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Gauss Diagram Formulas for Vassiliev Invariants

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Although the original definitions of Vassiliev knot invariants are rather far from explicit formulas, recently several types of formulas for them have been found; see [1], [2], [7], [4], [5]. However, most of them were designed more for the sake of general theory than for actual computations. Our initial goal was to provide more practical formulas.

We were motivated by the well-known case of the linking number. It is the simplest of Vassiliev invariants for links. It can be computed in many different ways; see, e.g., [8]. Integral formulas of Bar-Natan [2] and Kontsevich [1] for Vassiliev invariants generalize the Gauss integral formula for the linking number. As is known, the Gauss integral formula has simple combinatorial counterparts. In this paper we present a similar transition to combinatorial formulas for higher-order Vassiliev invariants.

As in the case of the linking number, both integral and combinatorial formulas may be obtained from an interpretation of the invariants as degrees of some maps. It was this viewpoint that motivated the whole of our investigations and appeared to be a rich source of various special formulas. We plan to discuss this phenomenon in detail in a forthcoming paper.

1 Gauss diagrams

Initial data for a formula may be of various sorts. We start from a knot diagram. A knot diagram is a generic immersion of a circle to a plane enhanced by information on

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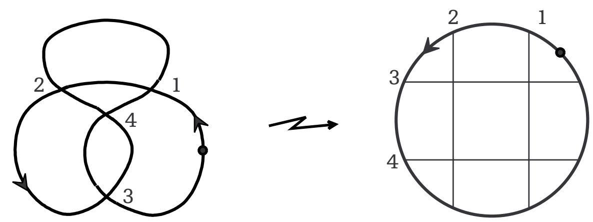


Figure 1

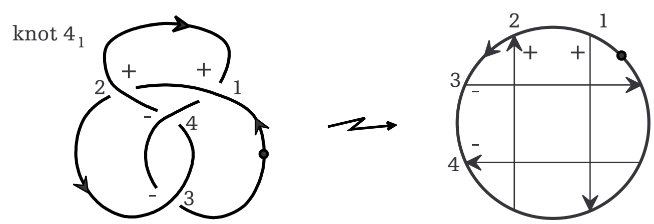


Figure 2

overpasses and underpasses at double points. It is a nice visual description of knots, but we need to reorganize it in a more combinatorial fashion.

A generic immersion of a circle to a plane is characterized by its *Gauss diagram*. The Gauss diagram is the immersing circle with the preimages of each double point connected with a chord. See Figure 1. (Gauss studied the question of which chord diagrams correspond to immersions.)

To incorporate the information on overpasses and underpasses, we orient chords from the upper branch to the lower one. Furthermore, each chord is equipped with the sign of the corresponding double point (local writhe number). See Figure 2. We call the result a *Gauss diagram* of the knot.

2 Arrow diagrams

By an *arrow diagram* we mean an (oriented) circle with several oriented chords having distinct end points and equipped with multiplicities 1 or 2. In a picture, the multiplicity 2 of a chord is shown by a double arrowhead.

Two arrow diagrams are said to be *isomorphic* if there exists a homeomorphism between them mapping the circle to the circle and preserving all orientations and multiplicities. The isomorphism class of an arrow diagram A is denoted by $[A]$.

Let G be a Gauss diagram of a knot and A an arrow diagram. By a *representation* of A in G we mean an embedding of A to G which maps the circle of A to the circle of G preserving the orientation and each chord of A to a chord of G, also preserving orientation.

For a representation $\varphi : A \to G$, we define the *sign* by

$$\operatorname{sign}(\varphi) = \prod \operatorname{sign}(\varphi(C))^{\mu(C)}, \tag{1}$$

where the product is taken over all chords C of A, $\mu(C)$ denotes the multiplicity of C in A, and $\operatorname{sign}(\varphi(C))$ is the sign of the chord $\varphi(C)$ in G.

Denote by $\langle A, G\rangle$ the sum

$$\langle A, G\rangle = \sum_{\varphi:A\to G} \operatorname{sign}(\varphi) \tag{2}$$

over all representations of A in G. Obviously, $\langle A, G\rangle$ does not change when A changes inside the same isomorphism class. Set $\langle [A], G\rangle = \langle A, G\rangle$.

Example. If G is the Gauss diagram of a knot diagram D, then $\langle$ , G$\rangle$ is the writhe of D.

It follows immediately from the definition of the writhe: it is the sum of signs of all double points of D.

