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# Positive polynomials for ample vector bundles

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## Introduction

Our purpose is to describe all numerically positive polynomials in the Chern classes of an ample vector bundle. In particular, we complete a program initiated by Griffiths [16].

Let $P \in \mathbf{Q}[c_1, \ldots, c_e]$ be a weighted homogeneous polynomial of degree $n$, the variable $c_i$ being assigned weight $i$. We say that $P$ is *numerically positive for ample vector bundles* if for every projective variety $X$ of dimension $n$, and every ample vector bundle $E$ of rank $e$ on $X$, the Chern number

$$\int_X P(c_1(E), \ldots, c_e(E))$$

is strictly positive. (We follow Hartshorne's definition [20] of ample vector

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bundles.) For example, a theorem of Bloch and Gieseker [1] asserts that the Chern class $c_n$ is numerically positive for ample bundles provided that $n \leq e$.

Denote by $\Lambda(n, e)$ the set of all partitions of $n$ by non-negative integers $\leq e$. Thus an element $\lambda \in \Lambda(n, e)$ is specified by a sequence

$$e \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad \text{with} \quad \sum \lambda_i = n.$$

Each $\lambda \in \Lambda(n, e)$ gives rise to a *Schur polynomial* $P_\lambda \in \mathbf{Q}[c_1, \ldots, c_e]$ of degree $n$, defined as the $n \times n$ determinant

$$P_\lambda = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+n-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+n-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{\lambda_n-n+1} & c_{\lambda_n-n+2} & \cdots & c_{\lambda_n} \end{vmatrix},$$

where by convention $c_0 = 1$ and $c_i = 0$ if $i \notin [0, e]$.* Schur polynomials have been extensively studied in connection with symmetric functions and representations of the symmetric group (cf. [25], [28], [29]). Geometrically, if $Q$ is the universal quotient bundle on the Grassmannian $G(m-e, m)$ of codimension $e$ subspaces of an $m$-dimensional vector space, then $P_\lambda(c(Q))$ is represented by the Schubert cycle $\sigma_\lambda = \sigma_{\lambda_1, \ldots, \lambda_n}$ (in the notation of [18], Chapter 1, §5). The classes $P_\lambda(c(Q))$ $(\lambda \in \Lambda(n, e))$ span the cone of effective codimension $n$ cycles on $G(m-e, m)$, and are independent if $m \geq n + e$ (cf. [18], [23], [29]).

The Schur polynomials $P_\lambda$ $(\lambda \in \Lambda(n, e))$ form a basis for the $\mathbf{Q}$-vector space of weighted homogeneous polynomials of degree $n$ in $e$ variables. Given such a polynomial $P$, write

$$P = \sum_{\lambda \in \Lambda(n, e)} a_\lambda(P) P_\lambda \qquad (a_\lambda(P) \in \mathbf{Q}).$$

Our main result is the following

THEOREM I. *The polynomial $P$ is numerically positive for ample vector bundles if and only if $P$ is non-zero and*

$$a_\lambda(P) \geq 0 \quad \textit{for all} \quad \lambda \in \Lambda(n, e).$$

For example, taking $\lambda = (n, 0, \ldots, 0)$ we recover the theorem of Bloch and Gieseker [1] that $c_n$ is numerically positive when $e \geq n$. For $\lambda = (1, \ldots, 1)$, $P_\lambda(c(E))$ is the $n$th Segre class (i.e., inverse Chern class) of $\check{E}$; the positivity of these classes for ample bundles was proved in [13]. We remark that it follows from the theorem and the theory of Schur polynomials that *the product of*

*This is a slight abuse of standard terminology: strictly speaking, $P_\lambda$ is the result of setting the variables $c_{e+1}, c_{e+2}, \ldots$ equal to zero in the $\lambda^{\text{th}}$ Schur polynomial.

*numerically positive polynomials is again numerically positive.* (We shall give a direct proof.) In particular, any monomial in the Chern classes $c_i$ $(i \leq e)$ is numerically positive for ample vector bundles. This was proved by Gieseker [13] for quotients of direct sums of ample line bundles. The case $n = 2$ of the theorem is due to Kleiman [24].

Theorem I is closely related to some results and conjectures of Griffiths ([16], [17]). Specifically, Griffiths gave an essentially analytic definition of a cone $\Pi(e) \subseteq \mathbf{Q}[c_1, \ldots, c_e]$ of what he termed positive polynomials, and proved that they are in fact numerically positive for certain classes of vector bundles. He conjectured the numerical positivity of these polynomials for positive, and more generally, for ample vector bundles ([16], Conjecture (0.7); [17], p. 38). We use a result of Usui and Tango [30], who proved Griffiths' conjecture for ample bundles generated by their global sections, to show that any $P \in \Pi(e)$ is a non-negative linear combination of the Schur polynomials. Thus the general case of the conjecture follows from Theorem I. To complete the picture, we prove that the Schur polynomials $P_\lambda$ are in the Griffiths cone $\Pi(e)$, which thus consists *precisely* of the numerically positive polynomials for ample bundles. It was through the work just mentioned of Usui and Tango that we became aware of the fundamental role of the Schur polynomials (although they do not appear in [30] explicitly).

We may explain the idea of the proof of Theorem I. Given a polynomial $P = \Sigma a_\lambda(P)P_\lambda$ with $a_\mu(P) < 0$ for some $\mu \in \Lambda(n, e)$, one easily constructs a variety $X$ and an ample vector bundle $E$ with $\int_X P(c(E)) < 0$. Thus the real issue is to show that the Schur polynomials are numerically positive for ample bundles, and to this end our approach is to focus on the numerical properties of cones in vector bundles. Specifically, let $E \to X$ be a vector bundle of rank $e$, and let $C \subseteq E$ be a cone in $E$ (i.e., a subvariety stable under the natural $\mathbf{C}^*$ action on $E$). Assume that $C$ has pure dimension $e$. Then the intersection class of $C$ with the zero-section $X \hookrightarrow E$ is a well-defined zero-dimensional homology (or rational equivalence) class on $X$, which we denote by $z(C, E) \in H_0(X)$. These cone classes arise extensively in the intersection theory developed by R. MacPherson and the first author (cf. [10], [11], [7]). In the present context, our basic technical result is

THEOREM II. *If $E$ is ample, then the cone class $z(C, E)$ has strictly positive degree.*

To understand the connection with Theorem I, consider the problem of proving that $\int_X c_n(E) > 0$ when $e \geq n = \dim X$. Suppose for the moment that $E$ is generated by its global sections. Then $\int_X c_n(E)$ is just the number of points of $X$ at which $e - n + 1$ sufficiently general sections of $E$ become linearly dependent.

To say the same thing differently, let $V$ be a trivial vector bundle of rank $e - n + 1$, and let $H$ be the vector bundle $\mathrm{Hom}(V, E)$. Inside $H$ one has the cone $\Omega$, of codimension $n$, whose fibre over $x \in X$ consists of homomorphisms $\sigma\colon V(x) \to E(x)$ of rank $\leq e - n$. Giving $e - n + 1$ sections of $E$ amounts to choosing a section $s\colon X \hookrightarrow H$, and the resulting dependency locus is just $s^{-1}(\Omega)$. If $s$ is sufficiently general, then $s^{-1}(\Omega)$ represents the class $z(\Omega, H)$, and we see that

$$z(\Omega, H) = c_n(E) \cap [X].$$

The point is that this formula holds for *any* vector bundle $E$. So provided only that $E$ (and hence $H$) are ample, the positivity of $\int_X c_n(E)$ follows from Theorem II. A similar argument handles all Schur polynomials. As for Theorem II, it quickly reduces to proving the numerical positivity of the top Chern class of a certain (non-ample) vector bundle. This is accomplished by use of the results and techniques of Bloch and Gieseker [1].

The definitions and results required from intersection theory are reviewed in Section 1, along with some preliminary lemmas; Section 2 is devoted to the proof of Theorem II. Applications to numerically positive polynomials occupy Section 3. Besides proving Theorem I (§ 3a), we establish the numerical positivity of various products involving possibly more than one ample vector bundle (§ 3c). We also give a simple application of the theorem to the degeneration of vector bundle maps (§ 3b). The relation of our work to that of Griffiths is discussed in some detail in Appendix A. Finally, in Appendix B we sketch a proof of the theorem of Bloch and Gieseker valid in all characteristics. In their original argument, Bloch and Gieseker needed resolution of singularities in order to apply the hard Lefschetz theorem. Hironaka's theorem can now be circumvented by use of the Goresky-MacPherson-Deligne intersection homology groups on singular varieties, for which the hard Lefschetz theorem has recently been proved by Gabber. In the classical case, this approach also leads to a simplification of the proof of the Bloch-Gieseker theorem.

