NON-DIFFEOMORPHIC BUT HOMEOMORPHIC KNOTTINGS OF SURFACES IN THE 4-SPHERE

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§ 1. Introduction

The main result of this paper is the following theorem, which was announced in [29].

THEOREM. There exists an infinite series $S_1, S_2, \dots$ of smooth submanifolds of S^4 such that:

(1) for any i,j the pairs (S^4, S_i), (S^4, S_j) are homeomorphic via a map restricting to a diffeomorphism between appropriate neighborhoods of the surfaces.

(2) for any $i \neq j$ the pairs (S^4, S_i), (S^4, S_j) are not diffeomorphic;

(3) each S_n is homeomorphic to the connected sum $\# \mathbb{R}P^2$ of 10 copies of the real projective plane;

(4) $\pi_1(S^4 \setminus S_n) = \mathbb{Z}_2$.

(5) the normal Euler number (with local coefficients) of S_n in S^4 is 16.

There is no analogous result in other dimensions. Let N be a closed smooth submanifold of a closed manifold M of dimension $\neq 4$. Let $\mathcal{U}$ be its smooth tubular neighbourhood. Then there are only finitely many diffeomorphism types rel. boundary of smooth manifolds X with $\partial X = \partial \mathcal{U}$ and homeomorphic rel. ∂ to $M \setminus \operatorname{Int} \mathcal{U}$. If $\dim M \leq 3$ the number of diffeomorphism types is 1 and if $\dim M \geq 5$ the number of smoothings of $M \setminus \operatorname{Int} \mathcal{U}$ rel. boundary (which is an upper bound for the number of diffeomorphism types) is finite, see [12].

In fact we describe an infinite family $F_1, F_2, \dots$ of smooth submanifolds of S^4 satisfying conditions (2) - (5) of the Theorem, and we prove that there are only finitely many homeomorphism types of (S^4, F_n) in the sense described above.

The F_n's are obtained from a fixed smooth submanifold $F \subset S^4$ by a family of new knotting constructions. F is the obvious simplest submanifold satisfying conditions (3), (4) and (5): the pair (S^4, F) is the connected sum of the standard pair $(S^4, \mathbb{R}P^2)$ (with normal Euler number -2) and nine copies of it with the orientation of reversed.

Our knotting constructions can be applied to "smaller" submanifolds, e.g. the Klein bottle with normal Euler number 0 and the torus which are standardly embedded in S^4. The real difficulty in this situation is to prove the non-existence of diffeomorphisms.

The construction of F_n is motivated by the recent work of S.Donaldson [5], [6] C.Okonek / A. van de Ven [2], resp. R.Friedman / J.Morgan [10]. They considered the Dolgachev surfaces [3], [4], which are complex elliptic surfaces organized in families $\mathcal{D}_{p,q}$ with $p,q \in \mathbb{N}$, $(p,q) = 1$. The $\mathcal{D}_{p,q}$-surfaces are 1-connected and permit an elliptic fibration over the 2-sphere $\mathbb{C}P^1$ with two multiple fibres of multiplicity p and q . Any $\mathcal{D}_{p,q}$-surface can be obtained from some rational elliptic surface diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ (blow up 9 points of $\mathbb{C}P^2$) by logarithmic transformations [1] of multiplicity p and q along two non-singular fibres. The rational elliptic surfaces themselves are included in this system as $\mathcal{D}_{1,1}$. By Freedman's classification of 1-connected closed 4-manifolds [8] ,all Dolgachev surfaces are homeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$. But S.Donaldson [5] , [6] proved that no $\mathcal{D}_{2,3}$ -surface is diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$, and this was extended in [10] , [11] , showing that no $\mathcal{D}_{2,q}$ surface is diffeomorphic to a $\mathcal{D}_{1,1}$-surface or a $\mathcal{D}_{2,r}$-surface with odd $r \neq q$.

Now we give an outline of the proof of our Theorem.

PROPOSITION 1. For any p, q there exists a $\mathcal{D}_{p,q}$-surface M which admits an antiholomorphic involution c with M/c diffeomorphic to S^4 .

In the case of $\mathcal{D}_{1,1}$ such involution can easily be constructed via the usual complex conjugation $\mathrm{conj}: \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2 : (z_0 : z_1 : z_2) \mapsto (\bar{z}_0, \bar{z}_1, \bar{z}_2)$. The orbit space $\mathbb{C}P^2/\mathrm{conj}$ is diffeomorphic to S^4 ([20] , [14] , [18]). The fixed point set $\mathbb{R}P^2$ is standardly embedded in it with normal Euler number -2. For any $\mathcal{D}_{1,1}$ -surface M obtained by blowing up 9 real points (i.e. points in $\mathbb{R}P^2$) of $\mathbb{C}P^2$ the involution conj induces an antiholomorphic involution on with M/c diffeomorphic to S^4 $(= \#_{10} S^4)$. The fixed point set is

$$F = \mathbb{R}P^2 \# 9\mathbb{R}P^2 = \#_{10} \mathbb{R}P^2 \hookrightarrow S^4$$

with normal Euler number $-2 + 9 \cdot 2 = 16$, the standardly embedded $\#_{10} \mathbb{R}P^2$ with this normal Euler number. Proposition 1 is easily deduced from this and the following Lemma closely related to surgery of two-fold branched coverings considered by O.Viro [26] and J.M.Montesinos [21] .

LEMMA (on real logarithmic tramsformation). Let $E \to B$ be an elliptic fibration commuting with antiholomorphic involutions $c: E \to E$ and $\sigma: B \to B$. Let F be a non-singular fibre with $F \cap \mathrm{fix}(c) \neq \emptyset$. Then there exists a logarithmic transform E' of E along F of any given multiplicity which admits an antiholomorphic involution extending $c|_{E \setminus F}$ with orbit space diffeomorphic to E/c .

For any involution c of a $\mathcal{D}_{p,q}$-surface M with M/c diffeomorphic to S^4 the topology of the fixed point set and its normal Euler number are determined by the topology of M , and thus $\mathrm{fix}(c)$ is again $\#_{10} \mathbb{R}P^2$ embedded in S^4 with normal Euler number 16. Under appropriate conditions we can also control the fundamental group of the complement of the fixed point set in S^4:

PROPOSITION 2. For any odd q there exists a $\mathcal{D}_{2,q}$-surface M and an involution c as in Proposition 1 with abelian $\pi_1((M/c) \setminus \mathrm{fix}(c))$ (implying $\pi_1((M/c) \setminus \mathrm{fix}(c) = \mathbb{Z}_2)$.

For $q = 2n+1$ let us take such M and c and denote by F_n the image of $\mathrm{fix}(c)$ under some diffeomorphism $M/c \to S^4$. Since M can be obtained back from F_n as a 2-fold covering of S^4 branched along F_n , the results of [10] , [22] imply that for $i \neq j$ pairs $(S^4, F_i), (S^4, F_j)$ are not diffeomorphic. This result together with the following proposition implies our Theorem.

PROPOSITION 3. Consider the class of all smooth submanifolds S of S^4 with a fixed normal Euler number, $\pi_1(S^4 \setminus S) = \mathbb{Z}_2$ and S homeomorphic to a fixed closed connected non-orientable surface. Then the number of ambient homeomorphism types (in the sense of (1) in the Theorem) is finite.

We hope to prove that all (S^4, F_n) are homeomorphic, and moreover that under the conditions of Proposition 3 all the pairs (S^4, S) are homeomorphic extending a result of T.Lawson [15] who showed that for S homeomorphic to $\mathbb{R}P^2$ there is a unique homeomorphism type of pairs (S^4, S) with $\pi_1(S^4 \setminus S) = \mathbb{Z}_2$. However now in the proof of Proposition 3. which uses the surgery method of [13] (applicable in dimension 4 by Freedman's results [9]) there occur several obstructions sitting in non-trivial finite groups. We don't see an obvious reason for these obstructions to be trivial.

Notice that Proposition 3 and the Whitney conjecture proved by W. Massey [19] straightforwardly imply the following

COROLLARY. The number of homeomorphism types of pairs (S^4, S) with $\pi_1(S^4 \setminus S) = \mathbb{Z}_2$ and S a smooth closed connected non-orientable 2-submanifold of S^4 of a fixed topological type is finite.

The following result extends a Theorem of Massey [30]. It is a by-product of the proof of Prop. 3.

PROPOSITION: If $\pi_1(S^4 - S) = \mathbb{Z}_2$, then $S^4 - S$ is homotopy equivalent to $\mathbb{R}P^2 \vee (1 - \chi(S)) \bullet S^2$ where $\chi(S)$ is the Euler characteristic.

Now we describe our knotting constructions and indicate the proof of Proposition 2. Let X be a smooth 4-manifold and F a smooth closed 2-submanifold of X. Let $\mathfrak{M} \subset X$ be a membrane homeomorphic to $S^1 \times I$ with $\partial \mathfrak{M} = \mathfrak{M} \cap F$ and let $\mathfrak{M}$ have index 0 or, equivalently, there exists a diffeomorphism of a regular neighbourhood of $\mathfrak{M}$ in X, $\varphi: N \longrightarrow S^1 \times \mathcal{D}^3$, mapping $N \cap F$ onto $S^1 \times (I \amalg I)$, and such that the segments $I \amalg I$ are embedded unknotted and unlinked into $\mathcal{D}^3$ as in Fig. 1 ($\amalg$ means the disjoint sum operation). For arbitrary relatively prime p, q denote by $K_{p,q}(F, \mathfrak{M}, \varphi)$ the new smooth submanifold of X obtained from F by replacing the embedded segments $I \amalg I \hookrightarrow \mathcal{D}^3$ drown in Fig.1 by the two embedded segments drown in Fig. 2.

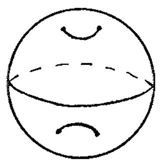


Fig. 1

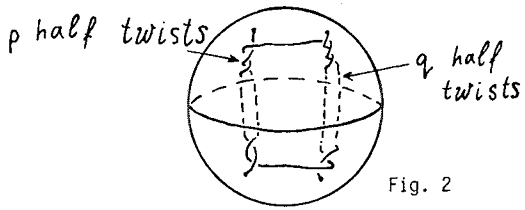


Fig. 2

PROPOSITION 4. If $F \setminus \partial\mathfrak{M}$ is connected and $\pi_1(X \setminus (F \cup \mathfrak{M}))$ is abelian, then $\pi_1(X \setminus K_{2,q}(F, \mathfrak{M}, \varphi))$ is abelian for any odd q.

PROPOSITION 5. Let T be a non-singular fibre of a real (i.e. equivariant with respect to the standard complex conjugations $c: M \longrightarrow M$ and $\mathrm{conj}: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$) elliptic fibration $M \longrightarrow \mathbb{C}P^1$, M a $\mathcal{D}_{1,1}$-surface. Let T intersect the fixed point set F of c in two disjoint circles. Then the 2-fold covering of $S^4 = M/c$ branched over $K_{p,q}(F, T/c, \varphi)$ for some φ is equivariantly diffeomorphic to a $\mathcal{D}_{p,q}$-surface.

In the situation of Proposition 5 the conditions of Proposition 4 is satisfied, as the following Proposition 6 implies.

PROPOSITION 6. Let T be a non-singular fibre of a real elliptic fibration $M \longrightarrow \mathbb{C}P^1$ with M a $\mathcal{D}_{1,1}$-surface. Suppose T intersects the real point set F of M (i.e. the fixed point set of c). Then $\pi_1((M \setminus (T \cup F))/c)$ is abelian.

This Proposition is related to results (and is proved by the methods) of S.M.Finashin's work [7] . The Propositions 4, 5 and 6 obviously imply Proposition 2.

We finish the Introduction with an open question. The knotting construction $K_{p,q}$ corresponds to a pair of logarithmic transformations of multiplicity p and q in the 2-fold branched covering space, see Section 2.5 below. As was mentioned above, $K_{p,q}$ can be constructed in simpler situations for they do not require an elliptic fibration, but only a torus equivariantly embedded into the covering space with zero self intersection number. Such a torus can easily be found in $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, thus there are candidates for exotic $S^2 \times S^2$'s and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$'s as well as for corresponding exotic knottings of $S^1 \times S^1$ and the Klein bottle in S^4. Are they really exotic ?

REMARK: It was shown by Friedman and Morgan [10] that their result about $D_{2,q}$ holds also after arbitrarily blow ups. Thus in our main Theorem one can replace $\#_{10}\, \mathbb{R}P^2$ by $\#_r\, \mathbb{R}P^2$ for a fixed $r \geq 10$ (the normal Euler number under (5) changes into $2r - 4$).

