MAT 540, Homework 5, due Wednesday, Oct 18, in class

Part of this homework is a refresher on the van Kampen theorem. I believe that most students are already familiar with tis material, and also that although van Kampen is sometimes a useful tool, in most situations π_1 is best computed by other methods. For this reason, van Kampen will be discussed in the lectures *only* if people have any trouble with the reading and the exercises.

Please read in Hatcher, Chapter 1, section 1.2, and Fomenko-Fuchs, Lecture 7, the setup, statement, proof and applications of the van Kampen theorem. Much of the work goes into a careful statement on the algebraic side: the groups are often represented by generators and relators.

Use van Kampen in questions 1 and 2a (and the optional question on the knot group.) Please pay attention to the open set requirement of the theorem: you usually need to consider larger open sets that are homotopy equivalent to the original "pieces". (For example, in 2a you'll need to use contractible neighborhoods of x_0 and y_0.)

1. Use van Kampen to compute the fundamental groups of:

(a) the Klein bottle K, represented as the union of two Möbius bands glued along their boundary circle.

Note: when the Klein bottle is given as the square whose top and bottom sides are glued after a twist, and left and right sides are glued without a twist, you can see the decomposition of K into two Möbius bands as above: one Möbius band M is obtained from the vertical middle third of the square (top and bottom identified after a twist), and the second Möbius band is obtained from the rest of the square.

(b) the CW complex Y obtained from S^1 by attaching two 2-cells, via maps $z \mapsto z^2$ and $z \mapsto z^3$, respectively. (Think of a 2-cell as unit disk $D \subset \mathbb{C}$, so that $\partial D = \{|z| = 1\}$).

(c) the space Z obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

2. Let (X, x_0), (Y, y_0) be two CW complexes, x_0, y_0 0-cells, $X \vee Y = X \sqcup Y / x_0 \sim y_0$ as usual.

(a) Suppose that the fundamental groups $\pi_1(X, x_0) = \langle g_\alpha \mid r_\beta \rangle$ and $\pi_1(Y, y_0) = \langle h_\gamma \mid q_\delta \rangle$ are given by generators and relations. Show that $\pi_1(X \vee Y, x_0) = \langle g_\alpha, h_\gamma \mid r_\beta, q_\delta \rangle$.
We say that $\pi_1(X \vee Y, x_0)$ is the *free product* of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

(b) Explain how to compute $\pi_1(X \vee Y, x_0)$ via the CW-structure, and how it relates to the calculations for $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$. What is the 1-skeleton of $X \vee Y$, and how are the 2-cells attached? You should get the same result as in (a).

(c) For a concrete example, compute $\pi_1(\mathbb{RP}^3 \vee \mathbb{RP}^2 \vee T^2)$ via the CW-structure.

Optional but strongly recommended: Read sections 7.3 and 7.4 of Fomenko-Fuchs to learn about the knot group (the fundamental group of the complement of the knot) and the Wirtinger presentation; see also Hatcher Exercise 22 p.55, section 1.2. This is one situation where the van Kampen thorem is really needed. Try to work through some of the exercises in Fomenko–Fuchs but do not submit anything.

3. A pair (X, A) is called n-**connected** if $\pi_i(X, A, x_0) = 0$ for every $x_0 \in A$, $i = 1, 2, \ldots, n$, and every path-component of X contains points of A.

Prove that every n-connected CW pair (X, A) is homotopy equivalent to a CW pair (X', A') such that A' contains all cells of X' of dimensions n or less (that is, A' contains the n-th skeleton of X').

4. (Do not submit.) We proved three of the statements required to prove exactness of the homotopy sequence of the pair. Prove the remaining three. Also, check the statements at the end of the sequence (π_0 requires a bit of special attention). See Fomenko-Fuchs, Chapter 1 section 8.7 if you get stuck, but these proofs are not difficult and it's best if you do them yourself. Do exercise 10 section 8.7 (exact sequence of a triple) or read this material in Hatcher Chapter 4.

Please also do and **submit Questions 9, 10, 14** from section 1.2 of Hatcher (pp.53-54 online).