## 0. Notation and conventions

(0.1) We work over an algebraically closed field of arbitrary characteristic. A *variety* is reduced and irreducible.

(0.2) If $E$ is a vector bundle on a variety $X$, $\mathbf{P}(E)$ is the projective bundle of one-dimensional *subspaces* of $E$. $S^k(E)$ denotes the $k$th symmetric power of $E$. Recall that $E$ is *ample* [20] if for every coherent sheaf $\mathcal{F}$ on $X$ there is an integer $k(\mathcal{F}) > 0$ such that $S^k(E) \otimes \mathcal{F}$ is generated by its global sections for all $k \geq k(\mathcal{F})$.

We refer to [20] for alternative characterizations and the general theory of ample bundles. We will often use without explicit mention the most basic facts, notably that a quotient or direct sum of ample bundles is ample, and that amplitude is preserved under pulling back by finite maps.

(0.3) Pending the appearance of [7], we refer to [5] for the theory of rational equivalence on possibly singular varieties. If $X$ is a variety, $A_k(X)$ denotes the Chow group of $k$-dimensional cycles on $X$ modulo rational equivalence. (When $X$ is complete, the reader so inclined may replace $A_k(X)$ by $H_{2k}(X)$.) Suppose that $X$ is complete, of dimension $n$. Then the *degree* of a class $\alpha \in A_0(X)$ is well-defined. If $L$ is a line bundle on $X$, and $\alpha \in A_k(X)$, then the *L-degree of* $\alpha$, written $\deg_L(\alpha)$, is the degree of the zero-dimensional class $c_1(L)^k \cap \alpha$. Let $P \in \mathbf{Q}[c_1, \ldots, c_e]$ be (weighted) homogeneous of degree $n$, the grading on $\mathbf{Q}[c_1, \ldots, c_e]$ being given as always by assigning weight $i$ to the variable $c_i$. If $E$ is a vector bundle of rank $e$ on $X$, then the cap product

$$P(c_1(E), \ldots, c_e(E)) \cap [X]$$

is a class in $A_0(X)_{\mathbf{Q}} = A_0(X) \otimes \mathbf{Q}$. We write

$$\int_X P(c_1(E), \ldots, c_e(E)), \quad \text{or} \quad \int_X P(c(E))$$

to denote its degree. By abuse of notation we write the same thing if $E$ is a vector bundle on some ambient variety containing $X$, instead of the more cumbersome $\int_X P(c(E|X))$. Finally, note that if $f\colon Y \to X$ is a finite surjective map, then

$$f_*(P(c(f^*E)) \cap [Y]) = P(c(E)) \cap f_*([Y])$$

by the projection formula. In particular,

$$\int_Y P(c(f^*E)) = (\deg f) \int_X P(c(E)).$$

## I. Preliminary lemmas; Cone classes

**Extracting roots of line bundles.** We shall make repeated use of the following lemma of Bloch and Gieseker, which roughly speaking has the effect of allowing one to deal with fractional powers of a line bundle.

LEMMA I.1 ([1], §2). *Let $X$ be a projective variety and $\xi$ a line bundle on $X$. Fix a positive integer $k$. Then there exists a variety $Y$, a finite surjective map*

$$f\colon Y \to X,$$

*and a line bundle* $\eta$ *on* $Y$ *such that*

$$f^*\xi = \eta^{\otimes k}.$$

*Moreover, we can take* $\eta$ *to be very ample if* $\xi$ *is.*

*Proof.* We first suppose $\xi$ is very ample, so that $\xi = \mathcal{O}_{\mathbf{P}^N}(1)|X$ for some embedding $X \subseteq \mathbf{P}^N$. Choose a branched covering $\bar{f}$: $\mathbf{P}^N \to \mathbf{P}^N$ with $\bar{f}^*\mathcal{O}_{\mathbf{P}^N}(1) = \mathcal{O}_{\mathbf{P}^N}(k)$. Then simply take $Y$ to be an irreducible component of the inverse image $\bar{f}^{-1}(X)$; $f$: $Y \to X$ is the natural projection, and $\eta = \mathcal{O}_{\mathbf{P}^N}(1)|Y$. Note that $\eta$ is very ample by construction. In general, choose a line bundle $\xi_1$ such that $L = \xi \otimes \xi_1^{\otimes k}$ is very ample, and apply the case just treated to $L$. □

Suppose in the situation of the lemma that $X$ has dimension $n$ and that $P \in \mathbf{Q}[c_1,\ldots,c_e]$ is a weighted homogeneous polynomial of degree $n$. If $E$ is a vector bundle of rank $e$ on $X$, then we set, for $t \in \mathbf{Z}$:

$$F_\xi(t) = \int_X P\big(c_1(E \otimes \xi^{\otimes t}),\ldots, c_e(E \otimes \xi^{\otimes t})\big)$$

and

$$F_\eta(t) = \int_Y P\big(c_1(f^*E \otimes \eta^{\otimes t}),\ldots, c_e(f^*E \otimes \eta^{\otimes t})\big).$$

Then there are polynomials $P_i \in \mathbf{Q}[c_1,\ldots,c_e]$ of degree $i$, depending only on $P$, such that

$$F_\xi(t) = \sum_{i=0}^{n}\left\{\int_X c_1(\xi)^{n-i}P_i(c(E))\right\}t^{n-i}$$

and

$$F_\eta(t) = \sum_{i=0}^{n}\left\{\int_Y c_1(\eta)^{n-i}P_i(c(f^*E))\right\}t^{n-i}.$$

LEMMA 1.2. *Viewing* $F_\xi(t)$ *and* $F_\eta(t)$ *as polynomials in* $t$, *one has*

$$F_\eta(t) = (\deg f)F_\xi(t/k).$$

*Proof.*

$$(\deg f)\int_X c_1(\xi)^{n-i}P_i(c(E)) = \int_Y c_1(f^*\xi)^{n-i}P_i(c(f^*E))$$

$$= k^{n-i}\int_Y c_1(\eta)^{n-i}P_i(c(f^*E)).$$ □

**Cone classes.** We briefly review here the definitions and facts from intersection theory required in the proofs of our main results. Everything we need will appear in the forthcoming book [7]; in the meantime, the reader may consult [10], [11], and [31] for details.

Let $X$ be a variety, and $p_E: E \to X$ a vector bundle of rank $e$ on $X$. Denote by $0_E: X \hookrightarrow E$ the zero-section. A *cone* in $E$ is a subscheme $C \subseteq E$ stable under the natural $\mathbf{G}_m$-action on $E$. A cone $C \subseteq E$ gives rise in the obvious way to a subscheme $\mathbf{P}(C) \subseteq \mathbf{P}(E)$.

If $C \subseteq E$ is a cone of pure dimension $c$, then one may intersect its cycle $[C]$ with the zero-section of $E$: the result is a well-defined rational equivalence class of dimension $c - e$ on $X$. We denote this class by

$$z(C, E) \in A_{c-e}(X).$$

In case there is a section $s: X \hookrightarrow E$ which meets $C$ properly (i.e., $\dim(C \cap s(X)) = c - e$ if $C \cap s(X) \neq \varnothing$), then $z(C, E)$ is represented by the intersection cycle $[X] \cdot_s [C]$ defined as in Serre [27] (cf. [10], §6). If $C$ is a sub-bundle of $E$, of rank $f$, then $z(C, E) = c_{e-f}(E/C) \cap [X]$ (cf. [10], §1).

In the general case, there are several equivalent constructions of $z(C, E)$. First,

$$z(C, E) = 0_E^*[C], \tag{1.3}$$

where $0_E^*: A_c(E) \to A_{c-e}(X)$ is the Gysin homomorphism determined by the zero-section. Alternatively, since $0_E^*$ is the inverse isomorphism to the flat pull-back $p_E^*: A_{c-e}(X) \to A_c(E)$ (defined on the cycle level by $[V] \mapsto [p_E^{-1}(V)]$), $z(C, E)$ is the unique cycle class on $X$ such that

$$p_E^*(z(C, E)) = [C] \quad \text{in } A_c(E). \tag{1.4}$$

(The equivalence of (1.3) and (1.4) uses the functoriality $0_E^* \circ p_E^* = (p_E \circ 0_E)^* = \mathrm{id}$; cf. [31].)