We wish to express our gratitude to V.M. Kharlamov for a stimulating conversation.

§ 2. Real Dolgachev surfaces

2.1. Complex elliptic fibrations.

Here we remind some basic facts on complex elliptic fibrations in the form used below, for details see e.g. [1]. A complex elliptic fibration is a regular map $\pi : E \longrightarrow B$, if E is a non-singular complex surface, B a non-singular complex curve and a generic fibre of π is a non-singular elliptic curve. A surface E which admits such a fibration is called an elliptic surface.

To construct the simplest elliptic fibration with simply connected total space one takes two non-singular complex plane projective cubic curves transversal to each other. They intersect in 9 points, say $p_1, \dots, p_9$, and determine a pencil of cubic curves passing through these points. If the initial curves are defined by equations $f_0(x_0, x_1, x_2) = 0$ and $f_1(x_0, x_1, x_2) = 0$, then the curves of the pencil are defined by equations $f_u(x_0, x_1, x_2) = 0$, where $u = (u_0, u_1) \in \mathbb{C}^2 \setminus 0$ and $f_u = u_0 f_0 + u_1 f_1$. Blow up $\mathbb{C}P^2$ at $p_1, \dots, p_9$. The result M of these σ-processes admits a natural projection $\pi : M \longrightarrow \mathbb{C}P^1$ which assigns to $p \in M$ the unique point $u = (u_0 : u_1) \in \mathbb{C}P^1$ such that the proper preimage of the curve $f_u(x_0, x_1, x_2) = 0$ contains p. A generic fibre of π is naturally isomorphic with a non-singular plane cubic curve and therefore it is a non-singular elliptic curve. Obviously, M is diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$.

Let $\pi : E \longrightarrow B$ be an elliptic fibration, F a non-singular fibre of it, Δ a neighbourhood of the point $\pi(F)$ in B equipped with a biholomorphism $\varphi : \Delta \longrightarrow \mathcal{D}^2$, and $\mathcal{U} = \pi^{-1}(\Delta)$. Suppose that Δ is so small that $\mathcal{U}$ contains no singular fibre of π. Then $\mathcal{U}$ can easily be uniformized in the following sense. There exists a covering $u : \mathbb{C} \times \mathcal{D}^2 \longrightarrow \mathcal{U}$ such that:

(i) the preimage $u^{-1}\pi^{-1}(x)$ of the fibre $\pi^{-1}(x)$ over any $x\in\Delta$ is the fibre $\mathbb{C}\times\varphi(x)$

(ii) there is a holomorphic function $\omega:\mathcal{D}^2\longrightarrow\mathbb{C}$ with $\operatorname{Im}\omega(t)\neq 0$ such that $u(z,t)=u(z',t)$, iff $z\equiv z' \bmod (\mathbb{Z}+\omega(t)\mathbb{Z})$

Furthermore, such a uniformization is uniquely determined by fixing $u|_{0\times\mathcal{D}^2}$ (which can be any lift of $\varphi^{-1}:\mathcal{D}^2\longrightarrow\Delta$) and the homology class of the loop $I\longrightarrow F:\tau\longmapsto u(\tau,0)$ (which can be any primitive element of $H_1(F)$).

A logarithmic transformation of multiplicity m of an elliptic fibration $\pi:E\longrightarrow B$ along a non-singular fibre F still requires for its definition also a natural number μ relatively prime to and a primitive class $\delta\in H_1(F)$. We call this μ a supplementary multiplicity, δ a direction of the logarithmic transformation. Now we describe the logarithmic transformation with these data. Consider a uniformization $u:\mathbb{C}\times\mathcal{D}^2\longrightarrow\mathcal{U}$ of some neighbourhood $\mathcal{U}$ of F in E of the kind described above. Suppose that the class δ is realized by the loop $I\longrightarrow F:\tau\longmapsto u(\tau,0)$. We form the smooth quotient $\mathcal{U}^*$ of $\mathbb{C}\times\mathcal{D}^2$ by the action of $\mathbb{Z}\oplus\mathbb{Z}$ defined by

$$(a,b)(z,s)=(z+a+b\omega(s^m),s)$$

for $(a,b)\in\mathbb{Z}\oplus\mathbb{Z}$ and $(z,s)\in\mathbb{C}\times\mathcal{D}^2$. Further, we form the smooth quotient $\mathcal{U}'$ of $\mathcal{U}^*$ by the cyclic group of order m generated by the transformation

$$(z,s)\longmapsto\left(z+\frac{\mu}{m},\ e^{\frac{2\pi\sqrt{-1}}{m}}\cdot s\right)$$

Thus, $\mathcal{U}'$ is a smooth complex manifold with a natural uniformization $u':\mathbb{C}\times\mathcal{D}^2\longrightarrow\mathcal{U}'$ such that $u'(z_1,s_1)=u'(z_2,s_2)$ iff $s_1=e^{\frac{2\pi\sqrt{-1}k}{m}}\cdot s_2$ and $z_1=z_2+a+b\omega(s_2^m)+\frac{\mu k}{m}$ for some $a,b,k\in\mathbb{Z}$ [$\mathcal{U}'$ can be obtained also as the smooth quotient of $\mathbb{C}\times\mathcal{D}^2$ by the action of $\mathbb{Z}\oplus\mathbb{Z}$ defined by

$$(a,b)(z,s)=\left(z+\frac{a}{m}+b\omega(s^m),\ e^{\frac{2\pi\sqrt{-1}\,\nu a}{m}}\cdot s\right)$$

for $(a,b)\in\mathbb{Z}\oplus\mathbb{Z}$, $(z,s)\in\mathbb{C}\times\mathcal{D}^2$ and ν a natural number with $\nu\mu\equiv 1 \bmod m$]

The map $\mathcal{U}'\longrightarrow\mathcal{D}^2$ defined by $u'(z,s)\longmapsto s^m$ is an elliptic fibration with the only singular fibre $F'=\{u'(z,s)\in\mathcal{U}' \mid s=0\}$, which is a non-singular elliptic curve, but, as fibre, has multiplicity m. The formula

$$u'(z,s)\longmapsto u\left(z+\frac{\mu\sqrt{-1}}{2\pi}\log s,\ s^m\right)$$

defines a biholomorphic fibre-preserving map $\gamma:\mathcal{U}'\setminus F'\longrightarrow\mathcal{U}\setminus F$. We put $E':=(E\setminus F)\cup_\gamma\mathcal{U}'$. It is a complex non-singular surface. The map $\pi':E'\longrightarrow B$ defined by

$$\pi'(x)=\begin{cases}\pi(x), & \text{if } x\in E'\setminus F' \ (=E\setminus F)\\ \pi(F), & \text{if } x\in F'\end{cases}$$

is an elliptic fibration. This is the result of the logarithmic transformation under consideration.

From the point of view of differential topology, E' can be described in a simpler manner. In fact it is easy to check that the maps

$$\mathbb{R}\times\mathbb{R}\times\mathcal{D}^2\longrightarrow\mathcal{U}:(x,y,z)\longmapsto u(y+x\omega(z),z),$$

$$\mathbb{R}\times\mathbb{R}\times\mathcal{D}^2\longrightarrow\mathcal{U}':(x,y,z)\longmapsto u'\left(\frac{y}{m}+x\omega(e^{2\pi\sqrt{-1}\nu y}\cdot z^m),\ e^{\frac{2\pi\sqrt{-1}\nu y}{m}}\cdot z\right)$$

induce diffeomorphisms $\chi:\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\mathcal{D}^2\to\mathcal{U}$, $\chi':\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\mathcal{D}^2\to\mathcal{U}'$ and therefore we can substitute $\mathcal{U}$ for $\mathcal{U}'$ in the construction of Thus E' is diffeomorphic to the result of gluing $E\setminus\operatorname{Int}\mathcal{U}$ and $\mathcal{U}$ by a diffeomorphism $\partial\mathcal{U}\xrightarrow{\gamma_*}\partial\mathcal{U}$. The latter is the composition

$$\partial\mathcal{U}\xrightarrow{(\chi|)^{-1}}\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\partial\mathcal{D}^2\xrightarrow{\chi'|}\partial\mathcal{U}'\xrightarrow{\gamma|}\partial\mathcal{U}$$

To describe it more clearly, let us introduce the diffeomorphism

$$\chi_*: S^1\times S^1\times \mathcal{D}^2 \longrightarrow \mathcal{U}:(x,y,z)\longmapsto u(\frac{1}{2\pi\sqrt{-1}}(\log y+\omega(z)\log x), z)$$

that is the composition of the natural diffeomorphism $S^1\times S^1\times\mathcal{D}^2\longrightarrow \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\mathcal{D}^2$ and χ. The gluing diffeomorphism $r_*:\partial\mathcal{U}\to\partial\mathcal{U}$ acts by the formula

$$\chi_*(x,y,z)\longmapsto \chi_*(x, y^n z^{-\mu}, y^\nu z^m),$$

where $nm+\nu\mu=1$

2.2. Real elliptic fibrations.

A real elliptic fibration is a complex one $\pi:E\to B$ endowed with antiholomorphic involutions $c:E\to E$ and $\sigma:B\to B$ such that $\pi\circ c=\sigma\pi$. The fixed point set of c and σ are denoted by $\mathbb{R}E$ and $\mathbb{R}B$ and are called the real point sets of E and B.

The construction of the elliptic fibering via a pencil of plane cubic curves described in the preceding section gives a real elliptic fibration if the initial cubics $f_0(x_0,x_1,x_2)=0$ and $f_1(x_0,x_1,x_2)=0$ are real. Then σ is the usual $conj:\mathbb{C}P^1\to\mathbb{C}P^1:(u_0:u_1)\mapsto(\bar u_0,\bar u_1)$ and $c:M\to M$ is induced by $conj:\mathbb{C}P^2\to\mathbb{C}P^2$. The initial real cubics can be chosen in such a way that all their intersection points $p_1,\dots,p_9$ would be real (i.e. $p_i\in\mathbb{R}P^2$). It is the case, in particular, if each of the initial cubics is obtained by a small perturbation of the union of three real lines and no 3 of these 6 lines pass through one point.

Let $\pi:E\to B$ be a real elliptic fibration with antiholomorphic involutions c and σ, let F be a non-singular fibre of it with $F\cap\mathbb{R}E\neq\emptyset$, let Δ be a neighbourhood of $\pi(F)$ in B with $\sigma(\Delta)=\Delta$ equipped with a biholomorphism $\varphi:\Delta\to\mathcal{D}^2$ equivariant with respect to σ and $conj$. Suppose that Δ is so small that $\mathcal{U}=\pi^{-1}(\Delta)$ contains no singular fibre of π. Then $\mathcal{U}$ can be uniformized equivariantly. We need a somewhat unexpexted notion of equivariance. We

say that a uniformization $u:\mathbb{C}\times\mathcal{D}^2\to\mathcal{U}$ of the sort considered in 2.1 is equivariant, if $cu(z,t)=u(-\bar z,\bar t)$. To obtain such we take an equivariant lift of $\varphi^{-1}:\mathcal{D}^2\to\Delta$ to $\mathcal{U}$ as $u|_{0\times\mathcal{D}^2}$ (it exists since $F\cap\mathbb{R}E\neq\emptyset$) and a skew-invariant (with respect to c) homology class in $H_1(F)$ as the class of the loop $\tau\mapsto u(\tau,0)$.

Functions $\omega:\mathcal{D}^2\to\mathbb{C}$ related to u (see 2.1) must satisfy the condition $\mathbb{Z}+\omega(t)\mathbb{Z}=\mathbb{Z}+\overline{\omega(\bar t)}\mathbb{Z}$, since the fibres over $\varphi^{-1}(t)$ and $\varphi^{-1}(\bar t)$ [isomorphic to $\mathbb{C}/\mathbb{Z}+\omega(t)\mathbb{Z}$ and $\mathbb{C}/\mathbb{Z}+\omega(\bar t)\mathbb{Z}$] are transposed by the antiholomorphic involution. Thus $\omega(t)\equiv\pm\overline{\omega(\bar t)}\bmod 1$. If $\omega(t)\equiv\overline{\omega(\bar t)}\bmod 1$ then $\operatorname{Im}\omega(t)=0$ for $t\in\mathbb{R}$, but $\operatorname{Im}\omega(t)\neq0$, see 2.1. Hence $\omega(t)\equiv-\overline{\omega(\bar t)}\bmod 1$. We put $\omega(t):=\omega(t)-[\frac{\omega(t)+\overline{\omega(\bar t)}}{2}]$. (We can add to $\omega(t)$ any integer, since it does not change $\mathbb{Z}+\omega(t)\mathbb{Z}$.) Now $\omega(t)=-\overline{\omega(\bar t)}$ or $\omega(t)=1-\overline{\omega(\bar t)}$. These two cases differ from each other in the type of invariant fibres: if $\omega(t)=-\overline{\omega(\bar t)}$, then $F\cap\mathbb{R}E$ consists of two circles, if $\omega(t)=1-\overline{\omega(\bar t)}$, then it consists of one circle.

