Chapter 4 Review Exercises

51. \( f(x) = e^x - 2x^{-1/2} \), hence the antiderivative is \( F(x) = e^x - 2x^{-1/2} + C = e^x - 4x^{1/2} + C = e^x - 4\sqrt{x} + C \).

52. \( g(t) = \frac{1 + t}{\sqrt{t}} = \frac{1}{\sqrt{t}} + \frac{t}{\sqrt{t}} = t^{-1/2} + t^{1/2} \), hence the antiderivative is

\[
G(t) = \frac{t^{-1/2} + 1}{2} + \frac{t^{1/2} + 1}{2} + C = 2\sqrt{t} + 2t^{3/2}/3 + C.
\]

55. \( f''(x) = 1 - 6x + 48x^2 \), hence \( f'(x) = x - 3x^2 + 16x^3 + C \) (the antiderivative of \( f''(x) \)). Since \( f'(0) = 2 \), we have \( 0 - 3(0)^2 + 16(0)^3 + C = 2 \), that is \( C = 2 \). \( f(x) = \frac{x^2}{2} - x^3 + 4x^4 + 2x + D \). Since \( f(0) = 1 \), we have \( \frac{0^2}{2} - 0^3 + 4(0)^4 = 2(0) + D = 1 \), that is \( D = 1 \). Therefore, \( f(x) = 1 + 2x + \frac{x^2}{2} - x^3 + 4x^4 \).

57. \( s(t) \) is an antiderivative of \( v(t) = 2t - \frac{1}{1+t^2} \), that is \( s(t) = t^2 - \tan^{-1} t + C \). Since \( s(0) = 1 \), \( 0^2 - \tan^{-1} 0 + C = 1 \). Thus \( C = 1 \) and \( s(t) = t^2 - \tan^{-1} t + 1 \).

Chapter 5 Review Exercises

1. In both parts \( \Delta x = 1 \) (the interval \([0,6]\) is divided into six parts).

   (a) Left endpoints: 0, 1, 2, 3, 4, 5, so the sum is 
   \[
   (f(0) + f(1) + f(2) + f(3) + f(4) + f(5))\Delta x = (2 + 3.5 + 4 + 2 - 1 - 2.5)1 = 8
   \]

   (b) Midpoints: 0.5, 1.5, 2.5, 3.5, 4.5, 5.5, so the sum is 
   \[
   (f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5))\Delta x = 3 + 3.8 + 3.5 + 0.5 - 2 - 2.9 = 5.9
   \]

   The sums represent the total areas of rectangles drawn below ( (a) on the left; (b) on the right):
3. \( \int_{0}^{1} (x + \sqrt{1 - x^2}) \, dx = \int_{0}^{1} x \, dx + \int_{0}^{1} \sqrt{1 - x^2} \, dx \). The first integral is the area of the triangle formed by the \( x \)-axis, the line \( y = x \) and the vertical line at \( x = 1 \), thus it equals 1/2 (alternatively, you can compute by using anti-derivatives). The second integral of the function \( y = \sqrt{1 - x^2} \) (i.e. \( x^2 + y^2 = 1 \) and the graph of the function is the upper semi-circle of radius 1) is the area of the quarter of the circle of radius 1, i.e. \( \pi/4 \). Answer: \( 1/2 + \pi/4 \).

4. This is the sum of the areas of rectangles of width \( \Delta x \) and height determined by the function \( \sin x \). Hence it equals
\[
\int_{0}^{\pi} \sin x \, dx = -\cos x \bigg|_{0}^{\pi} = -\cos \pi - (-\cos 0) = 2.
\]

5. \( \int_{4}^{6} f(x) \, dx = \int_{0}^{6} f(x) \, dx - \int_{0}^{4} f(x) \, dx = 10 - 7 = 3. \)

9. \( \int_{1}^{2} (8x^3 + 3x^2) \, dx = 8x^4 + 3x^3 \bigg|_{1}^{2} = 2x^4 + x^3 \bigg|_{1}^{2} = 2 \cdot 2^4 + 2^3 - (2 \cdot 1 + 1) = 37 \)

13. \( \int \left( \frac{1 - x}{x} \right)^2 \, dx = \int \frac{1 - 2x + x^2}{x^2} \, dx = \int \frac{1}{x^2} - 2\frac{1}{x} + 1 \, dx = \int x^{-2} - 2x^{-1} + 1 \, dx \\
= -\frac{1}{x} - 2 \ln x + x + C \)

