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Progress in the topology of real algebraic varieties over the last six years

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In memory of my teacher Vladimir Abramovich Rokhlin

CONTENTS

Introduction

The topology of real algebraic varieties as a separate field appeared a little more than a hundred years ago. Nowadays its appearance is commonly dated from 1876 (see, for example, [72], [11], [69]), when the famous paper of Harnack [57] on the number of components of a plane projective real algebraic curve was published. Indeed, before that paper appeared questions on the topology of algebraic curves were not separated from other geometrical questions, more subtle ones as a rule. Because of this, topological questions were considered in unreasonably simple situations only. The attention of the world mathematical community was attracted to the topology of real algebraic varieties by Hilbert in 1900, when he included its main problems in his famous list of problems ([17], 16-th problem). Since it is not my aim in this paper (as the title obviously indicates) to present the whole history of the topology of real algebraic varieties (the reader can get some knowledge of it, for example, from the survey articles by Oleinik [72], Gudkov [11], and Arnol'd and Oleinik [69]), I cannot fail to mention here the most important papers that have determined the character of this field. In the initial period the basic role had been played by the papers of such masters as Hilbert [58], [59], Rohn [61], [62], and Klein [60]. Ragsdale's paper [29] also dates from that time. In that work conjectures based on the analysis of Harnack's and Hilbert's results were made. In many aspects these conjectures anticipated the subsequent development of the topology of curves. The main one of these conjectures is still the most important unsolved problem (see §3 below). An essential contribution was then made by members of the Italian algebraic-geometry school, in particular, by Brusotti [53] and Comessatti [55], [56] (see Brusotti's survey [54]). A new stage was opened with Petrovskii's papers [63], [25]. In them he obtained deep results on the topology of plane curves and created the technical base of many subsequent papers. In 1949 Petrovskii and Oleinik [70] obtained the first general results on the topology of real algebraic varieties of arbitrary dimension, namely the famous Petrovskii–Oleinik inequalities, which are estimates of such an important topological invariant as the Euler characteristic. Analogous results for curves lying on an arbitrary surface in three-dimensional space were obtained by Oleinik [71]. In 1969 Gudkov [65] completed the isotopy classification of non-singular plane projective real algebraic curves of degree 6 and made a number of conjectures, which produced a highly stimulating effect on the development of the subject. In 1971 Arnol'd [1] proved a weakened version of one of Gudkov's conjectures and a number of new restrictions on the topology of plane curves and discovered deep connections between the topology of real plane algebraic curves and the topology of four-dimensional manifolds. This conjecture of Gudkov was completely proved by Rokhlin [31]; in fact, Rokhlin proved its generalization to the case of varieties of arbitrary dimension. Other conjectures of Gudkov (generalized to the same extent) were proved by Kharlamov [67] and Gudkov and Krakhnov [66]. Here I interrupt the list of the most remarkable papers. During the last 13 years the topology of real algebraic varieties has developed especially intensively due to the involvement of new topological methods, initiated by Arnol'd [1] and Rokhlin [31]. There are several survey articles on this development. Its first stage (up to 1974) was described by Gudkov [11]. The state of the topology of plane curves in 1978 was described by Wilson [47] and Rokhlin [33]. The topology of surfaces was the subject of Kharlamov's address [14] at the International Congress of Mathematicians in Helsinki. A wide survey of the whole subject, with a list of unsolved problems, was published in 1979 by Arnol'd and Oleinik [69]. In my address [45] at the International Congress of Mathematicians in Warsaw I aimed to describe the development of the subject from 1978 to 1983. This paper is a version of that address, considerably enlarged and supplemented by information on the latest achievements. As in [45], I do not attempt to give here a complete survey and confine myself to the following themes:

(i) complex topological characteristics of non-singular plane projective real algebraic curves (see §§1–5);

(ii) complex topological characteristics of non-singular real algebraic surfaces, §6;

(iii) new restrictions on the topology of non-singular plane real algebraic curves, §4;

(iv) classification of curves and surfaces up to rigid isotopy, §7;

(v) construction of real algebraic varieties with prescribed topological properties, §8.

Here I do not touch at all on the following themes, in which considerable work was done in 1978–1984. Without a consideration of these themes this survey is, to my regret, very far from being complete.

(i) Indices of singularities of polynomial vector fields, see Khovansky [18], Varchenko [64], and Gusein-Zade [68].

(ii) New restrictions on the topology of non-singular real algebraic surfaces, found by Nikulin [23] (in §4, however, we consider restrictions on the topology of plane curves, which are consequences of those restrictions).

(iii) Curves on surfaces and, in particular, on quadrics, see Gudkov [12] and Zvonilov [49], [50].

(iv) Singular curves, see Zvonilov [48].

(v) Points of inflexion and bitangents of curves of degree 4 with arbitrary singularities, see Gudkov and Nebukina [52].

I am deeply graveful to V.A. Rokhlin, G. Wilson, V.M. Kharlamov, V.V. Nikulin, and V.I. Arnol'd for valuable discussions and suggestions.

I dedicate this survey to the memory of my teacher Vladimir Abramovich Rokhlin. The topology of real algebraic varieties, in which he worked actively during the last thirteen years of his life, is indebted to him not only for a number of first-class results, but also for the essential enlargement of the stock of technical methods and for the formation of a new, wider point of view on its principal objects and problems. His elegant and deep work has had, and will have for a long time, a decisive influence on the development of this field.

§1. Real algebraic curves as complex objects

We first consider non-singular plane projective real algebraic curves. For short let us call them simply *curves*. The set of real points of a curve A will be denoted by $\mathbf{R}A$. It is a smooth closed one-dimensional subvariety of the real projective plane $\mathbf{R}P^2$. Each component of it is homeomorphic to a circle. If the degree of the curve is even, then the components are all positioned in $\mathbf{R}P^2$ two-sidedly. If the degree is odd, then there is exactly one one-sided component. The two-sided components are called *ovals*. The isotopy type of $\mathbf{R}A \hookrightarrow \mathbf{R}P^2$ (or, equally, the topological type of the pair $(\mathbf{R}P^2, \mathbf{R}A)$) is determined by the scheme of mutual position of the components of $\mathbf{R}A$, which is called the *real scheme* of the curve A.

It is a tradition going back to Hilbert to regard the question of which real schemes are realized by curves of a given degree as the main question of the topology of real algebraic curves. However, Klein [20] had already posed a wider question. He had been interested in the connection between the real scheme of a curve A and the embedding of $\mathbf{R}A$ in the set $\mathbf{C}A$ of complex points of A.

$\mathbf{C}A$ is an oriented smooth connected closed two-dimensional submanifold of the complex projective plane $\mathbf{C}P^2$. It is invariant under the involution conj: $\mathbf{C}P^2 \to \mathbf{C}P^2$: $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$. The curve $\mathbf{R}A$ is the set of fixed points of the restriction of conj to $\mathbf{C}A$. It may or may not divide $\mathbf{C}A$. In the first case A is said to be *dividing* or of *type* I, in the second case it is said to be *non-dividing* or of *type* II. In the first case $\mathbf{R}A$ divides $\mathbf{C}A$ into two connected parts. Their natural orientations define on $\mathbf{R}A$ (as their common boundary) two opposite orientations, which are called the *complex orientations* of the curve. The real scheme of a curve enriched by an indication of the type and, in the case of type I, by a description of the complex orientations, is called the *complex scheme* of the curve.

The real scheme of a curve of degree m is said to be of *type* I (*type* II) if any curve of degree m with this real scheme is of type I (type II). Otherwise (that is, if there are curves of both types with this real scheme) it is said to be of *indefinite type*.

The division of curves into types was introduced by Klein [20]. The complex orientations were introduced into the topology of real algebraic curves by Rokhlin [32]. As I recently learned, they appeared in a paper by Petrovskii [26] on lacunas of partial differential equations. Complex schemes were introduced six years ago by Rokhlin [33]. In the latest development of the field they occupy the central position. Recently there has been a wider understanding of the problems of the topology of real algebraic varieties, in which the basic object is a real variety together with its embedding in its complexification, rather than the real variety in itself (see Rokhlin [35]).

To resolve the question of which real and complex schemes are realizable by curves of a given degree, it is necessary to work in two directions: firstly, it is necessary to find restrictions imposed on the schemes by the algebraic nature of the curves; secondly, it is necessary to find methods of constructing curves of a given degree with a prescribed scheme.

A substantial part of the known restrictions follows from a comparatively small number of purely topological properties of algebraic curves. Thus, in parallel with algebraic curves it is useful to consider objects that topologically imitate them. Let M be an oriented smooth connected closed two-dimensional submanifold of $\mathbf{C}P^2$. We say that M is a *flexible curve of degree* m if

(i) it realizes $m[\mathbf{C}P^1] \in H_2(\mathbf{C}P^2)$;
(ii) its genus is equal to $(m-1)(m-2)/2$;
(iii) it is invariant under conj;
(iv) the field of its tangent planes on $M \cap \mathbf{R}P^2$ can be deformed, in the class of planes invariant under conj, to the field of lines of $\mathbf{C}P^2$ tangent to $M \cap \mathbf{R}P^2$.

