MAT 126: PRACTICE FOR THE FINAL
SOLUTIONS

Chapter 6 Review Exercises

2. Intersection points: between \( y = 0 \) and \( y = x^2 \): at (0, 0);
   between \( y = 1/x \) and \( y = x^2 \): \( 1/x = x^2 \) implies that \( x^3 = 1 \), that is, \( x = 1 \).
   The bottom of the region is the line \( y = 0 \); the top is made up of two curves, \( y = x^2 \) and \( y = 1/x \). Hence the area splits into two (each having a different “top”):
   \[
   \int_0^1 x^2 \, dx + \int_1^e \frac{1}{x} \, dx = \frac{x^2}{2} \bigg|_0^1 + \ln x \bigg|_1^e = \frac{1}{2} - 0 + (\ln e - \ln 1) = \frac{1}{2} + 1 = \frac{3}{2}
   \]

4. First, find the intersection points of \( x = -y, x = y^2 + 3y \):
   \[
   -y = y^2 + 3y
   \]
   \[
   y^2 + 4y = 0
   \]
   \[
   y = 0, -4
   \]
   For \(-4 < y < 0, -y > y^2 + 3y\) (check, e.g. the values at \( y = -1 \)). So, we integrate
   \[
   \int_{-4}^{0} -y - (y^2 + 3y) \, dy = \left( -\frac{y^3}{3} - \frac{4y^2}{2} \right) \bigg|_{-4}^{0} = \left( -\frac{(-4)^3}{3} - \frac{4(-4)^2}{2} \right) = \frac{4^3}{6} = \frac{32}{3}
   \]

6. Note that the curves \( y = 1 + x \) and \( y = e^{-2x} \) intersect at the point \((0, 1)\)
   \((e^{-2\cdot0} = 1 = 1 + 0)\).
   On the interval \([0, 1]\), \( 1 + x > e^{-2x} \) (because \( e^{-2x} < 1 \)).
   Hence, the volume is
   \[
   \pi \int_0^1 (1 + x)^2 - (e^{-2x})^2 \, dx = \pi \int_0^1 1 + 2x + x^2 - e^{-4x} \, dx = \pi \left( 1 + x^2 + \frac{x^3}{3} + \frac{1}{4} e^{-4x} \right) \bigg|_0^1 = \pi \left( 1 + 1 + \frac{1}{3} + \frac{e^{-4}}{4} - \frac{1}{4} \right) = \frac{(29 - 3e^{-4})\pi}{12}
   \]
   (The antiderivative of \( e^{-4x} \), i.e. \( \int e^{-4x} \, dx \), can be computed by taking the substitution \( u = -4x, du = -4dx \) and \( dx = -(1/4)du \). Then \( \int e^{-4x} \, dx = -(1/4) \int e^u \, du = -e^u/4 + C = -e^{-4x}/4 + C \).)

8. Points of intersection are:
   \[
   x^3 = 2x - x^2
   \]
   \[
   x^3 + x^2 - 2x = 0
   \]
   \[
   x(x^2 + x - 2) = 0
   \]
   \[
   x = 0, 1, -2
   \]
   Since the region is taken in the first quadrant, we consider only the region between \( x = 0 \) and \( x = 1 \).
Also, note that in this region $x^3 < 2x - x^2$

(a) $\int_0^1 2x - x^2 - x^3 \, dx = \left( x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$

(b) $\int_0^1 \pi(2x - x^2)^2 - \pi(x^3)^2 \, dx = \pi \int_0^1 4x^2 - 4x^3 + x^4 - x^6 \, dx = \pi \left( \frac{4x^3}{3} - x^4 + \frac{x^5}{5} - \frac{x^7}{7} \right) \bigg|_0^1 = \pi \left( \frac{4}{3} - 1 + \frac{1}{5} - \frac{1}{7} \right) = \frac{41\pi}{105}$

9. Points of intersection: $x = x^2$, hence $x = 0, 1$.

(a) $\int_0^1 \pi(x)^2 - \pi(x^2)^2 \, dx = \pi \int_0^1 x^2 - x^4 \, dx = \pi \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \bigg|_0^1 = \pi \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}$

(b) If $y = x^2$, then $x = \sqrt{y}$. Also note that for $0 \leq y \leq 1, \sqrt{y} \geq y$.