3 Based diagrams

We will use also a version of the definitions above with *base points*. *Based* Gauss and arrow diagrams are obtained from the diagrams considered above by marking a point on the circle of the diagram. The point should be distinct from the endpoints of chords. For a based arrow diagram A and a based Gauss diagram, the number $\langle A, G\rangle$ is defined as above, but the summation is taken over all representations of A in G mapping the base point of A to the base point of G.

Example. Let G be the Gauss diagram of a knot diagram D with some base point. Then the winding number of D is equal to

$\langle$ , G$\rangle$ $-$ $\langle$ , G$\rangle$ $+ i_1 + i_2$ (3)

where i_1 and i_2 are the indices with respect to D of the components of the complement of D adjacent to the base point.

This is a straightforward reformulation of a well-known Whitney formula [10].

Theorem 1. If G is any based Gauss diagram of a knot K, then

$\nu_2(K) = \Big\langle$  $, G \Big\rangle,$ (4)

where $\nu_2(K)$ is the Vassiliev invariant of degree 2 which takes values 0 on the unknot and 1 on a trefoil.

Example. As it is easy to see from Figure 2, $\nu_2(4_1) = -1$.

Corollary of Theorem 1. If G is any based Gauss diagram of a knot K, then the Arf invariant of K is equal to the number of representations of the arrow diagram  in G modulo 2.

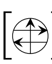
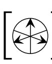
4 Linear combinations of arrow diagrams

Denote by $\mathcal{A}$ the vector space over $\mathbb{Q}$ generated by isomorphism classes of arrow diagrams. By linearity, $\langle A, G \rangle$ is naturally extended to $A \in \mathcal{A}$.

This allows us to rewrite the formula 3 as

$\langle [$  $] - [$  $], G \rangle + i_1 + i_2.$

Theorem 2. If G is a Gauss diagram of a knot K, then

$\nu_3(K) = \Big\langle \frac{1}{2} \Big[$  $\Big] + \Big[$  $\Big], G \Big\rangle,$ (5)

where $\nu_3(K)$ is the Vassiliev invariant of degree 3 which takes values 0 on the unknot, $+1$ on the right trefoil, and -1 on the left trefoil.

Example. As it is easy to see from Figure 2, $\nu_3(4_1) = 0$.

Combinatorial formulas for ν_2 and ν_3 in terms of the Gauss diagram were discovered recently by Lannes [5]. His formulas look more complicated than 4 and 5.

5 Arrow diagrams algebra

The vector space $\mathcal{A}$ has a natural structure of filtered algebra. To define the filtration, we define the degree of an arrow diagram to be the number of chords. Denote by $\mathcal{A}_n$ the subspace of $\mathcal{A}$ generated by classes of diagrams of degree $\leq n$. To define the multiplication, first we have to introduce some preliminary notions.

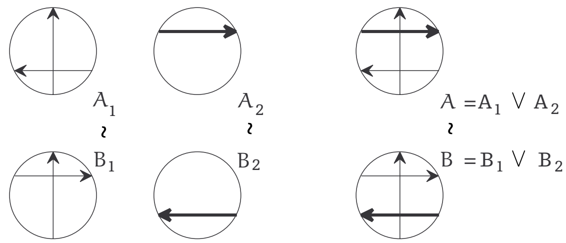


Figure 3

Let A be an arrow diagram. By a decomposition of the multiplicity $\mu(C)$ of a chord C of A, we mean a pair $\mu_1(C)$, $\mu_2(C)$ such that $\mu_i = 0, 1$, or 2, $\mu_1 + \mu_2 \neq 0$ and $\mu = \mu_1 + \mu_2 \bmod 2$. A decomposition like that gives rise to a decomposition of A into 2 arrow diagrams. Namely, denote by A_i, $i = 1, 2$ the arrow diagram obtained from A by erasing all chords C with $\mu_i(C) = 0$ and assigning multiplicity $\mu_i(C)$ to each remaining chord C. Then we will say that there is a *decomposition* $A = A_1 \vee A_2$ of A into arrow diagrams A_1, A_2.

Decompositions $A = A_1 \vee A_2$ and $B = B_1 \vee B_2$ of arrow diagrams A and B are said to be isomorphic if there is an isomorphism $A \to B$ mapping A_i to B_i for $i = 1, 2$. In general, isomorphism classes of arrow diagrams involved do not determine the isomorphism class of the decomposition: it may happen that A is isomorphic to B, A_i is isomorphic to B_i, but the decompositions $A = A_1 \vee A_2$ and $B = B_1 \vee B_2$ are not isomorphic. See Figure 3.