(Maybe this definition is not final; for example, future investigations may lead us to add new conditions.) The intersection of a flexible curve with $\mathbf{R}P^2$ is a smooth one-dimensional submanifold, which is called the real part of the curve. The set of complex points of an algebraic curve of degree m is obviously a flexible curve of degree m. Everything stated above concerning algebraic curves and their schemes extends without alteration to the case of flexible curves. Restrictions on schemes of curves of degree m that are proved for schemes of flexible curves of degree m are said to have *topological origin*. A well-known classification of schemes of curves of degree ≤ 6 (see §8 below) is provided by the restrictions of topological origin stated in §3 below, so that for $m \leq 6$ all restrictions have topological origin.

§2. Numerical characteristics and encoding of schemes of curves

To formulate restrictions on the schemes, let us introduce some notions and numerical characteristics connected with schemes of curves. Two ovals are said to constitute an *injective pair* if one of them is enclosed by the other. A set of ovals, each pair of which is injective, is called a *nest*. An injective pair of ovals of a dividing curve is said to be *positive* if the orientations of the ovals determined by the complex orientation are induced by some orientation of the annulus bounded by the ovals. The ovals of a dividing curve of odd degree are separated into positive and negative ones. Namely, consider the Möbius strip that is the complement of the interior of an oval in $\mathbf{R}P^2$. If the integer homology classes realized in it by the oval and the doubled one-sided component with orientations determined by the complex orientation differ in sign, then the oval is said to be *positive*, otherwise it is said to be *negative*. In the case of a dividing curve of even degree only non-exterior ovals are separated into positive and negative. Namely, a non-exterior oval is *positive* if it constitutes a positive pair with the exterior oval enclosing it, and *negative* otherwise. An oval is said to be *even* if it lies inside an even number of other ovals. The Euler characteristic of a component of the complement of a curve is called the *characteristic* of the outer bounding oval of the compnent. A component of the complement of a curve is said to be *even* if each of its inner bounding ovals encloses an odd number of ovals.

The most important numerical characteristics of a real scheme are the following numbers: l is the number of ovals; p is the number of even ovals; n is the number of odd ovals; l^+, l^0, and l^- are the numbers of ovals with positive, zero, and negative characteristics; p^+, p^0, p^- and n^+, n^0, n^- are the analogous numbers of even and odd ovals; π and ν are the numbers of even and odd non-empty exterior bounding ovals of even components of the complement of the curve; h_r is the greatest number of ovals in a union of at most r nests; h_r' is the greatest number of ovals in a set contained in a union of at most r nests and not containing an oval that encloses all other ovals of the set. The following numbers are characteristics of the complex scheme of a dividing curve: Π^+ and Π^- are the numbers of positive and negative injective pairs, Λ^+ and Λ^- are the numbers of positive and negative ovals.

For the description of real schemes of curves we shall use the following system of notation. A connected one-sided curve is encoded by the symbol $\langle J\rangle$, a curve consisting of one oval by the symbol $\langle 1\rangle$, the empty curve by the symbol $\langle 0\rangle$. If the symbol $\langle A\rangle$ encodes some set of ovals, then the set obtained from it by adjoining one oval enclosing all the rest is encoded by the symbol $\langle 1\langle A\rangle\rangle$. A curve presented as the union of two non-intersecting sets of ovals encoded by the symbols $\langle A\rangle$ and $\langle B\rangle$ and such that no oval of one set is enclosed by an oval of the other, is encoded by the symbol $\langle A \amalg B\rangle$. We shall use two abbreviations: firstly, if $\langle A\rangle$ is the code of a set of ovals, then a fragment of another code having the form $A \amalg \dots \amalg A$, where A is repeated n times, is denoted for short by $n \times A$; secondly, fragments of a code having the form $n \times 1$ are denoted for short by n.

This coding system is transformed into a coding system for complex schemes in the following way. According to the type of the curve, its code is supplied with a subscript I or II. In the case of type I, codes of the positive ovals are supplied with a superscript +, and codes of the negative ovals with a superscript −.

§3. Old restrictions on schemes of curves

In this section the restrictions obtained before 1978 are considered. We first consider the restrictions of topological origin on real schemes of curves of degree m (see [33] and [47]).

(3.1) ***The Harnack inequality.*** $l \leqslant (m^2 - 3m + 3 + (-1)^m)/2$.

Curves with $l = (m^2 - 3m + 3 + (-1)^m)/2$ are called *M-curves*, and curves with $l = (m^2 - 3m + 3 + (-1)^m)/2 - a$ are called *$(M-a)$-curves*.

In the following restrictions (3.2)–(3.9) the degree m is even, $m = 2k$.

Extremal properties of the Harnack inequality.

(3.2) *In the case of an M-curve (that is, if* $l = (m^2-3m+4)/2$) $p-n \equiv k^2 \bmod 8$.

(3.3) *In the case of an $(M-1)$-curve (that is, if* $l = (m^2-3m+2)/2$) $p-n \equiv k^2 \pm 1 \bmod 8$.

The strengthened Petrovskiĭ inequalities.

(3.4) $p - n^- \leqslant (3k^2 - 3k + 2)/2$,

(3.5) $n - p^- \leqslant (3k^2 - 3k)/2$.

The strengthened Arnol'd inequalities.

(3.6) $p^- + p^0 \leqslant (k^2 - 3k + 3 + (-1)^k)/2$,

(3.7) $n^- + n^0 \leqslant (k^2 - 3k + 2)/2$.

Extremal properties of the strengthened Arnol'd inequalities.

(3.8) *If k is even and $p^- + p^0 = (k^2-3k+4)/2$, then $p^- = p^+ = 0$.*

(3.9) *If k is odd and $n^- + n^0 = (k^2-3k+2)/2$, then $n^- = n^+ = 0$ and there is only one exterior oval.*

Besides the Harnack inequality there was only one known restriction of topological origin applicable to the case of odd m.

(3.10) *If $m \neq 4$, then $l^- + l^0 \leqslant (m-3)^2/4 + (m^2 - h_2)/4h^2$, where h is any integer that divides m and is a power of an odd prime.*

For even m this follows from (3.6)–(3.9), for odd m it is the Viro-Zvonilov inequality ([33], 1.3).

Extremal property of (3.10).

(3.11) *If $l^- + l^0 = (m-3)^2/4 + (m^2 - h^2)/4h^2$, where h divides m and is a power of an odd prime p, then there are $\alpha_1, \dots, \alpha_r \in \mathbf{Z}_p$ and components $B_1, \dots, B_r$ of $\mathbf{R}P^2 \setminus \mathbf{R}A$ with $\chi(B_1) = \dots = \chi(B_r) = 0$ such that the boundary of the chain $\sum_{i=1}^{r}\alpha_i[B_i] \in C_2(\mathbf{R}P^2;\ \mathbf{Z}_p)$ is $[\mathbf{R}A]$.*

Now let us consider the restrictions of topological origin on *complex* schemes of curves of degree m considered in [33].

(3.12) *If the curve is of type* I, *then* $l \equiv [m/2] \bmod 2$.

The Rokhlin formulae.

(3.13) *If m is even and the curve is of type* I, *then*

$$2(\Pi^+ - \Pi^-) = l - m^2/4.$$

(3.14) *If m is odd and the curve is of type* I, *then*

$$\Lambda^+ - \Pi^- + 2(\Pi^+ - \Pi^-) = l - (m^2 - 1)/4.$$

Extremal properties of the Harnack inequality.

(3.15) *Any M-curve is of type* I.

(3.16) ***The Kharlamov*** ([33], 3.4)***–Marin*** [21] ***congruence.*** *Any $(M-2)$-curve of even degree m with $p - n \equiv m^2/4 + 4 \bmod 8$ is of type* I.

Extremal properties of the strengthened Arnol'd inequalities.

(3.17) *If $m \equiv 0 \bmod 4$ and $p^- + p^0 = (m^2 - 6m + 16)/8$, then the curve is of type* I.

(3.18) *If $m \equiv 2 \bmod 4$ and $n^- + n^0 = (m^2 - 6m + 8)/8$, then the curve is of type* I.

Restrictions that do not have topological origin are difficult to outline, as a rule. Most of them are corollaries of Bezout's theorem, that is, topological consequences of the fact that two irreducible curves of degrees m and q either coincide or intersect in at most mq points. Let us formulate some of them. For more general statements, see [11] and [33]. For $m \leqslant 11$ they follow from those formulated below.

(3.19) $$h_2 \leqslant m/2.$$

In particular, if $h_1 = [m/2]$, then $l = [m/2]$.

(3.20) $$h'_2 \leqslant m.$$

In particular, if $h'_4 = m$, then $l = m$.

(3.21) *If* $r \leqslant 8$, *then* $h_r + [(8-r)/2] \leqslant 3m/2$.

In particular, if $h_7 = [3m/2]$, then $l = [3m/2]$.

(3.22) *If* $r \leqslant 13$, *then* $h'_r + [(13-r)/2] \leqslant 2m$.

In particular, if $h'_{12} = 2m$, then $l = 2m$.

Besides the restrictions of this type, there were only two restrictions in [33] (formulated below as (3.23) and (3.24)) that had not been proved for schemes of flexible curves.

(3.23) ***Zvonilov's inequality.*** *If m is odd, then*

$$l^- + l^0 \leqslant (m^2 - 4m + 3)/4.$$

In many cases it is weaker than (3.10), but in some cases it is stronger. The smallest value of m for which (3.23) is stronger than (3.10) is 693.

(3.24) *If* $h_1 = [m/2]$, *then the curve is of type* I.

