WHAT IS an amoeba?

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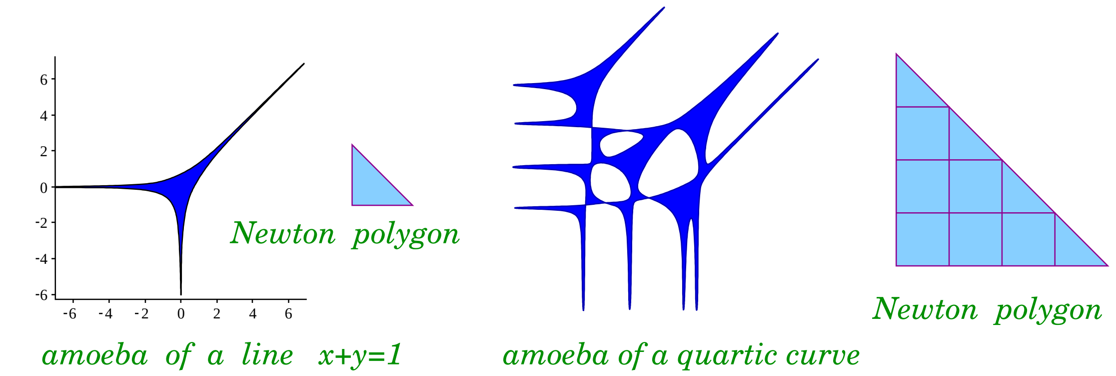

In mathematical terminology the word *amoeba* is a recent addition.[1] It was introduced by I.M.Gelfand, M.M.Kapranov and A.V.Zelevinsky in their book [2] in 1994. A mathematical amoeba falls short of similarity to its biological prototype. In the simplest case, it is a region in $\mathbb{R}^2$ that may pretend to be a picture of an amoeba: a body with several holes (vacuoles) and straight narrowing tentacles (pseudopods) going to infinity.

A planar amoeba is the image of the zero locus of a polynomial in two variables under the map $\operatorname{Log} : (\mathbb{C} \smallsetminus 0)^2 \to \mathbb{R}^2 : (z, w) \mapsto (\log|z|, \log|w|)$. The zero locus of a polynomial in two variables is called a *plane complex algebraic curve*. This is a surface in the 4-space $\mathbb{C}^2$ defined by the equation $f(z, w) = 0$, where f is a polynomial $\sum c_{pq} z^p w^q$ with complex coefficients c_{pq}. The minimal convex polygon Δ that contains all points $(p, q) \in \mathbb{R}^2$ corresponding to non-zero coefficients of the equation is called the *Newton polygon* of f. It represents geometry of the equation, and, as we will see, its geometry is closely related to the geometry of the corresponding complex curve $\mathcal{C} \subset \mathbb{C}^2$ and its amoeba $\mathcal{A} \subset \mathbb{R}^2$.

An amoeba reaches infinity by several tentacles. Each tentacle accommodates a ray and narrows exponentially fast towards it. Thus there is only one ray in a tentacle. The ray is orthogonal to a side of the Newton polygon and directed along an outward normal of the side. For each side of Δ there is at least one tentacle associated to it. The maximal number of such tentacles is a sort of lattice length of the side: the number of pieces into which the side is divided by integer lattice points (i.e., points with integer coordinates).

Each connected component of an amoeba's complement $\mathbb{R}^2 \smallsetminus \mathcal{A}$ is convex. Besides components lying between tentacles, there can be bounded components. The number of bounded components is at most the number of interior integer lattice points of Δ, and hence the total number of components of $\mathbb{R}^2 \smallsetminus \mathcal{A}$ is at most the number of all integer lattice points of Δ. Each component corresponds to some integer lattice point of Δ.

To establish this correspondence, take a point in a component of $\mathbb{R}^2 \smallsetminus \mathcal{A}$, and consider its preimage under the map Log. The preimage is a torus and consists of points whose complex coordinates have fixed absolute values, but varying arguments. On the torus there are circles: meridians, along which z is fixed, and parallels, along which w is fixed. Consider a meridian, and call the disk it bounds D. Let us count the intersections, with multiplicities, between D and the complex curve $\mathcal{C}$ (so this is the homological intersection number $D \circ \mathcal{C}$, or, if you like, the linking number $\operatorname{lk}(m, \mathcal{C})$). Denote the intersection number by q. In the same way, consider a parallel and the disk it bounds, count (with multiplicities) the intersections of the disk with $\mathcal{C}$, and denote the intersection number by q. The point $(p, q) \in \mathbb{R}^2$

[1]I was told that in mathematical logic amoebas have been known for more than twenty years. However, they belong to an entirely different class of mathematical microbes, and have never bitten me, so I cannot tell you about them.

belongs to Δ and corresponds to the component of $\mathbb{R}^2 \setminus \mathcal{A}$ we started with. (The numbers p and q are independent of the choice of meridian or parallel and depend only on the connected component of $\mathbb{R}^2 \setminus \mathcal{A}$.) Different components of $\mathbb{R}^2 \setminus \mathcal{A}$ give rise to different integer lattice points of Δ. It may happen that some integer lattice points of Δ do not correspond to any component. Only vertices of Δ necessarily correspond to components. Any collection of integer lattice points of Δ, which includes all vertices, is realizable by the amoeba of an appropriate algebraic curve with this Newton polygon Δ.

