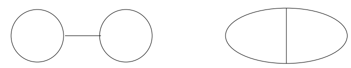
MAT 540, Homework 1, due Wednesday, Sept 13, in class

0. *(do not submit but make sure you understand this)*
The two subspaces of $\mathbb{R}^2$ shown below are homotopy equivalent, as can be seen, for example, by representing them as a deformation retract of a third space. Explain how to construct explicit homotopy equivalences between these spaces (and why they are homotopy inverses of one another). Arguing by picture is fine but you need to understand what the maps and the homotopies do.



1. The *cone* CX over a topological space X is defined as $X \times I/(X \times \{0\})$, where $I = [0,1]$. The point of CX obtained by collapsing $X \times \{0\}$ is called the vertex of the cone; $X \times \{1\}$ is called the base of the cone. (The base can be identified with X.)
The *suspension* ΣX is obtained by collapsing the base of the cone CX to one point, that is,

$$\Sigma X = CX/(X \times \{1\}).$$

Alternatively, you can obtain ΣX from $X \times I$ by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point, or by gluing two cones over X together along their bases. Suspensions are very important in homotopy theory.
(a) What is ΣS^n?
(b) Prove that the base of a cone CX is a retract of CX if and only if X is contractible.
(c) Show that if $X \sim Y$, then $\Sigma X \sim \Sigma Y$.

2. Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path components of X and the set of path components of Y. (Explain this directly, although this can be extracted from a more general statement we will prove in class.)

3. The Gram-Schmidt othogonalization process can be interpreted to give a deformation retraction of one space of matrices onto another. Give a precise statement (what is a deformation retract of what?) and prove it.

4. Let S^2 be a 2-sphere, T^2 a torus, $a, b, c \in S^2$ three distinct points, $d \in T^2$. Show that $S^2 \setminus \{a, b, c\}$ is homotopy equivalent to $T^2 \setminus \{d\}$.

5. Let $f, g : X \to S^n$ be two continuous maps. Show that either there exists a point $x_0 \in X$ such that $f(x_0) = g(x_0)$, or $g \sim -f$. Here, $-f : X \to S^n$ is a map given by $x \mapsto -f(x)$.

6. Let P be the space of quadratic polynomials of the form $p(x) = x^2 + ax + b$, with complex coefficients a, b, such that p has two distinct complex roots. Show that P is homotopy equivalent to S^1.
The space P has induced topology as a subset of $\mathbb{C}^2$, where the coordinates are given by coefficients a, b.

7. A topological group G is a topological space which is also a group, such that the group operation $(x, y) \mapsto xy$ and taking the inverse $x \mapsto x^{-1}$ are continuous (as maps $G \times G \to G$ and $G \to G$, respectively).
Let G be a path-connected topological group.
(a) Show that $\pi_1(G, x_0)$ is abelian.
(b) Show that G is n-simple for every n, that is, the action of $\pi_1(G, x_0)$ on $\pi_n(G, x_0)$ is trivial for every n.
Remember what we mean by the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$: a loop u in the class $[u] \in \pi_1(X, x_0)$ gives an isomorphism $u_\# : \pi_n(X, x_0) \to \pi_n(X, x_0)$ as constructed in class. This isomorphism only depends on the homotopy class $[u]$, not on the choice of its representative. In general, $u_\#$ is a non-trivial automorphism, different elements of $\pi_1(X, x_0)$ give different automorphisms, and $u_\# v_\# = (u * v)_\#$, where $u * v$ is a product of paths. The action is trivial if it happens that $u_\# = id$ for all $[u] \in \pi_1(X, x_0)$.

8. Let X, Y be topological spaces. The *compact-open topology* on the space of continuous functions $C(X,Y)$ is generated by the pre-basis given by sets

$$\mathcal{O}(K,U) = \{f : X \to Y | f(K) \subset U\},$$

where $K \subset X$ is compact, $U \subset Y$ is open. The basis of this topology is given by the collection $\mathcal{B}$ of all finite intersections of such sets.

(a) Check that $\mathcal{B}$ satisfies the conditions that ensure that $\mathcal{B}$ is a basis for some topology on the space.

(b) Suppose that X is compact, $Y = \mathbb{R}$. Show that in this case, the compact-open topology coincides with the *topology of uniform convergence*. The latter is the topology on $C(X,Y)$ induced by the metric

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

(It's obvious that the formula above gives a metric; no need to write up checking the metric space axioms.)

Your proof should work for any metric space Y, with only notational changes.

(c) Let $X = \mathbb{R}$, $Y = [-1,1]$. Show that on the space $C(X,Y) = \{f : \mathbb{R} \to [-1,1]\}$, the metric topology as in (b) is *different* from the compact-open topology.

In general, when Y is a metric space and X may be non-compact, the compact-open topology is the *topology of uniform convergence on compact sets*, which is typically different from uniform convergence on the entire X for bounded functions. (Think about this, but you don't need to submit any explanations for this more general case.)

(d) Show that the set $\pi(X,Y)$ of homotopy classes of continuous maps $f : X \to Y$ is the set of path components of $C(X,Y)$, with respect to the compact-open topology.

In part (d), assume that X is *locally compact*, in the following strong sense:

for every $x \in X$ and every open $U \ni x$, there is an open set V such that $x \in V \subset Cl\,V \subset U$, and the closure $Cl\,V$ of V is compact.

You can also assume that all spaces are Hausdorff if you want.