Rokhlin observed ([33], 3.6) that some other known restrictions could be obtained by modifying the proof of (3.24), but he missed the following restriction, which can also be obtained in that way and does not follow from the theorems listed above.

(3.25) *If* $h'_4 = m$, *then the curve is of type* I.

To conclude this section I should mention the oldest conjecture of the topology of plane real algebraic curves. It was made by Ragsdale [29] in 1906 and says that

$$p \leqslant (3m^2 - 6m + 8)/8 \quad \text{and} \quad n \leqslant (3m^2 - 6m)/8$$

for any curve of even degree m. In 1938 Petrovskii [25] made the following weaker conjecture:

$$p \leqslant (3m^2 - 6m + 8)/8 \quad \text{and} \quad n \leqslant (3m^2 - 6m + 8)/8.$$

For any $m \geqslant 8$, $m \equiv 0 \bmod 4$ I constructed [40] in 1980 a curve of degree m with the real scheme $\langle (m^2 - 6m)/8 \amalg 1\langle (3m^2 - 6m + 8)/8\rangle\rangle$. Thus the second Ragsdale inequality is false for $m \geqslant 8$, $m \equiv 0 \bmod 4$. The question of whether the Petrovskii conjecture is true remains open. It admits a wider formulation: is it true that if X is the set of fixed points of an antiholomorphic involution of a non-singular simply-connected compact complex surface $\mathscr{X}$, then $\dim H_1(X; \mathbf{Z}_2) \leqslant h^{1,1}(\mathscr{X})$.

§4. New restrictions on schemes of curves

We first consider the restrictions of topological origin.

The Rokhlin inequalities [34].

(4.1) *If the curve is of type* I *and* $m \equiv 0 \bmod 4$, *then*

$$4\nu + p - n \leqslant (m^2 - 6m + 16)/2.$$

(4.2) *If the curve is of type* I *and* $m \equiv 2 \bmod 4$, *then*

$$4\pi + n - p \leqslant (m^2 - 6m + 14)/2.$$

Comparison of (4.1) and (4.2) with (3.15) and (3.16) gives restrictions on real schemes containing new information for $m \geqslant 10$, see [34]. The proofs of (4.1) and (4.2) are similar to the proofs of (3.6) and (3.7) but, in contrast to the previous proofs in the field, they involve non-algebraic branched coverings, namely, two-sheeted coverings of $\mathbf{C}P^2$ branched over surfaces consisting of half of $\mathbf{C}A$ and half of $\mathbf{R}P^2$. Recently Fiedler [8] has found new restrictions analogous to (4.1) and (4.2), which are proved in a similar manner. The innovation in these proofs is the use of auxiliary imaginary lines and conics. (It makes the proofs unsuitable for flexible curves.)

In 1982, independently and by different methods, Fiedler [7] and Nikulin [23] obtained restrictions that are close to each other. Fiedler's proofs are related to Marin's proofs [21] of the extremal properties of the Harnack inequality. They are based on the application of the Guillou-Marin generalization [13] of the Rokhlin congruence [30], §3. Fiedler applied it to surfaces in $\mathbf{C}P^2$, while Marin worked in S^4. Nikulin's proofs are based on an investigation of the arithmetic role played by the homology classes of the real cycles in the intersection form of the two-sheeted covering of $\mathbf{C}P^2$ branched over $\mathbf{C}A$. Nikulin obtained his results as consequences of theorems on real algebraic surfaces, which are the only restrictions on the topology of real algebraic surfaces found during the last six years. For lack of space we do not consider them, see [23]. In both papers there are results on curves of all even degrees, but for some values of the degree the stronger results are in one paper, for other values in the other paper. Below I formulate the strongest restrictions without discussing each paper separately.

New extremal properties of the Harnack inequality.

(4.3) Fiedler [7]. *If* $m \equiv 4 \mod 8$, *the curve is an M-curve, and the characteristic of each even oval is even, then* $p-n \equiv -4 \mod 16$.

(4.4) Nikulin [23]. *If* $m \equiv 0 \mod 8$, *the curve is an M-curve, and the characteristic of each even oval is divisible by* 2^r, *then either* $p-n \equiv 0 \mod 2^{r+3}$ *or* $p-n = 4^q\chi$, *where* $q \geq 2$ *and* $\chi \equiv 1 \mod 2$.

(4.5) Nikulin [23]. *If* $m \equiv 2 \mod 4$, *the curve is an M-curve, and the characteristic of each odd oval is divisible by* 2^r, *then* $p-n \equiv 1 \mod 2^{r+2}$.

Theorems (4.3)–(4.5) contain new information for $m \geq 12$ (see Fiedler [7]; Nikulin [23] asserted that (4.5) contains new information for $m = 10$, but his example, like any similar example of degree 10, does not satisfy the old restriction (3.9)). It is curious that if the Ragsdale-Petrovskii conjecture is true, then the conditions of (4.3)–(4.5) are not realized (see Fiedler [7]).

New congruences for the real schemes of curves of type I.

(4.6) *If* $m \equiv 0 \mod 2$, *the curve is of type* I, *and the characteristic of each odd oval is even, then* $p-n \equiv m^2/4 \mod 8$.

(4.7) *If* $m \equiv 0 \mod 4$, *the curve is of type* I, *and the characteristic of each even oval is even, then* $p-n \equiv 0 \mod 8$.

Theorem (4.6) was proved by Slepyan [36] in 1980. He showed that it is a formal consequence of Rokhlin's formula (3.13), thus strictly speaking it is not new. Theorem (4.7) and a special case of (4.6) were proved by Nikulin [23]. Both theorems can also be proved by Fiedler's method [7]. Theorem (4.7) contains new information for $m \geq 8$: the complex scheme $\langle 1 \langle 6^+ \amalg 7^- \rangle\rangle_I$ of degree 8 satisfies all the old restrictions but does not satisfy (4.7).

We now consider restrictions of non-topological origin. In the period under review a new class of such restrictions was discovered. They are less easy to state compactly than the consequences of Bezout's theorem. Satisfactory general statements of them have not been found. We therefore restrict ourselves to discussion of their origins, to some special statements, and to a general reference to [6] and [41].

Most of these new restrictions are proved by constructing auxiliary curves of type I (of degrees 1 or 2, as a rule) or families of such curves and applying Bezout's theorem together with the following theorems (4.8), (4.9) and the old restrictions formulated in §3.

(4.8) ***The Rokhlin formula for a pair of curves.*** *Let* A_1, A_2 *be dividing curves of degrees* m_1, m_2 *transversal to each other and intersecting in r real points. Let C be a curve of degree* $m = m_1 + m_2$ *obtained from* $A_1 \cup A_2$ *by a small perturbation such that some complex orientations of* A_1 *and* A_2 *determine an orientation of* $\mathbf{R}C$. *For* $\mathbf{R}C$ *with this orientation let*

$$\sigma = \begin{cases} m^2/4 - l + 2(\Pi^+ - \Pi^-) & \text{if } m \text{ is even,} \\ (m^2-1)/4 - l + 2(\Pi^+ - \Pi^-) + \Lambda^+ - \Lambda^- & \text{if } m \text{ is odd.} \end{cases}$$

Then σ *is even and* $0 \leq \sigma \leq m_1 m_2 - r$. *The curve C is of type* I *if and only if* $\sigma = 0$ *(that is, if and only if the Rokhlin formula is valid for the orientation of* $\mathbf{R}C$ *determined by the complex orientations of* A_1 *and* A_2*). In that case this orientation is complex*

Special cases of (4.8) were found by Fiedler ([33], 3.7), Marin [21], and Polotovskii, and the final version by Zvonilov and the author. It is a special case of the generalization of Rokhlin's formula to singular curves, found by Zvonilov [48]. Discovered as a basis for the construction of curves with a prescribed complex scheme, Theorem (4.8) proved to be a powerful restriction on the mutual position of two curves of type I; by auxiliary constructions it was made the origin of restrictions on schemes of curves.

Let $\mathcal{L} = \{L_t\}_{t \in \mathbf{R}P^1}$ be a pencil of real lines. (The line $L_{(t_0:t_1)}$ is determined by the equation $t_0\lambda_0(x) + t_1\lambda_1(x) = 0$, where $\lambda_0(x) = 0$ and $\lambda_1(x) = 0$ are the equations of the lines $L_{(1:0)}$ and $L_{(0:1)}$. All the lines L_t pass through one point P.) Let A be a non-singular curve of degree m. Suppose that no real line passing through P touches A at an imaginary point or is an inflexional tangent to A. Then the intersection $\mathbf{C}A \cap (\bigcup_{t \in \mathbf{R}P^1} \mathbf{C}L_t)$ consists of $\mathbf{R}A$ and a finite number of open smooth arcs. The closure of $(\mathbf{C}A \cap (\bigcup_{t \in \mathbf{R}P^1} \mathbf{C}L_t)) \setminus \mathbf{R}A$ is denoted by $S_P A$. It is a smooth closed one-dimensional submanifold of $\mathbf{C}A$ and intersects $\mathbf{R}A$ in the points of tangency of $\mathbf{R}A$ and $\mathbf{R}L_t$.

