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CONSTRUCTION OF MULTICOMPONENT REAL ALGEBRAIC SURFACES

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1. In 1973 V. I. Arnol'd [1] conjectured that the number of components of the complement of a real algebraic hypersurface of degree m in $\mathbf{R}P^q$ does not exceed $1 + C^q_{m+q-2}$. This conjecture is known to be true for $q = 1$ and 2. For nonsingular surfaces of degree m in $\mathbf{R}P^3$ this means that the number of components of the surface does not exceed $(m^3 - m)/6$ for m even and $(m^3 - m + 6)/6$ for m odd. For $m \leqslant 4$ these estimates are known to be correct, and they are sharp for $m = 1, 2$ and 4.

Arnol'd's conjecture is connected with the incorrect (according to Arnol'd [1]) theorem of Courant-Herman on the zeros of linear combinations of eigenfunctions of a Laplace operator: the validity of Arnol'd's conjecture would follow from the validity of the Courant-Herman theorem for the sphere S^q with its standard metric; see [1]. The main result of this paper is Theorem 1, which shows that Arnol'd's conjecture is false for $q = 3$ and any even $m \geqslant 6$, so that the Courant-Herman theorem is false for the Laplace operator on the sphere S^3 with its standard metric.

THEOREM 1. *For any even m there exists in $\mathbf{R}P^3$ a nonsingular real algebraic surface of degree m with $(m^3 - 2m^2 + 4)/4$ components; it is homeomorphic to*

$$(3m^3-2m^2-8m)/16\, S_0 \amalg (m^3-6m^2+8m)/16\, S_1 \amalg S_{(m^3-8m^2+16)/16}.$$

Here S_p is the sphere with p handles, and $\amalg$ is the disjoint sum.

The surfaces whose existence is asserted by Theorem 1 are not M-surfaces. For $m \equiv 2 \bmod 4$ the following theorem gives M-surfaces of degree m, and for $m \geqslant 10$ they have a large number of components.

THEOREM 2. *For any $m \equiv 2 \bmod 4$, in $\mathbf{R}P^3$ there exists a nonsingular real algebraic surface of degree m with $(7m^3 - 24m^2 + 32m)/24$ components; it is homeomorphic to*

$$(11m^3-30m^2+28m-24)/48\, S_0 \amalg (m^3-6m^2+12m-8)/16\, S_1 \amalg S_{(7m^3-30m^2+44m+24)/48}.$$

These theorems are proved by using new forms of the method of small variation.

Usually, a real algebraic variety with prescribed properties is constructed by means of a small variation of the union of several algebraic varieties; as a rule one is restricted to the case where these varieties are transversal. All the isotopy types of nonsingular real algebraic plane curves in the literature have been obtained in this way. This method has also been applied to the construction of surfaces (see [13]), but here its possibilities are more modest. For

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example, surfaces of degree 4 in $\mathbf{R}P^3$ that are gotten in this way have no more than three components, whereas in $\mathbf{R}P^3$ there are surfaces of degree 4 with 10 components.

In this paper we present an extension of the method of small variation, consisting mainly in replacing the initial union of varieties by an arbitrary real divisor. Variation of divisors which are doubled nonsingular varieties is an especially simple method of constructing surfaces of even degree. The proofs of Theorems 1 and 2 are also based on this method. Its possibilities are also illustrated by the following theorem.

THEOREM 3. *By small variations of doubled nonsingular quadrics in* $\mathbf{R}P^3$ *it is possible to realize all isotopy types of nonsingular real algebraic surfaces of degree* 4 *in* $\mathbf{R}P^3$, *except for the isotopy type of a surface homeomorphic to* $S_1 \amalg 9S_0$.

The proofs of the realizability of these types that exist in the literature are less elementary. The isotopy type of $S_1 \amalg 9S_0$ was realized by Utkin [11] by means of the method of Rohn. The isotopy classification of surfaces of degree 4 in $\mathbf{R}P^3$ was completed by Harlamov [4], [5].

2. Variations of a divisor. Let B be a real algebraic variety, D a locally principal effective real divisor on it, and $\gamma\colon \mathbf{R} \longrightarrow \mathscr{L}(D)$ a smooth mapping of the line $\mathbf{R}$ into the space $\mathscr{L}(D)$ of real rational functions on B that is associated with D (for the definition see, for example, [10]). A family of locally principal effective real divisors $D_t = D + (1 + \gamma(t))$ with $t \in \mathbf{R}$ is called a *variation* of the divisor D if $\gamma(0) = 0$. If $\gamma(t) = t\varphi$, where $\varphi \in \mathscr{L}(D)$, then the variation is called *linear*. If for some $\epsilon > 0$ the topological type of the pair (B, D_t) does not depend on t for $t \in (0, \epsilon]$, then the family of divisors D_t with $t \in [0, \epsilon]$ is called a *small variation* of the divisor D.

