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A SHARP CASTELNUOVO BOUND FOR SMOOTH SURFACES

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Introduction. Consider a smooth irreducible complex projective variety $X \subset \mathbb{P}^r$ of degree d and dimension n, not contained in any hyperplane. There has been a certain amount of interest recently in the problem of finding an explicit bound, in terms of n, d and r, on the degrees of hypersurfaces that cut out a complete linear system on X. At least for $r \geqslant 2n+1$, the best possible linear inequality would be:

$$(*) \qquad H^1\big(\mathbb{P}^r, I_{X/\mathbb{P}^r}(k)\big) = 0 \quad \text{for } k \geqslant d+n-r.$$

This was established for (possibly singular) curves in [GLP], completing classical work of Castelnuovo [C]. For X of arbitrary dimension, Mumford (cf. [BM]) showed that $H^1(I_{X/\mathbb{P}^r}(k)) = 0$ for $k \geqslant (n+1)(d-2)+1$. Pinkham [P] subsequently obtained the sharper estimate that if X is a surface, then hypersurfaces of degree $\geqslant d-2$ [resp. $\geqslant d-1$] cut out a complete series when $r \geqslant 5$ [resp. $r = 4$], but he left open the question of whether or not (*) holds. Recently Gruson has extended Pinkham's theorem to threefolds.

The purpose of this note is to complete Pinkham's result by establishing the optimal bound (*) for surfaces:

THEOREM. *Let $X \subset \mathbb{P}^r$ be a smooth irreducible complex projective surface of degree d, not contained in any hyperplane. Then hypersurfaces of degree $d+2-r$ or greater cut out a complete linear series on X.*

By the theory of Castelnuovo–Mumford [M, Lecture 14], the theorem has implications for the equations defining X in $\mathbb{P}^r$:

COROLLARY. *In the situation of the theorem, the ideal sheaf $I_{X/\mathbb{P}^r}$ is $(d+3-r)$—regular in the sense of Castelnuovo-Mumford. In particular, the homogeneous ideal of X is generated by forms of degrees $(d+3-r)$ or less.*

Bounds on the regularity of an ideal sheaf are important in connection with algorithms for computing syzygies (cf. [BS]), and this accounts for some of the recent interest in these questions.

The proof of the theorem revolves around a technique used by Gruson and Peskine in their work on space curves [GP1, GP2]. As in the arguments of

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Castelnuovo, Mumford and Pinkham, one starts by taking a general projection to $\mathbb{P}^3$. The construction of Gruson–Peskine, which we review in §1, then reduces the question to proving the vanishing of some cohomology groups of a certain vector bundle on $\mathbb{P}^3$. This is in turn accomplished (§2) by means of regularity considerations and an Eagon–Northcott complex. We remark that the same argument (slightly simplified) gives a quick proof of the regularity theorem of [GLP], at least for smooth curves. For the most part the proof is formal, and the hypothesis that X is a surface essentially comes into play just to verify some general position statements. We hope that this approach—which involves only more or less standard techniques—may prove useful in the case of higher dimensions.

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§1. A construction of Gruson and Peskine. We begin with some notation and conventions. All varieties will be defined over the complex numbers. Throughout the paper $X \subset \mathbb{P}^r$ is a smooth irreducible projective surface of degree d, not contained in any hyperplane. The theorem being trivial when $r = 3$, we assume always that $r \geqslant 4$. Finally, in some cohomological arguments it will be convenient to let $\mathbb{P}$ denote a projective space whose precise dimension is unimportant.

Our purpose in this section is to review in the present context a construction used by Gruson and Peskine in their studies of space curves (cf. [GP1], [GP2] and [S2]). The first step is to take a generic projection to $\mathbb{P}^3$. Fix to this end a linear space $\Lambda \subset \mathbb{P}^r$ dimension $r - 4$, disjoint from X, and let $p\colon M \to \mathbb{P}^r$ be the blowing-up of $\mathbb{P}^r$ along Λ. Denoting by $q\colon M \to \mathbb{P}^3$ the natural projection, one obtains for each $k \in \mathbb{Z}$ a homomorphism

$$w_k\colon q_*\left(p^*\mathcal{O}_{\mathbb{P}^r}(k)\right) \to q_*\left(p^*\mathcal{O}_X(k)\right)$$

of sheaves on $\mathbb{P}^3$.