(4.9) (Fiedler [6]). *Let* $\{L_t\}_{t \in U}$ *be the set of the lines of the pencil* $\mathcal{L}$ *intersecting a component C of* $S_P A$ *and let* L_{t_1}, L_{t_2} *be the extreme lines of this set (thus* L_{t_1}, L_{t_2} *touch* $\mathbf{R}A$ *at* $C \cap \mathbf{R}A$*). Let the lines* L_t, $t \in U$, *be compatibly oriented. If A is of type* I *and the orientation of* L_{t_1} *is compatible with a complex orientation of A at the point of contact, then the orientation of* L_{t_2} *is also compatible with this complex orientation of A at the point of contact* (see Fig. 1).

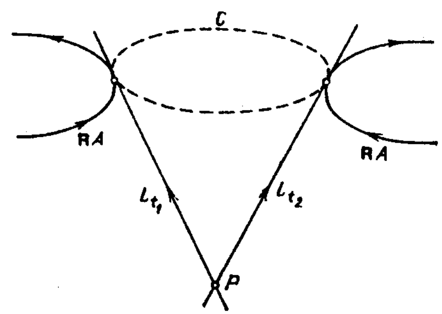

Fig. 1.

By (4.9), the complex orientations of the chain of ovals connected by the components of S_pA alternate (see Fig. 2). If any line $L_t \in \mathcal{L}$ intersects $\mathbf{R}A$ in at least $m-2$ points, then it is easy to discover such chains (see Fig. 2, where $m = 8$). In that case (4.9) is a simple consequence of (4.8) (see [41], 1.4).

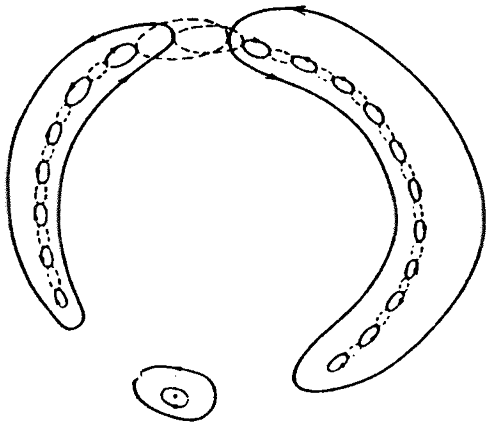

Fig. 2.

Let us consider some special restrictions obtained by using (4.8) and (4.9).

(4.10) *There does not exist a curve of degree 7 with real scheme* $\langle J \amalg 1\langle 14\rangle\rangle$.

(4.11) *Let* $\langle \alpha \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle \amalg 1\langle\delta\rangle\rangle$ *be the real scheme of a curve of degree* 8 *with non-zero* β, γ, *and* δ.

(i) *If* $l = 22$ (*that is*, $\alpha+\beta+\gamma+\delta = 19$), *then* β, γ, *and* δ *are odd.*

(ii) *If* $l = 20$ *and* $p-n \equiv 4 \bmod 8$, *then two of the numbers* β, γ, *and* δ *are odd and one is even.*

Theorem (4.10) was announced in my note [40] and published with different proofs in [41] and [6]. Theorem (4.11(i)) for $\alpha = 0$, $\beta = 1$ was found by Fiedler [6]. In the form stated above, Theorem (4.11) was proved in [41]. The first restrictions of this sort were found by Fiedler. The idea of using (4.8) instead of Fiedler's alternating of orientations is due to Rokhlin.

Another application of (4.8) and (4.9) was the discovery of the connection between the position of a curve in $\mathbf{R}P^2$ relative to lines and conics on the one hand, and the complex scheme of the curve on the other hand. For example, the type of a curve of degree 5 with $l = 4$ is determined by its position with respect to lines (see Fig. 3 or Fiedler [6]). Analogous connections of the type of a curve with its other geometrical invariant, the number of its real θ-characteristics, were discovered by Gross and Harris [10].

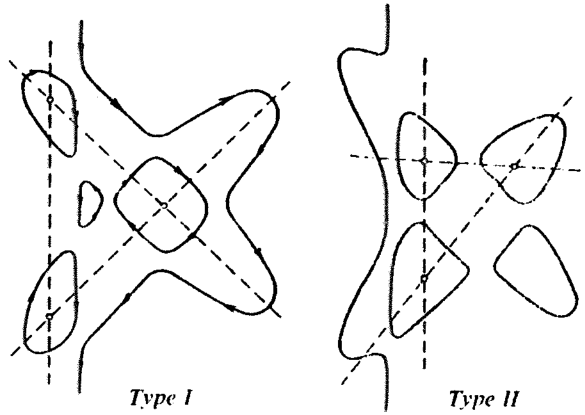

Fig. 3.

In 1984 I found a new possibility of obtaining restrictions of non-topological origin. It is based on the construction of membranes in $\mathbf{C}P^2$ with boundaries in S_pA. Here I announce one special restriction obtained by this method.

(4.12) *If* $\langle 1\langle\alpha\rangle \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle\rangle$ *is the real scheme of an M-curve of degree* 8, *then the triple* (α, β, γ) *cannot be* (1, 3, 15), (1, 5, 11), (1, 9, 9), (3, 3, 13), (3, 5, 11), (3, 7, 9), *or* (5, 5, 9). *There does not exist a curve of degree* 8 *with the real scheme* $\langle 4 \amalg 1\langle 3\rangle \amalg 1\langle 3\rangle \amalg 1\langle 9\rangle\rangle$.

Thus by (4.11) and (4.12) the real scheme of an M-curve of degree 8 that has the form $\langle 1\langle\alpha\rangle \amalg 1\langle\beta\rangle \amalg 1\langle\gamma\rangle\rangle$ can only be $\langle 1\langle 1\rangle \amalg 1\langle 1\rangle \amalg 1\langle 17\rangle\rangle$, $\langle 1\langle 1\rangle \amalg 1\langle 7\rangle \amalg 1\langle 11\rangle\rangle$, or $\langle 1\langle 5\rangle \amalg 1\langle 7\rangle \amalg 1\langle 7\rangle\rangle$. Recently Shustin showed that these three schemes are in fact realized.

§5. Klein's assertion

More than 100 years ago Klein ([20], 155) wrote in a slightly unclear manner that a curve of type I *does not permit any development.*

In 1978 Rokhlin ([33], 3.9), referring to this phrase of Klein and the actual material, made a conjecture that any curve of a given degree with a given real scheme is of type I if and only if the scheme is not part of a bigger real scheme of a curve of the same degree.

Polotovskii [27] observed that truth of the "if" part of this conjecture and Theorem (4.11(i)) would imply new restrictions on real schemes of curves of degree 8, and Shustin [37] constructed curves of degree 8 with real schemes $\langle 10 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle \amalg 1\langle 4\rangle\rangle$ and $\langle 6 \amalg 1\langle 2\rangle \amalg 1\langle 4\rangle \amalg 1\langle 5\rangle\rangle$ that do not satisfy these restrictions. Thus there exists a real scheme of type II of degree 8 (for example, either $\langle 10 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle \amalg 1\langle 4\rangle\rangle$ or $\langle 11 \amalg 1\langle 1\rangle \amalg 1\langle 2\rangle \amalg 1\langle 4\rangle\rangle$) that is not part of a bigger real scheme of a curve of degree 8. It is not known whether any maximal real scheme of a given degree is of type I. It is interesting whether Rokhlin's conjecture is true for flexible curves.

It seems to me that in spite of the attractiveness and fundamental nature of the question of the relation between the maximality of a real scheme and its belonging to type I, Klein's words are to be understood more literally. Namely, there is the following simple theorem, which is close to Klein's pioneering work in spirit and in proof.

(5.1) *Let A_t be a continuous family of real algebraic curves (not necessarily plane). If A_0 has just one singular point, which is a non-degenerate double point, the other A_t are non-singular, and if A_t with $t < 0$ are of type* I, *then the number of components of* $\mathbf{R}A_t$ *with $t > 0$ is not more than the number of components of* $\mathbf{R}A_t$ *with $t < 0$.*

This theorem seems to be a special case of a theorem on varieties of arbitrary dimension (see the next section). Recently Marin (private communication) rediscovered (5.1) and proved by using it that *any pencil of plane curves of even degree $m \geqslant 4$ contains a curve whose number of components is at most* $(m^2-3m-2)/2$ (the latter for $m \equiv 0 \bmod 4$ with an incorrect proof was published by Cheponkus [2], and a correct proof for $m = 4$ was found by Chislenko [4]). For $m = 4$ this implies that any 13 real points of the plane lie on some connected real curve of degree 4; this, in turn, implies (3.22). We can probably look forward to new progress in the topological investigation of real pencils of curves, which is interesting both in itself and as a non-topological origin of restrictions on schemes of curves.

§6. Complex topological characteristics of surfaces

The transference of the theory of complex topological characteristics of curves to the case of varieties of higher dimensions is just beginning, and it is too early for a survey. I consider only some definitions and facts concerning the two-dimensional case.

There are three types of non-singular real algebraic surfaces: I abs (I absolute), I rel (I relative) and II. A surface A is of type I abs if its real part $\mathbf{R}A$ realizes $0 \in H_2(\mathbf{C}A;\ \mathbf{Z}_2)$, it is of type I rel if $\mathbf{R}A$ and the plane sections of $\mathbf{C}A$ realize the same element of $H_2(\mathbf{C}A;\ \mathbf{Z}_2)$, and it is of type II in the other cases.