2.3. Real logarithmic transformation.

Here we give a more precise version of the Lemma on real logarithmic transformation stated in § 1.

2.3.A. Let $\pi:E\to B$ be a real elliptic fibration with antiholomorphic involutions c and σ. Let F be a non-singular fibre with $F\cap\mathbb{R}E\neq\emptyset$ and $\delta\in H_1(F)$ be a primitive class with $(c|_F)_*(\delta)=-\delta$. Then for any relatively prime natural m and μ the result $\pi':E'\to B$ of the logarithmic transformation of $\pi:E\to B$ along F of multiplicity m, supplementary multiplicity μ, and direction δ admits an antiholomorphic involution c' with $\pi'c'=\sigma\pi'$, $c'|_{E'\setminus F'}=c|_{E\setminus F}$ and E'/c' diffeomorphic to E/c. If $F\cap\mathbb{R}E$ consists of two components, then the presentation of E' as $(E\setminus\operatorname{Int}\mathcal{U})\cup_{r_*}\mathcal{U}$ (see the end of 2.1) can be chosen to be equivariant with respect to c' and c and there is a diffeomorphism $\chi_*: S^1\times S^1\times \mathcal{D}^2 \longrightarrow \mathcal{U}$ such that $c\chi_*(x,y,z)=\chi_*(x,\bar y,\bar z)$ and $\gamma_*\chi_*(x,y,z)=\chi_*(x,y^n z^{-\mu}, y^\nu z^m)$.

PROOF. In the construction of E' one can use an equivariant uniformization of a neighbourhood of F. Let $u: \mathbb{C}\times\mathcal{D}^2\longrightarrow\mathcal{U}$ be such a uniformization (see 2.2). Then the tubular neighbourhood $\mathcal{U}'$ of the new fibre F', as constructed in 2.1, admits an antiholomorphic involution $c_{\mathcal{U}'}$ such that $c_{\mathcal{U}'}(u'(z,s))=u'(-\bar z,\bar s)$, where $u':\mathbb{C}\times\mathcal{D}^2\longrightarrow\mathcal{U}'$ is the natural uniformization (see 2.1). In fact, the involution $\mathbb{C}\times\mathcal{D}^2\longrightarrow\mathbb{C}\times\mathcal{D}^2:(z,s)\longmapsto(-\bar z,\bar s)$ defines such an involution of $\mathcal{U}'$, since if $u'(z,s)=u'(z_1,s_1)$, then $s_1=e^{\frac{2\pi\sqrt{-1}k}{m}}\cdot s$ and $z_1=z+a+b\omega(s^m)+\frac{\mu k}{m}$ for $a,b,k\in\mathbb{Z}$ (see 2.1) and then $\bar s_1 = e^{\frac{2\pi\sqrt{-1}(-k)}{m}}\bar s$ and $-\bar z_1=-\bar z-a-b\overline{\omega(s^m)}-\frac{\mu k}{m}=$

$$=\begin{cases}(-\bar z)-a+b\omega(\bar s^m)+\dfrac{\mu(-k)}{m}, & \text{if } \omega(t)=-\overline{\omega(\bar t)}\\ (-\bar z)-a-b+b\omega(\bar s^m)+\dfrac{\mu(-k)}{m}, & \text{if } \omega(t)=1-\overline{\omega(\bar t)}\end{cases}$$

Therefore $u'(-\bar z,\bar s)=u'(-\bar z_1,\bar s_1)$. The gluing map $\gamma:\mathcal{U}'\setminus F'\longrightarrow\mathcal{U}\setminus F$ is equivariant with respect to $c_{\mathcal{U}'}$ and c:

$$\gamma c_{\mathcal{U}'} u'(z,s)=\gamma u'(-\bar z,\bar s)=u\left(-\bar z+\frac{\mu\sqrt{-1}}{2\pi}\log\bar s,\bar s^m\right)=$$
$$=u\left(-\left(z+\frac{\mu\sqrt{-1}}{2\pi}\log s\right),\bar s^m\right)=cu\left(z+\frac{\mu\sqrt{-1}}{2\pi}\log s, s^m\right)=$$
$$= c\gamma u'(z,s).$$

We define $c': E'\longrightarrow E'$ by

$$c'(x)=\begin{cases}c(x), & \text{if } x\in E'\setminus F' (=E\setminus F)\\ c_{\mathcal{U}'}(x), & \text{if } x\in\mathcal{U}'\end{cases}$$

Since γ is equivariant, c' is well defined. Clearly, $\pi'c'=\sigma\pi'$

and $c'|_{E'\setminus F'}=c|_{E\setminus F}$. Now let us show that E'/c' is diffeomorphic to E/c.

Firstly, consider the case of a two-component $F\cap\mathbb{R}E$. In this case $\omega(t)=-\overline{\omega(\bar t)}$. Then as a straightforward calculation shows

$$c\chi(x,y,z)=\chi(x,-y,\bar z),$$
$$c_{\mathcal{U}'}\chi'(x,y,z)=\chi'(x,-y,\bar z)$$

for $x,y\in\mathbb{R}/\mathbb{Z}$ and $z\in\mathcal{D}^2$, and $c\chi_*(x,y,z)=\chi_*(x,\bar y,\bar z)$ for $x,y\in S^1$, $z\in\mathcal{D}^2$.

Thus the identification of $\mathcal{U}'$ and $\mathcal{U}$ by $\chi'\circ\chi^{-1}$ is equivariant, and in the presentation of E' as $E\setminus\operatorname{Int}\mathcal{U}\cup_{\gamma_*}\mathcal{U}$ the c' can be described by

$$c'(x)=\begin{cases}c(x), & \text{if } x\in E\setminus\operatorname{Int}\mathcal{U}\\ c(x), & \text{if } x\in\mathcal{U}\end{cases}$$

Therefore $E'/c'=(E\setminus\operatorname{Int}\mathcal{U})/c\cup_{\gamma_*/c}\mathcal{U}/c$. The diffeomorphism χ_* induces the diffeomorphism $(S^1\times S^1\times\mathcal{D}^2)/_{(x,y,z)\sim(x,\bar y,\bar z)}\longrightarrow\mathcal{U}/c$. Thus $\mathcal{U}/c$ is diffeomorphic to $S^1\times(S^1\times\mathcal{D}^2/_{(y,z)\sim(\bar y,\bar z)})$. The involution $(y,z)\longmapsto(\bar y,\bar z)$ of the solid torus $S^1\times\mathcal{D}^2$ can be thought as the symmetry with respect to a line intersecting the solid torus standardly embedded in $\mathbb{R}^3$ in two segments, see fig. 3. The factor-space is diffeomorphic to $\mathcal{D}^3$. The projection $S^1\times\mathcal{D}^2\longrightarrow\mathcal{D}^3$ is a two-fold covering branched over two unknotted and unlinked segments (the images of $(S^1\cap\mathbb{R})\times(\mathcal{D}^2\cap\mathbb{R})$), see fig. 3. Consequently $\mathcal{U}/c$ is diffeomorphic to $S^1\times\mathcal{D}^3$. The gluing map γ_*/c is the product of id_{S^1} and some diffeomorphism of S^2. Therefore it is diffeotopic to the identity, and E'/c' is diffeomorphic to $(E\setminus\operatorname{Int}\mathcal{U})/c\cup_{id}\mathcal{U}/c=E/c$.

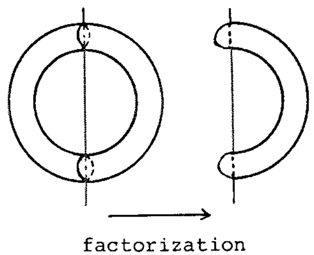


Fig. 3

The case $\omega(t)=1-\overline{\omega(\bar t)}$ is more subtle. In this case

$$c\,\chi(x,y,z)=\chi(x,-y-x,\bar{z}),$$
$$c_{\mathcal{U}'}\,\chi'(x,y,z)=\chi'(x,-y-mx,ze^{2\pi\sqrt{-1}\,\nu x})$$

for $x,y\in\mathbb{R}/\mathbb{Z}$, $z\in\mathcal{D}^2$. Indeed,

$$c\,\chi(x,y,z)=c\,u(y+x\omega(z),z)=u(-\bar{y}-\bar{x}\,\overline{\omega(z)},z)=$$
$$=u(-y-x+x\omega(\bar{z}),\bar{z})=\chi(x,-y-x,z),$$
$$c_{\mathcal{U}'}\chi'(x,y,z)=c_{\mathcal{U}'}\,u'\Big(\frac{y}{m}+x\omega(e^{2\pi\sqrt{-1}\,\nu y}z^m),\,e^{\frac{2\pi\sqrt{-1}\,\nu y}{m}}z\Big)=$$
$$=u'\Big(-\frac{\bar{y}}{m}-\bar{x}\,\overline{\omega(e^{2\pi\sqrt{-1}\,\nu y}z^m)},\,e^{\frac{-2\pi\sqrt{-1}\,\nu y}{m}}\bar{z}\Big)=$$
$$=u'\Big(-\frac{y}{m}-x+x\omega(e^{2\pi\sqrt{-1}\,\nu(-y)}\bar{z}^m),\,e^{\frac{2\pi\sqrt{-1}\,\nu(-y)}{m}}\bar{z}\Big)=$$
$$=u'\Big(\frac{-y-mx}{m}+x\omega(e^{2\pi\sqrt{-1}\,\nu(-y-mx)}z^{-m}e^{2\pi\sqrt{-1}\,\nu mx}),\,e^{\frac{2\pi\sqrt{-1}\,\nu(-y-mx)}{m}}\times$$
$$\times\,\bar{z}\,e^{2\pi\sqrt{-1}\,x\nu}\Big)=\chi'(x,-y-mx,\bar{z}e^{2\pi\sqrt{-1}\,\nu x})$$

Thus the identification of $\mathcal{U}'$ and $\mathcal{U}$ by $\chi'\circ\chi^{-1}$ is not equivariant in this case, and in the presentation of E' as $E\setminus\operatorname{Int}\mathcal{U}\cup_{\gamma_*}\mathcal{U}$ the involution c' is described by

$$c'(a)=\begin{cases}c(a) & ,\text{ if } a\in E\setminus\operatorname{Int}\mathcal{U}\\ \chi\circ(\chi')^{-1}\circ c_{\mathcal{U}'}\circ\chi'\circ\chi^{-1}(a) & \text{if } a\in\mathcal{U}\end{cases}$$

Let us denote $\chi\circ(\chi')^{-1}\circ c_{\mathcal{U}'}\circ\chi'\circ\chi^{-1}$ by τ. For $a=\chi(x,y,z)$ we have $\tau(a)=\chi(x,-y-mx,\bar{z}e^{2\pi\sqrt{-1}\,\nu x})$ and for $a=\chi_*(x,y,z)$ $\tau(a)=\chi_*(x,\bar{y}\bar{z}^m,\bar{z}x^\nu)$. So τ preserves the fibres $\chi_*(pt\times S^1\times\mathcal{D}^2)$. In each of the fibres, τ is conjugated with the involution

$$S^1\times\mathcal{D}^2\longrightarrow S^1\times\mathcal{D}^2:(y,z)\longmapsto(\bar{y},\bar{z})$$

considered above. Thus $\mathcal{U}/\tau$ is fibred over S^1 with fibre $\mathcal{D}^3$ and therefore $\mathcal{U}/\tau$ is diffeomorphic to $S^1\times\mathcal{D}^3$ (for it is obviously orientable). The gluing map $\gamma_*:\partial\mathcal{U}\longrightarrow\partial\mathcal{U}$ also preserves the fibres

$\chi_*(pt\times S^1\times S^1)$ and the factor-map γ_*/τ can be extended to $\mathcal{U}/\tau$ sinse $\pi_1(\mathcal{D}iff\,S^2,SO(3))=0$. Thus $E'/c'=(E\setminus\operatorname{Int}\mathcal{U})/c\cup_{\gamma_*/\tau}\mathcal{U}/\tau$ is diffeomorphic to E/c.