Although a planar amoeba is not bounded, its area is finite. Moreover,

$$Area(\mathcal{A}) \leq \pi^2 Area(\Delta).$$

Complex curves whose amoebas have the extremal area are very special. In particular, $\mathbb{R}^2 \setminus \mathcal{A}$ has the maximal number of components. A mapping $\mathbb{C}^2 \to \mathbb{C}^2 : (z, w) \mapsto (az, bw)$ with appropriate $a, b \in \mathbb{C}$ makes such a curve *real,* i.e., defined by a polynomial equation with real coefficients. The geometry of the real part of this curve is also very special. Real algebraic curves of this kind were discovered by A.Harnack in 1876 when he constructed real algebraic plane projective curves with the maximal number of components for each degree. Only one component of a Harnack curve meets the coordinate axes (including the line at infinity), and the intersections with the axes lie on disjoint arcs of this component. Consideration of amoebas allowed G.Mikhalkin to prove that any real curve with these properties must be topologically isotopic to a Harnack curve.

One of the main analytic tools used in the study of amoebas is the remarkable *Ronkin function* $N_f : \mathbb{R}^2 \to \mathbb{R}$. For a polynomial f, it is defined by

$$N_f(x, y) = \int_{\text{Log}^{-1}(x,y)} \log|f(z, w)|\, d\frac{z}{2\pi i|z|}\, d\frac{w}{2\pi i|w|}.$$

If f is a monomial $az^p w^q$, then N_f is a linear function, $N_f(x, y) = px + qy + \log|a|$ with gradient (p, q). For a general f, the Ronkin function is convex. On each component of $\mathbb{R}^2 \setminus \mathcal{A}$, the function N_f behaves like the Ronkin function of a monomial: it is linear, and its gradient is the corresponding integer point of Δ. The maximum of these linear functions is a piecewise linear convex function. The set where it is not differentiable is a union of segments and rays that are contained in the amoeba and that constitute its deformation retract. This set is called the *spine* of $\mathcal{A}$.

Logarithmic coordinates and amoebas disclose a piecewise linear stream in the nature of algebraic geometry. There is a non-archimedian version of amoebas that brings these ideas to algebraic varieties over other fields. There is also a similar theory in higher dimensions. The notion of an algebraic curve is replaced by the notion of an algebraic variety, and the Newton polygon becomes a Newton polytope. Amoebas provide a new way to visualize complex algebraic varieties. Looking at an amoeba, one can see handles of complex curves and cycles in high dimensional varieties, watch degenerations, and build more complicated varieties from simple ones.

The theory of amoebas is a fresh and beautiful field of research, still quite accessible to a newcomer, where exciting discoveries are still ahead. The impressive results described above were obtained during a short period of about 8 years by various people. The definition and initial fundamental observations are due to I.M.Gelfand, M.M.Kapranov and A.V.Zelevinsky. Relations between components of $\mathbb{R}^2 \setminus \mathcal{A}$ and integer lattice points of Δ were discovered by M.Forsberg, M.Passare, and A.Tsikh. The spine of an amoeba, the Ronkin function, and the estimate of the area are due to H.Rullgård and M.Passare. Homological interpretations and relations to real algebraic geometry are due to G.Mikhalkin. I enjoyed the feast. About 20 years ago I found a way to construct real algebraic curves by sort of gluing curves to each other. I heard that this gluing and the use of logarithmic coordinates in its description, after being replanted to the complex soil, motivated the introduction of amoebas. A version of the gluing is used to glue amoebas.

References

[1] Mikael Forsberg, Mikael Passare, August Tsikh, *Laurent determinants and arrangements of hyperplane amoebas,* Advances in Math. **151** (2000), 45–70.

[2] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994. x+523 pp.

[3] Grigory Mikhalkin, *Real algebraic curves, moment map and amoebas,* Ann. of Math. 151 (2000), no. 1, 309–326.

[4] G.Mikhalkin, H.Rullgård, *Amoebas of maximal area.* Internat. Math. Res. Notices 2001, no. **9**, 441–451.