We shall be particularly interested in the map w_2. It may be interpreted very concretely as follows. Let

$$f = f_\Lambda\colon X \to \mathbb{P}^3$$

be the linear projection of X centered at Λ, so that $f_*\mathcal{O}_X(k) = q_*(p^*\mathcal{O}_X(k))$, and choose homogeneous coordinates on $\mathbb{P}^r$ in such a way that Λ is defined by $T_0 = T_1 = T_2 = T_3 = 0$. Then $T_4, \dots, T_r$ determine sections in $H^0(\mathbb{P}^r, \mathcal{O}_X(1)) = H^0(\mathbb{P}^3, f_*\mathcal{O}_X(1))$, and similarly the monomials T_iT_j $(4 \leqslant i, j \leqslant r)$ give sections of $H^0(\mathbb{P}^3, f_*\mathcal{O}_X(2))$. Combining these with the canonical map $\mathcal{O}_{\mathbb{P}}^3 \to f_*\mathcal{O}_X$ one deduces a homomorphism

$$(1.1) \qquad w\colon \mathcal{O}_{\mathbb{P}^3}(-2)^{N(r)} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{r-3} \oplus \mathcal{O}_{\mathbb{P}^3} \to f_*\mathcal{O}_X$$

of $\mathcal{O}_{\mathbb{P}^3}$-modules, where $N(r) = (r-2)(r-3)/2$. A moment's thought shows that this map may be identified with w_2.

The essential use of the hypothesis that X is a surface occurs in

LEMMA 1.2. *For sufficiently general $\Lambda \subset \mathbb{P}^r$, w_2 is surjective if $r \geqslant 4$.*

Proof. It is enough to prove surjectivity fibre by fibre over $\mathbb{P}^3$, i.e. it suffices to show that $w_2 \otimes \mathbb{C}(y)$ is surjective for every point $y \in \mathbb{P}^3$. To this end, let $L_y = p(q^{-1}(y))$ be the $(r-3)$-plane through Λ corresponding to $y \in \mathbb{P}^3$. Then, denoting by X_y the scheme-theoretic intersection $X \cap L_y$, $w_k \otimes \mathbb{C}(y)$ is identified with the restriction homomorphism

$$H^0\big(L_y, \mathcal{O}_{L_y}(k)\big) \to H^0\big(L_y, \mathcal{O}_{X_y}(k)\big).$$

Hence the question is reduced to showing that for generic Λ,

$$(*) \qquad H^1\big(L_y, I_{X_y/L_y}(2)\big) = 0$$

for every $L_y \supset \Lambda$. To this end, choose Λ so that the projection $f\colon X \to \mathbb{P}^3$ has only ordinary singularities (cf. [GH, p. 616] or [Mo]), i.e. a curve of double points, and finitely many pinch-points and triple points. Then (*) is clear except perhaps when y is a pinch-point or triple point of f. But X_y is a scheme of length 2 in the first case, and X_y consists of three distinct points in the latter case. And for such schemes, (*) certainly holds. □

Remark. In a draft of this paper, we asserted that w_1 is surjective when $r \geqslant 5$, which amounts to saying that one can arrange for $X \cap L_y$ to consist of three non-collinear points whenever y is a triple point of f. However Grinberg pointed out that the argument given was faulty. He also observed that one could bypass the issue by working directly with w_2, as we originally had to do in any event when $r = 4$.

Fix now once and for all an $(r-4)$-plane $\Lambda \subset \mathbb{P}^r$ for which the assertion of Lemma 1.2 holds, and let $E = \ker(w_2)$. In view of (1.1), E fits into an exact sequence

$$(1.3) \qquad 0 \to E \to \mathcal{O}_{\mathbb{P}^3}(-2)^{N(r)} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{r-3} \oplus \mathcal{O}_{\mathbb{P}^3} \to f_*\mathcal{O}_X \to 0$$

of sheaves on $\mathbb{P}^3$. Since $f_*\mathcal{O}_X$ is a sheaf of two-dimensional Cohen–Macaulay modules over $\mathcal{O}_{\mathbb{P}^3}$, E is locally free. One has

$$(1.4) \quad rk(E) = r - 2 + N(r) \quad \text{and} \quad c_1(E) = -d - r + 3 - 2 \cdot N(r).$$

[The second assertion is a consequence of the fact that the vector bundle map in (1.3) drops rank on a surface of degree d.]