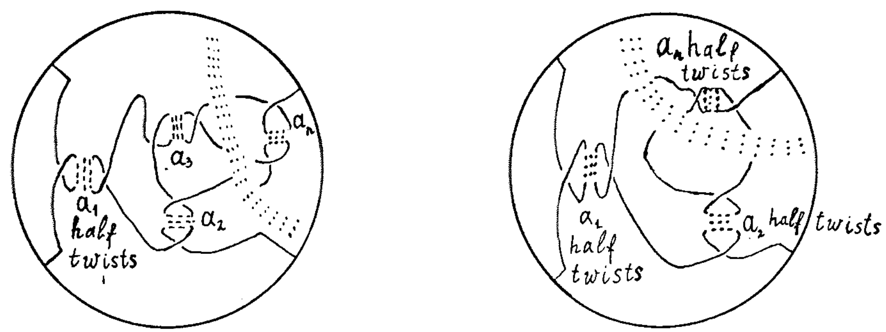
2.4. Branch locus after real logarithmic transformation.

Here we investigate the effect of the logarithmic transformation on the position of the branching locus (the image of the real point set) in the factor-space. We restrict ourselves to the case of the fibre F (along which the transformation is done) having the real part $F\cap\mathbb{R}E$ consisting of two circles (i.e. F as a real curve is an M-curve). The case of a one-component $F\cap\mathbb{R}E$ is more interesting on its own, but it is more subtle, and on the other hand the case of two-component $F\cap\mathbb{R}E$ is sufficient for our main purpose.

Let us remind that to any pair α,β of relatively prime integers a pair of smooth arcs in $\mathcal{D}^3$ is assigned. It is called a β/α-tangle. These arcs interest us only up to diffeomorphism of $\mathcal{D}^3$ fixed on $S^2=\partial\mathcal{D}^3$. The end-points of the arcs are in fixed standard position. Denote the set of the end-points by $\mathcal{E}$. If

$$\frac{\beta}{\alpha}=a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\dots+\cfrac{1}{a_n}}}$$

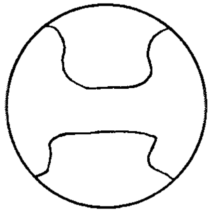
then the β/α-tangle is described by the following picture



the case of even n the case of odd n

Fig. 4.

As it was remarked by Conway [2], it is well defined, i.e. it does not depend on the representation of β/α as a continued fraction. The two-fold covering of $\mathcal{D}^3$ branched over the 0-tangle (shown in fig. 5) emerged in the preceeding section. Its covering space is the solid torus $S^1 \times \mathcal{D}^2$. As it is easy to show (see fig. 4), any β/α-tangle can be obtained from the 0-tangle by a diffeomorphism $\mathcal{D}^3 \to \mathcal{D}^3$. This diffeomorphism restricted to S^2 preserves the set of the four end-points of the arcs. It induces the autodiffeomorphism of the two-folds covering space of S^2 branched over these 4 points. The latter autodiffeomorphism, up to composing with a diffeomorphism extendible over $S^1 \times \mathcal{D}^2$, is determined by the homology class of the image of meridians $pt \times \partial\mathcal{D}^2$, which is



$\frac{0}{1} = 0$, $n = 1$, $a_1 = 0$

Fig. 5.

$$\alpha \text{ (the class of meridians) } + \beta \text{ (the class of longitudes)}$$

see e.g. [24].

2.4.A. Under the assumptions of Lemma 2.3.A let $F \cap \mathbb{R}E$ consists of two components. There is a regular neighbourhood $\mathcal{U}_/$ of F/c in E/c with $(\mathcal{U}_/, fix(c) \cap \mathcal{U}_/)$ diffeomorphic to $(S^1 \times \mathcal{D}^3,\ S^1 \times (0\text{-tangle}))$. There exists a diffeomorphism $E'/c' \to E/c$ such that it maps $fix(c')$ onto the surface that can be obtained from $fix(c)$ by substituting $S^1 \times (-\mu/m)$-tangle) for $fix(c) \cap \mathcal{U}_/ = S^1 \times (0\text{-tangle})$.

PROOF. For $\mathcal{U}_/$ we can get the image of a c-invariant tubular neighbourhood $\mathcal{U}$ of F, i.e. $\mathcal{U}_/ := \mathcal{U}/c$, cf. 2.3.A. By 2.3.A the presentation of E' as $(E \setminus Int\,\mathcal{U}) \cup_{\gamma_*} \mathcal{U}$ can be found to be equivariant with respect to c' and c. The diffeomorphism χ_* from 2.3.A induces a diffeomorphism $\chi_*/c : (S^1 \times \mathcal{D}^3,\ S^1 \times (0\text{-tangle})) \to (\mathcal{U}_/, fix(c) \cap \mathcal{U}_/)$. The diffeomorphism $E'/c' \to E/c$ can be obtained as

an extension of $id_{(E/c \setminus Int\,\mathcal{U}/c)}$ by some diffeomorphism $\mathcal{U}/c \to \mathcal{U}/c$ extending (γ_*/c) (compare the proof of 2.3.A). The latter can be taken in form $(\chi_*/c) \circ (id_{S^1} \times h) \circ (\chi_*/c)^{-1}$ for some $h : \mathcal{D}^3 \to \mathcal{D}^3$. Since γ_* maps the meridianal loop $I \to \partial\mathcal{U} : t \mapsto \chi_*(x, y, e^{2\pi\sqrt{-1}t})$ onto the loop $I \to \partial\mathcal{U} : t \mapsto \chi_*(x, y^n e^{-2\pi\sqrt{-1}\mu t}, y^\nu e^{2\pi\sqrt{-1}mt})$, h maps the 0-tangle onto the $(-\mu/m)$-tangle.

2.5. Branch locus after a pair of real logarithmic transformations.

A pair of disjoint smooth arcs in $\mathcal{D}^3$ with end-points constituting a fixed set $\mathcal{E} \subset S^2$ is called a tangle. The (β/α)-tangles above are the simplest tangles, they are called rational tangles. The tangle shown in fig. 2 is not rational provided $p \neq 1$ and $q \neq 1$. However there is a natural sum operation of tangles described by fig. 6. such that the tangle of fig. 2 can be obtained as the sum of two rational tangles, see fig. 7. The summands are $(1/p)$- and $(1/q)$-tangles

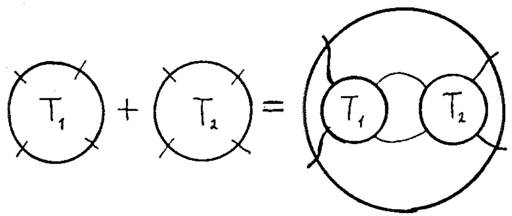


Fig. 6.

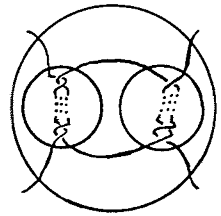


Fig. 7.

The following proposition is a straight-forward consequence of 2.4.A.

2.5.A. Let $\pi : E \to B$ be a real elliptic fibration with antiholomorphic involutions c and σ. Let F_1, F_2 be its non-singular fibres with $F_i \cap fix(c)$ consisting of two components. Suppose that $\pi(F_1)$ and $\pi(F_2)$ are joined in $fix(\sigma)$ by an arc ℓ that does not contain the image of a singular fibre. Let $\delta_i \in H_1(F_i)$ be primitive classes with $(c|_{F_i})_*(\delta_i) = -\delta_i$ which maps by the isomorphisms $in_*: H_1(F_i) \longrightarrow H_1(\pi^{-1}(\ell))$ onto the same element of $H_1(\pi^{-1}(\ell))$. Let $(m_1,\mu_1),(m_2,\mu_2)$ be two pairs of relatively prime numbers, and let $\pi': E' \longrightarrow B$ be the result of the pair of the logarithmic transformations of $\pi: E \longrightarrow B$ along F_1 and F_2 of multiplicity m_1 and m_2, supplementary multiplicity μ_1 and μ_2 and directions δ_1 and δ_2. Let $c': E' \longrightarrow E'$ be the antiholomorphic involution which coincides with c on $E \setminus (F_1 \cup F_2)$. Then there is a regular neighbourhood $\mathcal{U}_/$ of $\pi^{-1}(\ell)/c$ in E/c with $(\mathcal{U}_/, fix(c) \cap \mathcal{U}_/)$ diffeomorphic to $(S^1 \times \mathcal{D}^3$, $S^1 \times ($ 0 -tangle$))$ and there exists a diffeomorphism $E'/c' \longrightarrow E/c$ such that it maps $fix(c')$ onto the surface that can be obtained from $fix(c)$ by substituting

$S^1 \times$(sum of $(-\mu_1/m_1)$-tangle and $(-\mu_2/m_2)$ -tangle)

for $fix(c) \cap \mathcal{U}_/ = S^1 \times ($ 0 -tangle)

This proposition obviously implies Proposition 5 stated in Introduction, since the substitution in the case of $\frac{\mu_1}{m_1} = \frac{-1}{p}$, $\frac{\mu_2}{m_2} = \frac{-1}{q}$ coincides with the knotting construction $K_{p,q}$ of Introduction.

§ 3. Commutativity of π_1 throughout the knotting

3.1. Knottings along an annulus.

Our main purpose in § 3 is to prove Proposition 4. A big part of the arguments is naturally extended to the wider situation suggested by 2.5.A. We investigate this situation up to the point where it would require more complicated calculations than just the proof of Proposition 4 (i.e. up to Section 3.3). Some possibilities of extending it are discussed in Section 3.6.

First we describe the corresponding generalization of the knotting construction $K_{p,q}$. Let X be a smooth 4-manifold and F a

smooth closed connected 2-submanifold of X. Let $\mathfrak{M} \subset X$ be a membrane homeomorphic to $S^1 \times I$ with $\partial\mathfrak{M} = \mathfrak{M} \cap F$, let N be a regular neighbourhood of $\mathfrak{M}$ in X. Suppose that $\mathfrak{M}$ has index 0. Let $\varphi: N \longrightarrow S^1 \times \mathcal{D}^3$ be a diffeomorphism mapping $N \cap F$ onto $S \times$ $\times($ 0 -tangle). For a tangle τ denote by $K_\tau(F,\mathfrak{M},\varphi)$ a new smooth submanifold of X obtained from F by substituting $S^1 \times \tau$ for $F \cap N = \varphi^{-1}(S^1 \times ($ 0 -tangle$))$. If τ is the sum of $\frac{1}{p}$ -tangle and $\frac{1}{q}$ -tangle, then $K_\tau(F,\mathfrak{M},\varphi) = K_{p,q}(F,\mathfrak{M},\varphi)$, cf. 2.5 above.

3.2. The problem and its reduction.

Let $X, F, \mathfrak{M}, \varphi, N$ and τ be as in 3.1 and let $x_0 \in \partial N \setminus F$. We begin with the problem: under what conditions on τ commutativity of $\pi_1(X \setminus (F \cap \mathfrak{M}), x_0)$ implies commutativity of $\pi_1(X \setminus K_\tau(F, \mathfrak{M}, \varphi))$.

Let $\varphi(x_0) = (s_0, d_0) \in S^1 \times S^2$ and k be the kernel of the homomorphism $\pi_1(S^2 \setminus \mathcal{E}, d_0) \longrightarrow \pi_1(X \setminus (F \cap \mathfrak{M}), x_0)$ induced by the composition

$$s_0 \times (S^2 \setminus \mathcal{E}) \xrightarrow{\varphi^{-1}} \partial N \setminus F \hookrightarrow X \setminus (F \cap \mathfrak{M})$$

Denote by G the factor-group of $\pi_1(\mathcal{D}^3 \setminus \tau, d_0)$ by the normal subgroup generated by the image of k under

$$in_*: \pi_1(S^2 \setminus \mathcal{E}, d_0) \longrightarrow \pi_1(\mathcal{D}^3 \setminus \tau, d_0).$$

3.2.A. If G and $\pi_1(X \setminus (F \cup \mathfrak{M}))$ are abelian then $\pi_1(X \setminus K_\tau(F, \mathfrak{M}, \varphi))$ is abelian.