Let A_1 and A_2 be arrow diagrams. Take a maximal family $B_1, \dots, B_n$ of pairwise nonisomorphic arrow diagrams such that each B_i is decomposable into A_1 and A_2. Denote by m_i, $i = 1, \dots, n$ the number of isomorphism classes of decompositions of B_i into A_1 and A_2. Define the product of the isomorphism classes of A_1 and A_2 by

$$[A_1][A_2] = \sum_{i=1}^{n} m_i [B_i].$$

The multiplication is extended by linearity to the whole $\mathcal{A}$, and turns it into associative commutative algebra with unit. The unit of this algebra is the isomorphism class of arrow diagrams with no chords. It is clear that $\mathcal{A}_n \cdot \mathcal{A}_m \subset \mathcal{A}_{n+m}$, which means that this algebra is filtered.

As follows immediately from the definitions, for any $a, b \in \mathcal{A}$ and a Gauss diagram G,

$$\langle ab, G\rangle = \langle a, G\rangle\langle b, G\rangle.$$

We call an element $A \in \mathcal{A}_n$ the *arrow polynomial of degree* $\leq n$, and $\langle A, G\rangle$ the *value* of the arrow polynomial A on the Gauss diagram G.

A similar construction can be applied to based arrow diagrams. The algebra obtained will be denoted by $\mathcal{B}$.

6 First general questions

The notions introduced above naturally give rise to the following questions.

(1) Which arrow polynomials define knot invariants, i.e., take values invariant under the Reidemeister moves of knot diagrams?

(2) Are the knot invariants obtainable in that way also Vassiliev invariants? If yes, what is the degree?

(3) Can any Vassiliev invariant be calculated as a function of arrow polynomials evaluated on the knot diagram?

(4) Are presentations of knot invariants by arrow polynomials unique?

(5) What are generalizations to similar situations: links, framed knots, framed links, etc.?

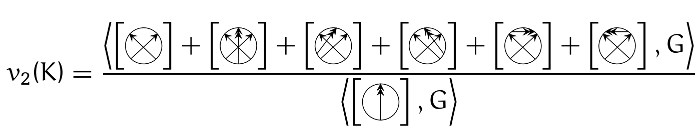
We can immediately answer the second question. The answer is positive.

Theorem 3. Let A be an arrow polynomial of degree n. If A defines a knot invariant (i.e., $\langle A, G\rangle$ depends only on the isotopy type of the knot) then this knot invariant is a Vassiliev invariant of degree $\leq n$.

The first question admits an answer in the form of an algorithm. There are transformations of Gauss codes corresponding to the Reidemeister moves, and one can check if a given arrow polynomial is invariant under these transformations.

However, this answer does not seem to be satisfactory. It does not suggest how to produce all invariant arrow polynomials of a given degree. Furthermore, it does not help to answer to the third question.

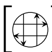
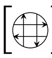
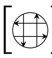
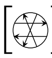
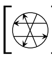
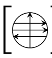
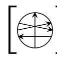
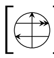
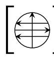
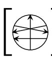
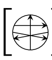
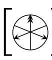
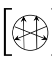
Conjecturally, the answer to the third question is positive. However, we have to allow based diagrams and/or functions of arrow polynomials. For example, by Theorem 1 the Vassiliev invariant of degree 2 can be presented by based arrow diagram. The base point can be excluded, but at the cost of using a ratio of arrow polynomials. Namely, if G is a Gauss diagram of a knot K with a nonzero number of arrows, then

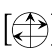
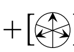
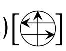
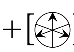
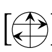
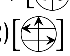


It is proved by an obvious symmetrization of formula (4).

Using skein relations, one can find formulas for the Vassiliev invariants arising from the expansions of the corresponding quantum polynomial invariants. However, these formulas involve a large number of terms. Here is the next example.

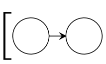
Theorem 4. Let ν_4 be the Vassiliev invariant of degree 4 which is additive, is invariant under mirror reflection, takes value 3 on the trefoil, and takes value 2 on the figure 8 knot. If G is a Gauss diagram of a knot K, then $\nu_4(K) = \langle A, G\rangle$ where

$A =$ [] $+ 6$ [] $+ 2$ [] $+ 3$ [] $+$ [] $+ 2$ [] $+ 2$ [] $+$ [] $-$ []
$+$ [] $+$ [] $+$ [] $+ 2$ [] $+ 2$ [] $+$ [] $+$ []. (6)

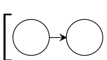
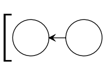
The answer to the fourth question is negative: in fact all Vassiliev invariants presented above by arrow polynomials can be presented by other arrow polynomials. The invariant of degree 2 is presented not only by [], but also by [], and thus by each arrow polynomial a[] $+ (1-a)$[] with $a \in \mathbb{C}$. Similarly, ν_3 of Theorem 2 can be presented not only by the arrow polynomial $(1/2)$[]$+$[], but also by $(1/2)$[]$+$[], and thus by arrow polynomials $(a/2)$[] $+ ((1-a)/2)$[] $+$ [] with $a \in \mathbb{C}$.