The point of this construction in the present situation is that it reduces one to proving some vanishings for E:

LEMMA 1.5. *Suppose that $H^1(\mathbb{P}^3, E(k)) = 0$ for some integer k. Then hypersurfaces of degree k cut out a complete linear series on $X \subset \mathbb{P}^r$.*

Proof. With notation as before, we may take $T_0, \ldots, T_4$ as homogeneous coordinates on the target of the projection $f\colon X \to \mathbb{P}^3$. Then the image of the map

$$(*) \qquad H^0\big(\mathbb{P}^3, \mathcal{O}(k-2)^{N(r)} \oplus \mathcal{O}(k-1)^{r-3} \oplus \mathcal{O}(k)\big)$$
$$\to H^0\big(\mathbb{P}^3, f_*\mathcal{O}_X(k)\big) = H^0(\mathbb{P}^r, \mathcal{O}_X(k))$$

determined by (1.3) consists of the restriction to X of all homogeneous polynomials of the form $\Sigma P_{ij}T_iT_j + \Sigma Q_iT_i + R$ ($4 \leqslant i, j \leqslant r$), where $P_{ij} \in H^0(\mathcal{O}_{\mathbb{P}^3}(k-2))$, $Q_i \in H^0(\mathcal{O}_{\mathbb{P}^3}(k-1))$, and $R \in H^0(\mathcal{O}_{\mathbb{P}^3}(k))$. In particular, the image of the canonical map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathbb{P}^r, \mathcal{O}_X(k))$ contains the image of (*), and the Lemma follows. □

§2. Cohomological computations. Keeping notation and assumptions as in §1, we now study the cohomological properties of the vector bundle E defined by (1.3). The first step is to realize E as the kernel of a surjective map of vector bundles on $\mathbb{P}^3$. Recall [M, Lecture 14] that a coherent sheaf F on some projective space $\mathbb{P}$ is said to be m-regular if $H^i(\mathbb{P}, F(m-i)) = 0$ for $i > 0$. As above, we set $N(r) = (r-2)(r-3)/2$.

LEMMA 2.1. *There is an exact sequence*

$$(2.2) \qquad 0 \to E \to B \to A \to 0$$

of vector bundles on $\mathbb{P}^3$, where A^* is (-1)-regular, and B^* is (-2)-regular. Furthermore, one has $rk(B) - rk(A) = r - 2 + N(r)$, and

$$c_1(A) - c_1(B) = d + r - 3 + 2 \cdot N(r).$$

Proof. Choosing coordinates as in §1, consider the graded module

$$M = \oplus H^0\big(\mathbb{P}^3, f_*\mathcal{O}_X(n)\big) = \oplus H^0(\mathbb{P}^r, \mathcal{O}_X(n))$$

over the homogeneous coordinate ring $\mathbb{C}[T_0, \ldots, T_3]$ of $\mathbb{P}^3$. The exact sequence (1.1) gives rise to generators of M; specifically, it determines one in degree 0, $r-3$ in degree 1 (viz. $T_4, \ldots, T_r$), and $N(r)$ generators in degree 2 (to wit, the T_iT_j). These can be expanded to a full set of generators of M by adding (say) n more generators, in degrees $a_1, \ldots, a_n \geqslant 1$. Setting $A = \oplus \mathcal{O}_{\mathbb{P}^3}(-a_i)$, this system of generators determines upon sheafifying an exact sequence

$$(2.3) \quad 0 \to B \to A \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^{N(r)} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{r-3} \oplus \mathcal{O}_{\mathbb{P}^3} \to f_* \mathcal{O}_X \to 0,$$

which defines a vector bundle B on $\mathbb{P}^3$ of rank $n + r - 2 + N(r)$. Comparing (1.3) with (2.3), one sees that E is isomorphic to the kernel of the (surjective) map $B \to A$ appearing in (2.3). This yields the desired sequence (2.2).

Since A^* is a direct sum of line bundles of degrees $\geqslant 1$, it is evidently (-1)-regular. To show that B^* is (-2)-regular, it is equivalent by duality to prove that

$$(*) \qquad H^i(\mathbb{P}^3, B(1-i)) = 0 \quad \text{for} \quad i = 0, 1, \text{and } 2.$$

When $i = 0$, (*) follows from the observation that since $X \subset \mathbb{P}^r$ is non-degenerate, there are no syzygies in degree one among the generators of M used to construct (2.3). The vanishing of $H^1(B)$ is clear, since in fact $H^1(\mathbb{P}^3, B(k)) = 0$ for all k by construction. For $i = 2$, note first that (2.3) gives $H^2(\mathbb{P}^3, B(-1)) = H^1(\mathbb{P}^3, f_* \mathcal{O}_X(-1))$. But $H^1(\mathbb{P}^3, f_* \mathcal{O}_X(-1)) = H^1(X, \mathcal{O}_X(-1)) = 0$ by Kodaira vanishing, and (*) is proved. Finally, the last assertion of the Lemma follows from (1.4). □