PROOF. We apply the Van Kampen theorem to the triad $(X \setminus K_\tau(F, \mathfrak{M}, \varphi)$; $X \setminus (K_\tau(F,\mathfrak{M},\varphi) \cup \text{Int } N)$; $N \setminus K_\tau(F, \mathfrak{M}, \varphi))$. The space $X \setminus (K_\tau(F,\mathfrak{M},\varphi) \cup \text{Int } N) = X \setminus (F \cup \text{Int } N)$ is a deformation retract of $X \setminus (F \cup \mathfrak{M})$. On the other hand $N \setminus K_\tau(F, \mathfrak{M}, \varphi)$ is homeomorphic to $S^1 \times (\mathcal{D}^3 \setminus \tau)$. Thus $\pi_1(X \setminus K_\tau(F, \mathfrak{M}, \varphi), x_0)$ is isomorphic to a factor-group of the free product $\pi_1(X \setminus (F \cup \mathfrak{M}, x_0)) * \pi_1(\mathcal{D}^3 \setminus \tau, d_0)$.

To obtain it we must adjoin the relations which say that the images of elements of $\pi_1(S^2\setminus\mathcal{E}, d_0)$ under $in_*: \pi_1(S^2\setminus\mathcal{E}, d_0) \to \pi_1(\mathcal{D}^3\setminus\tau, d_0)$ are equal to the images of the same elements under the homomorphism $\pi_1(S^2\setminus\mathcal{E}, d_0) \to \pi_1(X\setminus(F\cup\mathfrak{M}), x_0)$ above. Certainly we can factorize $\pi_1(\mathcal{D}^3\setminus\tau, d_0)$ by k beforehand and substitute G for $\pi_1(\mathcal{D}^3\setminus\tau, d_0)$. Since $H_1(S^2\setminus\mathcal{E}) \to H_1(\mathcal{D}^3\setminus\tau)$ is onto, the homomorphism $\pi_1(S^2\setminus\mathcal{E}, d_0) \to G$ is onto too and the factor-group of $\pi_1(X\setminus(F\cup\mathfrak{M}), x_0) * G$ is generated by the image of $\pi_1(X\setminus(F\cup\mathfrak{M}), x_0)$. Therefore the factor-group is abelian.

3.3. Estimation of k.

In the situation considered in 3.2 suppose that $\pi_1(X\setminus(F\cup\mathfrak{M}), x_0)$ is abelian. Here we estimate k using only information on $F\cup\mathfrak{M}$ regardless of position of $F\cup\mathfrak{M}$ in X. First, k obviously contains the kernel of the natural homomorphism $\pi_1(S^2\setminus\mathcal{E}, d_0) \xrightarrow{i} H_1((T\cup N)\setminus(F\cup\mathfrak{M}), x_0)$ where T is a tubular neighbourhood of F in X. Thus if the factor-group of $\pi_1(\mathcal{D}^3\setminus\tau, d_0)$ by the normal subgroup generated by the image of $\operatorname{Ker} i$ is abelian, then G is abelian too. Now we study what $\operatorname{Ker} i$ can be.

The group $\pi_1(S^2\setminus\mathcal{E}, d_0)$ is a free group of rank 3, but it is more convenient for us to consider 4 generators shown in fig. 8. They satisfy the relation $xu = yv$. Additional relations satisfied by the images of x, y, u, v under i to some extend are determined by the topology of the pair $(F, \partial\mathfrak{M})$. In fact $i(x)$, $i(y), i(u), i(v)$ are the classes of the boundaries of fibres of the tubular fibration $T \to F$. Two such boundaries are homologous modulo 2 in $(T\cup N)\setminus(\mathfrak{M}\cup F)$, if the corresponding points of F are in one component of $F\setminus\partial\mathfrak{M}$. Moreover, if this is the case,

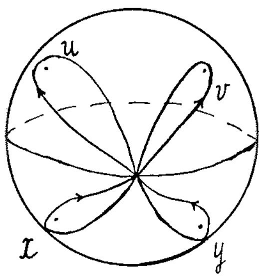


Fig. 8.

then the classes are equal or differ in sign. The sign depends on the first Stiefel-Whitney class. For example, if F is orientable and $F\setminus\partial\mathfrak{M}$ is connected, then $i(x) = i(y)$ and $i(u) = i(v)$. If, conversely, $\partial\mathfrak{M}$ realizes the homology class dual to $w_1(F)$ and $F\setminus\partial\mathfrak{M}$ is still connected, then $i(x) = -i(y)$ and $i(u) = -i(v)$ [those interested just in the proof of the Theorem could restrict themselves to the latter case]. If $F\setminus\partial\mathfrak{M}$ is non-orientable and connected then $i(x) = i(y) = i(u) = i(v)$ and $2i(x) = 0$. If $F\setminus\partial\mathfrak{M}$ is orientable and connected, then in the equality $i(x) = \pm i(u)$ the sign depends on φ. All the facts concerning dependence of the signs on topology of F and $F\setminus\partial\mathfrak{M}$ which are stated above are not proved here. This is not necessary as it is enough to consider those cases needed to prove Proposition 4. By 3.2.A it is sufficient to prove that the groups $\pi_1(\mathcal{D}^3\setminus\tau, d_0)$ with the corresponding τ's become abelian by adjoining relations $x = y^\alpha = u^\beta = v^\gamma$ with $\alpha, \beta, \gamma = \pm 1$.

3.4. The group of the sum of a $\frac{1}{2}$-tangle and a $\frac{1}{q}$-tangle.

In fig.9 we show (as arrows) the loops representing the Wirtinger generators of the fumdamental group of the complement in $\mathcal{D}^3$ of the sum of a $\frac{1}{2}$-tangle and a $\frac{1}{5}$-tangle. Clearly, the generators are subject to the following relations:

$$u = x^{-1}z^{-1}x^{-1}zx,$$

$$v = zyzyz^{-1}y^{-1}z^{-1}y^{-1}z^{-1},$$

$$x^{-1}zx = zyzyzyz^{-1}y^{-1}z^{-1} \times y^{-1}z^{-1}$$

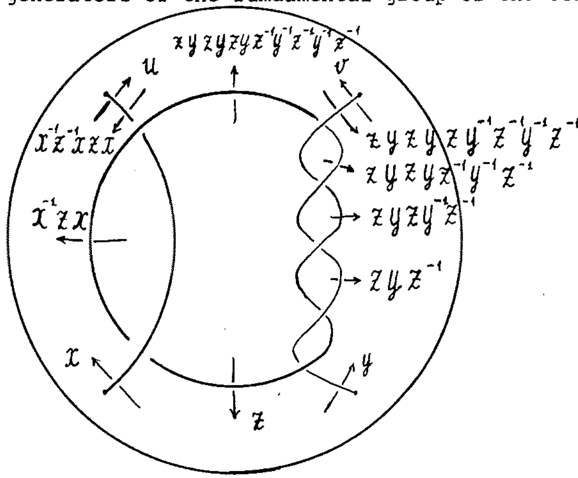


Fig. 9.

It is clear that in the case of the sum of a $\frac{1}{2}$-tangle and a $\frac{1}{q}$-tangle with any odd q the picture is analogous and the corresponding relations are:

$$u = x^{-1} z^{-1} x^{-1} z x$$

$$v = (zy)^{\frac{q-1}{2}} z^{-1} (y^{-1} z^{-1})^{\frac{q-1}{2}}$$

$$x^{-1} z x = (zy)^{\frac{q+1}{2}} z^{-1} (y^{-1} z^{-1})^{\frac{q-1}{2}}$$

3.5. Completion of the proof of Proposition 4. We must show that the group

$$\langle x, y, z, u, v \mid u = x^{-1} z^{-1} x^{-1} z x,\ v = (zy)^{\frac{q-1}{2}} z^{-1} (y^{-1} z^{-1})^{\frac{q-1}{2}},$$
$$x^{-1} z x = (zy)^{\frac{q+1}{2}} z^{-1} (y^{-1} z^{-1})^{\frac{q-1}{2}},\ xu = yv \rangle$$

by adjoining relations $x = y^{\alpha} = u^{\beta} = v^{\gamma}$ with $\alpha, \beta, \gamma = \pm 1$ is caused to become abelian.

In the case $\beta = -1$ this is obvious. In fact, the first relation turns into $z^{-1} x^{-1} z x = 1$ and y, u, v can be removed.

Thus consider the case $\beta = 1$. Then the first relation gives $x^{-1} z^{-1} x^{-1} z = 1$. The latter works almost like the commutativity relation: it implies $xz = zx^{-1}$, $x^{-1} z = zx$ etc. The third relation gives

$$x^{-1} z x = (z x^{\alpha})^{\frac{q+1}{2}} z^{-1} (x^{-\alpha} z^{-1})^{\frac{q-1}{2}}$$

From this using the first relation we obtain

$$z x^2 = z^0 x^{\tau}$$

with some τ and therefore $z = x^{\tau - 2}$. Since the generators y, u, v can also be removed the group is cyclic and thus abelian

3.6. Related results.

In Section 3.5 we do not use the relation $x = v^{\gamma}$. We could prove analogously the commutativity using the relation $x = v^{\gamma}$ instead of $x = u^{\beta}$. Thus we can strengthen Proposition 4. The hypothesis of the

connectedness of $F \setminus \partial \mathfrak{M}$ can be replaced by the following weaker hypothesis: $F \setminus \partial \mathfrak{M}$ has at most two connected components and the components of $\partial \mathfrak{M}$ are not homologous (in F).

Some other similar proposition can be proved. For example, it can easily be proved that if $F \setminus \partial \mathfrak{M}$ is connected, F is orientable, and $\pi_1(X \setminus (F \cup \mathfrak{M}))$ is abelian, then $\pi_1(X \setminus K_{p,q}(F, \mathfrak{M}, \varphi))$ is abelian for any relatively prime p, q. We don't know whether this statement remains true without the hypothesis of orientability of F.

§ 4. Commutatively of π_1 before the knotting

4.1. ℓ-curves and generalization of Proposition 6.

The main purpose of § 4 is to prove Proposition 6. In fact we prove a certain generalization, which is a general theorem on the topology of real algebraic curves of some special sort. This theorem extends some results of Finashin's work [7].

We begin with describing the problem on topology of curves which we deal with. Let $A \subset \mathbb{C}P^2$ be the set of complex points of a real algebraic plane projective curve. Let M be the complex surface obtained from $\mathbb{C}P^2$ by blowing up points $p_1, \dots, p_k \in \mathbb{R}P^2$. Let $c: M \to M$ be the involution induced by the standard conjugation involution $conj: \mathbb{C}P^2 \to \mathbb{C}P^2$. Denote by A^M the proper pre-image of A under the projection $M \to \mathbb{C}P^2$. Further, denote by $M_{\mathbb{R}}$ the fixed point set of $c: M \to M$ as well as the image of it in $M/c\ (\cong S^4)$. Put

$$R^M_A = M_{\mathbb{R}} \cup A^M/c \subset M/c .$$

The problem is: what conditions imply commutativity of $\pi_1(M/c \setminus R^M_A)$

For our purpose (to prove Proposition 6) it is sufficient to prove that it is the case if A is a non-singular cubic curve and at least one of the p_i lies on A . We work with a wider class of curves - with ℓ-curves considered by Finashin in [7] . A non-singular real algebraic curve $A \subset \mathbb{C}P^2$ is called an ℓ-curve if it can be obtained by a perturbation of a curve $A_0 = L_1 \cup \ldots \cup L_m$, where L_i (with $i = 1, \ldots, m$) is (the set of complex points of) a real projective line and $L_1, \ldots, L_m$ are in general position (i.e. no 3 of them have a common point). By a perturbation of a singular curve A_0 we mean a path $\mathfrak{z}:[0,1] \to \mathbb{R}C_m$ in the space $\mathbb{R}C_m$ of all the real algebraic plane projective curves of degree m such that $\mathfrak{z}(0) = A_0$ and curves $\mathfrak{z}(t)$ with $t \in (0,1]$ are non-singular.

Any non-singular real plane projective cubic is an ℓ-curve. Actually, according to the well known classification of real plane projective cubics, non-singular cubics up to rigid isotopy are of two types. Cubics of one of the types have connected real part, cubics of the other type have real part consisting of two components. Both types are obviously presented by ℓ-curves and therefore consist of them.

4.1.A. In the notations introduced in the beginning of this section if A is an ℓ-curve and at least one of the points $p_1, \ldots, p_k$ belongs to A then the group $\pi_1(M/c \setminus R^M_A)$ is cyclic.

The rest of § 4 is aimed at the proof of 4.1.A.