For surfaces of types I abs and I rel the author [42] defined structures analogous to complex orientations of dividing curves. It is convenient to formulate the definition in the following analytical situation, which generalizes the cases of surfaces of types I abs and I rel. Let $\mathcal{X}$ be a non-singular complex surface with $H_1(\mathcal{X};\ \mathbf{Z}_2) = 0$, let $c\colon \mathcal{X} \to \mathcal{X}$ be an antiholomorphic involution, and let $\mathcal{Y} \subset \mathcal{X}$ be a non-singular curve (possibly empty), invariant under c. We put $X = \operatorname{fix}(c)$ and $Y = X \cap \mathcal{Y}$. Let X and $\mathcal{Y}$ realize the same element of $H_2(\mathcal{X};\mathbf{Z}_2)$ (if $\mathcal{Y} = \varnothing$ this means that X realizes zero). Then $X \setminus Y$ has two distinguished opposite orientations and a distinguished spin-structure, which are defined by the following properties and are said to be *complex*.

Let a_0 and a_1 be points of $X \setminus Y$. The complex orientation of $X \setminus Y$ and the natural orientation of $\mathcal{X}$ determine orientations of the fibres D_0, D_1 of a tubular neighbourhood of X in $\mathcal{X}$ lying over a_0 and a_1. Let $b_i \in \partial D_i$, let u_i be a path connecting b_i with $c(b_i)$ in ∂D_i and compatible with the orientation of D_i, and let $s\colon I \to \mathcal{X} \setminus (X \cup \mathcal{Y})$ be a path connecting b_0 with b_1. Then the loop $su_1(c \circ s)^{-1}u_0^{-1}$ realizes $0 \in H_1(\mathcal{X} \setminus (X \cup \mathcal{Y});\ \mathbf{Z}_2)$.

Here is another way to define these complex orientations. It is easy to show that there is a unique two-fold covering of $\mathcal{X} \setminus \mathcal{Y}$, branched over $X \setminus Y$, and that the composition of this covering and the natural projection $\mathcal{X} \setminus \mathcal{Y} \to (\mathcal{X} \setminus \mathcal{Y})/c$ is a 4-fold cyclic covering, branched over the image of $X \setminus Y$. The group of automorphisms of this 4-fold branched covering has two generators, which turn the fibres of the tubular neighbourhood of $X \setminus Y$ in the covering space through an angle $\pi/2$ in two opposite directions. Thus the generators determine two opposite orientations of the normal bundle of $X \setminus Y$. Together with the natural orientation of the ambient variety, these orientations determine the complex orientations of $X \setminus Y$.

Let $\mathcal{Y}'$ be another curve invariant under $c\colon \mathcal{X} \to \mathcal{X}$ and realizing the same element of $H_2(\mathcal{X};\mathbf{Z}_2)$ and let $Y' = \mathcal{Y}' \cap X$. Then $Y \cup Y'$ divides X into two parts, and a complex orientation of $X \setminus Y$ and a complex orientation of $X \setminus Y'$ coincide on one of these parts and are opposite on the other. Thus for a real non-singular surface in $\mathbf{R}P^3$ we actually have two opposite complex orientations of $\mathbf{R}A$ in the case of type I abs and two opposite orientations of the inverse image of $\mathbf{R}A$ under the covering $S^3 \to \mathbf{R}P^3$ in the case of type I rel.

The quadratic form $H_1(X \setminus Y;\ \mathbf{Z}_2) \to \mathbf{Z}_2$ corresponding (see [19]) to the distinguished spin-structure of $X \setminus Y$ maps the class realized by a smooth circle $S \subset X \setminus Y$ to 1+ (the linking number in $\mathcal{X}$ of $X \cup \mathcal{Y}$ and the circle obtained from S by a small shift along a vector field obtained from the tangent vector field of S by multiplying by $\sqrt{-1}$) (modulo 2). Otherwise this spin-structure can be described by its values on framed circles. Its value on a circle $S \subset X \setminus Y$ supplied with a field V of vectors tangent to X is equal to the linking number (modulo 2) in $\mathcal{X}$ of $X \cup \mathcal{Y}$ and the circle obtained from S by a small shift along the vector field $\sqrt{-1}\,V$.

The complex orientations and spin-structures of surfaces permit applications similar to those of complex orientations of curves. I state here just one theorem proved by using them, an analogue of Klein's assertion, that was proved by Kharlamov and the author.

(6.1) *Let A_t be a continuous family of real algebraic surfaces of* $\mathbf{R}P^3$. *If A_0 has just one singular point, and this is a non-degenerate double point, the other A_t are non-singular and if A_t with $t < 0$ are of type* I abs, *then the number* $\dim H_*(\mathbf{R}A_t;\ \mathbf{Z}_2)$ *with $t > 0$ is not more than* $\dim H_*(\mathbf{R}A_t;\ \mathbf{Z}_2)$ *with $t < 0$.*

Under suitable conditions the real part $\mathbf{R}A$ of an algebraic surface A has some other additional structures determined by the position of $\mathbf{R}A$ in $\mathbf{C}A$, which have no analogues in the case of curves. If $\mathbf{R}A$ realizes in $\mathbf{C}A$ the homology class dual to $w_2(\mathbf{C}A)$, then $\mathbf{R}A$ has a distinguished spin-structure.(1) Essentially it was introduced by Rokhlin [30] and Guillou and Marin [13]: for an orientable $\mathbf{R}A$ the corresponding quadratic form $H_1(\mathbf{R}A;\ \mathbf{Z}_2) \to \mathbf{Z}_2$ was introduced by Rokhlin [30], and in the general case the corresponding quadratic form $H_1(\mathbf{R}A;\ \mathbf{Z}_2) \to \mathbf{Z}_4$ was introduced by Guillou and Marin [13].

If the first Chern class $c_1(\mathbf{C}A) \in H^2(\mathbf{C}A;\ \mathbf{Z})$ is divisible by n, then there is a distinguished element of $H^1(\mathbf{R}A;\ \mathbf{Z}_n)$. The value of this cohomology class on an oriented circle S, which bounds a compact oriented smooth surface $F \subset \mathbf{C}A$ tangent to $\mathbf{R}A$ along S, is equal (modulo n) to the obstruction to extending the tangent vector field and the normal vector field of S in $\mathbf{R}A$ to a pair of vector fields on F linear independent over $\mathbf{C}$. This class is closely related to the Maslov class of a Lagrangian manifold. It was introduced by Netsvetaev. He observed also that for $n = 2$ this class and the two spin-structures mentioned above are related: when a surface possesses any two of these three additional structures, it possesses the third, and the sum of the quadratic forms $H_1(\mathbf{R}A;\ \mathbf{Z}_2) \to \mathbf{Z}_2$ corresponding to these two spin-structures and the linear form $H_1(\mathbf{R}A;\ \mathbf{Z}_2) \to \mathbf{Z}_2$ corresponding to the Netsvetaev class is equal to zero.

§7. Isotopies

A rigid isotopy of a curve of degree m is an isotopy in the class of (non-singular) curves of degree m (that is, a path in the space of non-singular curves of degree m). This notion is naturally introduced for other analogous classes of real algebraic varieties, for example for surfaces in $\mathbf{R}P^3$.

Complex schemes of curves of degree ⩽4

m	Complex schemes of curves of degree m
1	$\langle J\rangle_I$
2	$\langle 1\rangle_I$, $\langle 0\rangle_{II}$.
3	$\langle J \amalg 1^-\rangle_I$, $\langle J\rangle_{II}$.
4	$\langle 4\rangle_I$, $\langle 3\rangle_{II}$, $\langle 1\langle 1^-\rangle\rangle_I$, $\langle 2\rangle_{II}$, $\langle 1\rangle_{II}$, $\langle 0\rangle_{II}$.

Types of surfaces of degree ⩽3

(here P_r is the projective plane with r handles, S_r is the sphere with r handles, $\amalg$ is the disjoint sum)

m	Topological types of surfaces of degree m
1	P_0
2	S_1, S_0, $\varnothing$.
3	P_3, P_2, $P_0 \amalg S_0$, P_1, P_0.

(1) If $\mathbf{R}A$ is not orientable, then it is not a spin-structure in the usual sense, but a special 4-fold cyclic covering of the space of unit tangent vectors of $\mathbf{R}A$.

The classification of curves of degree ⩽4 and of surfaces of degree ⩽3 up to rigid isotopy was known in the 19th century. Up to rigid isotopy a curve of degree ⩽4 is determined by its real scheme, and a surface of degree ⩽3 by the topological type of its real part.

The classification of curves of degrees 5 and 6 up to rigid isotopy was completed in 1978–1980 in the works of Rokhlin [33], Nikulin [22], and Kharlamov [15]. Rokhlin [33] showed that for curves of degree ⩾5 the class of a curve is not determined up to rigid isotopy by its real scheme: he asked, up to what degree is the class of a curve up to rigid isotopy determined by the complex scheme, and he gave a classification of complex schemes of curves of degrees 5 and 6, showing, in particular, that by (3.13) and (3.19) the complex scheme of a curve of degree ⩽6 is determined by its type and real scheme (see also Marin [21]). The fact that the real scheme and the type actually determine a curve up to rigid isotopy was proved for curves of degree 5 by Kharlamov [15] and for curves of degree 6 by Nikulin [22]. (Nikulin had arrived at this problem independently of Rokhlin's work [33].)