3. Variations of a doubled variety. Let A and L be closed real algebraic subvarieties of codimension 1 of a complete real algebraic variety B such that (i) the divisor $2A$ is equivalent to L, (ii) the variety A is nonsingular, (iii) the set of singular points of the variety L does not intersect A, and (iv) L is transversal to A (for the terminology see [10]). We put $D = 2A$ and $C = A \cap L$. Let $\varphi \in \mathscr{L}(D)$ be a function such that $D + (\varphi) = L$.

The variety A is separated by the variety C into two parts A_+ and A_- cut out on A by the closures of the sets $\{x \in B \mid \varphi(x) \geqslant 0\}$ and $\{x \in B \mid \varphi(x) \leqslant 0\}$. It is not hard to prove that there is an $\epsilon > 0$ such that the variation $D_t = D + (1 + t\varphi)$ with $t \in [0, \epsilon]$ is small. Then for $t \in (0, t]$ the divisor D_t is simple and represents a nonsingular real algebraic subvariety of B which is isotopic to the boundary of a regular neighborhood of the set A in B, and its complexification $\mathbf{C}D_t$ is a branched double cover of the complexification $\mathbf{C}A$ of A, branched over $\mathbf{C}C$ (the projection of the branched cover $\mathbf{C}D_t \longrightarrow \mathbf{C}A$ can be chosen to be real analytic, but it may happen that there is no holomorphic branched double cover $\mathbf{C}D_t \longrightarrow \mathbf{C}A$).

4. Sketch of the proof of Theorem 1. To construct the required surface it suffices to apply the construction of the preceding section, taking A to be any real algebraic surface of degree $m/2$ in $\mathbf{R}P^3$ that intersects $\mathbf{R}P^2$ along an M-curve with two bases of rank 1, and to take C to be an M-curve, constructed by the method of Brusotti [2], which is a regular complete intersection of A with some surface of degree m.

5. Sketch of the proof of Theorem 2. Let A be a nonsingular real algebraic surface of degree $2k + 1$ in $\mathbf{R}P^3$ that intersects $\mathbf{R}P^2$ transversally along a curve E. We assume that the nonorientable component A_0 of A is homeomorphic to a projective plane with p handles. Suppose E is an M-curve, that it has two bases of rank 1, that it lies in A_0, and that all its ovals bound nonintersecting domains in A_0 that are homeomorphic to the disc. By a small variation of the divisor $(4k + 2)E$ or by a construction analogous to Brusotti's [2], one can then obtain a nonsingular curve C in A whose components bound domains in A that are homeomorphic to the disc and are arranged in A in the following way: there are $2k^2 - k$ nests of depth $4k + 2$ lying outside one another and outside the other nests, $2k$ nests of depth $2k + 1$ lying outside one another and outside the other nests, $4k^3 + 2k^2 - k$ empty ovals lying outside the nests, and $12k^3 + 14k^2 + 3k$ empty ovals surrounded by a single oval within which there are no nests and which is not itself surrounded by the other ovals.

Applying the construction of §3 to the pair (A, C), one can obtain a surface of degree $m = 4k + 2$ which is homeomorphic to

$$2(A \setminus A_0) \amalg (12k^3 + 16k^2 + 2k)S_0 \amalg 4k^3 S_1 \amalg S_{2p+8k^3+8k^2+2}.$$

Taking A to be the M-surface A_{2k+1} constructed in [13], we get the surface whose existence is asserted by Theorem 2.

6. Construction of quartics in $\mathbf{R}P^3$ by variations of doubled quadrics. To realize the isotopy type of a surface of degree 4 in $\mathbf{R}P^3$ with the construction of §3, we must take A to be a nonsingular quadric in $\mathbf{R}P^3$ and C to be its regular complete intersection with some surface of degree 4 that bounds the subvariety A_- in A and such that the boundary of a regular neighborhood of it in $\mathbf{R}P^3$ belongs to the desired isotopy type

It is impossible to realize the type $S_1 \amalg 9S_0$ in this way: to do that we would have to take C to be a curve with 11 components, but its genus must be 9 and therefore the number of its components is $\leqslant 10$.

Before describing the construction of nonsingular quartics of the remaining 111 isotopy types, we make the following

DEFINITION. A real algebraic curve C' on a real algebraic surface A will be called a *reduction* of a curve $C \subset A$ if there exists a homeomorphism $h\colon A \longrightarrow A$ isotopic to the identity such that $h(C') \subset C$ and each component of the set $C \setminus h(C')$ bounds a domain in A which is homeomorphic to the disc and whose interior does not intersect C. In all the examples I know, evident modifications of the construction of the curve lead to the construction of its reductions of all isotopy types.