4.2. Genetic graph of an ℓ-curve.

Let $A_0 = L_1 \cup \ldots \cup L_m$, where $L_1, \ldots, L_m$ are real lines in general position. Let $\mathbb{R}P^{2\vee}$ be the projective plane dual to $\mathbb{R}P^2$ and $L^\vee_1, \ldots, L^\vee_m \in \mathbb{R}P^{2\vee}$ the points corresponding to $L_1, \ldots, L_m$. A perturbation of A_0 smooths out each singular point $p_{ij} = L_i \cap L_j$ of A_0 in one of the two ways presented by fig. 10. Consider the lines on $\mathbb{R}P^2$ which pass through p_{ij} and lie in the angles joined by the smoothing (the angles A and C in the left part of fig. 10 and B and $\mathcal{D}$ in the right part). These lines cor-

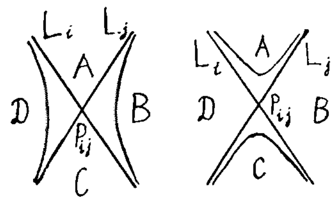


Fig. 10.

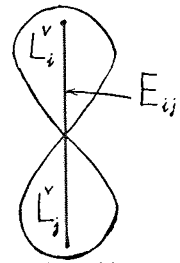


Fig. 11.

respond to the points of $\mathbb{R}P^{2\vee}$ constituting the segment of the line on $\mathbb{R}P^{2\vee}$ joining $L^\vee_i$ and $L^\vee_j$. Denote this segment by E_{ij}. Let Γ be the graph with vertices $L^\vee_1, \ldots, L^\vee_m$ and edges E_{ij} , $i,j = 1, \ldots, m$, $i \neq j$. We call it the genetic graph of the perturbation.

For each E_{ij} we take a map $S^1 \to \mathbb{R}P^{2\vee} \setminus \{L^\vee_1, \ldots, L^\vee_m\}$ which straight-forwardly parametrizes the figure-eight positioned in a regular neighbourhood of E_{ij} as shown in fig. 11. Let $C(\Gamma)$ denote the space obtained from the punctured plane $\mathbb{R}P^{2\vee} \setminus \{L^\vee_1, \ldots, L^\vee_m\}$ by gluing $\binom{m}{2}$ two-dimensional cells by these maps. Since the gluing maps are well described up to homotopy and self-homeomorphism of S^1 , the homotopy type of $C(\Gamma)$ is well defined. Put $R_A = \mathbb{R}P^2 \cup A/\mathrm{conj} \subset \mathbb{C}P^2/\mathrm{conj}$.

4.2.A. Let Γ be the genetic graph of a perturbation which yields an ℓ-curve A . Then $\mathbb{C}P^2/\mathrm{conj} \setminus R_A$ is homotopy equivalent to $C(\Gamma)$.

The proof of this assertion will be postponed to Section 4.5.

4.3. The bundle of semilines.

The set of complex points of a real projective line is homeomorphic to S^2 , the set of the real points is a great circle. The component of the set of real points consists of two connected components (= open hemispheres). Below the closure of such a component is called a semiline.

Denote the set of pairs (L, p), where L is a semiline in $\mathbb{C}P^2$

and $p \in L$ by $\tilde{\mathcal{Y}}$. It is a compact manifold. Its interior $\tilde{\mathcal{Y}} \setminus \partial\tilde{\mathcal{Y}}$ is naturally identified with $\mathbb{C}P^2 \setminus \mathbb{R}P^2$, since each point of $\mathbb{C}P^2\setminus\mathbb{R}P^2$ is contained in the complexification of the real line which is determined by the point. A point (L,p) of $\partial\tilde{\mathcal{Y}}$ can be reconstructed from the point $p\in\mathbb{R}P^2$ and the real line $L\cap\mathbb{R}P^2$ passing through p and supplied with the orientation induced by the natural orientation of L. The complex conjugation $conj : \mathbb{C}P^2 \to \mathbb{C}P^2$ induces a free involution of $\tilde{\mathcal{Y}}$. Let $\mathcal{Y}$ denote the orbit space. The interior of $\mathcal{Y}$ is naturally identified with $(\mathbb{C}P^2\setminus\mathbb{R}P^2)/conj$, the boundary with the space of linear elements of $\mathbb{R}P^2$ (= the flag space).

Let $\tilde{B}\subset\mathbb{C}P^2$ be the conic defined by the equation $x_0^2+x_1^2+x_2^2=0$ Since each semiline intersects $\tilde{B}$ in a single point, we have a bundle $\tilde{\xi}: \tilde{\mathcal{Y}}\to\tilde{B} : (L,p)\mapsto L\cap\tilde{B}$ with fibres naturally identified with semilines. [This gives a way of identifying $\tilde{B}$ with the space of oriented lines on $\mathbb{R}P^2$.] Let $B=\tilde{B}/conj$. The bundle $\tilde{\xi}:\tilde{\mathcal{Y}}\to\tilde{B}$ induces the bundle $\xi:\mathcal{Y}\to B$. A fibre of ξ is naturally interpret as the complexification of a real line factorized by $conj$. Thus we have the bijection $b: B\to \mathbb{R}P^{2\vee}$ of B onto the plane $\mathbb{R}P^{2\vee}$ dual to $\mathbb{R}P^2$. This projection maps a point of B which is a pair of conjugate points of $\tilde{B}$ to the line containing the points.

Let $A\subset\mathbb{C}P^2$ be a real algebraic curve. Denote by $A^{\tilde{\mathcal{Y}}}$ the closure in $\tilde{\mathcal{Y}}$ of the set $A\setminus\mathbb{R}P^2\subset\tilde{\mathcal{Y}}\setminus\partial\tilde{\mathcal{Y}}$. If A is non-singular then $A^{\tilde{\mathcal{Y}}}$ is a proper smooth 2-submanifold of $\tilde{\mathcal{Y}}$ with the boundary $\partial A^{\tilde{\mathcal{Y}}}$ that consists of points which are the oriented linear elements tangent to $A\cap\mathbb{R}P^2$. Let $A^{\mathcal{Y}}$ be the image of $A^{\tilde{\mathcal{Y}}}$ in $\mathcal{Y}$. Certainly, $A^{\mathcal{Y}}$ is the closure in $\mathcal{Y}$ of the set $A/conj\setminus\mathbb{R}P^2\subset\mathcal{Y}\setminus\partial\mathcal{Y}$. If A is non-singular, then $\partial A^{\mathcal{Y}}$ consists of nonoriented linear elements tangent to $A\cap\mathbb{R}P^2$. It is clear that the pair $(\mathcal{Y}, A^{\mathcal{Y}})$ is homeomorphic to the pair $(\mathbb{C}P^2/conj\setminus N, A/conj\setminus N)$, where N is a regular neighbourhood of $\mathbb{R}P^2$ in $\mathbb{C}P^2/conj$. Thus in our investigation of $\mathbb{C}P^2/conj\setminus(\mathbb{R}P^2\cup A/conj)$ we can substitute $\mathcal{Y}\setminus A^{\mathcal{Y}}$ for

$\mathbb{C}P^2/conj\setminus(\mathbb{R}P^2\cup A/conj)$.

The function $\mathbb{C}P^2\to[0,1]:(x_0:x_1:x_2)\mapsto \dfrac{|x_0^2+x_1^2+x_2^2|}{|x_0|^2+|x_1|^2+|x_2|^2}$ determines a function on $\mathcal{Y}$. We denote this function $\mathcal{Y}\to[0,1]$ by d. It is a smooth function, $d|_B=0$, $d|_{\partial\mathcal{Y}}=1$. For any fibre $L\subset\mathcal{Y}$ of ξ the restriction $d|_L$ is a Morse function with the only critical point $L\cap B$.

4.4. Reduction of 4.2.A to the case of a conic.

In the proof of 4.2.A we assume without loss of generality that A is obtained from $A_0=L_1\cup\ldots\cup L_m$ by a small perturbation. Then A is transversal to $\tilde{B}$ and intersects it in $2m$ points close to $\tilde{B}\cap A_0$. Therefore $A^{\mathcal{Y}}$ is transversal to B and intersects it in m points, say $Q_1,\ldots,Q_m$, which are close to $B\cap L_1^{\mathcal{Y}},\ldots, B\cap L_m^{\mathcal{Y}}$ respectively. The perturbation can be accompanied by an isotopy of B which moves $B\cap A_0^{\mathcal{Y}}$ to $A\cap A^{\mathcal{Y}}$. The inverse of b (see 4.3) and this isotopy give an embedding $\mathbb{R}P^{2\vee}\setminus\{L_1^\vee,\ldots,L_m^\vee\}\to\mathcal{Y}\setminus A^{\mathcal{Y}}$ well defined up to isotopy. We remind that $\mathbb{R}P^{2\vee}\setminus\{L_1^\vee,\ldots,L_m^\vee\}$ is a part of $C(\Gamma)$. We have constructed a map of it into $\mathcal{Y}\setminus A^{\mathcal{Y}}$. This map must be extended to construct a homotopy equivalence $C(\Gamma)\to\mathcal{Y}\setminus A^{\mathcal{Y}}$. But in advance we must understand the meaning of $\mathbb{R}P^{2\vee}\setminus\{L_1^\vee,\ldots,L_m^\vee\}$.

The homeomorphism $b^{-1}:\mathbb{R}P^{2\vee}\to B$ and the inclusion $B\hookrightarrow\mathcal{Y}$ induce a homotopy equivalence

$$\mathbb{R}P^{2\vee}\setminus\{L_1^\vee,\ldots,L_m^\vee\}\to\mathcal{Y}\setminus A_0^{\mathcal{Y}}.$$

In fact, $\xi:\mathcal{Y}\to B$ is a fibration with fibre D^2 and $A_0^{\mathcal{Y}}$ is the union of the fibres which are over the points $b^{-1}(L_1^\vee),\ldots,b^{-1}(L_m^\vee)$. The map $\mathbb{R}P^{2\vee}\setminus\{L_1^\vee,\ldots,L_m^\vee\}\to\mathcal{Y}\setminus A^{\mathcal{Y}}$ constructed above can be obtained from this map by a deformation going along with the perturbation. Thus to prove 4.2.A we must show that the homotopy effect of the perturbation is the gluing of 2-cells by the figure-eight loops along all

E_{ij} , see 4.2.

As it is well known, from the topological point of view the small perturbation is localized in a neighbourhood of the set of the points $p_{ij} = L_i \cap L_j$. This means that in the complement of the neighbourhood the perturbation is an isotopy. The Morse Lemma implies that in a neighbourhood of each p_{ij} the perturbation looks standardly, just as in the case of the perturbation

pair of lines $\longmapsto$ hyperbola

Thus it is sufficient to prove that the latter perturbation has the effect described above.

4.5. The case of a conic.

Let A be a real non-singular conic obtained by a small perturbation of the union $A_0 = L_1 \cup L_2$ of real lines L_1, L_2. Then A is transversal to the B and intersects it in 2 points, say Q_1 and Q_2, which are close to $B \cap L_1^{\mathcal{Y}}$ and $B \cap L_2^{\mathcal{Y}}$. There are unique real lines L'_1, L'_2 with $L'^{\mathcal{Y}}_i \cap B = Q_i$, these L'_i are close to L_i and A can be obtained from $L'_1 \cup L'_2$ by a small perturbation. Therefore without loss of generality we assume that $Q_i = B \cap L_i^{\mathcal{Y}}$ (i.e. $L'_i = L_i$).

Since $B \setminus A_0^{\mathcal{Y}} = B \setminus A^{\mathcal{Y}} = B \setminus \{Q_1, Q_2\}$ is a deformation retract of $\mathcal{Y} \setminus A_0^{\mathcal{Y}}$, the homotopy effect of the modification $\mathcal{Y} \setminus A_0^{\mathcal{Y}} \longmapsto \mathcal{Y} \setminus A^{\mathcal{Y}}$ coincides with that of the inclusion $B \setminus A^{\mathcal{Y}} \longrightarrow \mathcal{Y} \setminus A^{\mathcal{Y}}$. The latter will be investigated instead of the former.