There are 9 rigid isotopy classes of curves of degree 5. Their complex schemes are $\langle J \amalg 3^+ \amalg 3^-\rangle_I$, $\langle J \amalg 5\rangle_{II}$, $\langle J \amalg 1^+ \amalg 3^-\rangle_I$, $\langle J \amalg 4\rangle_{II}$, $\langle J \amalg 3\rangle_{II}$, $\langle J \amalg 1^-\langle 1^-\rangle\rangle_I$, $\langle J \amalg 2\rangle_{II}$, $\langle J \amalg 1\rangle_{II}$, $\langle J\rangle_{II}$.

There are 64 rigid isotopy classes of curves of degree 6. Their real schemes are

(i) $\langle 9 \amalg 1\langle 1\rangle\rangle$, $\langle 5 \amalg 1\langle 5\rangle\rangle$, $\langle 1 \amalg 1\langle 9\rangle\rangle$;
(ii) $\langle 10\rangle$, $\langle 8 \amalg 1\langle 1\rangle\rangle$, $\langle 5 \amalg 1\langle 4\rangle\rangle$, $\langle 4 \amalg 1\langle 5\rangle\rangle$, $\langle 1 \amalg 1\langle 8\rangle\rangle$, $\langle 1\langle 9\rangle\rangle$;
(iii) $\langle \alpha \amalg 1\langle \beta\rangle\rangle$ with $\alpha + \beta \leqslant 8$, $0 \leqslant \alpha \leqslant 7$, $1 \leqslant \beta \leqslant 8$;
(iv) $\langle \alpha\rangle$ with $0 \leqslant \alpha \leqslant 9$;
(v) $\langle 1\langle 1\langle 1\rangle\rangle\rangle$.

Six of these schemes, namely the schemes $\langle 9 \amalg 1\langle 1\rangle\rangle$, $\langle 5 \amalg 1\langle 5\rangle\rangle$, $\langle 1 \amalg 1\langle 9\rangle\rangle$, $\langle 6 \amalg 1\langle 2\rangle\rangle$, $\langle 2 \amalg 1\langle 6\rangle\rangle$, and $\langle 1\langle 1\langle 1\rangle\rangle\rangle$, are of type I. Eight schemes, namely $\langle 9\rangle$, $\langle 4 \amalg 1\langle 4\rangle\rangle$, $\langle 1\langle 8\rangle\rangle$, $\langle 5 \amalg 1\langle 1\rangle\rangle$, $\langle 3 \amalg 1\langle 3\rangle\rangle$, $\langle 1 \amalg 1\langle 5\rangle\rangle$, $\langle 2 \amalg 1\langle 2\rangle\rangle$, and $\langle 1\langle 4\rangle\rangle$, are of indefinite type. The remaining schemes are of type II.

For curves of degree 7 the problem of rigid isotopy classification is not solved. There are examples showing that neither the real scheme and type nor the complex scheme determine a curve of degree 7 up to rigid isotopy. Rokhlin [33] constructed two curves of degree 7 of type I with real scheme $\langle J \amalg 3 \amalg 1\langle 3\rangle\rangle$ differing from one another in complex orientation (their complex schemes are $\langle J \amalg 1^+ \amalg 2^- \amalg 1^-\langle 1^+ \amalg 2^-\rangle\rangle_I$ and $\langle J \amalg 3^- \amalg 1^+\langle 3^+\rangle\rangle_I$). Marin [21] and Fiedler [6] constructed pairs of curves of degree 7 that have a common complex scheme, but are not rigidly isotopic. In particular, Fiedler [6] constructed two curves of degree 7 and type II with the same real scheme $\langle J \amalg 3 \amalg 1\langle 3\rangle\rangle$ that are not rigidly isotopic. The reason why the curves in these examples of Fiedler and Marin are not rigidly isotopic is the following (see [21] and [6]). If the real scheme of a curve of degree m entails the existence of a line intersecting the curve in m real points, that is, if the inequality (3.19) becomes an equality, then the disposition of the ovals of the curve with respect to such a line is preserved by a rigid isotopy.

Flexible isotopy is isotopy in the class of flexible curves. Since the complex scheme is preserved by a flexible isotopy, for degree $\leqslant 6$ flexible isotopy is equivalent to rigid isotopy. Rokhlin [33] made the conjecture that curves of the same degree with the same complex scheme are flexibly isotopic (in [33] flexible isotopy is called equivariant isotopy). This conjecture together with the examples of Fiedler and Marin leads us to suppose that for curves of degree 7 flexible and rigid isotopies are not equivalent. The homotopy type of the space $\mathbf{C}P^2 \setminus (\mathbf{R}P^2 \cup \mathbf{C}A)$ may be an additional invariant under flexible isotopy. It was first investigated by Finashin [9]: for a large class of curves (in particular, for curves of degree $\leqslant 5$) he calculated the fundamental group of this space and constructed a pair of curves of degree 7 with common complex scheme $\langle J \amalg 2 \amalg 1\langle 2\rangle\rangle_{II}$ that have homotopy equivalent spaces $\mathbf{C}P^2 \setminus (\mathbf{R}P^2 \cup \mathbf{C}A)$ but are not rigidly isotopic. Maybe they are flexibly isotopic.

The problem of classification of surfaces of degree 4 up to rigid isotopy turned out to be more subtle than the corresponding problems for curves of degrees 5 and 6. At first a coarser problem was solved. Two non-singular real projective varieties are said to be *coarsely projectively equivalent* if one of them can be made rigidly isotopic to the other by projective transformation. This relation coincides with rigid isotopy if the dimension of the ambient projective space is even, because in this case the group of projective transformations is connected. In the case of odd dimension the coarse projective classification may be coarser than the rigid isotopy classification. However for surfaces of degrees $\leqslant 3$ these classifications were proved to be equivalent. At first surfaces of degree 4 were classified up to coarse projective equivalence [22]. Two surfaces of degree 4 are coarsely projectively equivalent if and only if they have the same type (see §6), and their real parts are homeomorphic and homotopic in $\mathbf{R}P^3$. This result was derived as a corollary of the following more general theorem of Nikulin [22]: *two real* K3 *surfaces embedded in* $\mathbf{R}P^N$ *by a complete linear system are coarsely projectively equivalent if and only if their intersection forms in* $H_2(\mathbf{C}A)$ *with the involution induced by complex conjugation and with the homology class of a hyperplane section are isomorphic.* (The rigid isotopy classification of curves of degree 6 mentioned above was also derived in [22] essentially as a corollary of this theorem.)

Nikulin's theorem (as well as the preceding results of Kharlamov on topological and real isotopy classification of surfaces of degree 4) is based on such fundamental facts of complex algebraic geometry as Torelli's theorem and the epimorphism period mapping theorem for $K3$-surfaces.

There are 134 coarse projective classes of surfaces of degree four. 62 of them consist of surfaces that are not contractible in $\mathbf{R}P^3$. Their topological types are:

(i) $S_{10} \amalg S_0$, $S_6 \amalg 5S_0$, $S_2 \amalg 9S_0$;
(ii) S_{10}, $S_9 \amalg S_0$, $S_6 \amalg 4S_0$, $S_5 \amalg 5S_0$, $S_2 \amalg 8S_0$, $S_1 \amalg 9S_0$;
(iii) $S_\alpha \amalg \beta S_0$ with $\alpha + \beta \leqslant 9$, $1 \leqslant \alpha \leqslant 9$, $0 \leqslant \beta \leqslant 8$;
(iv) $S_1 \amalg S_1$ (both components are not contractible).

Any non-contractible surface of degree 4 homeomorphic to $S_{10} \amalg S_0$, $S_6 \amalg 5S_0$, $S_2 \amalg 9S_0$, $S_7 \amalg 2S_0$, $S_3 \amalg 6S_0$, or $S_1 \amalg S_1$ is of type I abs. Any non-contractible surface of degree 4 homeomorphic to S_9, $S_5 \amalg 4S_0$, $S_1 \amalg 8S_0$, $S_8 \amalg S_0$, $S_4 \amalg 3S_0$, $S_2 \amalg 5S_0$, or $S_3 \amalg 2S_0$ is of type I abs or II (and for each of these topological types both possibilities are actually realized). The other non-contractible surfaces of degree 4 are of type II.

The topological types of surfaces of degree 4 contractible in $\mathbf{R}P^3$ are:
(i) $S_\alpha \amalg \beta S_0$ with $\alpha + \beta \leqslant 9$, $1 \leqslant \alpha \leqslant 9$, $0 \leqslant \beta \leqslant 8$;
(ii) αS_0 with $1 \leqslant \alpha \leqslant 10$;
(iii) $\varnothing$;
(iv) $S_1 \amalg S_1$.

Any contractible surface of degree 4 homeomorphic to $S_1 \amalg S_1$ or to $S_\alpha \amalg \beta S_0$ with $\alpha + \beta = 9$ and $\alpha \equiv 1 \bmod 2$ is of type I abs. Any contractible surface of degree 4 homeomorphic to $S_\alpha \amalg \beta S_0$ with $\alpha + \beta = 9$ and $\alpha \equiv 0 \bmod 2$ or consisting of two spheres enclosing one other is of type I rel. Any contractible surface of degree 4 homeomorphic to $S_6 \amalg S_0$, $S_4 \amalg 3S_0$, $S_2 \amalg 5S_0$, $3S_0$, $S_3 \amalg 2S_0$, or $S_4 \amalg 4S_0$ is of type I abs or II and for each of these topological types both possibilities are actually realized. Any contractible surface of degree 4 homeomorphic to S_7, $S_5 \amalg 2S_0$, $S_3 \amalg 4S_0$, $S_1 \amalg 6S_0$, $S_4 \amalg S_0$, $S_2 \amalg 3S_0$, S_3, or $S_1 \amalg 2S_0$ is of type I rel or II and for each of these topological types both possibilities are actually realized. The other contractible surfaces of degree 4 are of type II.