14. \( \int_{0}^{1} (\sqrt{u} + 1)^2 \, du = \int_{0}^{1} \sqrt{u} + 2\sqrt{u} + 1 \, du = \int_{0}^{1} u^{1/2} + 2u^{1/4} + 1 \, du = \frac{u^{3/2}}{3/2} + 2\frac{u^{5/4}}{5/4} + u \bigg|_{0}^{1} = \\
\frac{2}{3} + \frac{8}{5} + 1 - (0 + 0 + 0) = \frac{49}{15} \)

23. \( \int_{0}^{5} \frac{x}{x + 10} \, dx = \int_{0}^{5} \frac{x + 10 - 10}{x + 10} \, dx = \int_{0}^{5} 1 - \frac{10}{x + 10} \, dx = x - 10 \ln(x + 10) \bigg|_{0}^{5} = \\
5 - 10 \ln 15 - (0 - 10 \ln 10) = 5 + 10(\ln 10 - \ln 15) = 5 + 10 \ln(10/15) = \\
5 + 10 \ln(2/3), \)

using \( \int \frac{1}{x + 10} \, dx = \ln(x + 10) + C \) (check by differentiation).
37. Here is the rough estimate: (I basically use midpoints here; the extra bits in the rectangles more or less cancel the area under the graph that rectangles failed to cover)

The rough estimate is 0.3 + 2 + 4 + 6.7 = 13

The exact area is \[ \int_0^4 x\sqrt{x} \, dx = \int_0^4 x^{3/2} \, dx = \frac{x^{5/2}}{5/2} \bigg|_0^4 = \frac{4^{5/2}}{5/2} - 0 = \frac{64}{5} \]

39. \[ F'(x) = \frac{x^2}{1 + x^3} \]

40. Let \( F(x) = \int_1^x \frac{1 - t^2}{1 + t^4} \, dt \). Then \( F'(x) = \frac{1 - x^2}{1 + x^4} \)

\[ g(x) = F(\sin x). \] Then \( g'(x) = (F(\sin x))' = F'(\sin x)(\sin x)' = \frac{1 - \sin^2 x}{1 + \sin^4 x} \cos x = \frac{\cos^2 x}{1 + \sin^4 x} \cos x = \frac{\cos^3 x}{1 + \sin^4 x} \]

41. \[ y = \int_{\sqrt{x}}^{x} \frac{e^t}{t} \, dt = \int_0^{\sqrt{x}} \frac{e^t}{t} \, dt + \int_{\sqrt{x}}^{x} \frac{e^t}{t} \, dt = -\int_0^{\sqrt{x}} \frac{e^t}{t} \, dt + \int_0^{x} \frac{e^t}{t} \, dt \]

Let \( F(x) = \int_0^{x} \frac{e^t}{t} \, dt \). Then \( F'(x) = \frac{e^x}{x} \).
\[
\int_0^{\sqrt{x}} \frac{e^t}{t} \, dt = F(\sqrt{x}). \quad (F(\sqrt{x})' = F'(\sqrt{x})(\sqrt{x})' = \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}.
\]

Thus \( y' = -\frac{e^{\sqrt{x}}}{2x} + \frac{e^x}{x} = \frac{2e^x - e^{\sqrt{x}}}{2x} \).

63. (a) Displacement: \( \int_0^5 t^2 - t \, dt = \frac{t^3}{3} - \frac{t^2}{2} \bigg|_0^5 = \frac{5^3}{3} - \frac{5^2}{2} - 0 = \frac{175}{6} \) m.

(b) Distance: \( \int_0^5 |t^2 - t| \, dt \). To compute this integral, we need to figure out where \( t^2 - t \) is positive and negative:

\( t^2 - t = 0 \) at \( t = 0, 1 \). Hence, \( t^2 - t < 0 \) for \( 0 < t < 1 \) and \( t^2 - t > 0 \) on \( t > 1 \) (we don’t care about what happens for \( t < 0 \) because the integral is computed over \([0, 5]\)).

\[
\int_0^5 |t^2 - t| \, dt = \int_0^1 -(t^2 - t) \, dt + \int_1^5 |t^2 - t| \, dt = -\frac{t^3}{3} + \frac{t^2}{2} \bigg|_0^1 + \frac{t^3}{3} - \frac{t^2}{2} \bigg|_1^5 = \frac{1}{3} + \frac{1}{2} + \frac{5^3}{3} - \frac{5^2}{2} - \left( \frac{1}{3} - \frac{1}{2} \right) = \frac{177}{6} = 29.5 \) m.

65. \( \int_0^8 r(t) \, dt = R(8) - R(0) \), where \( R(t) \) is an antiderivative of \( r(t) \), i.e. the consumption of oil from \( t = 0 \) (the year 2000) to year \( t \). The integral represents the oil consumption between the years 2000 and 2008 measured in barrels.