If every irreducible component of $C$ maps to its support on $X$ with fibre dimension $\geq 1$, then

$$z(C, E) = \pi_*\big(c_{e-1}(Q_{\mathbf{P}(E)}) \cap [\mathbf{P}(C)]\big), \tag{1.5}$$

where $Q_{\mathbf{P}(E)} = \pi^*E/\mathcal{O}_{\mathbf{P}(E)}(-1)$ is the rank $e - 1$ universal quotient bundle on the projectivization $\pi: \mathbf{P}(E) \to X$. For an arbitrary cone $C \subseteq E$ one has the inclusion $\mathbf{P}(C \oplus 1) \subseteq \mathbf{P}(E \oplus 1)$ of projective completions, and

$$z(C, E) = \pi_*\big(c_e(Q_{\mathbf{P}(E\oplus 1)}) \cap [\mathbf{P}(C \oplus 1)]\big). \tag{1.6}$$

(The equivalence of (1.5) and (1.6) is in [10], §3. For (1.4) and (1.6), consider the embedding $E \subseteq \mathbf{P}(E \oplus 1)$, with complement $\mathbf{P}(E)$. If we define $z(C, E)$ by (1.4), the class $\beta = \pi^*z(C, E) - [\mathbf{P}(C \oplus 1)]$ restricts to zero in $A_c(E)$, and hence is represented by a cycle on $\mathbf{P}(E)$. Since the restriction of $Q_{\mathbf{P}(E\oplus 1)}$ to $\mathbf{P}(E)$ has a trivial summand, it follows that $c_e(Q_{\mathbf{P}(E\oplus 1)}) \cap \beta = 0$. Equation (1.6) is then a consequence of the elementary fact that $\pi_*(c_e(Q_{\mathbf{P}(E\oplus 1)}) \cap \pi^*\alpha) = \alpha$ for all $\alpha \in A_*(X)$.)

If $F$ is a vector bundle on $X$, let $C \oplus F$ denote the fibre product $C \times_X F$, which sits as a cone in $E \oplus F$. Then

$$z(C \oplus F, E \oplus F) = z(C, E). \tag{1.7}$$

Finally, if $F$ has rank $f$, then for the embedding $C = C \oplus 0 \subseteq E \oplus F$ one has

$$z(C, E \oplus F) = c_f(F) \cap z(C, E). \tag{1.8}$$

((1.7) follows immediately from (1.4). For (1.8) one uses the self-intersection formula (cf. [10], §5)

$$s^* s_*(\alpha) = c_f(p_E^* F) \cap \alpha$$

where $s\colon E \hookrightarrow E \oplus F$ is the embedding $e \mapsto (e, 0)$.)

## 2. Positivity of cone classes

This section is devoted to the proof of the following theorem, which is the basic technical result of the present paper.

THEOREM 2.1. *Let $E$ be an ample vector bundle of rank $e$ on a projective variety $X$, and let $C \subseteq E$ be a cone of pure dimension $e$. Then the cone class*

$$z(C, E) \in A_0(X)$$

*has strictly positive degree.*

As a simple consequence, we note

COROLLARY 2.2. *In the situation of the theorem, suppose that $C$ has pure dimension $c \geq e$. Then*

$$\deg_L(z(C, E)) > 0$$

*(see (0.3)) for any ample line bundle $L$ on $X$. In particular, $z(C, E) \neq 0$.*

*Proof.* Let $F$ be the direct sum of $c - e$ copies of $L$, and consider the embedding $C = C \oplus 0 \subseteq E \oplus F$. Then $\deg_L(z(C, E)) = \deg z(C, E \oplus F)$ by (1.8), and the theorem applies. □

*Remarks.* (1) It follows from the corollary that if $M$ is a variety, if $X \subseteq M$ is a projective local complete intersection of codimension $e$ with ample normal bundle, and if $Y \subseteq M$ is a subvariety of dimension $\geq e$ that meets $X$ (possibly improperly), then the intersection class $[X] \cdot [Y]$ is non-zero in $A_*(X)$. For instance if $M$ is acted on transitively by a connected algebraic group, then *any* subvariety $Z$ of dimension $\geq e$ must meet $X$. (Proof: by homogeneity one can find a translate $Y$ of $Z$ which meets $X$, and $[Y]$ is algebraically equivalent to $[Z]$.) This simplifies and somewhat extends a result of Lübke [26]. See [9] for this and other applications to intersection theory.

(2) If one assumes in addition that $E$ is generated by its global sections, then there is an extremely simple and elementary proof of Theorem 2.1. This also appears in [9] and [7].

Return now to the statement of the theorem. According to (1.7), it suffices to prove the result for the cone $C \oplus L \subseteq E \oplus L$, where $L$ is any ample line bundle on $X$. Since $z(C \oplus L, E \oplus L)$ may be computed by the formula (1.5), Theorem 2.1 is a consequence of the first assertion of

THEOREM 2.3. *Let $F$ be an ample vector bundle of rank $e + 1$ on a projective variety $X$, and consider the projectivized bundle*

$$\pi: \mathbf{P}(F) \to X$$

*with universal quotient bundle $Q_{\mathbf{P}(F)} = \pi^* F / \mathcal{O}_{\mathbf{P}(F)}(-1)$ of rank $e$. Then for any subvariety $T \subseteq \mathbf{P}(F)$ of dimension $e$, one has*

$$\int_T c_e(Q_{\mathbf{P}(F)}) > 0.$$

*More generally, let $S \subseteq \mathbf{P}(F)$ be a subvariety of dimension $n$, put*

$$a_S = \dim S - \dim \pi(S),$$

*and fix a very ample line bundle $\xi$ on $X$. Then, provided that $a_S \leq q$, one has*

$$\int_S c_1(\pi^*\xi)^{n-q} \cdot c_q(Q_{\mathbf{P}(F)}) > 0 \quad \text{for } q \leq e. \tag{2.4}$$

Note that if $\dim T = e$, the condition $a_T \leq e$ ($=$ relative dimension of $\pi$) is automatic.

We start by proving a non-negativity assertion:

LEMMA 2.5. *Let $E$ be a vector bundle of rank $e$ on a projective variety $S$ of dimension $n$. Assume that $S^m(E)$ is generated by its global sections for some $m > 0$. Then*

$$\int_S c_n(E) \geq 0.$$

*Proof.* If $n > e$ there is nothing to prove; so assume $n \leq e$. Fix an ample line bundle $\xi$ on $S$, and let

$$P_\xi(t) = \int_S c_n(E \otimes \xi^{\otimes t}),$$

which we view as a polynomial in $t$ (cf. §1). For any positive integer $k$, consider a corresponding Bloch-Gieseker covering

$$f\colon T \to S, \qquad f^*\xi = \eta^{\otimes k},$$

with $\eta$ ample, as in Lemma 1.1. Then setting $P_\eta(t) = \int_T c_n(f^*E \otimes \eta^{\otimes t})$, we have $P_\eta(t) = (\deg f)P_\xi(t/k)$ by Lemma 1.2.

By hypothesis, $S^m(f^*E)$ is generated by its global sections, and thus $S^m(f^*E \otimes \eta)$ is an ample vector bundle. Hence so too is $f^*E \otimes \eta$ ([20], (2.4)). But then

$$P_\xi(1/k) = \frac{1}{(\deg f)} \int_T c_n(f^*E \otimes \eta) > 0$$

by the theorem of Bloch and Gieseker [1]. As this holds for any $k > 0$, $P_\xi(0)$ is non-negative. □

*Remarks.* (1) Concerning our use of the theorem of Bloch and Gieseker in arbitrary characteristic on the possibly singular variety $T$, see Appendix B.

(2) This is the only point at which the Bloch-Gieseker theorem is used, and it would be interesting to give a proof of the lemma avoiding the hard Lefschetz theorem. Note that it would be enough to know the numerical non-negativity of the Chern classes of an ample vector bundle.