A surface is said to *amphicheiral* if it is rigidly isotopic to its mirror image. For amphicheiral surfaces and only for them the coarse projective classification coincides with the rigid isotopy classification. All surfaces of degree $\leqslant 3$ are amphicheiral. Kharlamov [16] showed that this is not the case for degree 4. Moreover, he found out which surfaces of degree 4 are not amphicheiral and thus completed the classification of surfaces of degree 4 up to rigid isotopy. A non-contractible surface of degree 4 is not amphicheiral if and only if it is homeomorphic to $S_\alpha \amalg \beta S_0$ with $\beta \geqslant 4$. A contractible surface of degree 4 is not amphicheiral if and only if it is homeomorphic to $S_\alpha \amalg \beta S_0$ with $\alpha \geqslant 3$ and $\beta \geqslant 3$. Thus there are 170 rigid isotopy classes of surfaces of degree 4. This result was first obtained by fairly complicated arguments connected with the moduli space involved. Recently Kharlamov simplified a large part of his proofs. He found simple geometric obstructions to a surface of degree 4 being amphicheiral. They are connected with my recent paper [46] on configurations of points and lines in $\mathbf{R}P^3$.

A set of disjoint lines in $\mathbf{R}P^3$ is called a *non-singular configuration of lines.* A set of points of $\mathbf{R}P^3$ of which any k with $k \leqslant 4$ do not lie in a

$(k-2)$-dimensional projective subspace of $\mathbf{R}P^3$ is called a *non-singular configuration of points*. An isotopy of a non-singular configuration (of lines or points) in the class of non-singular configurations is said to be *rigid*. A non-singular configuration is said to be *amphicheiral* if it is rigidly isotopic to its mirror image. In [46] the non-singular configurations of p lines with $p \leqslant 5$ are classified up to rigid isotopy. For $p = 3$ there are 2 classes, for $p = 4$ there are 3 classes and for $p = 5$ there are 7 classes. For $p \leqslant 5$ any two non-singular configurations of p lines that are not rigidly isotopic differ in the linking coefficients of the lines that occur in them. However, there are two non-singular configurations of 10 lines that cannot be distinguished by linking coefficients but are not rigidly isotopic.* In [46] I also proved that any non-singular configuration of p lines with $p \equiv 3 \bmod 4$ is not amphicheiral, that for any $p \not\equiv 3 \bmod 4$ there exists an amphicheiral non-singular configuration of p lines, and that any non-singular configuration of q points with $q \equiv 6 \bmod 8$ or with $q \equiv 3 \bmod 4$ and $q \geqslant 7$ is not amphicheiral.

The connection between these results and the rigid isotopy classification of surfaces of degree 4 is demonstrated by the following example. By Harnack's inequality there is no plane intersecting four spherical components of a non-contractible surface of degree 4. Since any non-singular configuration of 6 points is not amphicheiral, any non-contractible surface of degree 4 with 6 spherical components is not amphicheiral.

In conclusion I state an old result on rigid isotopy, which for a long time was not known to experts in the topology of real algebraic manifolds. In 1968 Nuij [24] proved that any two hypersurfaces of degree m in $\mathbf{R}P^n$ containing $[m/2]$ spheres totally ordered by inclusion are rigidly isotopic. Recently Dubrovin [5] obtained this result for the case of plane curves by a different method.

§8. Constructions

In classical papers on the topology of real algebraic curves constructions were carried out in the following manner. Firstly, a pair of non-singular curves transversal to each other was constructed, and then the union of the curves was slightly perturbed, to remove the singularities. For the construction of two curves of degree 6, Gudkov had to leave this framework and perturb not a reducible curve but the image of a non-singular curve under a quadratic transformation. However, as before, all the curves perturbed had only non-degenerate double singularities. There were two obstacles to the appearance of complicated singularities in the constructions: firstly, not very complicated singularities do not give anything new as compared with non-degenerate double points (the advantage of complicated

*Recently Mazurovskii succeeded in constructing such an example with configurations of 6 lines.

singularities appears only when we pass to non-degenerate 5-fold points and points of contact of three branches); secondly, a special technique was necessary for perturbing curves with complicated singularities.

In 1980 I proposed a construction for perturbing a curve with a semiquasihomogeneous singularity. It substitutes a curve fragment prepared beforehand for a small neighbourhood of the singularity. For some singularities, in particular for points of quadratic contact of three non-singular branches, I obtained a complete topological classification of their smoothings (that is, of the curve fragments that appear in place of the singularity after the perturbation), see [43]. For non-degenerate 5-fold points and points of quadratic contact of 4 non-singular branches an ample supply of smoothings was made (see [44]). Chislenko [3] continued this work and has constructed many smoothings for points of quadratic contact of 5 non-singular branches. Shustin [38] completed the topological classification of smoothings of non-degenerate 5-fold points and obtained new results on the smoothings of points of quadratic contact of 4 non-singular branches.

(8.1) *Any point of quadratic contact of* 3 *non-singular real branches can be smoothed so that in its place there appears one of the* 31 *fragments shown in Fig.* 4. *Any smoothing of such a point leads to the appearance of one of these fragments.*

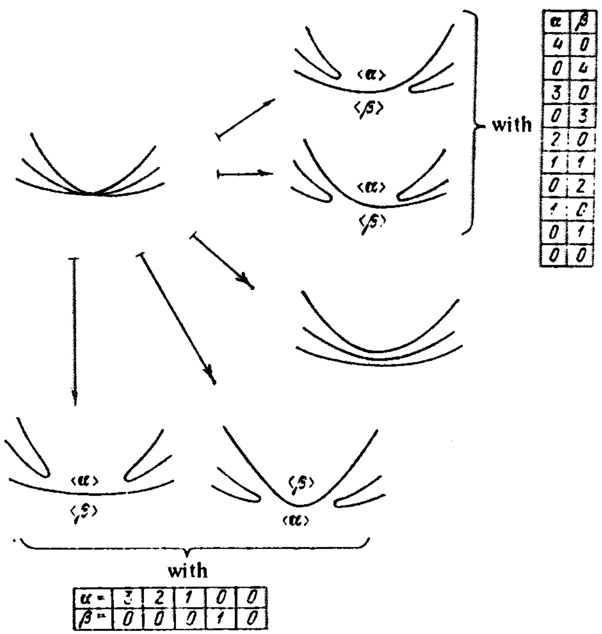

Fig. 4.

(8.2) *Any non-degenerate 5-fold point with 5 real branches can be smoothed so that in its place there appears one of the fragments shown in Fig. 5. Any smoothing of such a point leads to the appearance of one of these fragments.*

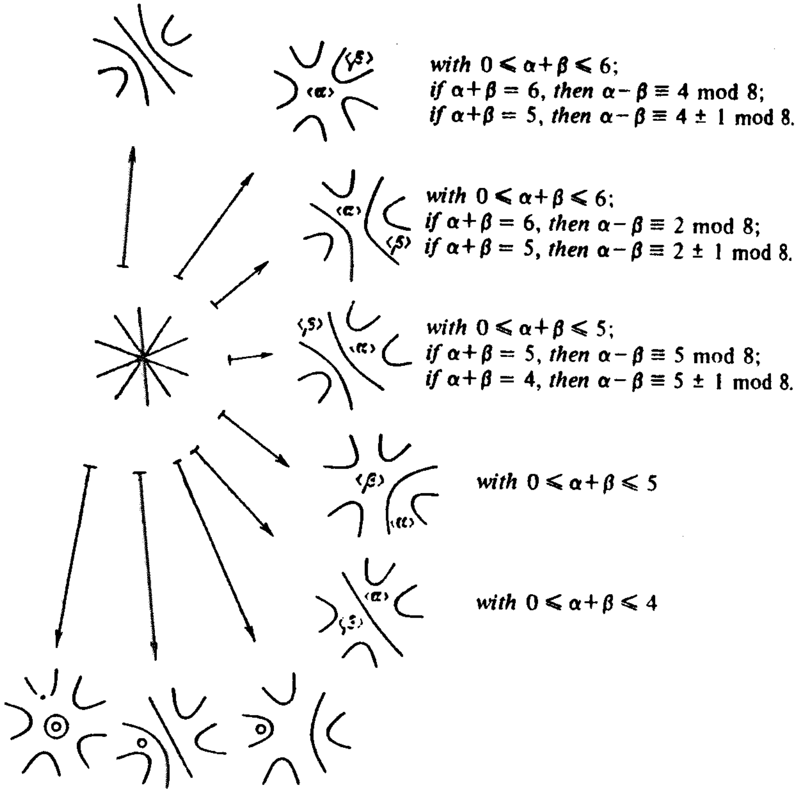

Fig. 5.

The new method of construction, which is a combination of the construction above and some traditional construction techniques with extensive use of Cremona transformations, proved to be useful in resolving problems inaccessible by the old method. Let me list the main results obtained by it. First of all, I succeeded in completing the classification of real schemes of curves of degree 7.