I. *Incontractible quartics homeomorphic to* $S_p \amalg qS_0$ *with* $2 \leqslant p \leqslant 10$ *and* $0 \leqslant q \leqslant 1$. If A is a one-sheeted hyperboloid and C is the M-curve of order 8 on it constructed by Hilbert [6], then (for an appropriate choice of the set A_-) one obtains an incontractible (in $\mathbf{R}P^3$) M-surface homeomorphic to $S_{10} \amalg S_0$. Hilbert's paper [7] is devoted to the construction of a homeomorphic quartic using the method of Rohn. The remaining isotopy types of family I are gotten by replacing Hilbert's M-curve by its reductions.

II. *Incontractible quartics homeomorphic to* $S_2 \amalg qS_0$ *with* $0 \leqslant q \leqslant 9$. If A is a one-sheeted hyperboloid and C consists of 10 ovals one of which bounds a domain in A which is homeomorphic to the disc and which contains the remaining ovals arranged within one another, then one obtains an M-surface homeomorphic to $S_2 \amalg 9S_0$. The existence of quartics of this type was proved by Harlamov [4] by a significantly less elementary method.

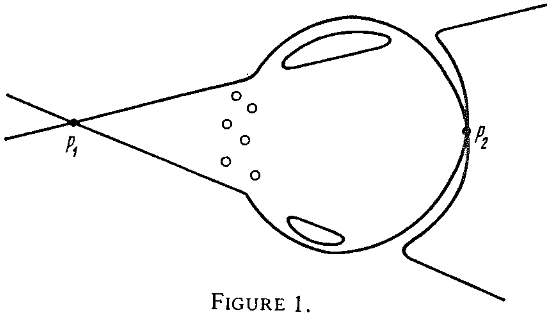

FIGURE 1.

Since I have not found the construction of such a curve in the literature, I present it here. We first construct a curve $\Pi \subset \mathbf{R}P^2$ of degree 6 with two ordinary double points p_1 and p_2, illustrated in Figure 1. This can be done by a small variation of the decomposable curve constructed by Polotovskiĭ which appears as the seventh from the end in his table in [8]. Let $A \subset \mathbf{R}P^3$ be a one-sheeted hyperboloid transversal to $\mathbf{R}P^2$ and passing through p_1 and p_2, and let q be the point of intersection of its two rectilinear generators that pass through p_1 and p_2. Then the proper preimage of the curve Π under the projection $A \longrightarrow \mathbf{R}P^2$ from the point q is a curve of order 8 with 9 components and unique double point q. One can take C to be the result of a small variation of this preimage having 10 components.

III. *Incontractible quartics homeomorphic to* $S_3 \amalg 6S_0$ *and* $S_p \amalg qS_0$ *with* $2 \leqslant p \leqslant 6$ *and* $0 \leqslant q \leqslant 5$. By constructing curves on a hyperboloid in the same way along plane curves of degree 6 resulting from small variations of a curve of Gudkov (see [3]) which decomposes into a line and an M-curve of degree 5, we can obtain surfaces homeomorphic to $S_3 \amalg 6S_6$ and $S_6 \amalg 5S_0$. The second is an M-surface; Utkin's paper [12] is devoted to constructing such surfaces by the method of Rohn. The reductions of these curves give the remaining types of family III.

IV. *Incontractible quartics homeomorphic to* $S_1 \amalg qS_0$ *with* $0 \leqslant q \leqslant 8$. By constructing a curve on a hyperboloid in the same way as the result of a small variation of the decomposable plane curve denoted by [1] (1) (2, 3, 4, 5, 6, 7, 8, 9) in Polotovskiĭ's paper [8], we obtain an incontractible quartic homeomorphic to $S_1 \amalg 8S_0$. The reductions of this curve give the remaining types of family IV.

V. *Contractible quartics homeomorphic to* $S_p \amalg qS_0$ *with* $p > 0, q \geqslant 0$ *and* $p + q \leqslant 9$. Proceeding in a similar way with the curves in Polotovskiĭ's table [8] that are denoted by Γ', one can construct contractible quartics homeomorphic to $S_p \amalg qS_0$ with $p > 0$, $q \geqslant 0$ and $p + q \leqslant 9$. The remaining types of family V are gotten from reductions of the corresponding curves.

VI. *Incontractible quartics homeomorphic to* $S_1 \amalg S_1$ *and* $S_p \amalg qS_0$ *with* $2 \leqslant p \leqslant 7$ *and* $0 \leqslant q \leqslant 2$, *and contractible quartics homeomorphic to* $S_1 \amalg S_1$ *and* $S_p \amalg qS_0$ *with* $p + q \leqslant 9$ *and with* $p \leqslant 2$ *or* $q \leqslant 2$. These and some other types are obtained if C is constructed as the preimage of a nonsingular plane curve of degree 4 under a projection $A \longrightarrow \mathbf{R}P^2$ from a point not lying in A. The isotopy classification of curves on quadrics that are constructed in this way can easily be extracted from Polotovskiĭ's [9] classification of plane curves that decompose into nonsingular transversal curves of degrees 2 and 4.

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