$\int_0^1 \pi(\sqrt{y})^2 - \pi(y)^2 \, dy = \pi \int_0^1 y - y^2 \, dy = \pi \left( \frac{y^2}{2} - \frac{y^3}{3} \right) \bigg|_0^1 = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}$

(c) $\int_0^1 \pi(x^2 - 2)^2 - \pi(x - 2)^2 \, dx = \pi \int_0^1 x^4 - 4x^2 + 4 - x^2 + 4x - 4 \, dx = \pi \int_0^1 x^4 - 5x^2 + 4x \, dx = \pi \left( \frac{x^5}{5} - \frac{5x^3}{3} + 2x^2 \right) \bigg|_0^1 = \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8\pi}{15}$

21. The area of a triangle with sides $a$ and $b$ and an angle $\theta$ between them is $\frac{1}{2}ab \sin \theta$. Thus, the area of the cross-section if $\frac{1}{2} \cdot 4 \cdot \frac{x}{2} \cdot \sin \frac{\pi}{3} = \frac{\sqrt{3}x^2}{64}$.

The volume is $\int_0^{20} \frac{\sqrt{3}x^2}{64} \, dx = \frac{\sqrt{3}x^3}{64 \cdot 3} \bigg|_0^{20} = \frac{125\sqrt{3}}{3} \, m^3$

23. $\frac{dx}{dt} = 6t; \frac{dy}{dt} = 6t^2$.

$L = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} \, dt = 6 \int_0^2 \sqrt{t^2 + t^4} \, dt = 6 \int_0^2 t \sqrt{1 + t^2} \, dt = \frac{1}{2} \int_1^5 \sqrt{u} \, du = \frac{2}{3} u^{3/2} \bigg|_1^5 = 2(5\sqrt{5} - 1)$

using the substitution $u = 1 + x^2, \, du = 2x \, dx$ (hence $x \, dx = \frac{1}{2} \, du$), and $u = 1$ when $x = 0, u = 5$ when $x = 2$.

25. $\frac{dy}{dx} = \frac{13}{6} (x^2 + 4)^{1/2} (2x) = \frac{1}{2} x (x^2 + 4)^{1/2}$

$L = \int_0^3 \sqrt{1 + \frac{1}{4} x^2 (x^2 + 4)} \, dx = \int_0^3 \sqrt{\frac{4 + x^4 + 4x^2}{4}} \, dx = \int_0^3 \sqrt{\frac{(x^2 + 2)^2}{4}} \, dx = \int_0^3 \frac{x^2 + 2}{2} \, dx = \frac{1}{2} \int_0^3 x^2 + 2 \, dx = \frac{1}{2} \left( \frac{x^3}{3} + 2x \right) \bigg|_0^3 = \frac{15}{2}$

28. To move the elevator up 30 ft, the work of $1600 \times 30 = 48,000 \text{ lb-ft}$ is required.
Since we are raising a 200 ft cable only 30 ft up, 170 ft of it will be raised all the way. The work required is $170 \times 10 \times 30 = 51,000$ lb-ft.

To compute the work required to lift the remaining 30 ft of cable, we split it into pieces. To lift a segment of the cable of length $\Delta x$ up $x$ feet, the work of $\Delta x \times 10 \times x$ is required. Hence, the total work is

$$\int_{30}^{0} 10x \, dx = 5x^2|_{0}^{30} = 4,500 \text{ lb-ft.}$$

Added up, the work is $48,000 + 51,000 + 4,500 = 103,500$ lb-ft.

29a. First, determine what kind of a parabola is rotated to obtain the tank's shape. We can assume that the parabola's vertex is at the origin, i.e. that it is of the form $y = ax^2$. Since when $x = 4$, $y = 4$ (at the top), we see that $4 = a\frac{1}{4}$, i.e. $a = 1/4$.

