Calculus IV with Applications MAT303
Solutions to Practice Problems for Midterm II

3.1, 26. If dependent, then $f = cg$ for a constant $c$, i.e., $2 \cos x + 3 \sin x = c(3 \cos x - 2 \sin x)$. Then comparing coefficients at $\cos x$ and $\sin x$, we get $2 = 3c$ and $3 = -2c$ at the same time, which is impossible. Therefore, $f$ and $g$ are linearly independent. (Another solution: compute the Wronskian.)

3.1, 38. Char. eq-n: $4r^2 + 8r + 3 = 0$. Solutions to char eq-n: $r = -3/2, -1/2$.

General sol-n: $c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$.

3.2, 10. $W(f, g, h) = \begin{vmatrix} e^{x} & x^{-2} & x^{-2}\ln x \\ e^{x} & -2x^{-3} & -2x^{-3}\ln x + x + x^{-3} \\ e^{x} & 6x^{-4} & 6x^{-4}\ln x - 5x^{-4} \end{vmatrix} = e^{x}(-2x^{-3}(6x^{-4}\ln x - 5x^{-4}) - (6x^{-4}\ln x + 6x^{-4}) - e^{x}((2x^{-2}-6x^{-4}\ln x - 5x^{-4}) - x^{-2}\ln 6x^{-4}) + e^{x}(x^{-2}-5x^{-3} - x^{-2}\ln x(-2x^{-3})) = e^{-x}(4x^{-7} + 5x^{-6} + x^{-5}) \neq 0$.

3.2, 15. General solution: $y(x) = c_1 e^{x} + c_2 xe^{x} + c_3 x^2e^{x}$. Then $y'(x) = (c_1 + c_2)e^{x} + (c_2 + 2c_3)xe^{x} + c_3 x^2e^{x}$ and $y''(x) = (c_1 + 2c_2 + 2c_3)e^{x} + (c_2 + 4c_3)xe^{x} + c_3 x^2e^{x}$.

Then we have

$$
\begin{cases}
2 = y(0) = c_1 \\
0 = y'(0) = c_1 + c_2 \\
0 = y''(0) = c_1 + 2c_2 + 2c_3
\end{cases}
$$

Hence, $c_1 = 2, c_2 = -2, c_3 = 1$. Solution: $2e^{x} - 2xe^{x} + x^2e^{x}$.

3.2, 23. General solution: $y(x) = y_1 + y_2 = c_1 e^{-x} + c_2 e^{3x} - 2$. Then $y'(x) = -c_1 e^{-x} + 3c_2 e^{3x}$. Then we get $3 = y(0) = c_1 + c_2 - 2$ and $11 = y'(0) = -c_1 + 3c_2$.

Then $c_1 = 1, c_2 = 4$. Solution: $e^{-x} + 4e^{3x} - 2$.

3.3, 12. Char eq-n: $r^4 - 3r^3 + 3r^2 - r = 0$ or, equivalently, $r(r^3 - 3r^2 + 3r - 1) = 0$ or, equivalently, $r(r - 1)^3 = 0$. Solutions to char eq-n: $r = 0, r = 1$ (with multiplicity 3). General sol-n: $c_1 e^{0x} + c_2 xe^{x} + c_3 x^2e^{x} + c_4 x^3e^{x} = c_1 + c_2 xe^{x} + c_3 x^2e^{x} + c_4 x^3e^{x}$.

3.3, 16. Char eq-n: $r^4 + 18r^2 + 81 = 0$. Put $r^2 = s$, then $s^2 + 18s + 81 = 0$ and $s = -9$, thus $r = \pm \sqrt{9} = \pm 3i$ (each root with multiplicity 2). Alternative approach: rewrite equation as $(r^2 + 9)^2 = 0$, then as $(r + 3i)^2(r - 3i)^2 = 0$. General sol-n: $c_1 \cos 3x + c_2 \sin 3x + c_3 x^2 \cos 3x + c_4 x^2 \sin 3x$.

3.4, 16. $3x^2 + 30x + 63 = 0$. The characteristic equation is $3r^2 + 30r + 63 = 0$, thus $r = \frac{-30 \pm \sqrt{30^2 - 4 \cdot 3 \cdot 63}}{2} = -7, -3$. The roots are real and distinct, therefore the system is overdamped. General solution: $x(t) = c_1 e^{-3t} + c_2 e^{-7t}$.