The natural homeomorphism $\mathfrak{b}: B \longrightarrow \mathbb{R}P^{2V}$ (see 4.3) maps $\xi(A^{\mathcal{Y}})$ onto the closure of the set of lines which do not intersect A on $\mathbb{R}P^2$. This set is bounded by the curve dual to A . It is a neighbourhood of E_{12}

The set $\xi^{-1}\xi(A^{\mathcal{Y}})$ is homeomorphic to $\mathcal{D}^2 \times \mathcal{D}^2$ and $A^{\mathcal{Y}}$ is situated in it as a section of the natural projection $\mathcal{D}^2 \times \mathcal{D}^2 \longrightarrow \mathcal{D}^2$ (= $\xi|$: $\xi^{-1}\xi(A^{\mathcal{Y}}) \longrightarrow \xi(A^{\mathcal{Y}})$). It intersects the section B transversally in points Q_1, Q_2 with intersection number +1. Those properties obviously

determine the position of $A^{\mathcal{Y}}$ in $\xi^{-1}\xi(A^{\mathcal{Y}})$ up to diffeomorphism rel. boundary. One can easily see that $\xi^{-1}\xi(A^{\mathcal{Y}}) \setminus A^{\mathcal{Y}}$ is homotopy equivalent to $\xi(A^{\mathcal{Y}}) \setminus \{Q_1, Q_2\}$ with a 2-disc glued along the figure-eight loop described above in 4.2. For example, this can be done in the following way. Consider a surface $\mathcal{O}$ of the isotopy type of $A^{\mathcal{Y}}$ shown in fig. 12 by a family of curves in $S^1 \times \mathcal{D}^2$, which are the intersections of it with levels of the function $d|_{\xi^{-1}\xi(A^{\mathcal{Y}})}$ (see 4.3). The surface $\mathcal{O}$ is chosen in such a way that this function restricted to it has minimal number of critical points: 2 minima (at Q_1 and Q_2) and 1 sadle point. Let $\mathcal{D}^t$ denote $d^{-1}[0,t] \cap \xi^{-1}\xi(A^{\mathcal{Y}})$ and d_0 be the value of d at the sadle point. The homotopy type of $\mathcal{D}^t \setminus \mathcal{O}$ does not change until t reaches d_0 . When t crosses d_0 the homotopy type changes as if a 2-cell were glued, cf. [23] . The corresponding cell in $\mathcal{D}^t \setminus \mathcal{O}$ is shown in fig. 12 in the level of d which is slightly higher than the sadle point level. The natural deformation retraction obviously maps the boundary circle of this cell onto the loop described above.

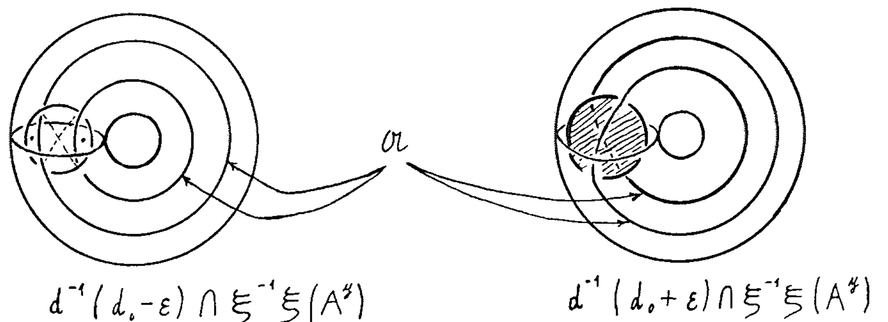


Fig. 12.

4.6. Generators of the group.

In this section we prove that in the case of a ℓ-curve the group $\pi_1(\mathbb{C}P^2/conj \setminus R_A)$ is generated by classes of arbitrarily small loops. To prove this we construct two generators of $\pi_1(C(\Gamma)) \cong$

$\cong \pi_1(\mathbb{C}P^2/conj \setminus R_A)$ and show that they can be realized in any neighbourhood of a point of A_R.

Let A be an ℓ-curve, let it be obtained by a small perturbation of $A_0 = L_1 \cup \dots \cup L_m$, where $L_1, \dots, L_m$ are lines in general position. Let Γ be the genetic graph of this perturbation, $Q \in \mathbb{R}P^{2\vee}$ be a point close to $L_1^\vee$, and $\ell \subset \mathbb{R}P^{2\vee} \setminus \{L_1^\vee, \dots, L_m^\vee\}$ be a line passing through Q. Denote by a the element of $\pi_1(C(\Gamma), Q)$ represented by a small loop encircling $L_1^\vee$, and by c the element represented by a parametrization of ℓ, see fig.13.

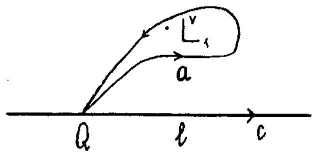


Fig. 13.

4.6.A. The elements a and c generate $\pi_1(C(\Gamma))$. This is evident. The loops realizing a and c described above realize the generators of the fundamental group even of the part of $C(\Gamma)$ constituted by $\mathbb{R}P^{2\vee} \setminus \{L_1^\vee, \dots, L_m^\vee\}$ and $m-1$ 2-cells corresponding to the edges $E_{12}, \dots, E_{1m}$.

4.6.B. For any regular neighbourhood $\mathcal{U}$ in $\mathbb{C}P^2/conj$ of a real point of A the homomorphism

$$in_*: \pi_1(\partial \mathcal{U} \setminus R_A) \longrightarrow \pi_1(\mathbb{C}P^2/conj \setminus R_A)$$

is surjective.

PROOF. Clearly, without loss of generality we can assume that the real point is close to the point $\ell^\vee$ dual to the line $\ell \subset \mathbb{R}P^{2\vee}$. Consider its neighbourhood in $\mathbb{C}P^2/conj$ containing $\ell^\vee$. The choice above of Q and ℓ makes it possible for this neighbourhood to be arbitrarily small. Let V be the pre-image of this neighbourhood under the natural map $\mathcal{Y} \longrightarrow \mathbb{C}P^2/conj$. It is easy to lift into V the loops realizing a and c (which are shown in fig. 13) with respect to the fibration $b \circ \xi : \mathcal{Y} \longrightarrow \mathbb{R}P^2$. For a we obtain a small loop that goes once around $L_1^{\mathcal{Y}}$, for c a loop parametrazing the boundary of a fibre over $\ell^\vee$ of a tubular fibration of $\mathbb{R}P^2$ in $\mathbb{C}P^2/conj$.

By a homotopy equivalence $\mathcal{Y} \setminus A^{\mathcal{Y}} \longrightarrow C(\Gamma)$ these loops are mapped into loops realizing a and c. Since $\mathcal{U}$ is a regular neighbourhood, $\partial \mathcal{U} \setminus R_A$ is a deformation retract of $\mathcal{U} \setminus R_A$.

4.7. The effect of a blow up.

In this section we prove 4.1.A. We consider only the case $k=1$. It is obvious that the other cases differ inessentially. Thus let A be an ℓ-curve, p its real point. Let M be the complex surface obtained from $\mathbb{C}P^2$ by blowing up a point p.

Then M can be presented as the connected sum of $\mathbb{C}P^2$ and $-\mathbb{C}P^2$. The proper pre-image A^M of A under the projection $M \longrightarrow \mathbb{C}P^2$ in this presentation is the connected sum of $A \subset \mathbb{C}P^2$ and the line $\mathbb{C}P^1 \subset (-\mathbb{C}P^2)$. Therefore $M/c \setminus R_A^M$ can be obtained in the following way. Denote some regular neighbourhood of p in $\mathbb{C}P^2/conj$ by $\mathcal{U}$ and some regular neighbourhood of some point $q \in \mathbb{R}P^1$ in $\mathbb{C}P^2/conj$ by V. Then $M/c \setminus R_A$ is diffeomorphic to the result of gluing of spaces $\mathbb{C}P^2/conj \setminus (\mathcal{U} \cup R_A)$ and $\mathbb{C}P^2/conj \setminus (V \cup \mathbb{C}P^1/conj \cup \mathbb{R}P^2)$ by some diffeomorphism $\partial \mathcal{U} \setminus R_A \longrightarrow \partial V \setminus (\mathbb{C}P^1/conj \cup \mathbb{R}P^2)$. Now we apply the van-Kampen theorem. The space $\mathbb{C}P^2/conj \setminus (V \cup \mathbb{C}P^1/conj \cup \mathbb{R}P^2)$ has the punctured projective plane $B \setminus (\mathbb{R}P^1)^\vee$ as a deformation retract, cf. 4.3 above. Therefore its fundamental group is cyclic. By 4.6.B the inclusion $\partial \mathcal{U} \setminus R_A \hookrightarrow \mathbb{C}P^2/conj \setminus (\mathcal{U} \cup R_A)$ induces an epimorphism. Thus $\pi_1(M/c \setminus R_A^M)$ is isomorphic to a quotient group of $\pi_1(\mathbb{C}P^2/conj \setminus (V \cup \mathbb{C}P^1/conj \cup \mathbb{R}P^2))$ and, consequently, it is cyclic.

§ 5. On the homeomorphism classification of knottings of a non-orientable surface in the 4-sphere

5.1. Reduction to a homotopy problem.

Consider the class of all smooth submanifolds S of S^4 with a fixed normal Euler number, $\pi_1(S^4\setminus S)=\mathbb{Z}_2$ and S homeomorphic to a fixed closed connected non-orientable surface. Choose for each S a smooth isomorphism of a tubular neighbourhood of S with a fixed 2-disc bundle and identify all boundaries of these tubular neighbourhoods by them. Our purpose in this § 5 is to prove that the number of homeomorphism types rel. boundary of the complements of the tubular neighbourhoods is finite (i.e. to prove Proposition 3 of the introduction).

REMARK. As it is mentioned in Introduction, for $S=\mathbb{R}P^2$ T.Lawson [15] had proved that the homeomorphism type is unique.

We denote the boundary of the fixed 2-dimensional disk bundle over S by M and the complements of the knottings by $C, C', \ldots$, which all have same boundary M.

Given two such knottings with complements C and C' we study the question whether the identity on $M=\partial C=\partial C'$ extends to a homeomorphism from C to C'. For this we apply the methods of [13]. We recall the relevant result ([13], Corollary 8.8), which can be applied in $\dim 4$ by Freedman's results [9]. Suppose that C and C' have same normal 2-type B and bordant relative normal 2-smoothings in B. This means there is a 3-connected fibration $B\to BO$ and 3-equivalences $f: C\to B$ lifting the normal bundle $\nu: C\to BO$ (and similarly for C') such that $f|_M = f'|_M$ and (C,f) and (C',f') are B-bordant rel. boundary. Then there is such a homeomorphism from C to C' (using the vanishing of $L_5^s(\mathbb{Z}_2)=\hat{L}_5^s(\mathbb{Z}_2)$ in the notation of [13] as proved by Wall [27]).

5.2. Normal 2-type of C.

The normal 2-type of C can easily be described. Let $C\to P$ be a 3-equivalence into the 2. stage of the Postnikov tower (i.e. P is the total space of the fibration over $K(\mathbb{Z}_2,1)$ with fibre $K(\pi_2(C),2)$ and k-invariant $k(C)\in H^3(\mathbb{Z}_2;\pi_2(C))$. We equip C with the restriction of the Spin-structure of S^4 to C. Then the map

$C\to P\times B\,Spin$ given by the 3-equivalence and the Spin-structure is a normal 2-smoothing in $B=P\times B\,Spin \xrightarrow{p_2} BO$. The corresponding B-bordism group is the 4-dimensional singular Spin bordism group $\Omega_4^{Spin}(P)$.

As P is determined by $[\pi_1(C)=\mathbb{Z}_2, \pi_2(C), k(C)]$ we have to compute this invariant. Denote the non-trivial Λ-module structure on $\mathbb{Z}$ with $\Lambda=\mathbb{Z}[\mathbb{Z}_2]$ by $\mathbb{Z}_-$.

5.2.A. LEMMA. $\pi_2(C)\cong\mathbb{Z}_-\oplus$ free Λ-module and $k(C)$ is the non-trivial element in $H^3(\mathbb{Z}_2;\pi_2(C))\cong\mathbb{Z}_2$.

PROOF. Consider the double $C\cup_M C$. The map $H_1(M)\to H_1(C)$ (with integer coefficients if no coefficient system is specified) is surjective as $H_1(C,M)\cong H_1(S^4,S)=0$. Thus the double is again a manifold with $\pi_1=\mathbb{Z}_2$. It is easy to see that $\pi_2(C\cup C)\cong$ $\cong\pi_2(C)\oplus\pi_2(C)^*$ and $k(C\cup C)=in_*k(C)$. But it is well known that $\pi_2(C\cup C)\cong\mathbb{Z}_-\oplus\mathbb{Z}_-\oplus$ free and $k(C\cup C)$ is non-trivial (compare [28], [11]). As cancellation holds for Λ-modules this completes the proof of the Lemma.

This Lemma has two consequences. It shows that all C have same homotopy type. For by [17] there is a map between C and C' inducing an isomorphism on π_1 and π_2 and thus a homotopy equivalence as the covering spaces have $H_3(\tilde{C})=H_3(\tilde{C}')=0$ by Poincaré duality. Moreover, we see that $C\simeq \mathbb{R}P^2\vee(1-\chi(S))\cdot S^2$ as this complex has same $[\pi_1, \pi_2, k]$ as C. This proves the Proposition mentioned in the introduction.