*Proof of Theorem* 2.3. Since $F$ is ample, the vector bundle $S^m(F) \otimes \check{\xi}$ is generated by its global sections for some $m > 0$. It is enough to prove the theorem after passing to a branched covering $f\colon Y \to X$, by replacement of $\mathbf{P}(F)$ by $\mathbf{P}(f^*F)$, and of $S$ by some subvariety $T \subseteq \mathbf{P}(f^*F)$ mapping onto $S$. Hence applying Lemma 1.1 with $k = m$, we may assume that

(2.6) $\quad S^m(F \otimes \check{\xi})$ is generated by its global sections for some $m > 0$.

We now proceed to prove the theorem by induction on $n$. Since the result is trivial when $n = 0$, we assume the theorem known for all subvarieties of dimension $< n$. Since $\xi$ is very ample, the condition that $a_S \le q$, i.e., the condition that $\dim \pi(S) \ge n - q$, implies that $c_1(\pi^*\xi)^{n-q} \cap [S]$ is represented by a non-zero effective $q$-cycle, with $q \le e$. For $q < n$, the induction hypothesis applies to the components of this cycle. Hence we may suppose that $n \le e$, and that (2.4) is known (if $a_S \le q$) for $q < n$; and it is enough to prove

$$(*) \qquad \int_S c_n(Q_{\mathbf{P}(F)}) > 0.$$

Observe next that $(*)$ is clear if $a_S = n$: for then $S$ lies in a fibre of $\pi$, and $c_n(Q_{\mathbf{P}(F)}) \cap [S]$ is represented by the intersection, in that fibre, of $S$ with a codimension $n$ linear space. So we assume that $a_S \leq n-1$, in which case the inequality (2.4) holds by hypothesis at least for $q = n-1$; i.e., we have

$$\int_S c_1(\pi^*\xi)c_{n-1}(Q_{\mathbf{P}(F)}) > 0. \tag{2.7}$$

Arguing by contradiction, suppose now that (*) is false. Set $L = \pi^*\xi$, and put

$$\begin{aligned} P_L(t) &= \int_S c_n\big(Q_{\mathbf{P}(F)} \otimes \check{L}^{\otimes t}\big) \\ &= \sum_{i=0}^{n} (-1)^i \binom{e-n+i}{i} \left\{ \int_S c_1(L)^i c_{n-i}\big(Q_{\mathbf{P}(F)}\big) \right\} t^i. \end{aligned}$$

Viewing $P_L(t)$ as a polynomial in $t$, we assert that $P_L(1/k) < 0$ for $k \gg 0$. Indeed, if $\int_S c_n(Q_{\mathbf{P}(F)}) < 0$ this is obvious, while if $\int_S c_n(Q_{\mathbf{P}(F)}) = 0$ it follows from (2.7). Fix an integer $k$ for which $P_L(1/k) < 0$, and use Lemma 1.1 to construct a finite surjective map $f\colon Y \to X$, plus a very ample line bundle $\eta$ on $Y$ such that $f^*\xi = \eta^{\otimes k}$. We consider the fibre square:

$$\bar{\pi}^*\eta = M \qquad \begin{array}{ccc} \mathbf{P}(f^*F) & \xrightarrow{\bar{f}} & \mathbf{P}(F) \\ {\scriptstyle\bar{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[f]{} & X \end{array} \qquad \pi^*\xi = L$$

with notation as indicated. Observe that $\bar{f}^*L = M^{\otimes k}$, and that $Q_{\mathbf{P}(f^*F)} = \bar{f}^*Q_{\mathbf{P}(F)}$. Choose any subvariety $T \subseteq \mathbf{P}(f^*F)$ mapping onto $S$. Then by Lemma 1.2 we have

$$\int_T c_n\big(Q_{\mathbf{P}(f^*F)} \otimes \check{M}\big) = \deg(T/S) \cdot P_L(1/k) < 0. \tag{2.8}$$

On the other hand, it follows from (2.6) that $S^m(f^*F \otimes \check{\eta})$ is a quotient of a direct sum of copies of $\eta^{\otimes m(k-1)}$, and hence, since $\eta$ is very ample, is generated by its global sections. Therefore $S^m(Q_{\mathbf{P}(f^*F)} \otimes \check{M})$ is also generated by its global sections, and so

$$\int_T c_n\big(Q_{\mathbf{P}(f^*F)} \otimes \check{M}\big) \geq 0$$

by Lemma 2.5. This contradicts (2.8), and proves the theorem. □

*Remark*. Observe that the proof does not depend very strongly on the fact that $P = \mathbf{P}(F)$ arises as a projective bundle. In the statement of Theorem 2.3, it would have been enough to assume, say, that $\pi\colon P \to X$ is a proper flat map of relative dimension $e$, and that $Q$ is a rank $e$ quotient of $\pi^*F$ (for some ample vector bundle $F$ on $X$) such that the Chern classes of $Q$ are numerically positive on each fibre.

## 3. Positive polynomials for ample vector bundles

**3a. Proof of Theorem I.** We use the notation established in the introduction, where Theorem I is stated.

THEOREM 3.1. *Each of the Schur polynomials* $P_\lambda \in \mathbf{Q}[c_1, \ldots, c_e]$ $(\lambda \in \Lambda(n, e))$ *is numerically positive for ample vector bundles.*

*Proof.* The method is to realize $P_\lambda$ as a cone class, and then to invoke Theorem 2.1. To this end, suppose that $\lambda$ is the partition $e \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$ of $n$. Let $V$ be a vector space of dimension $n + e$, and fix a flag of subspaces $0 \subset V_1 \subset \cdots \subset V_n \subseteq V$ with $\dim V_i = e + i - \lambda_i$. Given a vector bundle $E$ of rank $e$ on a projective variety $X$, let

$$p: H(E) \to X$$

be the vector bundle $\mathrm{Hom}(V_X, E)$, where $V_X$ denotes the trivial bundle $X \times V$ on $X$. Consider the cone

$$\Omega_\lambda(E) \subseteq H(E)$$

whose fibre over $x \in X$ consists of all $\sigma \in \mathrm{Hom}(V, E(x))$ such that $\dim(\ker \sigma \cap V_i) \geq i$. Note that $\Omega_\lambda(E)$ is flat (in fact, locally a product) over $X$, and has codimension $n = \Sigma\lambda_i$ in $H(E)$ (cf. [22], [23]). Alternatively, if $u$: $p^*V_X \to p^*E$ is the tautological homomorphism on $H(E)$, then $\Omega_\lambda(E)$ may be realized as the locus where each of the compositions

$$p^*(V_i)_X \hookrightarrow p^*V_X \xrightarrow{u} p^*E \qquad (1 \leq i \leq n) \tag{3.2}$$

has rank $\leq e - \lambda_i$. In particular, this defines a natural scheme structure on $\Omega_\lambda(E)$ (which turns out to be reduced and irreducible, cf. [22]).

Since $H(E)$ is ample if $E$ is, Theorem 3.1 is a consequence of Theorem 2.1 and

LEMMA 3.3. *The cone class defined by* $\Omega_\lambda(E)$ *in* $H(E)$ *is given by*

$$z(\Omega_\lambda(E), H(E)) = P_\lambda(c_1(E), \ldots, c_e(E)) \cap [X].$$

*Proof.* At least when $X$ is smooth, the determinantal formula of Kempf and Laksov ([23], Thm. 10) applies in the situation of (3.2) to show that the class of the cycle of $\Omega_\lambda(E)$ is expressed in $A_*(H(E))$ as

$$[\Omega_\lambda(E)] = \left\{\det\left(c_{\lambda_i - i + j}(p^*E)\right)_{1 \leq i, j \leq n}\right\} \cap [H(E)]. \tag{*}$$

Equivalently,

$$[\Omega_\lambda(E)] = p^*\{P_\lambda(c(E)) \cap [X]\},$$

where $p^*$: $A_*(X) \to A_{*+e(n-e)}H(E)$ denotes the flat pull-back. Since $z(\Omega_\lambda(E), H(E))$ is the unique class on $X$ pulling up to $[\Omega_\lambda(E)]$ (1.4), the assertion of the lemma follows. If $X$ is singular, the proof of Kempf and Laksov extends without essential change to show that (*) remains true. Details appear in Chapter 14 of the forthcoming book [7]. (One could also deduce the singular from the smooth case by choosing a closed embedding of $X$ in a non-singular variety $X'$ in such a way that $E$ is the restriction of a vector bundle $E'$ on $X'$ ([5], §3.2), and pulling back the desired formula from $H(E')$.) □

Conversely:

Proposition 3.4. *Let $P \in \mathbf{Q}[c_1, \ldots, c_e]$ be a weighted homogeneous polynomial of degree $n$, of the form*

$$P = \sum_{\lambda \in \Lambda(n,e)} a_\lambda(P) P_\lambda \qquad (a_\lambda(P) \in \mathbf{Q}),$$

*where*

$$a_\mu(P) < 0 \quad \text{for some } \mu \in \Lambda(n,e).$$

*Then there exists a projective variety $X$ of dimension $n$ and an ample vector bundle $E$ of rank $e$ on $X$ such that*

$$\int_X P(c(E)) < 0.$$

The construction will show that one can even take $E$ to be the quotient of a direct sum of very ample line bundles.