Then $v(t) = x'(t) = -3c_1 e^{-3t} - 7c_2 e^{-7t}$. From $x(0) = 2, v(0) = 2$, we have $c_1 + c_2 = 2, -3c_1 - 7c_2 = 0$. Thus $c_1 = 4, c_2 = -2$. Position function: $4e^{-3t} - 2e^{-7t}$.

In the undamped case, the equation is $3x'' + 63x = 0$. Then $\omega_0 = \sqrt{63/3} = \sqrt{21}$. General solution: $x(t) = C \cos(\omega_0t - \alpha) = C \cos(\sqrt{21}t - \alpha)$. Then $v(t) = x'(t) = -\sqrt{21} C \sin(\sqrt{21}t - \alpha)$. From $x(0) = 2, v(0) = 2$, we have $2 = C \cos(-\alpha)$ and $2 = -\sqrt{21} C \sin(-\alpha)$. $C = 2/\cos(-\alpha)$, thus $2 = -2\sqrt{21} \sin(-\alpha)/\cos(-\alpha) = -2\sqrt{21} \tan(-\alpha)$. It follows that $\alpha = \arctan(1/\sqrt{21})$. If tan $\alpha = 1/\sqrt{21}$, then cos $\alpha = 1/\sqrt{21}$. Hence $C = 2/\sqrt{21}$.

3.4, 20. $2x'' + 16x' + 40x = 0$. The characteristic equation is $2r^2 + 16r + 40 = 0$, thus $r = -16 \pm \sqrt{16^2 - 4 \cdot 40 \cdot 2} = -4 \pm 2i$. The roots are complex, therefore the system is underdamped. General solution $x(t) = Ce^{-4t} \cos(2t - \alpha)$ (i.e. $p = 4, \omega_1 = 2$). $x'(t) = -4Ce^{-4t} \cos 2t - \alpha - 2Ce^{-4t} \sin 2t - \alpha$. From
derivatives has the form $s$ must take $\cos x$ and undetermined coefficients, we get $\sin x$ are solutions of the associated equations, thus we take $s = 2$. Trial solution: $A x^2 e^x + B x^3 e^x$.

Since $f(x) = x e^x$, $f'(x) = x e^x + e^x$. A linear combination of $f(x)$ and its derivatives has the form $A e^x + B x e^x$. Trial solution: $x^4 (A e^x + B x e^x)$. Both $e^x$ and $x e^x$ are particular solutions of the associated equation, hence we take $s = 2$. Trial solution: $A x^2 e^x + B x^3 e^x$.

Plug in trial solution: $(A x^2 e^x + B x^3 e^x)' - 2 (A x^2 e^x + B x^3 e^x)'' + A x^2 e^x + B x^3 e^x = x e^x$.

$A (12 e^x + 8 x e^x + x^2 e^x) + B (24 e^x + 36 x e^x + 12 x^2 e^x + x^3 e^x) - 2 A (2 e^x + 4 x e^x + x^2 e^x) - 2 B (6 x e^x + 6 e^x + x^3 e^x) A x^2 e^x + B x^3 e^x = x e^x$.

$= 0$ or, equivalently, $(r - 1)^2 (r + 1)^2 = 0$. Solutions: $r = \pm 1$ (each with multiplicity 2). General solution:

Since $f(x) = \sin x$, $f'(x) = \cos x$. Thus the characteristic equation here is $r^2 + 1 = 0$. Solutions: $r = \pm i$. General solution: $y_c = c_1 \cos x + c_2 \sin x$.

Since $f(x) = \sin x + x \cos x$, we consider $\sin x$ and $x \cos x$ separately. Derivatives of $\sin x$ are $\pm \cos x$ or $\pm \sin x$, thus the trial solution for $\sin x$ has form $x^r (A \sin x + B \cos x)$. Both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. For $x \cos x$, the linear combinations of all its derivatives will have the form $C x \sin x + D x \cos x + E \sin x + F \cos x$. Thus the trial solution here is $x^r (C x \sin x + D x \cos x + E \sin x + F \cos x)$. Again, because both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. The sum of both trial solutions gives us the trial solution for $f(x)$: $x (A \sin x + B \cos x) + x (C x \sin x + D x \cos x + E \sin x + F \cos x)$. Combining similar terms together (and relabeling undetermined coefficients), we get $a x \sin x + b x \cos x + c x^2 \sin x + d x^2 \cos x$.