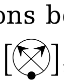
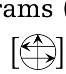
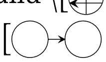
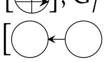
As was mentioned, our investigation was inspired by the well-known method for calculation of the linking number of 2 disjoint circles in $\mathbb{R}^3$. All the definitions given above admit the obvious generalization to the case of oriented links with an ordered set of components. In this terminology the classical formula for the linking number looks as follows.

Theorem 5. If G is any Gauss diagram of a 2-component oriented link L with components L_1 and L_2, then

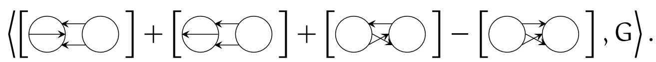
$lk(L_1, L_2) = \Big\langle \Big[$  $\Big], G \Big\rangle.$ (7)

As in the case of Vassiliev invariants of knots, this presentation of the linking number is not unique. Namely, for any $a \in \mathbb{C}$,

$lk(L_1, L_2) = \Big\langle a \Big[$  $\Big] + (1-a) \Big[$  $\Big], G \Big\rangle.$

Moreover, all other known relations between values of different arrow polynomials on Gauss diagrams (such as $\langle$[] $-$ []$, G\rangle = 0$ for any based Gauss diagram G of a knot and $\langle$[] $-$ []$, G\rangle = 0$ for any Gauss diagram G of a knot) follow from the relation $\langle$[] $-$ []$, G\rangle = 0$ for any Gauss diagram G of a 2-component oriented link.

The linking number is a Vassiliev invariant of degree 1 for 2-component links. In a sense the linking number is the only Vassiliev invariant of degree 1 of 2-component oriented links: any Vassiliev invariant of degree 1 of 2-component oriented links is a linear function of the linking number. Any invariant of degree 2 of 2-component oriented links is a function of the linking number and Vassiliev invariants of degree 2 of the components. An invariant of degree 3, which cannot be expressed in terms of components and the linking number, is defined by the following formula:



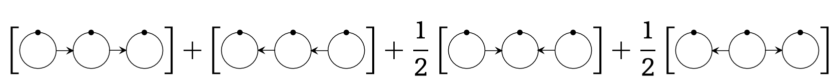
(8)

7 The Milnor invariant of a 3-component link

Although Milnor's μ-invariants of links do not satisfy the most straightforward generalization of the definition of Vassiliev invariants to the case of links, in a sense they are Vassiliev invariants, as was shown by Bar-Natan [3] and Lin [6].

Milnor's μ-invariants generalize the linking number and can be thought of as higher (secondary) invariants related to the linking number in the same sense as Massey products are higher products related to the cohomology cup-product. The simplest of Milnor's μ-invariants which is the next to the linking number is μ_{123}. It is a characteristic of a 3-component oriented ordered link. It is an integer defined modulo the greatest common divisor of the pairwise linking numbers of the link components. It is multiplied by -1 under reversing of the orientation of any component of the link or the orientation of the ambient space. A change of order of the components multiplies it by the sign of the permutation. The following theorem provides a combinatorial formula for calculation of μ_{123}.

Theorem 6. Let L be an oriented ordered 3-component link with components L_1, L_2, L_3, and let G be its based Gauss diagram. (Here based means that a point is marked on each component.) Denote by P the following based arrow polynomial



(9)

and let S be the result of antisymmetrization of P over all permutations of the 3 circles: $S = (1/6)\sum_{\sigma\in S_3} \operatorname{sign}(\sigma)P^\sigma$. Then

$$\mu_{123}(L) = \langle S, G\rangle \qquad \operatorname{mod} \gcd(lk(L_2, L_3), lk(L_1, L_3), lk(L_1, L_2)).$$

The integer $\langle S, G\rangle$ depends on the choice of base points on G: when a point on the ith circle passes the endpoint of an arrow connecting ith and jth circles, $\langle S, G\rangle$ changes by the linking number of ith and kth components of L, where (i, j, k) is a permutation of $(1, 2, 3)$.

Acknowledgments

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