subset X$ passing through x has L-degree $\geq s^2+3s+3$. In particular, if F is a component of $\overline{F}_k$ through x, then $ord_F(\overline{F}_k) < k$. Therefore Bertini's theorem implies that a general divisor $D \in |\mathcal{O}_X(kL)\otimes\mathcal{I}_x^{k(s+2)}|$ has multiplicity $< k$ in a punctured neighborhood of x, i.e. D has an almost isolated singularity of index $\geq (s+2)$ at x.

It remains to show that we can arrange to produce a divisor $D \in |kL|$ with an almost isolated singularity of index strictly greater than $s+2$. Had we proven a variant of Proposition 7.3 for **Q**-divisors, one would simply go through the argument just given with the divisor $f^*L-(s+2)E$ replaced by $f^*L-(s+2+\epsilon)E$ for small rational $\epsilon > 0$. To avoid **Q**-divisors in (7.3) one can equivalently choose a large integer $m \gg 0$ and work with

$$B = f^*(mL) - (m(s+2)+1)\,E.$$

We leave details to the reader. $\square$

Exercise 7.6. (**Criterion for Very Ample Adjoint Bundles**.) Let L be an ample line bundle on a smooth projective surface X such that $L^2 \geq 10$ and $L \cdot C \geq 7$ for all curves $C \subset X$. Show that then $\mathcal{O}_X(K_X + L)$ is very ample. [Given distinct points $x, y \in X$, construct a divisor $D \in |kL|$ ($k \gg 0$) with almost isolated singularities of index > 2 at x and y.] $\square$

Exercise 7.7. (**Effective Matsusaka Theorem on Surfaces**, [FdB2]). Using the results of [De1], Siu [Siu1] has recently obtained an effective version of Matsusaka's "Big Theorem" (cf. [LM], [KolM]) on varieties of all dimensions. However the constants that appear in his statement are exponential. Inspired by Siu's general result, Fernández del Busto shows in [FdB2] that on a surface one can use Proposition 7.3 to obtain the following considerably sharper bound (having a different shape than Siu's):

Theorem. *Let L be an ample line bundle on the smooth projective surface X, and set*

$$a = L^2 \quad , \quad b = (K_X + 4L) \cdot L.$$

If

$$m > \frac{(b+1)^2}{2a} - 1 \quad \left(\textit{resp. } m > \frac{(b+1)^2}{2a} + 1 \right),$$

then $\mathcal{O}_X(mL)$ is globally generated (resp. $\mathcal{O}_X(mL)$ is very ample).

(i). Let F and G be ample line bundles on X. Show that for $k \gg 0$:

$$h^0\left(X, \mathcal{O}_X(k(F-G))\right) \geq \left(F^2 - 2F \cdot G\right) \frac{k^2}{2} + o(k^2).$$

[In fact, if $F^2 > 2F \cdot G$, then $h^0(\mathcal{O}_X(k(F-G))) \geq (F-G)^2 \frac{k^2}{2} + o(k^2)$.] In particular, taking $F = (m+3)L$ and $G = K_X + 4L$, it follows that

$$h^0(\mathcal{O}_X(kB_m)) \geq \rho(m) \frac{k^2}{2} + o(k^2),$$

where:

$$B_m = (m-1)L - K_X,$$
$$\rho(m) = (m+3)^2 L^2 - 2(m+3)(K_X + 4L) \cdot L.$$

[Siu proves more generally that if F and G are ample line bundles on a smooth projective variety V of dimension n, and if $\rho = F^n - nF^{n-1} \cdot G > 0$, then $h^0(V, \mathcal{O}_V(k(F-G)) \geq \rho \frac{k^n}{n!} + o(k^n)$.]

(ii). Fix $x \in X$, and suppose that m is large enough so that one has the inequalities:

$$(*) \qquad \begin{gathered} \rho(m) > 4 \\ L \cdot B_m - \sqrt{(\rho(m) - 4)(L^2)} < 1. \end{gathered}$$

Show that then for $k \gg 0$ there exists a divisor $D \in |kB_m|$ with an almost isolated singularity of index ≥ 2 at x. Deduce that

$$H^1(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_x) = 0,$$

and conclude that $|mL|$ free at x. [For the existence of D, invoke Proposition 7.3 and argue as in the proof of Theorem 7.4. Then consider the **Q**-divisor:

$$M = mL - \frac{1}{k}D - K_X \equiv L.$$

Apply vanishing to f^*M on the blowing-up $f : Y \longrightarrow X$ of X at x.]

(iii). Show that the inequality $m > \frac{(b+1)^2}{2a} - 1$ implies (*), which proves the first assertion of the Theorem. Proceed similarly for very ampleness. $\square$

Exercise 7.8. (**Bounding singularities via Seshadri Constants.**) This exercise presents some further unpublished work of Geng Xu showing how to deduce a variant of Theorem 7.4 from his bound (Exercise 5.16) on the Seshadri constant of an ample line bundle along a finite set of general points.

Let L be an ample line bundle on a smooth projective surface X with $L^2 \geq 5$, and let $x \in X$ be a fixed point. The aim is to show that if $L \cdot C \geq 5$ for all irreducible curves $C \subset X$, then for $k \gg 0$ there exists a divisor $D \in |kL|$ with an almost isolated singularity of index > 2 at x.

(i). Consider as above the decomposition

$$|\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{2k+1}| = \Sigma_k + F_k$$

of $|\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{2k+1}|$ into its moving and fixed parts. Given any $r = L^2 - 5$ points $x_1, \ldots, x_r \in X$, show that for $k \gg 0$ there exists a divisor $D' \in \Sigma_k$ with $\mathrm{mult}_{x_i}(D') > k$ for all $1 \leq i \leq r$.

(ii). In the situation of (i), use Exercise 5.16 to prove that if one chooses the x_i sufficiently generally, then

$$L \cdot D' > kr = k(L^2 - 5).$$

Deduce that $L \cdot F_k < 5k$ for $k \gg 0$, and conclude that a general divisor $D \in |\mathcal{O}_X(kL) \otimes \mathcal{I}_x^{2k+1}|$ has an almost isolated singularity of index > 2 at x. $\square$

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