Plug in trial solution: $(ax \sin x + bx \cos x + cx^2 \sin x + dx^2 \cos x)'' = (ax + bx \cos x + cx^2 \sin x + dx^2 \cos x)''$. Therefore, $2c - 2b = 1$, $2a + 2d = 0$, $a - 4d = 0$, $-b + 4c = 1$. It follows that $a = d = 0$, $b = -1/3$, $c = 1/6$. 3.5, 38. Associated homogeneous equation: $y'' + 2y + 2y = 0$. Characteristic equation: $r^2 + 2r + 2 = 0$. Solutions: $r = -1 \pm i$. General solution: $y_e = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$.
Plug in trial solution: \((A \sin 3x + B \cos 3x)'' + 2(A \sin 3x + B \cos 3x)' + 2(A \sin 3x + B \cos 3x) = \sin 3x.
\]
\((-9A \sin 3x - 9B \cos 3x + 2(3A \cos 3x - 3B \sin 3x) + 2(A \sin 3x + B \cos 3x) = \sin 3x.
\]
\((-9A - 6B + 2A) \sin 3x + (-9B + 6A + 2B) \cos 3x = \sin 3x.
\]
Hence \(-7A - 6B = 1\) and \(-7B + 6A = 0\). Thus \(A = -7/85\), \(B = -6/85\), and
\(y_p = -\frac{7}{85} \sin 3x - \frac{6}{85} \cos 3x.
\]
The general solution is
\[y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x - \frac{7}{85} \sin 3x - \frac{6}{85} \cos 3x.
\]
Thus \(y'(x) = -c_1 e^{-x} \cos x - c_1 e^{-x} \sin x - c_2 e^{-x} \sin x + c_2 e^{-x} \cos x - \frac{21}{85} \cos 3x + \frac{18}{85} \sin 3x.
\]
\[2 = y(0) = c_1 - \frac{6}{85} \text{ and } 0 = y'(0) = (c_2 - c_1) - \frac{21}{85}. \text{ Then } c_1 = \frac{197}{85} \text{ and } c_2 = \frac{497}{85}.
\]
Solution: \(y(x) = (176e^{-x} \cos x + 197e^{-x} \sin x - 7 \sin 3x - 6 \cos 3x)/85.
\]

4.1, 6. Set \(z = x'\), \(w = y'\). Then \(x'' = z'\) and \(y'' = w\). Answer:

\[
\begin{align*}
z' - 5x + 4y &= 0 \\
w' + 4x - 5y &= 0 \\
z &= x' \\
w &= y'
\end{align*}
\]

4.1, 17. \(y = x'\), thus \(y' = x''\). From \(y' = 6x - y, x'' = 6x - x', i.e. x'' - x' - 6x = 0\). Char eq-n: \(r^2 + r - 6 = 0\). \(r = -3, 2\). General solution: \(x(t) = c_1 e^{-3t} + c_2 e^{2t}, y(t) = x'(t) = -3c_1 e^{-3t} + 2c_2 e^{2t}.
\]
\[1 = x(0) = c_1 + c_2, 2 = y(0) = -3c_1 + 2c_2. \text{ Hence } c_1 = 0, c_2 = 1. \text{ Solution: } x(t) = e^{2t}, y(t) = 2e^{2t}.
\]

4.2, 4. \(y = 3x - x'\), hence the second eq-n becomes \((3x - x')' = 5x - 3(3x - x')\). Then \(x'' - 4x = 0\). Char. eq-n: \(r^2 - 4 = 0\). \(r = \pm 2\), thus \(x(t) = c_1 e^{2t} + c_2 e^{-2t}\) and
\(y(t) = 3x - x' = c_1 e^{2t} + 5c_2 e^{-2t}.
\]
\[1 = x(0) = c_1 + c_2, -1 = y(0) = c_1 + 5c_2. \text{ Hence } c_1 = 3/2, c_2 = -1/2. \text{ Solution: } x(t) = \frac{3}{2} e^t - \frac{1}{2} e^{-t}, y(t) = \frac{9}{2} e^t - \frac{5}{2} e^{-t}.
\]