At the distance of $y$ ft from the bottom, the horizontal cross-section is a circle with radius $x = 2\sqrt{y}$ (because the tank's shape is determined by the graph of $y = x^2/4$). Hence, its area is $\pi x^2 = 4\pi y$. The cross-section can be thought of as a thin layer of volume $4\pi y \Delta y$. Its weight is $62.5(4\pi y \Delta y)$. The distance to the top is $4 - y$, hence the work required to lift the layer to the top is $62.5(4\pi y \Delta y)(4 - y)$.

$$W = \int_{0}^{4} 250\pi y(4 - y) \, dy = 250 \int_{0}^{4} 4y - y^2 \, dy = 250 \left(2y^2 - \frac{y^3}{3}\right)|_{0}^{4} = \frac{8000\pi}{3} \text{ lb-ft.}$$

31. Assume that the gate is made of strips with height $\Delta x$. If the area of the strip at the depth of $x$ feet is $A(x)$, then the hydrostatic force acting on the strip is $26.5A(x)\Delta x$ lb.

$A(x)$ is the area of a rectangle with width $\Delta x$ and length $3 + 2a$ (see below):

To determine, consider the triangle on the right. The smaller and the larger triangles are similar, hence $\frac{a}{1} = \frac{2 - x}{2}$. Thus $a = (2 - x)/2$. Therefore, $A(x) = (3 + 2a)\Delta x = (3 + (2 - x))\Delta x = (5 - x)\Delta x$.

The hydrostatic force acting on the strip is then $62.5(5 - x)\Delta x$. Summing up the forces and passing to the limit, we have

$$F = \int_{0}^{2} 62.5(5 - x) \, dx = 62.5 \int_{0}^{2} (5x - x^2) \, dx = 62.5 \left(\frac{5x^2}{2} - \frac{x^3}{3}\right)|_{0}^{2} =$$

$$62.5 \left(10 - \frac{8}{3}\right) = \frac{1375}{3} = 458\frac{1}{3} \text{ lb.}$$
37. (a) The function $f(x)$ is never negative. The only other condition to check is $\int_{-\infty}^{\infty} f(x) dx = 1$:

$$
\int_{-\infty}^{\infty} f(x) dx = \int_{0}^{10} \frac{\pi}{20} \sin \left( \frac{\pi x}{10} \right) dx = \int_{0}^{\pi} \frac{1}{2} \sin u \ du = \frac{1}{2} (-\cos u)\bigg|_{0}^{\pi} = \frac{1}{2} (-\cos \pi - (-\cos 0)) = \frac{1}{2}(-(-1) - (-1)) = 1
$$

using the substitution $u = \frac{\pi}{10} x$, $du = \frac{\pi}{10} dx$, and $u = 0$ when $x = 0$, $u = \pi$ when $x = 10$.

(b) $P(X < 4) = \int_{-\infty}^{4} f(x) dx = \int_{0}^{4} \frac{\pi}{20} \sin \left( \frac{\pi x}{10} \right) dx = \int_{0}^{2\pi/5} \frac{1}{2} \sin u \ du = \frac{1}{2} (-\cos u)\bigg|_{0}^{2\pi/5} = \frac{1 - \cos(2\pi/5)}{2}$

using same substitution as in (a)

(c) The mean is $\mu = \int_{0}^{10} x \frac{\pi}{20} \sin \left( \frac{\pi x}{10} \right) dx = -\frac{1}{2} x \cos \left( \frac{\pi x}{10} \right)\bigg|_{0}^{1} + \frac{1}{2} \int_{0}^{10} \cos \left( \frac{\pi x}{10} \right) dx = -\frac{1}{2} \cos \left( \frac{\pi x}{10} \right)\bigg|_{0}^{10} + \frac{10}{2} \sin \left( \frac{\pi x}{10} \right)\bigg|_{0}^{10} = -\frac{1}{2} (10 \cos \pi) = 5$

(using integration by parts with $x = u, du = dx, v = -\frac{1}{2} \cos \left( \frac{\pi x}{10} \right), dv = \frac{\pi}{20} \sin \left( \frac{\pi x}{10} \right) dx$)

That $\mu = 5$ is to be expected since $f(x)$ is symmetric about $x = 5$, hence this is where the mean value lies.