(8.3) *There exist curves of degree 7 with the following real schemes:*
(i) $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ *with* $\alpha+\beta \leqslant 14$, $0 \leqslant \alpha \leqslant 13$, $1 \leqslant \beta \leqslant 13$;
(ii) $\langle J \amalg \alpha\rangle$ *with* $0 \leqslant \alpha \leqslant 15$;
(iii) $\langle J \amalg 1\langle 1\langle 1\rangle\rangle\rangle$.
Any curve of degree 7 has one of these 121 real schemes.

Up to 1980 it remained unknown whether there exist curves of degree 7 with the schemes $\langle J \amalg 1\langle 14\rangle\rangle$, $\langle J \amalg 10 \amalg 1\langle 4\rangle\rangle$, and $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ with $13 \leqslant \alpha+\beta \leqslant 14$, $3 \leqslant \alpha$, $6 \leqslant \beta$. The non-realizability of the scheme $\langle J \amalg 1\langle 14\rangle\rangle$ is the content of Theorem (4.10) above. The schemes $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ with $6 \leqslant \alpha+\beta \leqslant 14$, $1 \leqslant \alpha$, $2 \leqslant \beta$ are realized as follows (see [43]). We first construct 4 curves of degree 7 having two singular points, at each of which three non-singular branches touch. They are shown in Fig. 6. Then we perturb the curves by the construction above (see (8.1)). The scheme $\langle J \amalg 4 \amalg 1\langle 10\rangle\rangle$ not only was not, but as Zvonilov and Fiedler observed, cannot be realized by the previous method.

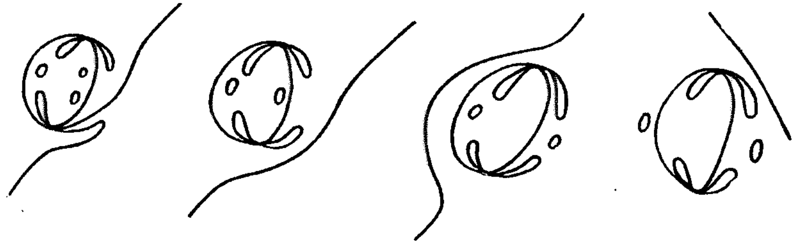

Fig. 6.

I also constructed counterexamples to one of the inequalities constituting the Ragsdale conjecture (see §3) and M-curves of degree 8 realizing 42 new real schemes (the old method gave 10 schemes) (see [44]). Recently Shustin [51] realized 6 new real schemes by M-curves of degree 8*. In the following table all real schemes of M-curves of degree 8 realized up to the end of 1984 are listed (see p.78).

These results led me to make the following conjecture [40]:

(8.4) (*Conjecture*). *Let* $\langle\alpha \amalg 1\langle\beta \amalg 1\langle\gamma\rangle\rangle\rangle$ *be the real scheme of an* M*-curve of degree* 8 *with* $\gamma \neq 0$. *Then* α *and* γ *are odd.*

Moreover, by the new method about 500 real schemes were realized by M-curves of degree 10 (see Chislenko [3]) (the old method gave 38 schemes), and 327 real schemes by $(M-2)$-curves of degree 8 (see Polotovskii [28]). Shustin's curves mentioned in §5 were also constructed by the new method.

We may look forward to the construction of surfaces by the new method. Now only the constructions from my note [39] can be added to the constructions of surfaces reviewed by Kharlamov [14]. In [39] surfaces of high degrees refuting a conjecture on the maximal number of components of a surface of given degree are constructed. All but one of the isotopy types of non-singular surfaces of degree 4 were realized by a method more elementary than the previous ones (see [14]).

*Recently Shustin has realized a new real scheme (namely $\langle 4 \amalg 3\langle 5\rangle\rangle$) by an M-curve of degree 8.

$p=19,\ n=3$	$p=15,\ n=7$	$p=11,\ n=11$	$p=7,\ n=15$	$p=3,\ n=19$
⟨18 ⨿ 1⟨3⟩⟩ ⟨17 ⨿ 1⟨1⟩ ⨿ 1⟨2⟩⟩	⟨14 ⨿ 1⟨7⟩⟩ ⟨13 ⨿ 1⟨1⟩ ⨿ 1⟨6⟩⟩ ⟨13 ⨿ 1⟨2⟩ ⨿ 1⟨5⟩⟩ ⟨13 ⨿ 1⟨3⟩ ⨿ 1⟨4⟩⟩	⟨10 ⨿ 1⟨11⟩⟩ ⟨9 ⨿ 1⟨1⟩ ⨿ 1⟨10⟩⟩ ⟨9 ⨿ 1⟨2⟩ ⨿ 1⟨9⟩⟩ ⟨9 ⨿ 1⟨3⟩ ⨿ 1⟨8⟩⟩ ⟨9 ⨿ 1⟨4⟩ ⨿ 1⟨7⟩⟩ ⟨9 ⨿ 1⟨5⟩ ⨿ 1⟨6⟩⟩	⟨6 ⨿ 1⟨15⟩⟩ ⟨5 ⨿ 1⟨1⟩ ⨿ 1⟨14⟩⟩ ⟨5 ⨿ 1⟨2⟩ ⨿ 1⟨13⟩⟩ ⟨5 ⨿ 1⟨3⟩ ⨿ 1⟨12⟩⟩ ⟨5 ⨿ 1⟨4⟩ ⨿ 1⟨11⟩⟩ ⟨5 ⨿ 1⟨5⟩ ⨿ 1⟨10⟩⟩ ⟨5 ⨿ 1⟨6⟩ ⨿ 1⟨9⟩⟩ ⟨5 ⨿ 1⟨7⟩ ⨿ 1⟨8⟩⟩	⟨2 ⨿ 1⟨19⟩⟩ ? ⟨1 ⨿ 1⟨2⟩ ⨿ 1⟨17⟩⟩ ? ⟨1 ⨿ 1⟨5⟩ ⨿ 1⟨14⟩⟩ ? ⟨1 ⨿ 1⟨8⟩ ⨿ 1⟨11⟩⟩ ?
⟨16 ⨿ 3⟨1⟩⟩	⟨12 ⨿ 2⟨1⟩ ⨿ 1⟨5⟩⟩ ⟨12 ⨿ 1⟨1⟩ ⨿ 2⟨3⟩⟩	⟨8 ⨿ 2⟨1⟩ ⨿ 1⟨9⟩⟩ ⟨8 ⨿ 1⟨1⟩ ⨿ 1⟨3⟩ ⨿ 1⟨7⟩⟩ ⟨8 ⨿ 1⟨1⟩ ⨿ 2⟨5⟩⟩ ⟨8 ⨿ 2⟨3⟩ ⨿ 1⟨5⟩⟩	⟨4 ⨿ 2⟨1⟩ ⨿ 1⟨13⟩⟩ ⟨4 ⨿ 1⟨1⟩ ⨿ 1⟨3⟩ ⨿ 1⟨11⟩⟩ ⟨4 ⨿ 1⟨1⟩ ⨿ 1⟨5⟩ ⨿ 1⟨9⟩⟩ ⟨4 ⨿ 1⟨1⟩ ⨿ 2⟨7⟩⟩ ⟨4 ⨿ 1⟨3⟩ ⨿ 1⟨5⟩ ⨿ 1⟨7⟩⟩	⟨2⟨1⟩ ⨿ 1⟨17⟩⟩ ⟨1⟨1⟩ ⨿ 1⟨7⟩ ⨿ 1⟨11⟩⟩ ⟨1⟨5⟩ ⨿ 2⟨7⟩⟩
⟨1 ⨿ 1⟨2 ⨿ 1⟨17⟩⟩⟩ ? ⟨9 ⨿ 1⟨2 ⨿ 1⟨9⟩⟩⟩ ⟨11 ⨿ 1⟨2 ⨿ 1⟨7⟩⟩⟩ ? ⟨17 ⨿ 1⟨2 ⨿ 1⟨1⟩⟩⟩	⟨1 ⨿ 1⟨6 ⨿ 1⟨13⟩⟩⟩ ? ⟨5 ⨿ 1⟨6 ⨿ 1⟨9⟩⟩⟩ ⟨7 ⨿ 1⟨6 ⨿ 1⟨7⟩⟩⟩ ⟨9 ⨿ 1⟨6 ⨿ 1⟨5⟩⟩⟩ ⟨11 ⨿ 1⟨6 ⨿ 1⟨3⟩⟩⟩ ⟨13 ⨿ 1⟨6 ⨿ 1⟨1⟩⟩⟩	⟨1 ⨿ 1⟨10 ⨿ 1⟨9⟩⟩⟩ ⟨3 ⨿ 1⟨10 ⨿ 1⟨7⟩⟩⟩ ⟨5 ⨿ 1⟨10 ⨿ 1⟨5⟩⟩⟩ ⟨7 ⨿ 1⟨10 ⨿ 1⟨3⟩⟩⟩ ⟨9 ⨿ 1⟨10 ⨿ 1⟨1⟩⟩⟩	⟨1 ⨿ 1⟨14 ⨿ 1⟨5⟩⟩⟩ ⟨3 ⨿ 1⟨14 ⨿ 1⟨3⟩⟩⟩ ⟨5 ⨿ 1⟨14 ⨿ 1⟨1⟩⟩⟩	⟨1 ⨿ 1⟨18 ⨿ 1⟨1⟩⟩⟩

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