*Proof.* Let $Q$ be the rank $e$ universal quotient bundle on the Grassmannian $G(m-e, m)$ of codimension $e$ subspaces of an $m$-dimensional vector space. Assuming that $m \geq n + e$, the classes $P_\lambda(c(Q)) \cap [G(m-e, m)]$ ($\lambda \in \Lambda(n,e)$) are represented by non-empty independent Schubert cycles $\Omega_\lambda$ of codimension $n$ (cf. [18], Chapter 1, §5). Let $Y \subseteq G(m-e, m)$ be an $n$-dimensional Schubert variety *dual* to $\Omega_\mu$, and let $E' = Q|Y$. Thus $\int_Y P_\lambda(c(E')) = \delta_{\mu\lambda}$, and hence

$$(*) \qquad \int_Y P(c(E')) = a_\mu(P) < 0.$$

Fix a very ample line bundle $\xi$ on $Y$, and consider the polynomial

$$F_\xi(t) = \int_Y P\big(c(E' \otimes \xi^{\otimes t})\big)$$

(cf. §1). By (*) we may choose a positive integer $k$ large enough so that $F_\xi(1/k) < 0$. Form the corresponding Bloch-Gieseker covering (Lemma 1.1):

$$f\colon X \to Y, \qquad f^*\xi = \eta^{\otimes k},$$

with $\eta$ very ample. Then

$$\int_X P(c(f^*E' \otimes \eta)) = (\deg f) F_\xi(1/k) < 0$$

by Lemma 1.2. But $E'$ is generated by its global sections, and so $E = f^*E' \otimes \eta$ is ample, as required. This completes the proof of Theorem I. □

*Remarks.* (1) The variety $X$ just constructed is usually singular, but with a little more care one can produce smooth examples, as follows. First replace the Schubert variety $Y$ by its canonical desingularization $\tilde{Y}$, and $E'$ by its pull-back to $\tilde{Y}$. Then the more precise version of Lemma 1.1 proved in [1] shows that if $k$ is prime to the characteristic of the ground field, one can find a Bloch-Gieseker covering of $\tilde{Y}$ by a *smooth* variety $X$, and the argument proceeds as before. (One can take $X$ non-singular even if $k$ is not prime to the characteristic, but then it is conceivably no longer possible to have $\eta$ very ample.)

(2) One of the motivations for early work on these questions had been to find a numerical criterion for ampleness analogous to the theorem of Nakai *et al.* for the line bundles. However it was shown in [6] that no such criterion exists.

(3) While Theorem I gives a good picture of the *numerical* properties of ample vector bundles, practically nothing seems to be known about the *effectivity* of the cycles in question, which suggests the following:

*Problem. If E is an ample vector bundle of rank e on an n-dimensional projective variety X, determine whether or not some multiple of a given Schur class*

$$P_\lambda(c(E)) \cap [X], \qquad \lambda \in \Lambda(k, e), k \leq n,$$

*is effective.*

This would already be interesting to know for algebraic equivalence. It is elementary that a sufficiently high multiple of the dual Segre class (corresponding to $\lambda = (1, \ldots, 1)$) is effective [6], but this is the only nontrivial result of which we are aware. When $E$ is generated by its global sections, then the Schur classes themselves are effective. Hence the problem has an affirmative solution whenever one can find a proper, surjective, generically finite map $f\colon Y \to X$ such that $f^*E$ is generated by its global sections. However, examples of Gieseker ([13], pp. 111–112), plus the Ramanujam vanishing theorem, show in characteristic zero that such a map need not exist. We remark that Gieseker's examples also give rise to counterexamples to a conjecture of Hartshorne ([21], III.4.4).

**3b. Application to degeneracy loci of vector bundle maps.** The positivity of cone classes may also be applied to the degeneration of vector bundle homomorphisms. Specifically, let $E$ and $F$ be vector bundles of ranks $e$ and $f$ on a projective variety $X$. One may consider three types of vector bundle maps:

(A) $u\colon E \to F$, $u$ arbitrary;

(B) $u\colon \check{E} \to E \otimes L$, $L$ a line bundle, $u$ symmetric;

(C) $u\colon \check{E} \to E \otimes L$, $L$ a line bundle, $u$ skew-symmetric.

In each instance, let

$$D_k(u) = \{x \in X \mid \operatorname{rank} u(x) \leq k\}.$$

Since a skew-symmetric matrix never has odd rank, we limit ourselves in case (C) to even values of $k$. The expected codimensions $m_k$ in $X$ of these degeneracy loci are

$$\text{for (A):}\quad m_k = (e-k)(f-k)$$

$$\text{for (B):}\quad m_k = \binom{e-k+1}{2}$$

$$\text{for (C):}\quad m_k = \binom{e-k}{2} \quad (\text{for } k \text{ even})$$

(cf. [19]). In general, of course, these loci may be empty even when their expected dimensions are non-negative. Under suitable positivity hypotheses, however, this cannot happen:

PROPOSITION 3.5. *Given a vector bundle homomorphism $u$ of type* (A), (B) *or* (C), *assume that* $\dim X \geq m_k$ *(for the appropriate choice of $m_k$). Suppose in addition that the vector bundle*

$$\begin{array}{ll} \operatorname{Hom}(E, F) & \textit{in case (A)}, \\ S^2(E) \otimes L & \textit{in case (B)}, \\ \Lambda^2(E) \otimes L & \textit{in case (C)} \end{array}$$

*is ample. Then $D_k(u) \neq \varnothing$. In fact, $D_k(u)$ meets any subvariety $Y \subseteq X$ of dimension $\geq m_k$.*

*Proof.* We treat case (A), the others being virtually identical. As in the proof of Theorem 3.1, there is a tautological degeneracy locus $\Omega_k \subset \operatorname{Hom}(E, F)$; $\Omega_k$ is a cone of codimension $m_k$. The given homomorphism $u$ determines a section $v\colon X \to \operatorname{Hom}(E, F)$, and $D_k(u) = v^{-1}(\Omega_k)$. In particular, if $D_k(u)$ has the expected codimension $m_k$, or is empty, then it is the support of a cycle representing the cone class $z(\Omega_k, \operatorname{Hom}(E, F))$ (cf. §1). But this class is non-zero if $\operatorname{Hom}(E, F)$ is ample, by Corollary 2.2, so $D_k(u) \neq \varnothing$. The last statement follows by restricting $u$ to $Y$. $\square$

*Remarks.* (1) If the degeneracy locus $D_k(u)$ is empty or has codimension $m_k$, then it supports a cycle given by a determinantal formula in the Chern classes of the bundles in question. When $X$ is smooth, this is the well-known formula of Porteous in case (A) (cf. [23]), and in cases (B) and (C) the corresponding expressions have been determined by Harris and Tu [19]; for singular $X$, see [7]. As in Lemma 3.3, the same formulae compute the cone classes $z(\Omega_k, \mathrm{Hom}(E, F))$, $z(\Omega_k, S^2(E) \otimes L)$, and $z(\Omega_k, \Lambda^2(E) \otimes L)$ arising in the proof of the proposition. Thus these determinantal expressions are in fact *numerically positive* for vector bundles satisfying the ampleness hypotheses of the proposition.

(2) In setting (A), the proposition was proved in [8]. It was shown there moreover that when $\dim X > (e - k)(f - k)$, then $D_k(u)$ is *connected* provided that $\mathrm{Hom}(E, F)$ is ample. As L. Tu points out, the proof in [8] does not seem to generalize to cases (B) or (C), and the connectedness of symmetric and skew-symmetric degeneracy loci when $\dim X > m_k$ is open. The natural conjecture to make here is that if $E$ is an ample vector bundle of rank $e$ on a projective variety $X$, and if $Z \subseteq E$ is any (irreducible) closed subvariety, then the intersection $Z \cap 0_E$ of $Z$ with the zero-section is connected when $\dim Z > e$. Applying this to a translate of $\Omega_k$ would give connectedness in cases (B) and (C). Corollary 2.2 and a deformation argument show that at least $Z \cap 0_E$ is non-empty when $\dim Z \geq e$.

