MAT 540, Homework 6, due Wednesday, Oct 25, in class

1. If X is path-connected, $p : \tilde{X} \to X$ a covering, and $x_1, x_2 \in X$, show that there is a (non-canonical) bijection between sets $p^{-1}(x_1)$ and $p^{-1}(x_2)$.

2. Find a 3-fold covering $p : \tilde{X} \to X$ of the space $X = S^1 \vee S^1$ and a loop $\gamma : I \to X$, $\gamma(0) = \gamma(1) = x_0$, such that when you lift the path γ to $\tilde{X}$, you can get a closed loop or an open path, depending on its initial point in $\tilde{X}$. In other words: consider $\tilde{x}_1$, $\tilde{x}_2 \in \tilde{X}$ such that $p(\tilde{x}_1) = p(\tilde{x}_2) = x_0$, and let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be two lifts of γ, such that $\tilde{\gamma}_1(0) = \tilde{x}_1$ and $\tilde{\gamma}_2(0) = \tilde{x}_2$. Find an example where $\tilde{\gamma}_1(1) = \tilde{x}_1$, so $\tilde{\gamma}_1$ is a closed loop, but $\tilde{\gamma}_2(1) \neq \tilde{x}_2$ (the path $\tilde{\gamma}_2$ is not a loop).

3. Let $k > 1, n \geq 1$, $f, g : \mathbb{R}\mathrm{P}^k \to T^n$ be two continuous maps. Show that f and g are homotopic.

4. (a) Prove that if $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, then $p_* : \pi_k(\tilde{X}, \tilde{x}_0) \to \pi_k(X, x_0)$ is an isomorphism for every $k > 1$.

(b) Compute $\pi_k(\mathbb{R}\mathrm{P}^\infty)$ for all $k \geq 1$, $\pi_k(\mathbb{R}\mathrm{P}^n)$ for $1 \leq k \leq n$, and $\pi_k(T^n)$ for all $k \geq 1$, where $T^n = (S^1)^n$ is the n-dimensional torus, $n \geq 1$. Assume $\pi_n(S^n) = \mathbb{Z}$ as known.

5. (a) Let X be a path-connected CW complex, $p : \tilde{X} \to X$ a covering. Prove that $\tilde{X}$ can be given the structure of a CW complex.

(b) If X has finitely many cells, and $p : \tilde{X} \to X$ is an n-fold covering (that is, $p^{-1}(x)$ consists of n points), compute the Euler characteristic $\chi(\tilde{X})$ in terms of the Euler characteristic $\chi(X)$.
Recall that the Euler characteristic of a finite CW complex X is defined as

$$\chi(X) = \sum (-1)^k c_k,$$

where c_k is the number of k-dimensional cells of X. The Euler characteristic is a topological invariant, it does not depend on the choice of the CW structure. (If X and X' are homeomorphic, or even just homotopy equivalent, then $\chi(X) = \chi(X')$ but you need homology to prove invariance.)

6. Does there exist a covering $S^2 \to T^2$?

Note that we didn't yet prove any theorems about classification of coverings or their equivalence. Please give a direct proof even if you already know those theorems.

7. A plane triangle is an unordered triple of points in $\mathbb{R}^2$ which are not collinear. Let T be the space of all plane triangles, and $R \subset T$ its subspace consisting of all right triangles. (What is the topology on T?)

(a) Find the fundamental groups of T and R. What is the homomorphism i_* induced by the inclusion $i : R \to T$?

(b) Does T retract to R?

There are some covering spaces lurking here.