The one remaining point is the following

PROPOSITION 2.4. *Let A and B be vector bundles on some projective space $\mathbb{P}$, with*

$$rk(B) - rk(A) = s \quad \text{and} \quad c_1(A) - c_1(B) = \delta.$$

Assume that A^* is (-1)-regular and that B^* is (-2)-regular, and suppose that E is the kernel of a surjective vector bundle map from B to A:

$$0 \to E \to B \to A \to 0.$$

Then E is $(\delta - 2s + 2)$-regular. In particular, $H^1(\mathbb{P}, E(k)) = 0$ for $k \geqslant \delta - 2s + 1$.

The theorem follows at once. In fact, applying the Proposition to the exact sequence (2.2), one finds that $H^1(\mathbb{P}^3, E(k)) = 0$ for

$$k \geqslant (d + r - 3 + 2 \cdot N(r)) - 2(r - 2 + N(r)) + 1 = d + 2 - r.$$

But by Lemma 1.5, this is what we need to show. As for the Corollary, it remains to prove that $H^i(\mathbb{P}^r, \mathcal{I}_{X/\mathbb{P}^r}(d + 3 - r - i)) = H^{i-1}(X, \mathcal{O}_X(d + 3 - r - i)) = 0$ for $i \geqslant 2$, and this follows from (1.3) and the $(d + 3 - r)$-regularity of E. (Alternatively one could apply [GLP] to a general hyperplane section of X.)

Turning to the proof of Proposition 2.4, we start with two lemmas.

LEMMA 2.5. *Consider an exact sequence*

$$(2.6) \qquad \cdots \to F_i \to \cdots \to F_1 \to F_0 \to F \to 0$$

of coherent sheaves on a projective space $\mathbb{P}$. *Suppose that for some integer* p, F_i *is* $(p+i)$*-regular for each* $i \geqslant 0$. *Then* F *is* p*-regular.* □

LEMMA 2.7. *Let* F *and* G *be vector bundles on a projective space* $\mathbb{P}$. *If* F *is* p*-regular and* G *is* q*-regular, then* $F \otimes G$ *is* $(p+q)$*-regular. Furthermore,* $S^k(F)$ *and* $\Lambda^k(F)$ *are* (kp)*-regular.*

Proof. The p-regularity of F is equivalent to the existence of an exact sequence of the form (2.6) with $F_i = \oplus \mathcal{O}_{\mathbb{P}}(-p-i)$. Tensoring through by G, and noting that $G(-p-i)$ is $(q+p+i)$-regular, the first assertion follows from (2.5). In particular, the k-fold tensor product $T^k(F)$ of F is (kp)-regular. But as we are working in characteristic zero, $S^k(F)$ and $\Lambda^k(F)$ are direct summands of $T^k(F)$. □

Proof of Proposition 2.4. Suppose that $rk(A) = n$, so that $rk(B) = n + s$, and let a and b denote the first Chern classes of A^* and B^* respectively. The given sequence $0 \to E \to B \to A \to 0$ determines an Eagon–Northcott complex, which in the case at hand is an exact sequence

$$\cdots \to L_2 \to L_1 \to L_0 \to E \to 0$$

with

$$L_i = \Lambda^{n+i+1}B \otimes S^iA^* \otimes \mathcal{O}_{\mathbb{P}}(a)$$

(c.f. [S1] or [GP3]). Since $rk(B) = n + s$ and $\det(B) = \mathcal{O}_{\mathbb{P}}(-b)$, one can rewrite this as

$$L_i = \Lambda^{s-i-1}B^* \otimes S^iA^* \otimes \mathcal{O}_{\mathbb{P}}(a-b).$$

In view of the regularity hypotheses on A^* and B^*, it then follows from Lemma 2.7 that L_i is $(b-a-2s+2+i)$-regular. But $\delta = b - a$, and so E is $(\delta - 2s + 2)$-regular thanks to (2.5). □

Remark. At least for $r \geqslant 5$, the theorem is the best possible in the sense that for any $d \geqslant r$ there exists a surface $X \subset \mathbb{P}^r$ of degree d such that hypersurfaces of degree $d+1-r$ fail to cut out a complete series on X. In fact, it is enough to take X to be a rational ruled surface having a $(d+3-r)$-secant line; such surfaces can be constructed as divisors in three-folds of minimal degree. When $r = 4$ the theorem is optimal for projections of the Veronese surface, but probably not for arbitrary values of d.

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