**3c. Cone cohomology classes and positivity of products.** As a final application of the results of Section 2, we will consider products of numerically positive polynomials in the Chern classes of possibly *different* ample vector bundles. We require first some general remarks on cone classes.

Consider as in Section 1 a vector bundle $E$ of rank $e$ on a projective variety $X$, and a cone $C \subseteq E$ of pure dimension $c \geq e$. At least when $X$ is singular, one cannot expect in general to be able to intersect the cone class $z(C, E) \in A_{c-e}(X)$ with arbitrary cycles on $X$. Roughly speaking, this reflects the fact that $z(C, E)$ lies in homology rather than cohomology. Certain cones, however, such as the determinantal varieties $\Omega_\lambda(E)$ arising in the proof of Theorem 3.1, do naturally define cohomology classes. These cohomology classes can be multiplied, and one can discuss the positivity of their product.

Specifically, suppose that $C$ is *flat* over $X$, of relative dimension $e'$, and let $n = e - e'$. Then $C$ determines a class

$$\mathrm{cl}(C, E) \in A^n(X),$$

where $A^*(X)$ is the operational Chow cohomology group constructed in [12], §9 (cf. also [7]). Before giving the formal definition, we note the salient points for our purposes. First, cap product with $\mathrm{cl}(C, E)$ gives a homomorphism

$$A_d(X) \to A_{d-n}(X)$$
$$\alpha \mapsto \mathrm{cl}(C, E) \cap \alpha,$$

defined on the cycle level as follows. Given a subvariety $V \subseteq X$ of dimension $d$, let $C|V = C \times_X V$ be the restriction of $C$ to $V$. Then

$$\text{(3.6)} \qquad \mathrm{cl}(C, E) \cap [V] = z(C|V, E|V) \in A_{d-n}(V),$$

which extends to $d$-cycles by linearity (and in fact passes to rational equivalence). Note in particular that $\mathrm{cl}(C, E)$ lifts the cone class $z(C, E)$ to cohomology in the sense that

$$\mathrm{cl}(C, E) \cap [X] = z(C, E).$$

Given another vector bundle $E'$, and a cone $C' \subseteq E'$ flat over $X$, of codimension $n'$ in $E'$, the product $\mathrm{cl}(C, E) \cdot \mathrm{cl}(C', E')$ acts according to the rule

$$\text{(3.7)} \qquad \{\mathrm{cl}(C, E) \cdot \mathrm{cl}(C', E')\} \cap \alpha = \mathrm{cl}(C, E) \cap \{\mathrm{cl}(C', E') \cap \alpha\}.$$

Finally, one has the cohomology analogue of (1.7) and (1.8): if $C \oplus C'$ denotes the fibre product $C \times_X C'$, which sits as a cone of codimension $n + n'$ in $E \oplus E'$, then

$$\text{(3.8)} \qquad \mathrm{cl}(C, E) \cdot \mathrm{cl}(C', E') = \mathrm{cl}(C \oplus C', E \oplus E')$$

in $A^{n+n'}(X)$.

More precisely, recall that an element $c \in A^n(X)$ is determined by specifying homomorphisms $A_*(X') \mapsto A_{*-n}(X')$, for every map $f\colon X' \to X$, satisfying various natural compatibilities ([12], §9.1). (One thinks of these homomorphisms as cap product with $f^*(c)$.) The class $\mathrm{cl}(C, E) \in A^n(X)$, $n = \mathrm{codim}(C, E)$, by definition operates as follows. Given $f\colon X' \to X$, set $E' = E \times_X X'$ and $C' = C \times_X X'$, which is a cone in $E'$. Let $\pi'\colon C' \to X'$ denote the projection (so that $\pi'$ is flat), let $i'\colon C' \hookrightarrow E'$ be the inclusion, and let $0_{E'}\colon X' \hookrightarrow E'$ be the zero-section. Then the homomorphism of $\mathrm{cl}(C, E)$ is defined as the composition

$$(*) \qquad A_l(X') \xrightarrow{\pi'^*} A_{l+e-n}(C') \xrightarrow{i'_*} A_{l+e-n}(E') \xrightarrow{0^*_{E'}} A_{l-n}X',$$

where $\pi'^*$ is flat pull-back, $i'_*$ is inclusion, and $0^*_{E'}$ is the Gysin homomorphism. The fact that these homomorphisms satisfy the compatabilities required to define an element in $A^k(X)$ follows from the corresponding assertions for flat pull-backs, proper push-forwards and Gysin maps proved in [12], §9. In view of (1.3), the formula (3.6) corresponds to the case $X' = V$, $l = d$ in $(*)$. Formula (3.7) is definitional. For (3.8), one must prove that both sides have the same effect on a class $[V]$ for $V$ a subvariety of $X'$, $X' \to X$ an arbitrary morphism. After pulling back the cones and bundles to $V$, one is reduced to considering $V = X' = X$, in which case (3.8) is equivalent to

$$(**) \qquad \mathrm{cl}(C, E) \cap z(C', E') = z(C \oplus C', E \oplus E')$$

in $A_{l-n-n'}(X)$, $l = \dim X$. Let $p'\colon E' \to X$, $q\colon E \oplus E' \to E'$ be the projections. By construction,

$$\begin{aligned} p'^*(\mathrm{cl}(C, E) \cap z(C', E')) &= \mathrm{cl}(C \times_X E', E \times_X E') \cap p'^*(z(C', E')) \\ &= \mathrm{cl}(C \oplus E', E \oplus E') \cap [C']. \end{aligned}$$

Similarly,

$$q^*(\mathrm{cl}(C \oplus E', E \oplus E') \cap [C']) = [C \oplus C'].$$

Hence

$$q^* p'^*(\mathrm{cl}(C, E) \cap z(C', E')) = [C \oplus C'].$$

Since $q^*p'^* = (p'q)^*$, and $p'q$ is the projection from $E \oplus E'$ to $X$, (**) follows.

These preliminaries out of the way, we may give the cohomology analogue of Theorem 2.1 on the positivity of cone classes:

THEOREM 3.9. *Let $E_1, \dots, E_r$ be ample vector bundles on a projective variety $X$. Let $C_i \subseteq E_i$ be cones, with* $\mathrm{codim}(C_i, E_i) = n_i$, *and suppose that each $C_i$ is flat over $X$. Set $n = n_1 + \cdots + n_r$. Then the product*

$$\mathrm{cl}(C_1, E_1) \cdot \cdots \cdot \mathrm{cl}(C_r, E_r) \in A^n(X)$$

*is numerically positive in the sense that*

$$\int_Y \mathrm{cl}(C_1, E_1) \cdot \cdots \cdot \mathrm{cl}(C_r, E_r) > 0$$

*for any subvariety $Y \subseteq X$ of dimension $n$.*

(For $\alpha \in A^n(X)$, $\int_Y \alpha$ denotes the degree of the zero-cycle $\alpha \cap [Y]$.)

*Proof.* Since a direct sum of ample vector bundles is ample, it suffices by (3.8) to treat the case $r = 1$. But in view of (3.6), this follows immediately from Theorem 2.1. □

Consider now, for $1 \leq i \leq r$, Schur polynomials

$$P_{\lambda_i} \in \mathbf{Q}[c_1, \dots, c_{e_i}], \qquad \lambda_i \in \Lambda(n_i, e_i),$$

(so that $P_{\lambda_i}$ has degree $n_i$) and let $n = n_1 + \cdots + n_r$. Then one has

COROLLARY 3.10. *For any projective variety $X$ of dimension $n$, and any ample vector bundles $E_1, \dots, E_r$ on $X$, with $rk(E_i) = e_i$,*

$$\int_X \prod_{i=1}^r P_{\lambda_i}(c(E_i)) > 0.$$

*Proof.* Let $\Omega_{\lambda_i}(E_i) \subseteq H(E_i)$ be the cone introduced in the proof of Theorem 3.1, which is flat over $X$. A minor extension of the proof of Lemma 3.3 shows that

$$P_{\lambda_i}(c(E_i)) = \mathrm{cl}(\Omega_{\lambda_i}(E_i), H(E_i)).$$

(The point to observe is that given a map $f\colon X' \to X$, the class of the cone $\Omega_{\lambda_i}(E_i') \subseteq H(E_i')$ is given by

$$p'^*\{P_{\lambda_i}(f^*c(E_i)) \cap [X']\},$$

where $p'\colon H(E_i') \to X'$ is the bundle map.) Hence the corollary follows from Theorem 3.9. □

For example, given ample vector bundles $E_1, \ldots, E_r$ on $X$, any monomial of the form $c_{j_1}(E_1) \cdot \cdots \cdot c_{j_r}(E_r)$ ($j_i \leq e_i$) is numerically positive. Of course the corollary also gives the positivity of products of non-negative linear combinations of Schur polynomials. In particular, one could replace $P_{\lambda_i}$ in the corollary by any numerically positive polynomial of degree $n_i$ for ample bundles of rank $e_i$. Note that Theorem 3.9 also applies to the cones arising in Section 3b.