The other consequence is that the normal 2-type is the same for all complements under consideration. The next step is to choose normal 2-smoothing $C\to P\times B\,Spin$ for all complements which on M restrict to the same map. By the choice of the Spin-structure on C its restriction to M is always equal.

5.3. Obstructions to a 3-equivalence.

Thus we have to look for 3-equivalences $f: C\to P$ which are all equal on M. In the following we will apply obstruction theory. For this we fix now one complement C and a 3-equivalence $f: C \to P$ and compare all other situations with this. If $f': C' \to P$ is another 3-equivalence we want to decide whether $f'|_M$ is homotopic to $f|_M$ rel. base points. The first obstruction is the difference of the induced maps on π_1 contained in $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2)$ which is a finite group. If this vanishes, the compositions $M \xrightarrow{f|_M} P \to K(\mathbb{Z}_2,1)$ and $M' \xrightarrow{f'|_M} P \to K(\mathbb{Z}_2,1)$ are homotopic. The only obstruction for lifting such a homotopy is contained in $H^2(M;\pi_2(P))$. We note that the map $H^2(M;\pi_2(P)) \to \mathrm{Hom}_\Lambda(H_2(\tilde{M}),\pi_2(P))$ has finite kernel and the image of the obstruction is the difference of the induced maps. Thus if we assume that the maps $H_2(\tilde{M}) \to H_2(\tilde{C}) \xrightarrow{f_*} H_2(\tilde{P}) = \pi_2(P)$ are equal for all complements there are only finitely many homotopy classes of maps $M \to P$ which are restrictions of such 3-equivalences.

The next step is to show that for every C' there exist a 3-equivalence f' inducing the same map $H_2(\tilde{M}) \to H_2(\tilde{P})$ as the fixed map $f: C \to P$. For a later argument we need the additional assumption that $(f'_*)^{-1} \circ f_*: H_2(\tilde{C}) \to H_2(\tilde{C}')$ is an isometry of the intersection form.

For this we assume for a moment that the equivariant intersection forms on $H_2(\tilde{C}')/H_2(\tilde{M}) = H_2(\tilde{C}')/\text{radical}$ and on $H_2(\tilde{C})/H_2(\tilde{M})$ are isomorphic under an isometry β and we ask for the existence of a homomorphism α making the following diagram commutative

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_2(\tilde{M}) & \longrightarrow & H_2(\tilde{C}') & \longrightarrow & H_2(\tilde{C}')/H_2(\tilde{M}) \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & {\scriptstyle \beta}\downarrow{\scriptstyle \cong} \\
 & & H_2(\tilde{M}) & \longrightarrow & H_2(\tilde{C}) & \longrightarrow & H_2(\tilde{C})/H_2(\tilde{M}) \\
 & & & & {\scriptstyle \cong}\downarrow{\scriptstyle f_*} & & \\
 & & & & H_2(\tilde{P}) & &
\end{array}$$

If the map α exists then $f_* \circ \alpha$ is an isomorphism $\pi_2(C') \to \pi_2(P)$ which maps $k(C')$ into $k(P)$ and thus by [17] can be realized by a 3-equivalence $f': C' \to P$ with the disired properties. The obstruction for the existence of α is an element of $\mathrm{Ext}^1_\Lambda(H_2(\tilde{C})/H_2(\tilde{M}), H_2(\tilde{M}))$.

We will see that this group is finite. For this we note that $H_2(\tilde{M}) \cong \mathbb{Z}_+^{\tau}$ and $H_2(\tilde{C}) = \mathbb{Z}_- \oplus \Lambda^{\tau}$ (same τ). This follows from 5.2.A and a simple calculation of Betti numbers and the fact that $H_2(\tilde{M}) \to H_2(M)$ is rationally an isomorphism. Thus $H_2(\tilde{C})/H_2(\tilde{M}) = \mathbb{Z}_-^{\tau+1}$. As $\mathrm{Ext}^1_\Lambda(\mathbb{Z}_-, \mathbb{Z}_+)$ is $\mathbb{Z}_2$, the group $\mathrm{Ext}^1_\Lambda(H_2(\tilde{C})/H_2(\tilde{M}), H_2(\tilde{M}))$ is finite.

Now we want to verify our assumption about the existence of β or at least show that again the obstructions for the existence of β are contained in a finite set. For this we consider the double ramified covering N of S^4 along S and the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_2(\tilde{M}) & \longrightarrow & H_2(\tilde{C}) & \longrightarrow & H_2(\tilde{C},\tilde{M}) \\
 & & & & \downarrow & & \downarrow{\scriptstyle \cong} \\
 & & x & & H_2(N) & \rightarrowtail & H_2(N,F) \\
 & & \downarrow & & \downarrow & & \\
 & & x \circ [s] & & H_2(N,\tilde{C}) = \mathbb{Z}_2 & &
\end{array}$$

It implies that there is an isometry

$$H_2(\tilde{C})/H_2(M) \xrightarrow{\cong} \mathrm{Ker}(H_2(N) \longrightarrow H_2(N,\tilde{C}) = \mathbb{Z}_2).$$

Thus up to finite ambiguity the form on $H_2(\tilde{C})/H_2(\tilde{M})$ is determined by the equivariant intersection form on $H_2(N)$. On the other hand the equivariant intersection form on $H_2(N)$ is up to finite ambiguity determined by the rank of $H_2(N)$ which is equal to $2-\chi(S)$.

We summarize what we have proved so far. If we fix a complement C and a 3-equivalence $f: C \to P$ then the obstruction for finding 3-equivalences $f': C' \to P$ such that $(f'_*)^{-1} \circ f_*$ preserves the intersection form and restricts to the same map on M are contained in a finite set. If these obstructions are equal for (S^4, S') and (S^4, S'') we can find maps on their complements with the disired properties.

5.4. Bordism obstruction. By the results of [13] mentioned above (in 5.1) the knottings $(S^4, S'), (S^4, S'')$ would be homeomorphic rel. tubular neighbourhood if the singular Spin manifold $(C' \cup_M - C'', f' \cup f'')$ is zero bordant in $\Omega_4^{Spin}(P)$. For a classification up to finite ambiguity it is enough to control the bordism class in $\Omega_4^{Spin}(P) \otimes \mathbb{Q} \cong$ $\cong \mathbb{Q} \oplus H_4(P; \mathbb{Q})$. The isomorphism is given by the signature and the image of the fundamental class. In our situation sign $(C') =$ sign $(C'') = 0$. Thus we are finished if the fundamental class $[C' \cup - C'']$ maps to zero under $f' \cup f''$ in $H_4(P)$. As the transfer is injective in rational homology it is enough to show this in the universal cover.

If we abbreviate $C' \cup - C''$ by X and $f' \cup f''$ by g we are finished with the proof of our proposition if $\tilde{g}_*[\tilde{X}] = 0$ in $H_4(\tilde{P})$. We note that $\tilde{P} = K(\pi_2(P), 2)$ and thus we are finished if for all $\alpha \in H^2(\tilde{P})$, $\langle \alpha^2, \tilde{g}_*[\tilde{X}] \rangle = 0$ or equivalently $\langle \tilde{g}^* \alpha^2, [X] \rangle = 0$. For this we consider the diagram

$$\begin{array}{ccccccc}
H^2(\tilde{P}, \tilde{M}) & \longrightarrow & H^2(\tilde{P}) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & 0 \\
\downarrow (f'^*, f''^*) & & \downarrow g^* & & \| & & \\
H^2(\tilde{C}', \tilde{M}) \oplus H^2(\tilde{C}'', \tilde{M}) & \longrightarrow & H^2(\tilde{X}) & \longrightarrow & H^2(\tilde{M}) & &
\end{array}$$

Because by assumption $(f'_*)^{-1} \circ f''_*$ preserves the intersection form, $\langle g^*(\alpha)^2, [\tilde{X}] \rangle = 0$ for all α coming from $H^2(\tilde{P}, \tilde{M})$. Next we want to construct a splitting $s: H^2(\tilde{M}) \to H^2(\tilde{P})$ and show for all $\alpha \in H^2(M)$ and $\beta \in H^2(\tilde{P}, \tilde{M})$:

$$\langle g^* s(\alpha)^2, [\tilde{X}] \rangle = 0 = \langle g^* s(\alpha) \cup g^* \beta, [\tilde{X}] \rangle .$$

Let $p: H^2(M) \to H^2(\tilde{M})$ be the map induced by projection. This map is an isomorphism in rational cohomology. Furthermore we note that $H^2(P)/Tors \xrightarrow{j^*} H^2(M)/Tors$ is injective with cokernel $\mathbb{Z}_2$ and we denote its image by A. Instead of constructing s on $H^2(\tilde{M})$ we can consider $s := p^* \circ (j^*)^{-1} : A \to H^2(\tilde{P})$

It is easy to see that for $\alpha \in A$ and $\beta \in H^2(\tilde{P}, \tilde{M})$, $\langle s(\alpha) \cup \beta, g_*[\tilde{X}] \rangle = 0$. The reason is that by construction $s(\alpha)$ is contained in the +1 -eigenspace of the involution and β in the (-1)-eigenspace. As the involution preserves $g_*[\tilde{X}]$ this implies the vanishing.

To check whether $\langle s(\alpha)^2, g_*[\tilde{X}] \rangle = 0$ or equivalently $\langle ((j^*)^{-1}(\alpha))^2, g_*[X] \rangle = 0$ is not so easy and in fact we don't know if it is always true. But again we can show that it is true modulo obstructions in a finite set. Note, that this difficulty doesn't occur if $S = \mathbb{R}P^2$ as then $H^2(\tilde{M}) = \{0\}$.

We denote $H_2(B)$ by H and consider a map $B \to K(H, 2) =: K$ inducing an isomorphism on H. If we denote the composition $X \xrightarrow{g} B \to K$ again by g we have to show for all $\alpha \in H^2(K)$: $\langle g^* \alpha^2, [X] \rangle = 0$. If we assume that $g: X \to K$ factors through the 2-skeleton of K which is a wedge of S^2's this follows automatically. We finish our proof by showing that the obstructions for such a factorization are contained in a finite set.

$X = C' \cup - C''$ and both C' and C'' are homotopy equivalent to a 2-complex, $\mathbb{R}P^2 \vee \vee_2 S^2$. Thus $g|_{C'}$ and $g|_{C''}$ factor through $K^{(2)} = \vee_2 S^2$. We are finished if the restriction of these factorizations to M are homotopic. As the image of $i \in H^2(K; H)$ in $H^2(M; H)$ is equal for both maps the only obstruction for a homotopy between them is contained in

$$H^3(M;\pi_3(V_2 S^2))/s\Delta(\theta,u)H^1(M;H) \qquad ([25], \text{ Theorem 10, p.451}).$$

Here θ is the cohomology operation corresponding to the first k-invariant k of $V_2 S^2$ contained in $H^4(K;\pi_3(V_2 S^2))$. $u \in H^2(M;H)$ is the image of $i \in H^2(K;H)$. $\pi_3(V_2 S^2) = \Gamma(H^*)$, the group of symmetric bilinear forms on H^* and also $H_4(K;\mathbb{Z}) = \Gamma(H^*)$ [16] . The k-invariant k is given by $j = id: H_4(K;\mathbb{Z}) = \Gamma(H^*) \longrightarrow \Gamma(H^*)$. It is not difficult to show that for $v \in H^1(M;H)$, $s\Delta(\theta,u)(v)$ is equal to $\sum(v \cup u)$, where $\sum: H \otimes H = Hom(H^*,H) \longrightarrow \Gamma(H^*)$ is the symmetrization map (compare the proof of [25] , Theorem 11, p.452).

If we denote the image of u in $Hom(H_2(M),H)$ by u' and identify $H^1(M;H)$ with $Hom(H_1(M),H) = Hom(H_2(M)^*,H)$, we see that for $v \in Hom(H_2(M)^*,H)$, $s\Delta(\theta,u)(v) = \sum(u' \circ v^*)$

By definition u' is the composition $H_2(M) \to H_2(C') \to H_2(K) = H$ and is by construction an isomorphism. Thus $s\Delta(\theta,u)H^1(M;H) \subset$ $\subset H^3(M;\pi_3(V_2 S^2)) = \Gamma(H^*)$ consists of all even forms and the obstruction group $H^3(M;\pi_3(V_2 S^2))/s\Delta(\theta,u)H^1(M;H)$ is finite.

q.e.d.

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