*Remark.* These results on the positivity of products seem rather striking in view of the fact that it is not generally true that the product of numerically positive classes is again numerically positive. For instance, Mumford has constructed a line bundle $L$ on a surface $X$ such that $c_1(L)$ is numerically positive; i.e.,

$$\int_Y c_1(L) > 0$$

for every effective curve $Y \subseteq X$, but with

$$\int_X c_1(L)^2 = 0.$$

(See Hartshorne [20], Chapter I, §10.)

## Appendix A. Relation to the work of Griffiths

Griffiths has defined in [16] a graded cone

$$\Pi(e) = \bigoplus_{n \geq 0} \Pi(e)_n \subseteq \mathbf{Q}[c_1, \ldots, c_e]$$

of *positive polynomials*,* and our main purpose in this appendix is to show that $\Pi(e)$ coincides with the cone of non-negative linear combinations of the Schur polynomials $P_\lambda$.

*To avoid confusion with our previous terminology, we will call these "Griffiths-positive."

To begin with, we recall, following [16], §5, the definition of $\Pi(e)$. Consider the coordinate ring $\mathbf{Q}[T_{ij}]$ $(1 \leq i, j \leq e)$ of the space of $e \times e$ matrices. $\mathrm{GL}(e, \mathbf{Q})$ acts on matrices by conjugation, and the corresponding graded ring of invariants

$$I(e) = \mathbf{Q}\left[T_{ij}\right]^{\mathrm{GL}(e, \mathbf{Q})}$$

is naturally isomorphic to the graded ring $\mathbf{Q}[x_1, \ldots, x_e]^{S_e}$ of symmetric polynomials in $e$ variables. Explicitly, the isomorphism is obtained by evaluating an invariant polynomial $P \in I(e)$ on the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_e)$. On the other hand, the ring of symmetric functions is isomorphic in the usual way to the graded ring $\mathbf{Q}[c_1, \ldots, c_e]$ ($c_i$ corresponds to the $i$th elementary symmetric function). Hence it suffices to specify the Griffiths-positive polynomials in $I(e)$.*

To this end, Griffiths first shows that any homogeneous $P \in I(e)_n$ can be written (non-uniquely) in the form

$$P = \sum_{\substack{\rho \in [1, e]^n \\ \pi, \tau \in S_n}} p_{\rho, \pi, \tau} T_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdot \cdots \cdot T_{\rho_{\pi(n)}\rho_{\tau(n)}} \tag{A.1}$$

for some numbers $p_{\rho, \pi, \tau}$, where $S_n$ denotes the symmetric group on $n$ objects. He then defines a non-zero polynomial $P$ to be *positive* if it can be expressed in the form (A.1) with

$$p_{\rho, \pi, \tau} = \sum_{j \in J} \lambda_{\rho, j} q_{\rho, j, \pi} \bar{q}_{\rho, j, \tau} \quad \text{for all } \rho, \pi, \tau \tag{A.2}$$

for some real $\lambda_{j, \rho} \geq 0$ and complex numbers $q_{\rho, j, \pi}$, and some finite set $J$. The positive cone $\Pi(e)$ consists by definition of all such Griffiths-positive polynomials. It is graded by degree.

Griffiths showed that $\Pi(e)$ contains Chern classes, dual Segre classes, and products of these. In fact, one can say more:

Proposition A.3. *Let*

$$P = \sum_{\lambda \in \Lambda(n, e)} a_\lambda(P) P_\lambda \qquad (a_\lambda(P) \in \mathbf{Q})$$

*be a non-zero weighted homogeneous polynomial in* $\mathbf{Q}[c_1, \ldots, c_e]$. *Then* $P$ *lies in the Griffiths cone* $\Pi(e)$ *if and only if each of the Schur coefficients* $a_\lambda(P)$ *is non-negative.*

*The significance of the ring $I(e)$ in the present context stems of course from the fact that if $\theta$ is an $e \times e$ matrix of 2-forms locally representing the curvature tensor of a Hermitian vector bundle $E$ of rank $e$ on a manifold $M$, and if $P \in I(e)$ is homogeneous of degree $n$, then $P((i/2\pi)\theta) \in H^{2n}_{DR}(M)$ represents the corresponding polynomial in the Chern classes of $E$. (Cf. [16] or [18], Chapter 3, §3).

*Remark.* Consider a vector bundle $E$ on a complex projective manifold $M$. Griffiths defines $E$ to be *numerically positive* if for any analytic subvariety $W \subseteq M$ of dimension $n$, and for any rank $q$ quotient $Q$ of $E|W$, one has

$$\int_W P(c(Q)) \quad \text{for all } P \in \Pi(q)_n.$$

Griffiths proved ([16], Theorem D) that $E$ is numerically positive if it is generated by its global sections, and if in addition the resulting second fundamental form is surjective. He conjectured ([16], Conjecture (0.7)) that any *positive* vector bundle* is numerically positive, and proved this in some special cases. In [17], § 7, after the work [1] of Bloch and Gieseker had appeared, Griffiths speculated on the possibility that arbitrary ample bundles are numerically positive. This was proved using Schubert calculus by Usui and Tango [30] for bundles generated by their global sections. Since restrictions and quotients preserve amplitude, the proposition shows that the numerical positivity of all ample vector bundles is in fact a consequence of our main result. At the same time, the proposition and Theorem I, one hopes, shed some light on the geometric significance of this concept.

LEMMA A.4. *For a polynomial $P$ as in Proposition* A.3, *the following are equivalent*:

(a) $a_\lambda(P) \geq 0$ *for every* $\lambda \in \Lambda(n, e)$;

(b) $P$ *is numerically non-negative for vector bundles generated by their global sections*;

(c) *For every $n$-dimensional subvariety $Y$ of the Grassmannian $G(m - e, m)$, $m \geq n + e$, with rank $e$ universal quotient bundle $Q$, one has*

$$\int_Y P(c(Q)) \geq 0.$$

*Proof.* That (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is clear, and (c) $\Rightarrow$ (a), which is all we need, is the first step in the proof of Proposition 3.4 above. □

*Proof of Proposition* A.3. Suppose first that $P$ is Griffiths-positive. It is the content of [30], Theorem 2.1, that condition (c) of the lemma holds. (See also [16], beginning of proof of Theorem D, p. 246. The gist of these computations is that $P(c(Q))$ is represented by a *non-negative* $(n, n)$-form.) Hence all the Schur coefficients $a_\lambda(P)$ are non-negative.

*This is a differential-geometric condition that in particular implies ampleness ([16], Theorem B). Note that ampleness (in the sense of Hartshorne [20], as we are using the term) is called "cohomological positivity" in [16].

Conversely, for $\lambda \in \Lambda(n, e)$, let $F_\lambda = F_\lambda(T_{ij}) \in I(e)_n$ denote the invariant polynomial corresponding to the Schur function $P_\lambda$ under the isomorphism $I(e) \cong \mathbf{Q}[c_1, \ldots, c_e]$. We wish first of all to express $F_\lambda$ in the form (A.1). To this end, recall that partitions of $n$ are in natural bijection with the irreducible representations of the symmetric group $S_n$ (cf. [2], §28). Given $\lambda \in \Lambda(n, e)$, let $\phi_\lambda$ denote the corresponding representation, and $\chi_\lambda$ its character. Then one has the formula

$$F_\lambda = \frac{1}{n!} \sum_{\substack{\tau \in S_n \\ \rho \in [1, e]^n}} \chi_\lambda(\tau) T_{\rho_1 \rho_{\tau(1)}} \cdots T_{\rho_n \rho_{\tau(n)}} \tag{A.5}$$

whose proof we recall below. This leads to

$$F_\lambda = \left(\frac{1}{n!}\right)^2 \sum_{\substack{\pi, \tau \in S_n \\ \rho \in [1, e]^n}} \chi_\lambda(\tau \pi^{-1}) T_{\rho_{\pi(1)} \rho_{\tau(1)}} \cdot \cdots \cdot T_{\rho_{\pi(n)} \rho_{\tau(n)}} \tag{A.6}$$

which is in the desired form (A.1). To see that (A.6) satisfies the requirement (A.2), choose a basis with respect to which the representation $\phi_\lambda$ is unitary, and write $\phi_\lambda(\tau) = (a_{kl}(\tau)) \in U(m)$, where $m = \dim \phi_\lambda$. Then

$$\begin{aligned} \chi_\lambda(\tau\pi^{-1}) &= \operatorname{Trace}\left(\phi_\lambda(\tau) \cdot {}^t\overline{\phi_\lambda(\pi)}\right) \\ &= \sum_{1 \le k, l \le m} a_{kl}(\tau) \overline{a_{kl}(\pi)}, \end{aligned}$$

which is of the form (A.2) (for $J = [1, m]^2$). Thus each Schur polynomial $P_\lambda$ is Griffiths-positive.

Finally, for lack of a suitable reference, we sketch the derivation of (A.5). For each positive integer $k$, let $u_k$ denote the symmetric polynomial $x_1^k + \cdots + x_e^k$. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition of $n$, set $u_\lambda = \Pi u_{\lambda_i}$. Given $\tau \in S_n$ we write $(\tau)$ for the partition of $n$ determined by the cycle structure of $\tau$, so that each $i$-cycle contributes a part $i$ to the partition $(\tau)$. Thinking of $c_i$ as the $i$th elementary symmetric function in $x_1, \ldots, x_e$, let $f_\lambda \in \mathbf{Q}[x_1, \ldots, x_e]$ be the symmetric polynomial $P_\lambda(c_1(x), \ldots, c_e(x))$. Then one has the beautiful formula of Frobenius:

$$f_\lambda(x_1, \ldots, x_e) = \frac{1}{n!} \sum_{\tau \in S_n} \chi_\lambda(\tau) u_{(\tau)}(x_1, \ldots, x_e). \tag{A.7}$$

(cf. [28], §13, or [25], Chapter 6.2). The formula (A.7) is sometimes stated under the assumption that the number of variables is at least $n$, but the general case follows by setting some equal to zero.

It remains only to rewrite (A.7) in the ring $I(e)$. The one point to note here is that the symmetric polynomial $u_k = x_1^k + \cdots + x_e^k$ corresponds to the polynomial

$$P = \mathrm{Trace}(T^k) \in I(e), \tag{A.8}$$

where $T$ is the matrix $(T_{ij})$. ($P$ is invariant and agrees with $u_k$ on $\mathrm{diag}(x_1, \dots, x_e)$.) The desired formula (A.5) now follows by using (A.8) to write out the right-hand side of (A.7). □

*Remark.* Perhaps the key feature of the definition (A.2) is that it implies that $P((i/2\pi)\theta)$ is a *non-negative* $(n, n)$-form when $\theta$ is the curvature matrix for the natural connection on a quotient of a trivial bundle and $P$ is Griffiths-positive (cf. the computations in [16] and [30] cited above). One may speculate that the definition was framed with this requirement in mind, and in this sense Griffiths-positivity appears as an essentially *analytic* notion. On the other hand, ampleness (in the sense of Hartshorne) seems not to be well understood from a differential-geometric point of view. So it might at first glance seem surprising that the Griffiths cone $\Pi$ exactly agrees with the cone of numerically positive polynomials for ample vector bundles. In fact, however, the thrust of our work is that the set of polynomials turning out to be positive or non-negative in a given context is not very sensitive to the particular class of vector bundles with which one deals. Thus the numerically positive polynomials for arbitrary ample bundles coincide with the positive polynomials for quotients of sums of very ample line bundles (Proposition 3.4), and these in turn essentially coincide with the non-negative polynomials for bundles generated by their global sections (Lemma A.4).

## Appendix B. The theorem of Bloch and Gieseker

Goresky and MacPherson ([14], [15]) have constructed *intersection homology groups* $IH^k(X)$ for a possibly singular complex projective variety $X$ which share many of the properties of homology-cohomology on a smooth variety. Deligne [4] has extended the construction to arbitrary characteristic (with coefficients in a field of characteristic zero), and Gabber has proved the hard Lefschetz theorem for these groups (cf. [3]):

(B.1) *If $\xi$ is the first Chern class of an ample line bundle on an n-dimensional projective variety $X$, then for all $i \le n$, the map*

$$IH^{n-i}(X) \xrightarrow{\cdot \xi^i} IH^{n+i}(X)$$

*obtained by cup-product with $\xi^i$ is an isomorphism.*

Using Gabber's theorem (B.1) in place of the classical Lefschetz theorem, the Bloch-Gieseker proof of the positivity of the Chern classes of an ample vector goes through in arbitrary characteristic. In fact, their proof becomes significantly shorter, since by working directly on possibly singular varieties one avoids the final part of the argument. We sketch their proof.

THEOREM (Bloch and Gieseker [1]). *Let $X$ be a projective variety of dimension $n$, and $E$ an ample vector bundle of rank $e$ on $X$. If $e \geq n$, then*

$$\int_X c_n(E) > 0.$$

*Proof.* We first show that $c_n(E) \neq 0$. To this end let $\mathbf{P} = \mathbf{P}(\check{E})$, and $\xi = c_1(\mathcal{O}_{\mathbf{P}(\check{E})}(1))$. The amplitude of $E$ is equivalent to the ampleness of the Serre line bundle $\mathcal{O}_{\mathbf{P}(\check{E})}(1)$ on $\mathbf{P}$ ([20], (3.2)). On the other hand, recall (cf. [4]) that there are canonical homomorphisms

$$H^k(\mathbf{P}) \to IH^k(\mathbf{P}) \to H_{2(n+e-1)-k}(\mathbf{P}),$$

compatible with multiplication by cohomology classes, whose composite is cap product with the fundamental class $[\mathbf{P}]$ of $\mathbf{P}$. Consider the class

$$\alpha = \xi^{n-1} - \pi^* c_1(E)\xi^{n-2} + \cdots + (-1)^{n-1}\pi^* c_{n-1}(E) \in H^{2n-2}(\mathbf{P}),$$

where $\pi\colon \mathbf{P} \to X$ denotes the bundle map. Since $\pi_*(\xi^{e-n} \cdot \alpha \cap [\mathbf{P}]) = [X]$, the image of $\alpha$ in $H_{2e}(\mathbf{P})$, and *a fortiori* in $IH^{2n-2}(\mathbf{P})$, is non-zero. But if $c_n(E) = 0$, then $\xi^{e+1-n} \cdot \alpha = 0$, which contradicts (B.1).

The proof proceeds by induction on $n$. Since the case $n = 0$ is trivial, we assume the result for all varieties of dimension $< n$. We assert that then $\int_X c_{n-1}(E)c_1(L) > 0$ for any ample line bundle $L$ on $X$. Indeed, after replacing $L$ by $L^{\otimes m}$ for $m \gg 0$, we may suppose that $L = \mathcal{O}_X(Y)$ for some Cartier divisor $Y \subseteq X$, in which case

$$\int_X c_{n-1}(E)c_1(L) = \int_Y c_{n-1}(E|Y).$$

Now suppose the theorem false, so that $\int_X c_n(E) < 0$. By the remarks above, we may choose an ample line bundle $L$ and a positive integer $k$ such that

$$\int_X c_n(E) + \frac{1}{k}\int_X c_{n-1}(E)c_1(L) = 0.$$

Consider a corresponding Bloch-Gieseker covering:

$$f\colon Y \to X, \qquad f^*L = M^{\otimes k}$$

(Lemma 1.1). Then $f^*E \oplus M$ is ample on $Y$, but

$$\int_Y c_n(f^*E \oplus M) = \int_Y c_n(f^*E) + \int_Y c_{n-1}(f^*E) \cdot c_1(M) = 0,$$

which contradicts the first step of the proof. □

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